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The Hjorth's IDB Generator of Distributions: Properties, Characterizations, Regression Modeling and Applications

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ABSTRACT

We introduce a new flexible class of continuous distributions via the Hjorth's IDB model. We provide some mathematical properties of the new family. Characterizations based on two truncated moments, conditional expectation as well as in terms of the hazard function are presented. The maximum likelihood method is used for estimating the model parameters. We assess the performance of the maximum likelihood estimators in terms of biases and mean squared errors by means of the simulation study. A new regression model as well as residual analysis are presented. Finally, the usefulness of the family is illustrated by means of four real data sets. The new model provides consistently better fits than other competitive models for these data sets.

1. INTRODUCTION

Hjorth [1] obtained a three-parameter distribution generalizing Rayleigh, exponential and linear failure rate distributions. Its cumulative distribution function (cdf) and probability density function (pdf) are given by

$$H_{Hj}(x;\alpha,\beta,\theta) = 1-(1+\beta x)^{-\theta/\beta}\exp\left\{-\alpha x^2/2\right\}, x \geq 0, \tag{1}$$

and

$$h_{Hj}(x;\alpha,\beta,\theta) = [\alpha x(1+\beta x)+\theta](1+\beta x)^{-(\theta/\beta+1)}\exp\left\{-\alpha x^2/2\right\}, x > 0,$$

respectively, where $\alpha > 0$ is the scale parameter and $\beta, \theta > 0$ are the shape parameters. Clearly, this distribution is reduced to Rayleigh, exponential and linear failure rate distributions for $\theta = 0$, $\alpha = \beta = 0$ and $\beta = 0$, respectively. Since this distribution has increasing ($\alpha \geq \theta\beta$), decreasing ($\alpha = 0$), constant ($\alpha = \beta = 0$) and bathtub-shaped ($0 < \alpha < \beta\theta$) hazard rate functions (hrfs), it has also been named increasing-decresing-bathtub (IDB) distribution by the author. The quantile function (qf), denoted by $Q(u)$, of the IDB distribution is the solution of the following equation:

$$(1-u)(1+\beta Q(u))^{\theta/\beta} - \exp\left\{-\frac{\alpha}{2}Q(u)^2\right\} = 0,\ \ u \in (0,1). \tag{2}$$

Hence, If U is a uniform random variable on (0, 1) then $Q_{Hj}(U)$ is an IDB random variable, where $Q_{Hj}(\cdot)$ is the solution of the Equation (2). On the other hand, Alzaatreh *et al.* [2] proposed a new technique to construct wider families by using any pdf as a generator. This generator

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called the T-X family of distributions has cdf defined by

$$F(x) = \int_a^{W\left[G\left(x;\xi\right)\right]} r(t)\, dt,$$

where $r(t)$ is the pdf of the random variable $T \in [a,b]$ for $-\infty < a < b < \infty$ and $W[G(x;\xi)]$ is a function of the baseline cdf which satisfies the following conditions: i) $W[G(x;\xi)] \in [a,b]$, ii) $W[G(x;\xi)]$ is the differentiable and monotonically non-decreasing, iii) $\lim_{x\to-\infty} W[G(x;\xi)] = a$ and $\lim_{x\to\infty} W[G(x;\xi)] = b$. The pdf of the T-X family is given by

$$f(x) = \left\{\frac{\partial}{\partial x} W[G(x;\xi)]\right\} r(W[G(x;\xi)]).$$

Based on the above transformer T-X generator, we propose a new wider family of continuous distribution, the Hjorth-G family by replacing $r(t)$ with the Hjorth density function and having cdf given by

$$F(x) = \int_0^{-\log\left[\overline{G}\left(x;\xi\right)\right]} h_{Hj}(t;\alpha,\beta,\theta)\, dt = 1 - \frac{\exp\left\{-\frac{\alpha}{2}\left[\log \overline{G}(x;\xi)\right]^2\right\}}{\left[1-\beta\log\overline{G}(x;\xi)\right]^{\theta/\beta}}, x\in\mathbb{R}, \tag{3}$$

where $\overline{G}(x;\xi) = 1 - G(x;\xi)$ is the baseline survival function depending on a $q\times 1$ vector ξ of unknown parameters, $G(x;\xi)$ is the baseline cdf and α,β and θ are the extra scale and shape parameters which are ensure flexibility to baseline distribution. The pdf corresponding to equation (3) is given by

$$f(x) = \frac{g(x;\xi)\left\{\alpha\left[\beta\log\overline{G}(x;\xi)-1\right]\log\overline{G}(x;\xi)+\theta\right\}}{\overline{G}(x;\xi)\left[1-\beta\log\overline{G}(x;\xi)\right]^{\theta/\beta+1}} \exp\left\{-\frac{\alpha}{2}\left[\log\overline{G}(x;\xi)\right]^2\right\}, x\in\mathbb{R}, \tag{4}$$

Hereafter, the random variable X with pdf (4) is denoted by $X \sim$Hj $-G(\alpha,\beta,\theta,\xi)$. Further, we can omit the dependence on the parameter vector ξ of the parameters and simply write $G(x;\xi) = G(x)$ and $g(x;\xi) = g(x)$. If T is an IDB random variable with cdf (1), then $X = G^{-1}\left(1-\mathrm{e}^{-T}\right)$ is the Hj-G random variable where $G^{-1}(\cdot)$ is the qf of the baseline distribution. Hence, qf of X is the solution of the non-linear equation $G^{-1}\left(1-\mathrm{e}^{-Q_{Hj}(u)}\right)$, $u\in(0,1)$. The hazard rate function (hrf) of X is given by

$$h(x) = \frac{g(x)\left[\alpha\left(\beta\log\overline{G}(x)-1\right)\log\overline{G}(x)+\theta\right]}{\overline{G}(x)\left[1-\beta\log\overline{G}(x)\right]}. \tag{5}$$

The goal of this work is to introduce a new flexible and wider family of the distributions based on T-X family using the IDB model. We are motivated to introduce the Hj $-G$ family because it exhibits increasing, decreasing, constant, upside down, unimodal then bathtub as well as bathtub hazard rates as shown in Figures 1 and 2. The members of the Hj $-G$ family can also be viewed as a suitable model for fitting the bimodal, unimodal, U-shaped and other shaped data. The Hj $-G$ family outperforms several of the well-known lifetime distributions with respect to three real data applications as illustrated in Section 9. The new log- regression model based on the Hj $-$Weibull provides better fits than the log-Topp-Leone odd log-logistic-Weibull [3] and log-Weibull regression models for Stanford heart transplant data set.

The paper is organized as follows: Some sub-families of the new family are introduced in Section 2. In Section 3, the series expansions for cdf and pdf of the new family are presented. In Section 4, some of its mathematical properties are derived. Section 5 deals with some characterizations of the new family. In Section 6, the maximum likelihood method is used to estimate the parameters. A new regression model as well as residual analysis are presented in Section 7. In Section 8, two simulation studies are performed to evaluate the efficiency of the maximum likelihood estimates. In Section 9, we illustrate the importance of the new family by means of three applications to real data sets. The paper is concluded in Section 10.

2. SPECIAL HJ-G DISTRIBUTIONS

The Hj-G family can extend to any baseline distribution due to its shape and scale parameters. So, the pdf (4) will generate more flexible distributions than baseline model. Also, Hj-G family includes some sub-families such as Rayleigh-G, exponential-G and linear failure rate-G families for $\theta = 0$, $\alpha = \beta = 0$ and $\beta = 0$, respectively. We note that the Rayleigh-G and exponential-G families are the special members of the Weibull-G family which was introduced by Corderio *et al.* [4]. Here, we obtain three special models of the Hj-G family. These special models extend some well-known distributions given in the literature.

2.1. The Hj-Normal (Hj-N) Distribution

The normal distribution is very useful model in statistics and related field. Since, it has increasing hrf shape, symmetrical and uni-modal pdf shape, its data modeling area can be limited. To extend the normal distribution, we consider Hj-N distribution as our first example by taking $G(x;\mu,\sigma) = \Phi\left(\frac{x-\mu}{\sigma}\right)$ and $g(x;\mu,\sigma) = \sigma^{-1}\phi\left(\frac{x-\mu}{\sigma}\right)$, the cdf and pdf, respectively, where $x,\mu \in \mathbb{R}$, $\sigma > 0$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively. We denote this distribution with Hj-N $(\alpha,\beta,\theta,\mu,\sigma)$. Some plots of the Hj-N density and hrf for selected parameter values are displayed in Figure 1. W note that the pdf shapes of the Hj-N can be skewed, bi-modal and uni-modal. Also, its hrf shapes are increasing and firstly increasing shape then bathtube shape.

2.2. The Hj-Weibull (Hj-W) Distribution

As our second example, we consider the Weibull distribution, which has monotone hrf and decreasing and uni-modal pdf, with shape parameter $\gamma > 0$ and scale parameter $\lambda > 0$. Its cdf and pdf are given by $G(x;\gamma,\lambda) = 1-\exp\left[-(\lambda x)^{\gamma}\right]$ and $g(x;\gamma,\lambda) = \lambda^{\gamma}x^{(\gamma-1)}\exp\left[-(\lambda x)^{\gamma}\right]$ for $x > 0$, respectively. We denote this distribution with Hj-W $(\alpha,\beta,\theta,\lambda,\gamma)$. Some plots of the Hj-W density and hrf for selected parameter values are displayed in Figure 2. Figure 2 shows that the pdf and hrf shapes of the Hj-W can be very flexible. For example, the new extended Weibull distribution has bi-modal, uni-modal, decreasing, firstly increasing shape then U shaped pdf. Nevertheless, it has both monotone and non-monotone hrf shape such as bathtube shape and firstly increasing shape then bathtube shape.

2.3. The Hj-Uniform (Hj-U) Distribution

As our third example, let the baseline distribution have an uniform distribution in the interval (a,b) Then $G(x;a,b) = (x-a)/(b-a)$ and $g(x;a,b) = 1/(b-a)$ for $a < x < b$, $a,b \in \mathbb{R}$. We denote this distribution with Hj-U $(\alpha,\beta,\theta,a,b)$. Some plots of the Hj-U density and hrf for selected parameter values are displayed in Figure 3. Figure 3 shows that the pdf can be increasing, decreasing, uni-modal, firstly decresing then uni-modal and U-shaped. Its hrf has increasing and bathtube shape. Hence, Hj-U distribution may be suggested for hrf model with bathtube shape.

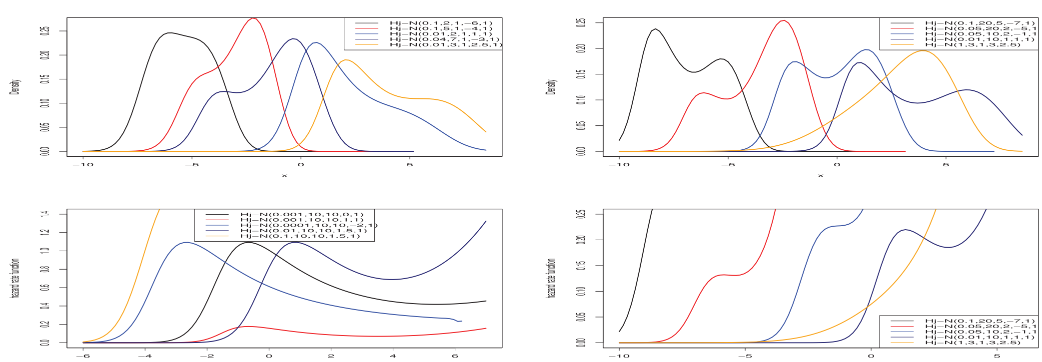

Figure 1 The probability density function (pdf) and hazard rate functions (hrf) of the Hj-Normal (Hj-N) distribution for selected parameter values.

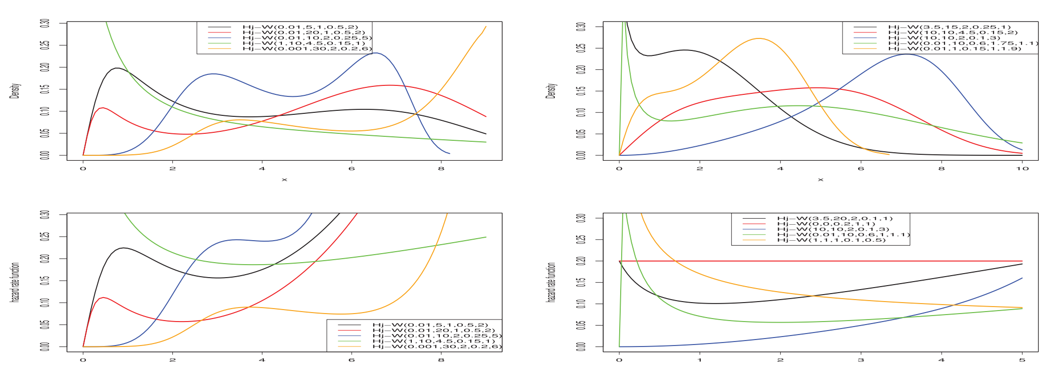

Figure 2 The probability density function (pdf) and hazard rate functions (hrf) of the Hj-Weibull (Hj-W) distribution for selected parameter values.

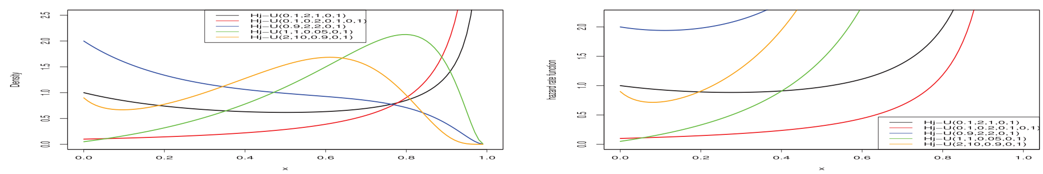

Figure 3 The probability density function (pdf) and hazard rate functions (hrf) of the Hj-Uniform (Hj-U) distribution for selected parameter values.

3. USEFUL EXPANSIONS

In this section, we provide a useful linear representation of the Hj-G cdf function

$$F(x) = 1 - \underbrace{\exp\left(-\frac{\alpha}{2}\left\{-\log\left[1-G(x)\right]\right\}^2\right)}_{A_i}\left\{1-\beta\log\left[1-G(x)\right]\right\}^{-\theta/\beta}.$$

Expanding the quantity A_i in power series, we can write

$$F(x) = 1 - \sum_{i=0}^{\infty}(-1)^{3i}\left\{\log\left[1-G(x)\right]\right\}^{2i}(i!)^{-1}(\alpha/2)^i\underbrace{\left\{1-\beta\log\left[1-G(x)\right]\right\}^{-\theta/\beta}}_{B_i},$$

and expanding the quantity B_i by via Taylor series

$$\tau^{\zeta} = \sum_{j=0}^{\infty}(j!)^{-1}(\tau-1)^j(\zeta)_j,$$

where j is a positive integer and $(\zeta)_j = \zeta(\zeta-1)\ldots(\zeta-j+1)$ is the descending factorial, we get

$$F(x) = 1 - \sum_{i,j=0}^{\infty}(-1)^{3i+j}(i!\,j!)^{-1}(\alpha/2)^i\beta^j(-\theta/\beta)_j\left\{\overbrace{\log\left[1-G(x)\right]}^{C_i}\right\}^{2i+j}.$$

Expanding the quantity C_i by using $\log(1-z)$ expansion

$$\log(1-\tau) = -\sum_{k=0}^{\infty}\tau^{k+1}(h+1)^{-1}, |\tau| < 1,$$

we obtain

$$F(x) = 1 - \sum_{i,j=0}^{\infty}(-1)^{5i+2j}(i!\,j!)^{-1}(\alpha/2)^i\beta^j(-\theta/\beta)_j\left[\sum_{k=0}^{\infty}\frac{G(x)^{k+1}}{(k+1)}\right]^{2i+j}.$$

Using the equation by Gradshteyn and Ryzhik [5], page 17, for a power series raised to a positive integer n

$$\left(\sum_{k=0}^{\infty}a_k u^k\right)^n = \sum_{k=0}^{\infty}c_{n,k}u^k,$$

where the coefficients $c_{n,k}$ (for $k = 1.2, \ldots$) are easily determined from the recurrence relation

$$c_{n,k} = (ka_0)^{-1}\sum_{m=1}^{k}\left[m(n+1)-k\right](a_m)\left(c_{n,k-m}\right),$$

where $C_{n,0} = a_0^n$, the coefficient $c_{n,k}$ can be calculated from $c_{n,0}, \ldots, c_{n,k-1}$ and hence from the quantities $a_0, \ldots, a_i$, we have

$$F(x) = 1 - \sum_{k=0}^{\infty}v_{k+1}\Pi_{k+1}(x), \tag{6}$$

where $\Pi_{k+1}(x) = G(x)^{k+1}$ is the cdf of the exponentiated-G (exp-G) class with power parameter $k+1$ and

$$\nu_{k+1} = \sum_{i,j=0}^{\infty} (-1)^{5i+2j} \beta^j \left(i!\, j!\right)^{-1} (\alpha/2)^i (-\theta/\beta)_j\, c_{2i+j,k}.$$

Upon differentiating (6), we obtain

$$f(x) = \sum_{k=0}^{\infty} \upsilon_{k+1} \pi_{k+1}(x), \tag{7}$$

where $\upsilon_{k+1} = -\nu_{k+1}$ and $\pi_{k+1}(x) = (k+1)\, g(x)\, G(x)^k$ denotes the exp-G class density with power parameter $k+1$.

4. SOME PROPERTIES OF THE HJ-G FAMILY

4.1. General Properties

The r^{th} ordinary moment of X is given by $\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\, dx$. Then, we obtain $\mu_r' = \sum_{k=0}^{\infty} \upsilon_{k+1} E(Y_{k+1}^r)$, where $Y_\gamma^r = \gamma \int_{-\infty}^{\infty} x^r\, G(x)^{\gamma-1} g(x)\, dx$, which can be computed numerically in terms of the baseline quantile function (qf) $Q_G(u;\xi) = G^{-1}(x;\xi)$ as $E(Y_\alpha^n) = \alpha \int_0^1 u^{\alpha-1} Q_G(u;\xi)^n\, du$. by setting $r=1$ in μ_r', we have the mean of X. The last integration can be computed numerically for most parent distributions. The rth central moment of X, say μ_n, follows as $\mu_r = E(X-\mu_1')^n = \sum_{h=0}^{n} (-1)^h \binom{n}{h} (\mu_1')^n \mu_{n-h}'$. For the skedwness and kurtosis coefficients, we have $\sqrt{\beta_1} = \sqrt{\frac{\mu_3^2}{\mu_2^3}}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$. The values for mean, variance, $\sqrt{\beta_1}$ and β_2 for selected Hj-W distributions are shown in Table 1. Table 1 shows that Hj-N distribution can be left skewed and right skewed as well as having different kurtosis values. Hence, the Hj-W model can be useful for data modeling in terms of skewness and kurtosis.

The n^{th} descending factorial moment of X (for $n = 1, 2, \ldots$) is

$$\mu_{(n)}' = E[X^{(n)}] = E\left[X(X-1) \times \ldots \times (X-n+1)\right] = \sum_{k=0}^{n} s(n,k)\, \mu_k',$$

where $s(n,k) = (k!)^{-1} \left[d^k k^{(n)} / dx^k\right]_{x=0}$ is the Stirling number of the first kind. Here, we provide two formulae for the moment generating function (mgf) $M_X(t) = E\left(e^{tX}\right)$ of X. Clearly, the first one can be derived from Equation (7) as $M_X(t) = \sum_{k=0}^{\infty} \upsilon_{k+1} M_{k+1}(t)$,where $M_k(t)$ is the mgf of Y_{k+1}. Hence, $M_X(t)$ can be determined from the exp-G generating function. A second formula for $M_X(t)$ follows from (7) as $M_X(t) = \sum_{j=0}^{\infty} \upsilon_{k+1}\, \tau(t,k)$,where $\tau(t,k) = \int_0^1 \exp\left[t\, Q_G(u)\right] u^k du$ and $Q_G(u)$ is the qf corresponding to $G(x;\xi)$, i.e., $Q_G(u) = G^{-1}(u;\xi)$. The r^{th} incomplete moment of X is defined by $m_r(y) = \int_{-\infty}^{y} x^r f(x) dx$. From (7) we can write $m_r(y) = \sum_{k=0}^{\infty} \upsilon_{k+1}\, m_{r,k}(y)$, where $m_{r,\gamma}(y) = E(Y_\gamma^r) = \int_0^{G(y;\xi)} Q_G^r(u;\xi)\, u^{\gamma-1}\, du$. The integral $m_{r,\gamma}(y)$ can be determined analytically for special models with closed-form expressions for $Q_G(u;\xi)$ or computed at least numerically for most baseline distributions.

Table 1 Mean, variance, coefficients of skewness and kurtosis for Hj-Weibull (Hj-W) distributions.

$(\alpha, \beta, \theta, \lambda, \gamma)$	μ_1'	*Variance*	$\sqrt{\beta_1}$	β_2
(0.5, 0.5, 0.5, 0.5, 0.5)	4.4975	37.1694	2.6572	13.5148
(1, 1, 1, 1, 1)	0.7706	0.3715	1.0395	3.9265
(2, 2, 0.5, 1, 2)	0.8161	0.0849	−0.1659	2.6427
(5, 5, 5, 5, 5)	0.1385	0.0011	−0.3015	2.6162
(0.5, 2, 0.5, 2, 0.5)	1.4083	3.1419	2.3539	11.1339
(1, 2, 3, 4, 5)	0.1918	0.0022	−0.1629	2.6292
(5, 4, 3, 2, 1)	0.1626	0.0188	1.1156	4.0671
(5, 10, 15, 20, 25)	0.0444	0.00001	−27.0075	31116

4.2. Order Statistics

Suppose $X_1, \dots, X_n$ is a random sample from any Hj-G distribution. Let $X_{i:n}$ denote the ith order statistic. The pdf of $X_{i:n}$ can be expressed as

$$f_{i:n}(x) = f(x)\,[B(i, n-i+1)]^{-1} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F(x)^{j+i-1}.$$

Following similar algebraic developments of Nadarajah *et al.* [6], we can write the density function of $X_{i:n}$ as

$$f_{i:n}(x) = \sum_{r,k=0}^{\infty} \upsilon_{r,k}\, \pi_{r+k+1}(x), \tag{8}$$

where

$$\upsilon_{r,k} = \frac{n!\ (r+1)\,(i-1)!\ \upsilon_{r+1}}{(r+k+1)} \sum_{j=0}^{n-i} \frac{(-1)^j f_{j+i-1,k}}{(n-i-j)!\ j!},$$

and υ_{r+1} is given in Section 3 and the quantities $f_{j+i-1,k}$ can be determined with $f_{j+i-1,0} = c_0^{j+i-1}$ and recursively for $k \geq 1$, $f_{j+i-1,k} = (k\,c_0)^{-1} \sum_{m=1}^{k} [m\,(j+i) - k]\, c_m f_{j+i-1,k-m}$. Equation (8) is the main result of this section. It reveals that the pdf of the Hj-G order statistics is a linear combination of Exp-G density functions. So, several mathematical quantities of the Hj-G-G order statistics such as ordinary, incomplete and factorial moments, mean deviations and several others can be determined from those quantities of the exp-G distribution.

5. CHARACTERIZATION

This section deals with various characterizations of the Hj-G distribution. These characterizations are based on (i) a simple relationship between two truncated moments; (ii) the hazard function and (iii) conditional expectation of a function of the random variable. It should be mentioned that for characterization (i) the cdf is not required to have a closed form. We present our characterizations (i) − (iii) in three subsections.

5.1. Characterizations Based on Two Truncated Moments

In this subsection we present characterizations of Hj-G distribution in terms of a simple relationship between two truncated moments. The first characterization result employs a theorem due to Glänzel [7] see Theorem 5.1.1 below. Note that the result holds also when the interval H is not closed. Moreover, as mentioned above, it could be also applied when the cdf F does not have a closed form. As shown in Glänzel [8], this characterization is stable in the sense of weak convergence.

Theorem 5.1.1. *Let* $(\Omega, \mathcal{F}, \mathbf{P})$ *be a given probability space and let* $H = [d, e]$ *be an interval for some* $d < e$ $(d = -\infty, e = \infty$ *might as well be allowed*$)$. *Let* $X : \Omega \to H$ *be a continuous random variable with the distribution function* F *and let* q_1 *and* q_2 *be two real functions defined on* H *such that*

$$\mathbf{E}\left[q_2(X) \mid X \geq x\right] = \mathbf{E}\left[q_1(X) \mid X \geq x\right] \eta(x), \quad x \in H,$$

is defined with some real function η. *Assume that* $q_1, q_2 \in C^1(H)$, $\eta \in C^2(H)$ *and* F *is twice continuously differentiable and strictly monotone function on the set* H. *Finally, assume that the equation* $\eta q_1 = q_2$ *has no real solution in the interior of* H. *Then* F *is uniquely determined by the functions* q_1, q_2 *and* η, *particularly*

$$F(x) = \int_a^x C \left| \frac{\eta'(u)}{\eta(u)\, q_1(u) - q_2(u)} \right| \exp(-s(u))\, du,$$

where the functions is a solution of the differential equation $s' = \dfrac{\eta' q_1}{\eta q_1 - q_2}$ *and* C *is the normalization constant, such that* $\int_H dF = 1$.

Proposition 5.1.1. *Let* $X : \Omega \to \mathbb{R}$ *be a continuous random variable and let,* $q_1(x) = -\dfrac{\left[\log \overline{G}(x)\right]\left[1 - \beta \log \overline{G}(x)\right]^{\frac{\theta}{\beta}+1}}{\alpha\left[\beta \log \overline{G}(x) - 1\right] \log \overline{G}(x) + \theta}$ *and* $q_2(x) = q_1(x) \exp\left\{-\frac{\alpha}{2}\left[\log \overline{G}(x)\right]^2\right\}$ *for* $x \in \mathbb{R}$. *The random variable* X *has pdf (4) if and only if the function* η *defined in Theorem 5.1.1 has the form*

$$\eta(x) = \frac{1}{\alpha} \exp\left\{-\frac{\alpha}{2}\left[\log \overline{G}(x)\right]^2\right\}, \quad x \in \mathbb{R}.$$

Proof. Let X be a random variable with pdf (4), then

$$(1-F(x))\,E\left[q_1(X)\mid X\geq x\right]=\frac{1}{\alpha}\exp\left\{-\frac{\alpha}{2}\left[\log\overline{G}(x)\right]^2\right\},\in\mathbb{R}$$

and

$$(1-F(x))\,E\left[q_2(X)\mid X\geq x\right]=\frac{1}{2\alpha}\exp\left\{-\alpha\left[\log\overline{G}(x)\right]^2\right\},\quad x\in\mathbb{R},$$

and finally

$$\eta(x)\,q_1(x)-q_2(x)=-\frac{q_1(x)}{2}\exp\left\{-\frac{\alpha}{2}\left[\log\overline{G}(x)\right]^2\right\}<0\quad for\quad x\in\mathbb{R}.$$

Conversely, if η is given as above, then

$$s'(x)=\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=-\frac{\alpha g(x)\left[\log\overline{G}(x)\right]}{\overline{G}(x)}\quad x\in\mathbb{R},$$

and hence

$$s(x)=\frac{\alpha}{2}\left[\log\overline{G}(x)\right]^2,x\in\mathbb{R}.$$

Now, in view of Theorem 5.1.1, X has density (4). □

Corollary 5.1.1. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable and let $q_1(x)$ be as in Proposition 5.1.1. The pdf of X is (4) if and only if there exist functions q_2 and η defined in Theorem 5.1.1 satisfying the differential equation*

$$\frac{\eta'(x)\,q_1(x)}{\eta(x)\,q_1(x)-q_2(x)}=-\frac{\alpha g(x)\left[\log\overline{G}(x)\right]}{\overline{G}(x)}\quad x\in\mathbb{R}.$$

The general solution of the differential equation in Corollary 5.1.1 is

$$\xi(x)=\exp\left\{\alpha\left[\log\overline{G}(x)\right]^2/2\right\}\left[\begin{array}{c}\int\alpha g(x)\left[\overline{G}(x)\right]^{-1}\left[\log\overline{G}(x)\right]\times\\ \exp\left\{-\alpha\left[\log\overline{G}(x)\right]^2/2\right\}\left(q_1(x)\right)^{-1}q_2(x)+D\end{array}\right],$$

where D is a constant. Note that a set of functions satisfying the above differential equation is given in Proposition 5.1.1 with $D=0$. However, it should be also noted that there are other triplets $\left(q_1,q_2,\eta\right)$ satisfying the conditions of Theorem 5.1.1.

5.2. Characterization Based on Hazard Function

It is known that the hazard function, h_F, of a twice differentiable distribution function, F, satisfies the first order differential equation

$$\frac{f'(x)}{f(x)}=\frac{h_F'(x)}{h_F(x)}-h_F(x).$$

For many univariate continuous distributions, this is the only characterization available in terms of the hazard function. The following characterization establishes a non-trivial characterization of Hj-G distribution in terms of the hazard function, which is not of the above trivial form.

Proposition 5.2.1. *Let $X:\Omega\to\mathbb{R}$ be a continuous random variable. The pdf of X is (4) if and only if its hazard function $h_F(x)$ satisfies the differential equation*

$$h_F'(x)-\frac{g'(x)}{g(x)}h_F(x)=g(x)\frac{d}{dx}\left\{\frac{\alpha\left[\beta\log\overline{G}(x)-1\right]\log\overline{G}(x)+\theta}{\overline{G}(x)\left[1-\beta\log\overline{G}(x)\right]}\right\},x\in\mathbb{R}.$$

Proof. If X has pdf (4), then clearly the above differential equation holds. Now, if the differential equation holds, then

$$\frac{d}{dx}\left\{\left(g(x)\right)^{-1}h_F(x)\right\}=\frac{d}{dx}\left\{\frac{\alpha\left[\beta\log\overline{G}(x)-1\right]\log\overline{G}(x)+\theta}{\overline{G}(x)\left[1-\beta\log\overline{G}(x)\right]}\right\},$$

or

$$h_F(x) = \frac{g(x)\left\{\alpha\left[\beta\log\overline{G}(x)-1\right]\log\overline{G}(x)+\theta\right\}}{\overline{G}(x)\left[1-\beta\log\overline{G}(x)\right]},$$

which is the hazard function of the Hj-G distribution. □

5.3. Characterizations Based on Conditional Expectation

The following proposition has already appeared in Hamedani [9], so we will just state it here which can be used to characterize the Hj-G distribution.

Proposition 5.3.1. *Let $X : \Omega \to (a,b)$ be a continuous random variable with cdfF. Let $\psi(x)$ be a differentiable function on (a,b) with $\lim_{x\to a^+}\psi(x)=1$. Then for $\delta \neq 1$,*

$$E[\psi(X) \mid X \geq x] = \delta\psi(x), \qquad x \in (a,b),$$

if and only if

$$\psi(x) = (1-F(x))^{\frac{1}{\delta}-1}, \qquad x \in (a,b).$$

Remark 5.3.1. (*a*) For $\psi(x) = \dfrac{\exp\left\{-\left[\log\overline{G}(x)\right]^2\right\}}{\left[1-\beta\log\overline{G}(x)\right]^{\frac{2\theta}{\alpha\beta}}}$, $\delta = \frac{\alpha}{2+\alpha}$ and $(a,b) = \mathbb{R}$, Proposition 5.3.1 provides a characterization of Hj-G distribution. (*b*) Of course there are other suitable functions than the one we mentioned above, which is chosen for simplicity.

6. MAXIMUM LIKELIHOOD ESTIMATION (MLE)

We consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood method. Let $x_1, \cdots, x_n$ be a random sample from the Hj-G family with a $(q+3)\times 1$ parameter vector $\Theta = (\alpha,\beta,\theta,\xi^\intercal)^\intercal$, where ξ is a $q\times 1$ baseline parameter vector. The log-likelihood function for Θ is given by

$$\begin{aligned}\ell(\Theta) = & \sum_{i=1}^n \log g(x_i;\xi) - \sum_{i=1}^n \log\overline{G}(x_i;\xi) + \left(\frac{\theta}{\beta}+1\right)\sum_{i=1}^n \log\left\{1-\beta\log\overline{G}(x_i;\xi)\right\} \\ & + \sum_{i=1}^n \log\left[\alpha\left[\beta\log\overline{G}(x_i;\xi)-1\right]\log\overline{G}(x_i;\xi)+\theta\right] - \frac{\alpha}{2}\sum_{i=1}^n\left[-\log\overline{G}(x_i;\xi)\right]^2.\end{aligned}$$

The components of the score vector, $U(\Theta) = \dfrac{\partial\ell}{\partial\Theta} = \left(\dfrac{\partial\ell(\Theta)}{\partial\alpha}, \dfrac{\partial\ell(\Theta)}{\partial\beta}, \dfrac{\partial\ell(\Theta)}{\partial\theta}, \dfrac{\partial\ell(\Theta)}{\partial\xi}^\intercal\right)^\intercal$, are

$$U_\alpha = \sum_{i=1}^n \frac{\left[\beta\log\overline{G}(x_i;\xi)-1\right]\log\overline{G}(x_i;\xi)}{\alpha\left[\beta\log\overline{G}(x_i;\xi)-1\right]\log\overline{G}(x_i;\xi)+\theta} - \frac{1}{2}\sum_{i=1}^n\left[-\log\overline{G}(x_i;\xi)\right]^2,$$

$$\begin{aligned}U_\beta = & \left(\frac{\theta}{\beta}+1\right)\sum_{i=1}^n \frac{-\log\overline{G}(x_i;\xi)}{1-\beta\log\overline{G}(x_i;\xi)} - \frac{\theta}{\beta^2}\sum_{i=1}^n \log\left\{1-\beta\log\overline{G}(x_i;\xi)\right\} \\ & + \sum_{i=1}^n \frac{\alpha\left[\log\overline{G}(x_i;\xi)\right]^2}{\alpha\left[\beta\log\overline{G}(x_i;\xi)-1\right]\log\overline{G}(x_i;\xi)+\theta},\end{aligned}$$

$$U_\theta = \frac{1}{\beta}\sum_{i=1}^n \log\left\{1-\beta\log\overline{G}(x_i;\xi)\right\} + \sum_{i=1}^n \frac{1}{\alpha\left[\beta\log\overline{G}(x_i;\xi)-1\right]\log\overline{G}(x_i;\xi)+\theta},$$

and

$$U_{\xi_r} = \sum_{i=1}^{n} \frac{g'\left(x_i;\xi\right)}{g\left(x_i;\xi\right)} + \sum_{i=1}^{n} \frac{G'\left(x_i;\xi\right)}{\overline{G}\left(x_i;\xi\right)} + \left(\frac{\theta}{\beta}+1\right) \sum_{i=1}^{n} \frac{\beta \frac{G'\left(x_i;\xi\right)}{\overline{G}\left(x_i;\xi\right)}}{1-\beta \log \overline{G}\left(x_i;\xi\right)}$$

$$+ \sum_{i=1}^{n} \frac{\{\beta \log \overline{G}\left(x_i;\xi\right) - \alpha[\beta \log \overline{G}\left(x_i;\xi\right)-1]\} \frac{G'\left(x_i;\xi\right)}{\overline{G}\left(x_i;\xi\right)}}{\alpha[\beta \log \overline{G}\left(x_i;\xi\right)-1] \log \overline{G}\left(x_i;\xi\right)+\theta} - \alpha \sum_{i=1}^{n} \frac{G'\left(x_i;\xi\right)}{\overline{G}\left(x_i;\xi\right)} \log \overline{G}\left(x_i;\xi\right),$$

where

$$g'\left(x_i;\xi\right) = \frac{\partial g\left(x_i;\xi\right)}{\partial \xi} \text{ and } G'\left(x_i;\xi\right) = \frac{\partial G\left(x_i;\xi\right)}{\partial \xi}.$$

Setting the nonlinear system of equations $U_\alpha = U_\beta = U_\theta = U_{\xi_r} = 0$ (for $r = 1, \ldots, q$) and solving them simultaneously yields the MLEs $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{\theta}, \hat{\xi}^{\intercal})^{\intercal}$. To solve these equations, it is more convenient to use nonlinear optimization methods such as the quasi-Newton algorithm to numerically maximize $\ell(\Theta)$. For interval estimation of the parameters, we can evaluate numerically the elements of the $(q+3) \times (q+3)$ observed information matrix $J(\Theta) = \left\{-\frac{\partial^2 \ell}{\partial \theta_r \theta_s}\right\}$. Under the standard regularity conditions, when $n \to \infty$, the distribution of $\hat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\hat{\Theta})^{-1})$ distribution to construct approximate confidence intervals for the parameters. Here, $J(\hat{\Theta})$ is the total observed information matrix evaluated at $\hat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method.

7. SIMULATION STUDY

In this section, the performance of the MLEs of Hj-W distribution is discussed via simulation study. The inverse transform method is used to generate random variables from Hj-W distribution. The performance of the MLEs is evaluated based on the following measures: biases, mean square error (MSE) and coverage probability (CP). $N = 1,000$ samples of sizes $n = 50, 55, \ldots, 1000$ is generated from the Hj-W distribution with $\alpha = 0.5, \beta = 0.5, \theta = 2, \lambda = 2, \gamma = 3$. The MLEs of the model parameters are obtained for each generated sample, say $(\widehat{\alpha}_i, \widehat{\beta}_i, \widehat{\theta}_i, \widehat{\lambda}_i, \widehat{\gamma}_i)$, for $i = 1, \ldots, N$. The standard errors of the MLEs are evaluated by inverting the observed information matrix, namely $(s_{\hat{\alpha}_i}, s_{\hat{\beta}_i}, s_{\hat{\theta}_i}, s_{\hat{\lambda}_i}, s_{\hat{\gamma}_i})$ for $i = 1, \ldots, N$. The estimated biases and MSEs and CPs are given by $\widehat{Bias}_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varepsilon}_i - \varepsilon)$, $\widehat{MSE}_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{N}(\hat{\varepsilon}_i - \varepsilon)^2$, and $CP_\varepsilon(n) = \frac{1}{N}\sum_{i=1}^{N} I(\hat{\varepsilon}_i - 1.95996 s\hat{\varepsilon}_i, \hat{\varepsilon}_i + 1.95996 s_{\hat{\varepsilon}_i})$ where $\varepsilon = \alpha, \beta, \theta, \lambda, \gamma$ and $s_{\hat{\varepsilon}_i}$ is standard error of $\hat{\varepsilon}_i$ for each generated sample.

Figure 4 displays the simulation results for above measures. As seen from Figure 4, the estimated biases and MSEs approach zero when the sample size increases. As expected, CPs are near the nominal value (0.95) for sufficiently large sample sizes. The simulation results verify the consistency property of MLE. The similar results can be obtained for different parameter vector.

8. LOG-HJ-W REGRESSION MODEL

Consider the Hj-W distribution with five parameters presented in Subsection 2.2. Henceforth, X denotes a random variable following the Hj-W distribution and $Y = \log(X)$. The density function of Y (for $y \in \mathbb{R}$) obtained by replacing $\gamma = 1/\sigma$ and $\lambda = 1/\exp(\mu)$, can be expressed as

$$f(y) = \frac{\frac{1}{\sigma}\exp\left[\left(\frac{y-\mu}{\sigma}\right) - \exp\left(\frac{y-\mu}{\sigma}\right)\right]\left\{\alpha\left[1+\beta\exp\left(\frac{y-\mu}{\sigma}\right)\right]\exp\left(\frac{y-\mu}{\sigma}\right)+\theta\right\}}{\left(\exp\left[-\exp\left(\frac{y-\mu}{\sigma}\right)\right]\right)\left[1+\beta\exp\left(\frac{y-\mu}{\sigma}\right)\right]^{\frac{\theta}{\beta}+1}} \exp\left\{-\frac{\alpha}{2}\left[\exp\left(\frac{y-\mu}{\sigma}\right)\right]^2\right\}, \quad (9)$$

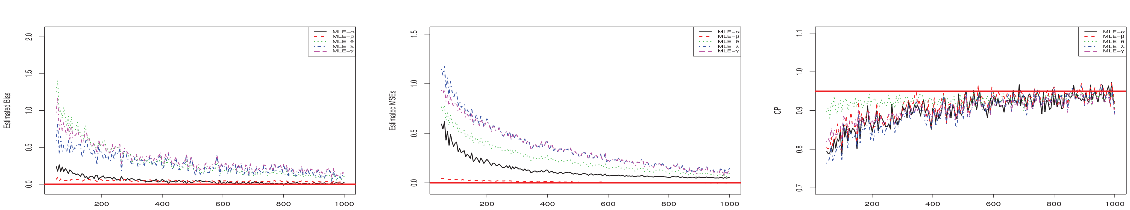

Figure 4 Estimated coverage probability (CPs), biases and mean square errors (MSEs) for the selected parameter vector.

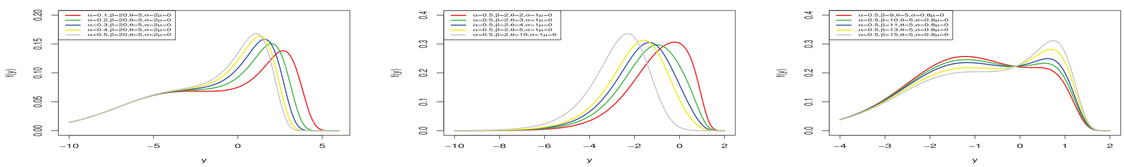

Figure 5 Plots of the LHj-W density function for some parameter values.

where $\mu \in \mathbb{R}$ is the location parameter, $\sigma > 0$ is the scale parameter and $\alpha > 0, \beta > 0$ and $\theta > 0$ are the shape parameters. We refer to equation (9) as the log-Hj-W (LHj-W) distribution, say $Y \sim$ LHj-W$(\alpha, \beta, \theta, \mu, \sigma)$. Figure 5 provides some plots of the density function (9) for selected parameter values. They reveal that this distribution is a good candidate for modeling left skewed and bimodal data sets.

The survival function corresponding to (9) is given by

$$S(z) = \frac{\exp\left[z - \exp(z)\right]\left\{\alpha\left[1 + \beta\exp(z)\right]\exp(z) + \theta\right\}}{\left(\exp\left[-\exp(z)\right]\right)\left[1 + \beta\exp(z)\right]^{\frac{\theta}{\beta}+1}} \exp\left\{-\frac{\alpha}{2}\left[\exp(z)\right]^2\right\}, \tag{10}$$

and the hrf is simply $h(y) = f(y)/S(y)$. The standardized random variable $Z = (Y - \mu)/\sigma$ has density function

$$f(z) = \frac{2\alpha\beta\exp\left[(z) - \exp(z)\right]\left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha\beta-1}\left[1 - \left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha}\right]^{\beta-1}}{\pi\left\{\left\{1 - \exp\left[-\exp(z)\right]\right\}^{2\alpha\beta} + \left[1 - \left\{1 - \exp\left[-\exp(z)\right]\right\}^{\alpha}\right]^{2\beta}\right\}}. \tag{11}$$

Based on the LHj-W density, we propose a linear location-scale regression model linking the response variable y_i and to explanatory variable vector $\mathbf{v}_i^T = \left(v_{i1}, \dots, v_{ip}\right)$ given by

$$y_i = v_i^T\beta + \sigma z_i, \mathrm{i} = 1, \dots, \mathrm{n} \tag{12}$$

where the random error z_i has the density function (11), $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^T$, $\sigma > 0$, $\alpha > 0$, $\beta > 0$ and $\theta > 0$ are unknown parameters. The parameter $\mu_i = \boldsymbol{v}_i^T\boldsymbol{\beta}$ is the location of y_i. The location parameter vector $\mu = (\mu_1, \dots, \mu_n)^T$ is represented by a linear model $\mu = V\beta$, where $V = (v_1, \dots, v_n)^T$ is a known model matrix. Consider a sample $(y_1, v_1), \dots, (y_n, v_n)$ of n independent observations, where each random response is defined by $y_i = \min\{\log(x_i), \log(c_i)\}$. We assume non-informative censoring such that the observed lifetimes and censoring times are independent. Let F and C be the sets of individuals for which y_i is the log-lifetime or log-censoring, respectively. The log-likelihood function for the vector of parameters $\tau = (\alpha, \beta, \theta, \sigma, \beta^T)^T$ from model (12) has the form $l(\tau) = \sum_{i \in F} l_i(\tau) + \sum_{i \in C} l_i^{(c)}(\tau)$, where $l_i(\tau) = \log[f(y_i)]$, $l_i^{(c)}(\tau) = \log[S(y_i)]$, $f(y_i)$ is the density (9) and $S(y_i)$ is the survival function (10) of Y_i. The total log-likelihood function for τ is given by

$$\begin{aligned} \ell(\tau) = {} & r\log\left(\frac{1}{\sigma}\right) + \sum_{i \in F} z_i + \sum_{i \in F}\log\left\{\alpha\left[1 + \beta u_i\right]u_i + \theta\right\} - \left(\frac{\theta + \beta}{\beta}\right)\sum_{i \in F}\log\left[1 + \beta u_i\right] - \frac{\alpha}{2}\sum_{i \in F} u_i^2 \\ & \sum_{i \in C}\log\left\{\frac{\exp\left\{\frac{\alpha}{2}\left[-u_i\right]^2\right\}}{\left[1 - \beta\left(-u_i\right)\right]^{\frac{\theta}{\beta}}}\right\}, \end{aligned} \tag{13}$$

where $u_i = \exp(z_i)$, $z_i = (y_i - \mu_i)/\sigma_i$, and r is the number of uncensored observations (failures). The MLE $\hat{\tau}$ of the vector of unknown parameters can be obtained by maximizing the log-likelihood function (13). The R software is used to estimate $\hat{\tau}$.

8.1. Residual Analysis

Residual analysis has critical role in checking the adequacy of the fitted model. In order to analyze departures from the error assumption, two types of residuals are considered: martingale and modified deviance residuals.

8.1.1. Martingale Residual

The martingale residuals is defined in counting process and takes values between +1 and $-\infty$ (see [10] for details). The martingale residuals for LHj-W model is

$$r_{M_i} = \begin{cases} 1 + \log\left(= \dfrac{\exp\left\{\frac{\alpha}{2}\left[-u_i\right]^2\right\}}{\left[1-\beta\left(-u_i\right)\right]^{\frac{\theta}{\beta}}} \right) & \text{if } i \in F, \\ \log\left(= \dfrac{\exp\left\{\frac{\alpha}{2}\left[-u_i\right]^2\right\}}{\left[1-\beta\left(-u_i\right)\right]^{\frac{\theta}{\beta}}} \right) & \text{if } i \in C, \end{cases}$$

where $u_i = 2\Phi\left[\exp\left(z_i\sqrt{2}\middle/2\right)\right.$ and $z_i = (y_i - \mu_i)/\sigma$.

8.1.2. Modified Deviance Residual

The main drawback of the martingale residual is that when the fitted model is correct, it is not symmetrically distributed about zero. To overcome this problem, modified deviance residual was proposed by Therneau *et al.* [11]. Th modified deviance residual for LHj-W model is

$$r_{D_i} = \begin{cases} sign\,(r_{M_i})\{-2\,[r_{M_i} + \log(1 - r_{M_i})]\}^{1/2}, & \text{if } i \in F \\ sign\,(r_{M_i})\{-2r_{M_i}\}^{1/2}, & \text{if } i \in C, \end{cases}$$

where $\hat{r}_{M_i}$ is the martingale residual.

9. REAL DATA APPLICATIONS

In this section, we consider three applications to real data sets to show the modeling ability of the Hj-N, Hj-W and Hj-U distributions. We compare these distribution models with both distributions of some members of the T-X family, where $W[G(x)]$ is equal to $-\log[1-G(x)]$, and some generalizations of ordinary normal, Weibull and uniform distributions. These families and generalized models are the Mc Donald-G (Mc-G) family [12], Gompertz-G (Gom-G) family [13], Generalized odd log logistic-G (GOLL-G) family [14], Weibull-G (W-G) family [4], Lomax-G (Lx-G) family [15], Lindley-G (Li-G) family [16], logistic-G (L-G) family [17], Kumaraswamy odd log logistic normal (KwOLLN) distribution [18], odd Burr normal (OBN) distribution [19], Zografos-Balakkrishnan odd log logistic Weibull (ZBOLLW) distribution [20], additive Weibull (AW) distribution [21] and gamma uniform (GU) distribution [22]. The cdfs of these distributions are available in the literature. To determine the best model, we also compute the estimated log-likelihood values $\hat{\ell}$, Akaike Information Criteria (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), Hannan–Quinn information criterion (HQIC), Cramer-von-Mises (W^*) and Anderson-Darling (A^*) goodness of-fit statistics for all distribution models. We note that the statistics W^* and A^* are described in detail in [23]. In general, it can be chosen as the best model the one which has the smaller the values of the AIC, CAIC, BIC, HQIC, W^* and A^* statistics and the larger the values of $\hat{\ell}$ and p-values. All computations are performed by the maxLik routine in the R programme. The details are given below.

9.1. Otis IQ Scores of Non-White Males Data Set

The first real data set is the data on the Otis IQ Scores of 52 non-white males hired by a large insurance company in 1971. This data set has been analyzed by [24–26] and [27]. On the data set, we compare the Hj-N model with Mc-N, Lx-N, W-N, KwOLLN, L-N, OBN, GOLL-N, Gom-N and Li-N models. Table 2 shows MLEs and standard erros of the estimates for the first dat set. Table 3 lists information criteria results and goodness-of-fits statistics. Table 3 clearly show that the Hj-N model has the smallest values AIC, CAIC, BIC, HQIC, W^* and A^* statistics and it has the largest values for $\hat{\ell}$ and two p-values among the fitted models. So, it can be chosen as the best model based on these criteria. For this data set, the plots of the fitted pdfs and cdfs for all models are shown in Figure 6. From this figure, we see that the Hj-N, Mc-N and KwOLLN models fit data as bi-modal shape whereas the OBN model fits data as uni-modal shape.

9.2. Failure Times Data Set

The second data set represents the times between successive failures (in thousands of hours) in events of secondary reactor pumps studied by [28,29] and [30]. This data set is also known as bathtub shaped. So, for this data set, we compare the Hj-W model with AW, Mc-W, Lx-W, W-W, ZBOLLW, L-W, GOLL-W, Gom-W and Li-W models. We fitted the Hj model and obtained its $\hat{\ell}$ value as -31.2520. Table 4 shows

Table 2 MLEs and standard erros of the estimates (in parentheses) for the first data set.

Data Set	Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\mu}$	$\hat{\sigma}$
I	Hj-N	190,363.1425	518,889.4549	152,083.8273	154.4301	11.9168
		(5.9316)	(380.4220)	(380.4954)	(0.00001)	(0.0862)
	KwOLLN	5.8735	84.4204	0.0221	113.0728	2.0037
		(0.2310)	(6.8585)	(0.0026)	(0.0308)	(0.0519)
	Mc-N	0.0160	0.0573	2.7977	111.0082	1.4883
		(0.0023)	(0.0132)	(0.0315)	(0.0001)	(0.0001)
	GOLL-N	0.7348	-	18.6911	84.2008	12.0843
		(0.1867)		(6.8108)	(4.7654)	(2.2345)
	OBN	1.7777	-	0.2647	98.9947	7.7191
		(0.8162)		(0.1183)	(1.9371)	(2.6981)
	W-N	0.1546	-	0.9030	96.8421	3.5746
		(0.0432)		(0.0972)	(0.0002)	(0.0002)
	Gom-N	0.0020	-	0.0921	94.9562	3.4364
		(0.0152)		(0.0196)	(0.0007)	(0.0007)
	Lx-N	118.0650	376.2148	-	99.5811	5.4795
		(93.3838)	(92.8438)		(1.0954)	(0.4381)
	Li-N	—	-	0.1470	90.5867	4.0007
				(0.0144)	(0.0001)	(0.0001)
	L-N	—	-	2.5576	101.6204	12.0705
				(0.9339)	(1.7310)	(4.3032)

Table 3 Information criteria results, A^*, W^* and $\hat{\ell}$ statistics ([·] and {·} denote their p-values) for the first data set.

Data Set	Model	*AIC*	*CAIC*	*BIC*	*HQIC*	A^*	W^*	$-\hat{\ell}$
I	Hj-N	366.8823	368.1866	376.6385	370.6226	0.2925	0.0428	178.4411
						[0.9434]	{0.9198}	
	KwOLLN	368.0000	369.3043	377.7562	371.7403	0.3144	0.0446	179.0001
						[0.9266]	{0.9098}	
	Mc-N	368.4839	369.7883	378.2401	372.2242	0.3422	0.0520	179.2420
						[0.9028]	{0.9030}	
	GOLL-N	372.5455	373.3966	380.3505	375.5378	0.4404	0.0683	182.2728
						[0.8072]	{0.7639}	
	OBN	371.8735	372.7245	379.6784	374.8657	0.3973	0.0584	181.9367
						[0.8508]	{0.8262}	
	W-N	372.5644	373.4154	380.3693	375.5566	0.4698	0.0750	182.2822
						[0.7770]	{0.7231}	
	Gom-N	371.0419	371.8929	378.8468	374.0341	0.5295	0.0856	181.5209
						[0.7160]	{0.6616}	
	Lx-N	373.2565	374.1076	381.0615	376.2488	0.6205	0.1031	182.6283
						[0.6280]	{0.5715}	
	Li-N	370.9549	371.4549	376.8087	373.1991	0.7868	0.1486	182.4775
						[0.4900]	{0.3947}	
	L-N	372.8924	373.3924	378.7461	375.1365	0.4423	0.0639	183.4462
						[0.8053]	{0.7911}	

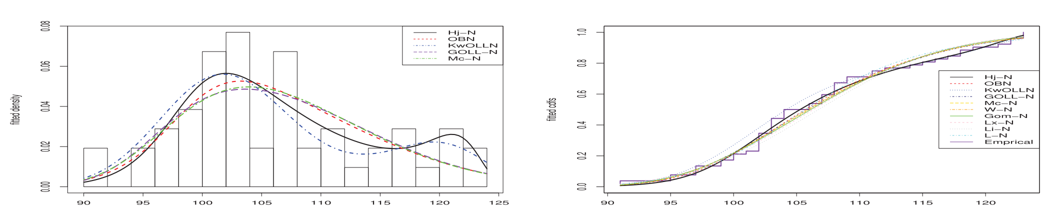

Figure 6 The fitted probability density functions (pdfs) and cumulative distribution functions (cdfs) for the first data set.

MLEs and standard erros of the estimates for the second data set. Table 5 lists information criteria results and goodness-of-fits statistics. The Hj-W model has the smallest values of the AIC, HQIC, W^* and A^* statistics and have the largest values for $\hat{\ell}$ and all p-values among the fitted models. For this data set, the plots of the fitted pdfs and cdfs for all models are shown in Figure 7. From this figure, we see that the Hj-W model fits the histograms of the data sets with more adequate fitting than Li-W and other models.

9.3. Student's Cognitive Skill Data

The third data set contains the student's cognitive skills for Organisation for Economic Co-operation and Development (OECD) countries. The score of student's cognitive skill represents the average score in reading, mathematics and science as assessed by the OECD's Programme

Table 4 MLEs and standard erros of the estimates (in parentheses) for the second data set.

Data Set	Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\gamma}$
II	Hj-W	0.0031	23.3597	6.9156	0.9910	1.8949
		(0.0018)	(6.5439)	(2.1775)	(0.0152)	(0.0007)
	ZBOLLW	2.5977	6.2815	-	41.9548	0.2567
		(3.3552)	(4.1634)		(12.3595)	(0.1468)
	Mc-W	87.0504	119.6097	2.3397	37.5696	0.0487
		(4.2657)	(1.4467)	(0.1265)	(2.5664)	(0.0064)
	GOLL-W	0.2619	-	10.3956	4.1457	0.8139
		(0.0579)		(0.2475)	(0.0893)	(0.7005)
	W-W	0.1100	-	1.1545	11.0445	0.6995
		(0.0523)		(0.1856)	(0.0051)	(0.0001)
	Gom-W	0.0023	-	0.0912	15.1874	0.7808
		(0.0230)		(0.0294)	(0.0003)	(0.0001)
	Lx-W	48.4187	99.9372	-	1.7814	0.8161
		(11.7869)	(17.9206)		(1.0382)	(0.1314)
	Li-W	–	-	27.1223	0.0126	0.1298
				(10.4406)	(0.0099)	(0.1298)
	L-W	–	-	21.5282	1.4143	0.0571
				(9.5853)	(0.4227)	(0.0272)
	AW	0.1576	11.1873	-	0.6779	0.7575
		(0.0172)	(10.4301)		(0.2075)	(0.1372)

Table 5 Information criteria results, A^*, W^* and $\hat{\ell}$ statistics ($[\cdot]$ and $\{\cdot\}$ denote their p-values) for the second data set.

Data Set	Model	AIC	CAIC	BIC	HQIC	A^*	W^*	$-\hat{\ell}$
II	Hj-W	68.7731	72.3025	74.4506	70.2010	0.1295	0.0157	29.3865
						[0.9997]	{0.9996}	
	ZBOLL-W	71.1057	73.3279	75.6476	72.2480	0.2213	0.0243	31.5528
						[0.9835]	{0.9923}	
	Mc-W	73.5915	77.1209	79.2690	75.0193	0.2264	0.0256	31.7957
						[0.9814]	{0.9898}	
	GOLL-W	69.9227	72.1450	74.4647	71.0650	0.2400	0.0384	30.9614
						[0.9754]	{0.9445}	
	W-W	73.0278	75.2500	77.5698	74.1701	0.4040	0.0617	32.5139
						[0.8433]	{0.8076}	
	Gom-W	72.3346	74.5568	76.8766	73.4770	0.3845	0.0569	32.1673
						[0.8626]	{0.8382}	
	Lx-W	73.0204	75.2427	77.5625	74.1628	0.3965	0.0602	32.5102
						[0.8509]	{0.8175}	
	Li-W	71.0288	72.2920	74.4354	71.8855	0.4047	0.0619	32.5144
						[0.8426]	{0.8066}	
	L-W	71.2273	72.4905	74.6338	72.0841	0.2326	0.0259	32.6136
						[0.9788]	{0.9891}	
	AW	70.5462	72.7684	75.0881	71.6884	0.3730	0.0600	31.2731
						[0.8738]	{0.8186}	

for International Student Assessment (PISA). The data set can be found in https://stats.oecd.org/index.aspx?DataSetCode=BL. By using this data set, we compare the Hj-U model with GU, L-U and W-U models. We note that since $a < x < b$, the MLE of the a and b are the minumum order statistic $x_{1:n}$ and maximum order statistic $x_{n:n}$ respectively. Hence, we assume that the parameters are $a = 416$ and $b = 529$ for all fitted models. Table 6 shows MLEs and standard erros of the estimates for the third data set. Table 7 lists information criteria results and goodness-of-fits statistics. The Hj-U model has the smallest values of the AIC, CAIC, HQIC, W^* and A^* statistics and have the largest values of the $\hat{\ell}$ and all p-values among the fitted models. So, it can be chosen as the best model based on these criteria. For this data set, the plots of the fitted pdfs and cdfs for all models are shown in Figure 8. From this figure, The Hj-U model has fitted the data as uni-modal shaped .

Finally, when we observe all results, we can say that the Hj-N, Hj-W and Hj-U models could be chosen as the best models for the three data sets via the above criteria.

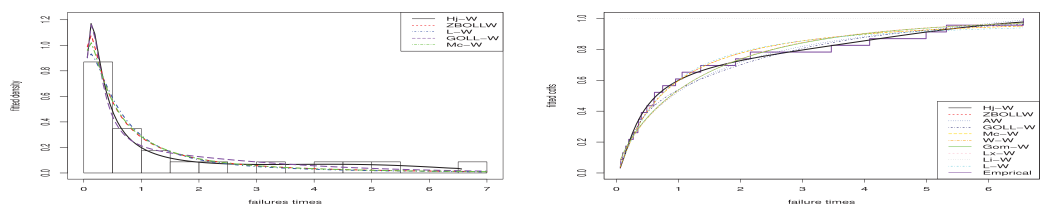

Figure 7 The fitted probability density functions (pdfs) and cumulative distribution functions (cdfs) for the second data set.

Table 6 MLEs and standard erros of the estimates (in parentheses) for the third data set.

Data Set	Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	a	b
III	Hj-U	0.8393	11.0082	0.1020	416	529
		(0.1537)	(4.2005)	(0.2336)		
	GU	1.0723	3.9112	-	416	529
		(0.2339)	(1.0792)			
	L-U	2.7437	-	-	416	529
		(0.4026)				
	W-U	0.4537	-	0.6334	416	529
		(0.1032)		(0.0930)		

Table 7 Information criteria results, A^*, W^* and $\hat{\ell}$ statistics ($[\cdot]$ and $\{\cdot\}$ denote their p-values) for the third data set.

Data Set	Model	AIC	CAIC	BIC	HQIC	A^*	W^*	$-\hat{\ell}$
III	Hj-U	295.7462	296.5738	300.2357	297.2568	0.6284	0.1031	144.8731
						[0.6202]	{0.5726}	
	GU	297.7327	298.1327	300.7257	298.7398	1.2184	0.2049	146.8684
						[0.2604]	{0.2588}	
	L-U	299.6478	299.7768	301.1443	300.1513	2.5881	0.5574	148.8239
						[0.0449]	{0.0279}	
	W-U	404.1662	404.5662	407.1592	405.1733	2.8695	0.5562	200.0831
						[0.0322]	{0.0281}	

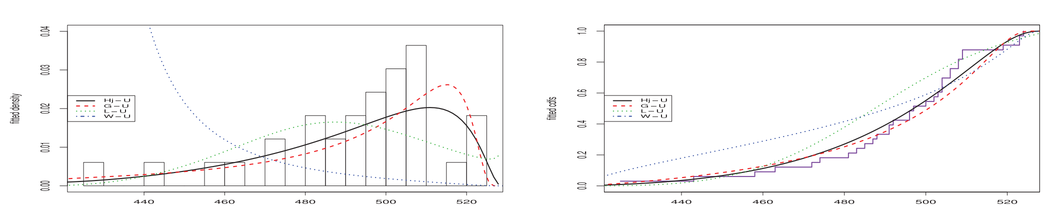

Figure 8 The fitted probability density functions (pdfs) and cumulative distribution functions (cdfs) for the third data set.

9.4. Stanford Heart Transplant Data

Recently, Brito *et al.* [3] introduced the Log-Topp-Leone odd log-logistic-Weibull (Log-TLOLL-W) regression model. Brito *et al.* [3] used the Stanford heart transplant data set to prove the usefulness of Log-TLOLL-W regression model. Here, we use the same data set to demonstrate the flexibility of LHj-W regression model against the Log-TLOLL-W and Log-Weibull regression models. These data set is available in *p3state.msm* package of R software. The sample size is $n = 103$, the percentage of censored observations is 27%. The goal of this study is to relate the survival times (t) of patients with the following explanatory variables: x_1- year of acceptance to the program; x_2- age of patient (in years); x_3- previous surgery status $\left(1 = yes, 0 = no\right)$; x_4- transplant indicator $\left(1 = yes, 0 = no\right)$; c_i - censoring indicator (0 = censoring, 1 = lifetime observed). The regression model fitted to the stanford heart transplant data is given by

$$y_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \beta_3 x_{i3} + \beta_4 x_{i4} + \sigma z_i,$$

respectively, where the random variable y_i follows the LHj-W distribution given in (9). The results for the above regression models are presented in Table 8. The MLEs of the model parameters and their SEs, p values and $-\ell$, AIC and BIC statistics are listed in Table 8. Based on the figures in Table 8, LHj-W model has the lowest values of the $-\ell$, AIC and BIC statistics. Therefore, it is clear that LHj-W regression model outperforms the others for this data set. In view of the results of LHj-W regression model, β_0, β_1 and β_2 are statistically significant at 1% level.

Finally, when we observe all results, we can say that the Hj-N, Hj-W and Hj-U models could be chosen as the best models for the three data sets via the above criteria.

9.4.1. Residual analysis of LHj-W model for Stanford heart transplant data set

Figure 9 displays the index plot of the modified deviance residuals and its Q-Q plot against $N(0, 1)$ quantiles for Stanford heart transplant data set. Based on Figure 9, we conclude that none of observed values appear as possible outliers. Therefore, the fitted model is appropriate for this data set.

10. CONCLUSIONS

In this work, we introduce a new flexible class of continuous distributions via the Hjorth's IDB model. We provide some mathematical properties of the new family. Characterizations based on two truncated moments, conditional expectation as well as in terms of the hazard

Table 8 MLEs of the parameters to Stanford Heart Transplant Data for Log-Weibull, Log-TLOLL-W and LHj-W regression models with corresponding SEs, p-values and $-\ell$, AIC and BIC statistics.

	Models								
	Log-Weibull			Log-TLOLL-W			LHj-W		
Parameters	Estimate	S.E.	p-value	Estimate	S.E.	p-value	Estimate	S.E.	p-value
α	-	-	-	2.34	3.546	-	1.662	0.541	-
β	-	-	-	-	-	-	5.382	6.910	-
θ	-	-	-	24.029	3.015	-	19.326	12.172	-
σ	1.478	0.133	-	9.68	12.526	-	1.223	0.141	-
β_0	1.639	6.835	0.811	−0.645	8.459	0.939	6.398	0.485	<0.001
β_1	0.104	0.096	0.279	0.074	0.097	0.448	0.234	0.096	0.014
β_2	−0.092	0.02	<0.001	−0.053	0.02	0.009	−0.066	0.018	<0.001
β_3	1.126	0.658	0.087	1.676	0.597	0.005	0.139	0.512	0.785
β_4	2.544	0.378	<0.001	2.394	0.384	<0.001	0.262	0.377	0.487
$-\ell$		171.2405			164.684			160.061	
AIC		354.481			345.368			338.122	
BIC		370.2894			366.4458			361.834	

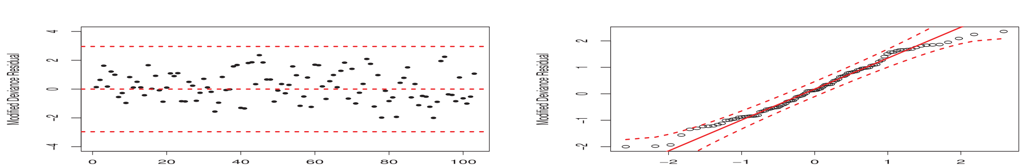

Figure 9 Index plot of the modified deviance residual (left) and Q-Q plot for modified deviance residual (right).

function are presented. The maximum likelihood method is used for estimating the model parameters. We assess the performance of the maximum likelihood estimators in terms of the biases and mean squared errors by means of two simulation studies. A new regression model as well as residual analysis are presented, Finally, the usefulness of the family is illustrated by means of three real data sets. The new model provides consistently better fits than other competitive models for these data sets.

CONFLICT OF INTEREST

Authors have no conflicts of interest to declare.

AUTHORS' CONTRIBUTIONS

The authors contributed equally to this work.

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