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PURSUIT DIFFERENTIAL-DIFFERENCE GAMES WITH PURE TIME-LAG

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To Professor Valery S. Mel'nik, in Memoriam

Abstract. The analytical approach for solution of pursuit differential-difference games with pure time-lag is considered. For the pursuit local problem with the fixed time the scheme of the method of resolving functions and Pontryagin's first direct method are developed. The integral presentation of game solution based on the time-delay exponential is proposed at first time. The guaranteed times of the game termination are found, and corresponding control laws are constructed. Comparison of the times of approach by the method of resolving functions and Pontryagin's first direct method for the initial problem are made.

1. **Statement of problem.** Conflict-controlled processes is a section of the mathematical control theory studying the manipulation of moving objects operated under conditions of conflict and uncertainty. We consider the game problems of approach, which are central to the theory of conflict-controlled processes. The evolution of an object can be described by systems of difference, ordinary differential, differential-difference, integral, integro-differential equations, systems of equations with distributed parameters, systems of equations with fractional derivatives, impulse influences and their various combinations (see [1, 2]). In this paper, the Method of Resolving Functions is chosen as the main tool for research, widely used to study conflict-controlled processes of various nature (see [3, 4]). The processes with fractional derivatives are studied in (see [5]), the general scheme of the method of resolving functions is given in (see [4]), the applied problem of soft meeting is solved in (see [6]), the nonstationary problems are considered in (see [7, 8, 9]), a variant of the matrix resolving functions are proposed in (see [10]), an approach games problem under the failure of controlling devices are considered in (see [11, 12]), and in (see [13]-[15]) the cases of integral constraints on control are examined.

Denote by $2^{\mathbb{R}^n}$ a set of all subsets of space $\mathbb{R}^n$, by $K(\mathbb{R}^n)$ a set of all nonampty compacts in $\mathbb{R}^n$, and by $coK(\mathbb{R}^n)$ a set of all nonempty convex compacts in $\mathbb{R}^n$. By a set-valued mapping is meant a mapping acting from $\mathbb{R}^n$ to $2^{\mathbb{R}^n}$ and transforming each element $x \in \mathbb{R}^n$ into a set in $\mathbb{R}^n$.

Consider a controlled object whose dynamics is described by the linear differential-difference system with pure time-lag $\tau = \text{const} > 0$ in an Euclidean space $\mathbb{R}^n$

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$$\dot{z}\,(t) = Bz\,(t-\tau) + \phi\,(u\,,\ v)\,, \quad z \in \mathbb{R}^n\,,\ u \in U\,,\ \ v \in V\,, \tag{1}$$

where B is a square constant matrix of order n; $U, V \in K\,(\mathbb{R}^n)$; $\phi\colon U \times V \to \mathbb{R}^n$ is jointly continuous in its variables; u and v are the control parameters of the pursuer and the evader; z is the state vector, involving geometric coordinates, velocities, accelerations of the pursuer and the evader.

The initial condition

$$z\,(t) = z^0\,(t)\,, \quad -\tau \le t \le 0 \tag{2}$$

is absolutely continuous on $[-\tau\,,\ 0]$.

Permissible controls $u(s)$, $v(s)$ are the Lebesgue measurable functions that take values from compacts U and V, respectively. Denote

$$\Omega_U = \{u\,(s):\ u\,(s) \in U,\ s \in [0,\ +\infty)\},$$
$$\Omega_V = \{v\,(s):\ v\,(s) \in V,\ s \in [0,\ +\infty)\}.$$

Function $u\,(\,\cdot\,) \in \Omega_U$ $(v\,(\,\cdot\,) \in \Omega_V)$, chosen by the pursuer (evader) on the basis of knowledge of the initial condition, will be called an open-loop control of the pursuer (evader). The function $v_t\,(\,\cdot\,) = \{v\,(s) : s \in [0,\ t]\,,\ v\,(\,\cdot\,) \in \Omega_V\}$, will be called a prehistory of the evader's control at time $t,\ t \ge 0$.

We define a qusistrategy of the pursuer as a mapping $U\left(t, z^0\,(\,\cdot\,)\,, v_t\,(\,\cdot\,)\right)$. To each moment of time $t \ge 0$, initial condition (2) and arbitrary prehistory $v_t\,(\,\cdot\,)$ of evader's control it assigns a Lebesgue measurable function $u\,(t) = U\left(t, z^0\,(\,\cdot\,)\,, v_t\,(\,\cdot\,)\right)$, $t \ge 0$, taking its values in control domain U.

Strobostrophic strategies (see [16]) is a special case of quasistrategies. To define them rigorously it suffices to substitute $U\left(z^0\,(\,\cdot\,)\,, v\,(t)\right)$ for $U\left(t, z^0\,(\,\cdot\,)\,, v_t\,(\,\cdot\,)\right)$ in the definition of quasistrategy with appropriate modifications. The pursuer's countercontrol is constructed on the basis of information on the initial state $z^0\,(\,\cdot\,)$ (2) of the process (1) and the instantaneous value of the evader's control $u\,(t) = U\left(z^0\,(\,\cdot\,)\,, v\,(t)\right),\ t \ge 0$. The evader's counter-strategy is defined in a similar way.

If in the course of the process (1) the information on a state vector $z\,(t)$ at the current time t is available to the pursuer we shall speak about positional strategies and control (see [17]). Let us identify positional strategies with the functions $u\,(z)\,,\ v\,(z)\,,\ u\,(z) \in U,\ v\,(z) \in V$. The players' controls selected in the form of the functions $u\,(z,\,v)\,,\,v\,(z,\,u)$ will be called the positional countercontrols. In this case the functions $u\,(t,z) = u\,(z, v\,(t)) \in U,\ v\,(t,z) = v\,(z, v\,(t)) \in V$ should be measurable in time.

Let $z\,(t)$ be a solution of equation (1) under the initial condition (2). It is known (see [18]) that if $z^0\,(t) \in C^0\,[-\tau\,,\ 0]\,,\ \phi \in C^0\,[0,\ +\infty)$, then the Cauchy formula for the system (1) implies the representation

$$z\,(t) = z^0\,(0)\,K\,(t) + B\int_{-\tau}^{0} K\,(t-\tau-s)\,b\,(s)\,\mathrm{d}s$$
$$+\int_0^t K\,(t-s)\,\phi\,(u\,(s)\,, v\,(s))\,\mathrm{d}s$$

where $K\,(t)$ is a matrix-valued function which satisfies such properties:

1) $K\,(t) = \Theta, \quad t < 0$, Θ is the null matrix of order n;
2) $K\,(0) = I$, I is the unit matrix of order n;

3) $K(t)$ is continuous in $[0, +\infty)$;
4) $K(t)$ satisfies the equation

$$\dot{K}(t) = BK(t-\tau), \quad t > 0. \tag{3}$$

The method of resolving functions for the pursuit problem (1), (2) was developed in (see [19]-[28]).

However, in practice the solution of the equation by the methods of sequential integration and mathematical induction is very stiff. Therefore, the question arises of finding another way to solve systems, perhaps for a narrower class of problems.

It is known that the solution of the system of the linear homogeneous differential equations with constant coefficients

$$\dot{z}(t) = Az(t), \quad z \in \mathbb{R}^n, \quad t \geq 0, \quad z(0) = z_0$$

can be represented in the form of matrix exponential $z(t) = z_0 exp(At)$,

$$exp(At) = I + A\frac{t}{1!} + A^2\frac{t^2}{2!} + \cdots + A^n\frac{t^n}{n!} + \ldots.$$

Consider the linear system of homogeneous differential equations with constant coefficients and the one constant pure time-lag

$$\dot{z}(t) = Bz(t-\tau), \quad z \in \mathbb{R}^n, \quad t \geq 0 \tag{4}$$

with the initial condition

$$z(t) = z^0(t), \quad -\tau \leq t \leq 0. \tag{5}$$

Here B is the quadratic matrix of order n with the constant elements; $\tau =$ const > 0; $z^0(t)$ is absolutely continuous vector function.

Definition 1.1. (see [29]-[31]). For each $k = 1, 2, \ldots$ the time-delay exponential is defined as follows

$$exp_\tau\{B, t\} = \begin{cases} \Theta, \ -\infty < t < -\tau; \\ I, \ -\tau \leq t < 0; \\ I + B\frac{t}{1!} + B^2\frac{(t-\tau)^2}{2!} + \cdots + B^k\frac{(t-(k-1)\tau)^k}{k!}, \ (k-1)\tau \leq t \leq k\tau. \end{cases} \tag{6}$$

Lemma 1.2. *(see [31]). Let $z(t)$ be a continuous solution to the system (4) under the initial condition (5). Then,*

$$z(t) = exp_\tau\{B, t\}z^0(-\tau) + \int_{-\tau}^{0} exp_\tau\{B, t-\tau-s\}\dot{z}^0(s)\,ds.$$

Consider the nonhomogeneous system with pure time-lag

$$\dot{z}(t) = Bz(t-\tau) + \phi(u, v). \tag{7}$$

The solution to the system (7) under the initial condition (5) consists of the sum of the solution $x_0(t)$ of the homogeneous system under the initial condition (5) and the solution $z_h(t)$ of the nonhomogeneous system under the initial condition $z(t) = \Theta$, $-\tau \leq t \leq 0$.

Lemma 1.3. *(see [31]). Let $z_h(t)$ be a continuous solution to the system (7) under the initial condition $z(t) = \Theta$, $-\tau \le t \le 0$. Then,*

$$z_h(t) = \int_0^t exp_\tau\{B, t - \tau - s\}\phi(u(s), v(s))\, ds$$

for $t \ge 0$.

Lemma 1.4. *(see [31]). Let $z(t)$ be a continuous solution to the system (7) with pure time-lag under the initial condition (5). Then,*

$$z(t) = exp_\tau\{B, t\}z^0(-\tau) + \int_{-\tau}^0 exp_\tau\{B, t - \tau - s\}\dot{z}^0(s)\, ds$$
$$+ \int_0^t exp_\tau\{B, t - \tau - s\}\phi(u(s), v(s))\, ds.$$

2. **Outline of method.** We consider the conflict-controlled processes (7), where $z \in \mathbb{R}^n$; $U, V \in K(\mathbb{R}^n)$.

The terminal set has cylindrical form, i.e.

$$M^* = M_0 + M, \tag{8}$$

where M_0 is a linear subspace in $\mathbb{R}^n$, and $M \in K(L)$, L is the orthogonal complement of M_0 in the $\mathbb{R}^n$.

The goal of the pursuer (u) is in the shortest time to bring a trajectory of the process to a certain closed set M^*; the goal of the evader (v) is to avoid a trajectory of the process from meeting with the terminal set (8) on a whole semi-infinite interval of time or it is impossible to maximally postpone the moment of meeting.

The game is evolving in the closed time interval $[0, T]$, T is a moment when a trajectory of the process brings to a terminal set (8), $T > 0$ such that $z(T) \in M^*$ or $\pi z(T) \in M$, where π is the orthogonal project, $\pi\colon \mathbb{R}^n \to L$.

If the game occurs on the interval $[0,\ T]$, then, according to the method of resolving functions (see [3], [4]), the interval is divided into two intervals and the pursuer chooses a control of the form

$$u(t) = \begin{cases} u_1\left(z^0(\,\cdot\,),\, t,\, v(t)\right), & t \in [0,\ t_*)\,; \\ u_2\left(z^0(\,\cdot\,),\, t,\, v(t)\right), & t \in [t_*\,,\ T]\,, \end{cases}$$

where $t_* = t_*(v(\,\cdot\,))$ is the moment of switching from one law of choosing a counter-control to another, depending on the history of the running of the evader. We will play the role of the pursuer and find sufficient conditions on the parameters of the problem (7), (8), insuring the game termination for certain guaranteed time.

Consider the set-valued mapping

$$\bar{W}(t,\, v) = \pi exp_\tau\{B, t\}\phi(U,\, v)\,,$$
$$\bar{W}(t) = \bigcap_{v \in V} \bar{W}(t,\ v)\,.$$

Condition 1 (Pontryagin's condition). The mapping $\bar{W}(t) \ne \emptyset$ for all $t \ge 0$.

By virtue of the assumptions on the process parameters and of the continuity function $exp_\tau\{B, T\}$, the set-valued mapping $\bar{W}(t,\, v)$ is continuous on the set $[0,\ +\infty) \times V$. Consequently, as it follows from Condition 1, the mapping $\bar{W}(t)$ is upper semi-continuous and therefore Borel measurable function (see [32]). Hence, there exists at least one Borelian selection $g(t)\,,\quad g(t)\in\bar{W}(t)$ (see [33]). Denote by $G=\{g(t):\ g(t)\in\bar{W}(t)\,,\quad t\geq 0\}$ the set of all Borelian selections of the set-valued mapping $\bar{W}(t)\,.$

By virtue of Lemma 3, the Cauchy formula for the system (7) implies the representation

$$\begin{aligned}z(t)=exp_\tau\{B,t\}z^0(-\tau)+\int_{-\tau}^{0}exp_\tau\{B,t-\tau-s\}\dot{z}^0(s)\,\mathrm{d}s\\+\int_0^t exp_\tau\{B,t-\tau-s\}\phi(u(s),v(s))\,\mathrm{d}s.\end{aligned}\tag{9}$$

For fixed $g(\,\cdot\,)\in G$ we put

$$\begin{aligned}&\xi\left(t\,,\ z^0(\,\cdot\,)\,,\ g(\,\cdot\,)\right)=\\&=\pi exp_\tau\{B,t\}z^0(-\tau)+\int_{-\tau}^{0}\pi exp_\tau\{B,t-\tau-s\}\dot{z}^0(s)\mathrm{d}s+\int_0^t g(s)\mathrm{d}s.\end{aligned}$$

Let $X\in(\mathbb{R}^n)$ and $0\in X$. Consider the Minkowski function (see [3], [34])

$$\mu_X(p)=\inf\{\mu\geq 0:\ p\in\mu X\}$$

and the function inverse to it

$$\alpha_X(p)=\sup\{\alpha\geq 0:\ \alpha p\in X\},\ p\in\mathbb{R}^n.$$

Denote

$$\alpha\left(t,s,z^0(\,\cdot\,),v,g(\,\cdot\,)\right)=\alpha_{\bar{W}(t-\tau-s,v)-g(t-\tau-s)}\left(m-\xi\left(t,z^0(\,\cdot\,),g(\,\cdot\,)\right)\right)$$

for all $t\geq s\geq 0$, $v\in V$, $g(\,\cdot\,)\in G$, $m\in M$, $x\in\mathbb{R}^n$.

By virtue of the properties of the superposition of set-valued mappings and functions, it is Borel measurable function in s,v (see [22]).

Finally, denote

$$\alpha\left(t,s,z^0(\,\cdot\,),v,g(\,\cdot\,)\right)=\max_{m\in M}\alpha\left(t,s,z^0(\,\cdot\,),m,v,g(\,\cdot\,)\right)$$

and then we obtain so-called resolving function

$$\begin{aligned}&\alpha\left(t,s,z^0(\,\cdot\,),v,g(\,\cdot\,)\right)=\sup\{\alpha\geq 0\ :\\&\left[\bar{W}(t-\tau-s,v)-g(t-\tau-s)\right]\cap\alpha\left[M-\xi\left(t,z^0(\,\cdot\,),g(\,\cdot\,)\right)\right]\neq\emptyset\}.\end{aligned}\tag{10}$$

It is easy to see that since $0\in\bar{W}(t-\tau-s,v)-g(t-\tau-s)$ for all $t\geq s\geq 0$, $v\in V$, then if $\xi\left(t,z^0(\,\cdot\,),g(\,\cdot\,)\right)\in M$ then function $\alpha\left(t,s,z^0(\,\cdot\,),v,g(\,\cdot\,)\right)=+\infty$. If $\xi\left(t,z^0(\,\cdot\,),g(\,\cdot\,)\right)\notin M$, then the resolving function (10) takes finite values, and it is uniformly bounded jointly in $s\in[0,\ t]$, $v\in V$.

Lemma 2.1. *(see [3]). Let the conflict-controlled process (7), (8) satisfy Condition 1 and for some $t>0$, $z^0(\cdot)\in\mathbb{R}^n$, $g(\cdot)\in G$ $\xi\left(t,z^0(\cdot),g(\cdot)\right)\notin M$. Then the resolving function (10) is Borel measurable function jointly in s,v, and upper semicontinuous in v, $s\in[0,\ t]$, $v\in V$.*

3. **Basic results.** Consider the function

$$
\begin{aligned}
&T = T\left(z^0(\,\cdot\,),\, g(\,\cdot\,)\right)\\
&= \inf\left\{ t \geq 0 \ :\ \int_0^t \inf_{v\in V} \alpha\left(t,\ s,\ z^0(\,\cdot\,)\ ,\ v,\ g(\,\cdot\,)\right) ds \geq 1 \right\}, \quad g(\,\cdot\,) \in G.
\end{aligned} \tag{11}
$$

Notice, that if inequality in curly brackets fails for all $t \geq 0$, then we set $T\left(z^0(\cdot), g(\cdot)\right) = +\infty$. If $\xi\left(t, z^0(\cdot), g(\cdot)\right) \notin M$, then the function $\inf_{v\in V} \alpha\left(t,\ s,\ z^0(\cdot)\ ,\ v,\ g(\cdot)\right)$ is measurable in s, and it is summable on the interval $[0;t]$. If $\xi\left(t, z^0(\cdot), g(\cdot)\right) \in M$, $t > 0$, then $\inf_{v\in V} \alpha\left(t,\ s,\ z^0(\,\cdot\,)\ ,\ v,\ g(\,\cdot\,)\right) = +\infty$ for $s \in [0,\ t]$, and it is natural to set the integral equal to $+\infty$. Then the inequality (11) automatically holds.

The value $T = T\left(z^0(\,\cdot\,),\, g(\,\cdot\,)\right)$ for the initial condition $z^0(\cdot)$ and some selection $g(\cdot) \in G$ is the guaranteed moment of the capture by the pursuer of the evader according to the method of the resolving function.

On the other hand, we set

$$
\begin{aligned}
&P\left(z^0(\,\cdot\,),\, g(\,\cdot\,)\right)\\
&= \min\left\{ t \geq 0 \ :\ \pi exp_\tau\{B,t\} z^0(-\tau) + \int_{-\tau}^0 \pi exp_\tau\{B, t-\tau-s\} \dot{z}^0(s) \mathrm{d}s \right.\\
&\left. \in M - \int_0^t \bar{W}(t-\tau-s) \mathrm{d}s \right\}.
\end{aligned} \tag{12}
$$

Let us show that the quantity (12) is the guaranteed moment of the end of the game of approach according to the First Direct Method of L.S. Pontryagin (see [36],[3]).

Theorem 3.1. *Let the conflict controlled process (7), (8) with the initial condition (5) satisfy Condition 1, the set M be convex, and for the given initial state $z^0(\cdot)$ $P\left(z^0(\,\cdot\,)\,\right) < +\infty$, when $P\left(z^0(\,\cdot\,)\,\right)$ is defined by formula (12).*

Then a trajectory of the process (7), (8) can be brought by the pursuer from $z^0(\,\cdot\,)$ to the terminal set M^* at the moment $P\left(z^0(\,\cdot\,)\,\right)$.

Proof. We denote by $P_0 = P\left(z^0(\,\cdot\,)\right)$. The following inclusion holds

$$
\begin{aligned}
&\pi exp_\tau\{B, P_0\} z^0(-\tau) + \int_{-\tau}^0 \pi exp_\tau\{B, P_0-\tau-s\} \dot{z}^0(s) \mathrm{d}s\\
&\in M - \int_0^{P_0} \bar{W}(P_0-\tau-s) \mathrm{d}s.
\end{aligned}
$$

Hence, there exist a point $m \in M$ and a selection $g(\,\cdot\,) \in G$ such that

$$
\begin{aligned}
&\pi exp_\tau\{B, P_0\} z^0(-\tau) + \int_{-\tau}^0 \pi exp_\tau\{B, P_0-\tau-s\} \dot{z}^0(s) \mathrm{d}s\\
=&m - \int_0^{P_0} g(P_0-\tau-s) \mathrm{d}s.
\end{aligned}
$$

Consider the set-valued mapping

$$\begin{aligned} U\left(s,\ v\right) = \{u \in U\ :\ \pi exp_{\tau}\{B, P_0 - \tau - s\}\phi\left(u,\ v\right) \\ -g\left(P_0 - \tau - s\right) = 0\},\quad s \in [0,\ P_0],\quad v \in V. \end{aligned} \tag{13}$$

It is a Borel measurable function in s, v. The selection $u\left(s,\ v\right) = \operatorname{lex}\min U\left(s,\ v\right)$ is Borel measurable function in its variables as well.

The pursuers control is constructed as follows

$$u\left(s\right) = u\left(s\ ,\ v\left(s\right)\right),\quad s \in [0,\ P_0],$$

where $v\left(s\right),\ v\left(s\right) \in V$, is a Borel measurable function of time.

By virtue (13) and (12), we obtain

$$\begin{aligned} \pi z\left(P_0\right) = \pi exp_{\tau}\{B, P_0\}z^0\left(-\tau\right) + \int_{-\tau}^{0} \pi exp_{\tau}\{B, P_0 - \tau - s\}\dot{z}^0\left(s\right)\mathrm{d}s \\ + \int_{0}^{P_0} \pi exp_{\tau}\{B, P_0 - \tau - s\}\phi\left(u\left(s\right),\ v\left(s\right)\right)\mathrm{d}s = m \in M. \end{aligned}$$

Finally, we have the inclusion $z\left(P_0\right) \in M^*$. The proof is therefore complete. □

Theorem 3.2. *Let the conflict controlled process (7), (8) with the initial condition (5) satisfy Condition 1.*

Then the inclusion

$$\begin{aligned} \pi exp_{\tau}\{B, t\}z^0\left(-\tau\right) + \int_{-\tau}^{0} \pi exp_{\tau}\{B, t - \tau - s\}\dot{z}^0\left(s\right)ds \\ \in M - \int_{0}^{t} \bar{W}\left(t - \tau - s\right)ds,\quad t \geq 0, \end{aligned}$$

holds if and only if a selection $g\left(\ \cdot\ \right) \in G$ *exists, such that* $\xi\left(t,\ z^0\left(\ \cdot\ \right),\ g\left(\ \cdot\ \right)\right) \in M$.

Proof. Let

$$\begin{aligned} \pi exp_{\tau}\{B, t\}z^0\left(-\tau\right) + \int_{-\tau}^{0} \pi exp_{\tau}\{B, t - \tau - s\}\dot{z}^0\left(s\right)\mathrm{d}s \\ \in M - \int_{0}^{t} \bar{W}\left(t - \tau - s\right)\mathrm{d}s. \end{aligned}$$

Then there exist a point $m \in M$ and a selection $g\left(\ \cdot\ \right) \in G$ such that

$$\begin{aligned} \pi exp_{\tau}\{B, t\}z^0\left(-\tau\right) + \int_{-\tau}^{0} \pi exp_{\tau}\{B, t - \tau - s\}\dot{z}^0\left(s\right)\mathrm{d}s \\ = m - \int_{0}^{t} g\left(t - \tau - s\right)\mathrm{d}s \end{aligned}$$

which is equivalent to $\xi\left(t,\ z^0\left(\ \cdot\ \right)\ \ g\left(\ \cdot\ \right)\right) = m \in M$.

Assuming that for some $g\left(\ \cdot\ \right) \in G$ $\xi\left(t,\ z^0\left(\ \cdot\ \right),\ g\left(\ \cdot\ \right)\right) \in M$, and arguing in the reverse order, we obtain the required result.

Thus, the case of equality of the resolving function $\alpha\left(t, s, z^0(\cdot), v, g(\cdot)\right) = +\infty$ corresponds to the first direct method of L.S. Pontryagin. In the future, the resolving function will play a key role in the method of resolving functions. □

Let X be a compact in $\mathbb{R}^n$. We denote by X_1 the set of vectors $x \in X$ the first component of which is smallest, by X_2 the set of vectors $x \in X_1$ the second component of which is smallest, and so on up to X_n. It is clear that set X_n consists of a single point x^* which is called a lexicographic minimum of compact X. Denote $x^* = lexmin X$.

Theorem 3.3. *Let the conflict controlled process (7), (8)) with the initial condition (5) satisfy Condition 1, and let the set M be convex, for the given initial state $z^0(\,\cdot\,)$ and some selection $g^0(\,\cdot\,) \in G$ $T = T\left(z^0(\,\cdot\,),\, g^0(\,\cdot\,)\right) < +\infty$.*

Then a trajectory of the process (7), (8) can be brought by the pursuer from $z^0(\,\cdot\,)$ to the terminal set M^* at the moment T.

Proof. Let $v(\,\cdot\,)$ be an arbitrary measurable function taking values from the control domain V. Moment T is the estimated time for the end of the pursuit game. Let us consider the case when $\xi\left(T,\ z^0(\,\cdot\,),\ g^0(\,\cdot\,)\right) \notin M$. For this we introduce the controlling function

$$h(t) = 1 - \int_0^t \alpha\left(T,\ s,\ z^0(\,\cdot\,),\ v(s),\ g^0(\,\cdot\,)\right) ds, \quad t \geq 0.$$

It is continuous, non-increasing, and $h(0) = 1$.

From the definition of time T, there exists a switching time $t_* = t_*\left(v(\,\cdot\,)\right), \quad 0 < t_* \leq T$, such that $h(t_*) = 0$.

The whole process of pursuit is divided into two time sections: active $[0\,,\ t_*)$ and passive $[t_*\,,\ T]$, where t_* is the moment of switching from one law of choosing the counter-control to another, depending on prehistory of running away. In the first section, the method of resolving functions actually works. When the integral of the resolving function becomes unity at the instant t_*, we switch the pursuit process to Pontryagin's first direct method.

In accordance with the foregoing, we define the following law of choice of the pursuer's control. To do this, we consider the set-valued mapping

$$\begin{aligned} U_1(s,v) = \big\{ u \in U \,:\, \pi exp_\tau\{B, T-\tau-s\}\phi(u,\,v) - g^0(T-\tau-s) \\ \in \alpha\left(T,\ s,\ z^0(\cdot),v(s),g^0(\cdot)\right)\left[M - \xi\left(T, z^0(\cdot), g^0(\cdot)\right)\right]\big\}. \end{aligned} \tag{14}$$

It follows from assumptions concerning the process (7), (8) parameters, with account of properties of the resolving function, that the mapping $U_1(s,\ v)$ is a Borel measurable function on the set $[0,\ T] \times V$. Then selection

$$u_1(s,\ v) = \operatorname{lex\,min} U_1(s,\ v)$$

appears as a jointly Borel measurable function in its variables (see [32]-[35]).

The pursuer's control on the interval $[0,\ t_*)$ is constructed in the following form

$$u(s) = u_1(s,\ v(s))$$

being a superposition of Borel measurable functions and it is also a Borel measurable function (see [32], [33]).

On the passive interval $[t_*,\ T]$ accumulate the resolving function no longer makes sense (see [3], [36]), so here the resolving function is assumed to be equal to zero:

$$\alpha\left(T,\ s,\ z^0(\,\cdot\,),\ v(s),\ g^0(\,\cdot\,)\right) \equiv 0$$

and we will now select the control of the pursuer in accordance with the first direct method of L.S. Pontryagin. To this end, we introduce the mapping

$$U_2(s,\ v) \\ = \left\{u \in U : \pi exp_\tau\{B, T-\tau-s\}\phi(u,\ v) - g^0(T-\tau-s) = 0\right\}, \quad s \in [t_*,\ T],\ v \in V$$

being a Borel measurable function in its variables, and its selection

$$u_2(s,\ v) = \operatorname{lex}\min U_2(s,\ v)$$

is Borel measurable function as well.

On the interval $[t_*,\ T]$ we set the pursuer's control equal to

$$u(s) = u_2(s,\ v(s)). \tag{15}$$

It is measurable function too.

In the case when $\xi\left(T,\ z^0(\,\cdot\,),\ g^0(\,\cdot\,)\right) \in M$, we choose the pursuers control on the interval $[0,\ T]$ in the form (15).

We will now show that if the pursuer follows these rules in the course of the game, a trajectory of process (7), (8) hits the terminal set at the time T under arbitrary admissible controls of the evader.

By virtue of Lemma 3, the Cauchy formula for the system (7) under the initial condition (5) implies the representation

$$\begin{aligned}\pi z(T) &= \pi exp_\tau\{B, T\}z^0(-\tau) + \int_{-\tau}^{0} \pi exp_\tau\{B, T-\tau-s\}\dot{z}^0(s)\mathrm{d}s \\ &+ \int_0^T \pi exp_\tau\{B, T-\tau-s\}\phi(u(s),\ v(s))\mathrm{d}s.\end{aligned} \tag{16}$$

We examine the case when $\xi\left(T,\ z^0(\,\cdot\,),\ g^0(\,\cdot\,)\right) \notin M$.

By adding and subtracting from the right-hand side of equation (16) the value $\int_0^T g^0(T-\tau-s)\,\mathrm{d}s$, one can deduce

$$\begin{aligned}\pi z(T) &\in \xi\left(T,\ z^0(\cdot),\ g^0(\cdot)\right)\left[1 - \int_0^{t_*} \alpha\left(T,\ s,\ z^0(\cdot),\ v(s),\ g^0(\cdot)\right)\mathrm{d}s\right] \\ &+ \int_0^{t_*} \alpha\left(T,\ s,\ z^0(\cdot),\ v(s),\ g^0(\cdot)\right) M\mathrm{d}s.\end{aligned} \tag{17}$$

By virtue $\int_0^{t_*} \alpha\left(T,\ s,\ z^0(\,\cdot\,),\ v(s),\ g^0(\,\cdot\,)\right)\mathrm{d}s = 1$ and the set M is convex then $\pi z(T) \in M$. Then, applying the rule of the pursuer control for the case when $\xi\left(T,\ z^0(\,\cdot\,),\ g^0(\,\cdot\,)\right) \in M$, we obtain the inclusion $\pi z(T) \in M$. The proof is therefore complete. □

Theorem 3.4. *Let the conflict-controlled process (7), (8) with the initial condition (5) satisfy Condition 1.*

Then for any initial state $z^0(\,\cdot\,)$ there exists a selection $g^0(\,\cdot\,) \in G$ such that

$$T\left(z^0(\,\cdot\,),\ g^0(\,\cdot\,)\right) \le P\left(z^0(\,\cdot\,)\right).$$

Proof. Let the game be completed at the moment $P\left(z^0(\,\cdot\,)\right)$. This means that the inclusion holds in relation (12). By virtue of Lemma 3 there exists a selection $g^0(\,\cdot\,) \in G$ such that $\xi\left(T,\ z^0(\,\cdot\,),\ g^0(\,\cdot\,)\right) \in M$. This implies that $\alpha\left(T,\ s,\ z^0(\,\cdot\,),\ v(s),\ g^0(\,\cdot\,)\right) = +\infty$. Then $\int_0^t \alpha\left(T,\ s,\ z^0(\,\cdot\,),\ v(s),\ g^0(\,\cdot\,)\right)\mathrm{d}s = +\infty > 1$. By virtue of Theorem 4 we can end the game of pursuit by the method of resolving functions in the time $T = T\left(z^0(\,\cdot\,)\,,\, g^0(\,\cdot\,)\right)$. In this case $T = P\left(z^0(\,\cdot\,)\right)$. This implies the inequality to be proved. $\square$

4. **Example.** Let the differential-difference game

$$\dot{z}(t) = bz(t-\tau) + u(t) - v(t)\,, \quad z \in \mathbb{R}^n\,, \;\; 0 < b < 1, \;\; \tau > 0,$$

be given, where the initial states

$$z(t) = z^0, \quad -\tau \le t \le 0\,,$$

$\|u\| \le 1$, $\|v\| \le 1 - b$, $\|z^0\| = 1$.

The game is considered complete if $x = y$.

The terminal set $M^* = M = M_0 = \{z \in \mathbb{R}^n : \; z = 0\}$, $L = \mathbb{R}^n$, π is the identity operator.

Consider the set-valued mapping

$$\bar{W}(t,\, v) = \pi exp_\tau\{bI, t\}\phi(U,\, v)$$

and verify the Pontryagin's condition:

$$\bar{W}(t,\, v) = exp_\tau\{bI, t\}(S - v)\,, \quad \bar{W}(t) = \{0\},$$

where S is the unit ball centered at zero in the space L and I is the unit matrix of order n.

The condition $\bar{W}(t) = \{0\}$ uniquely determines the selection $g(t) = 0$.

We say that in the game from the initial state $z^0 \notin M^*$ it is possible to avoid meeting the terminal set if there exists a measurable function $v(t) \in V$, $t \ge 0$, such that $z(t) \notin M^*$ for any $t \ge 0$.

It is shown in (see [24], [37], [38]) that if in this example we put in $v(t) = (b-1)z^0$, $t \in [0, T]$, then $\|z(t)\| \ge 1$, that is $z(t) \notin M^*$ for any $t \ge 0$. Thus, in such a game of two persons, it is possible to avoid meeting with the terminal set with any control of the pursuer, in spite of the fact that the dynamic capabilities of the pursuer are greater. But if the pursuers are several then it is shown that the pursuit game can be completed.

Thus, a scheme of the method of resolving functions for a class of differential-difference pursuit games with pure time-lag for the case of one evader and one pursuer has been developed. Sufficient conditions for the parameters of the process for guaranteed capture are found. A method for finding the fundamental matrix of a system and the method for constructing a resolving function are given. Since the inversion of the resolving function in $+\infty$ corresponds to the first direct method of L.S. Pontryagin, the guaranteed moment of the end of the pursuit game is obtained according to the first direct method of L.S. Pontryagin. Comparison of the approach times by the method of resolving functions and the first direct method of L.S. Pontryagin are made. An example is considered.

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