AN ULTRAPOWER CONSTRUCTION OF THE MULTIPLIER ALGEBRA OF A C^*-ALGEBRA AND AN APPLICATION TO BOUNDARY AMENABILITY OF GROUPS

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ABSTRACT. Using ultrapowers of C^*-algebras we provide a new construction of the multiplier algebra of a C^*-algebra. This extends the work of Avsec and Goldbring [Houston J. Math., to appear, arXiv:1610.09276.] to the setting of noncommutative and nonseparable C^*-algebras. We also extend their work to give a new proof of the fact that groups that act transitively on locally finite trees with boundary amenable stabilizers are boundary amenable.

1. INTRODUCTION

The multiplier algebra $\mathcal{M}(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is a C^*-algebra that contains $\mathcal{A}$ as an essential ideal and satisfies the following universal property: for every C^*-algebra $\mathcal{B}$ containing $\mathcal{A}$ as an ideal, there exists a unique $*$-homomorphism $\varphi : \mathcal{B} \to \mathcal{M}(\mathcal{A})$ such that φ is the identity on $\mathcal{A}$. If $\mathcal{A}$ is abelian, thus of the form $C_0(X)$ for some locally compact Hausdorff space X, then $\mathcal{M}(\mathcal{A})$ is isomorphic to $C_b(X)$ and this in turn can be identified with $C(\beta X)$, where βX is the Stone-Čech compactification of X (for more about multiplier algebras, see, for instance [4, 6]).

In the article [1], Avsec and Goldbring provided a new construction of the multiplier algebra of the abelian C^*-algebra $C_0(X)$ using ultraproducts of C^*-algebras, in the case when X is a second countable locally compact Hausdorff space. From there, they inferred a new construction of the Stone-Čech compactification of X, and they used it to give a new proof of the fact that groups that act properly and transitively on trees are boundary amenable. In section 2 of this note, we extend their work providing a construction of the multiplier algebra of any C^*-algebra $\mathcal{A}$ by means of ultraproducts of C^*-algebras. In section 3, we focus on the case of commutative and separable C^*-algebras, and compare our main technical tool with the main technical tool used in [1] to explain why the work done here is indeed a generalization of [1]. Finally, in section 4, we extend the techniques of [1] to show that groups that act transitively on locally finite trees having boundary amenable stabilizers are boundary amenable.

2010 *Mathematics Subject Classification.* Primary 46L05; Secondary 03C20, 20F65.

Key words and phrases. Multiplier algebra, ultraproduct of C^*-algebras, boundary amenable groups.

2. Ultraproducts and Multipliers

Let $\mathcal{I}$ be a set. An *ultrafilter* over $\mathcal{I}$ is a nonempty collection $\mathcal{U}$ of subsets of $\mathcal{I}$ with the following properties:

(1) finite intersection property: for every $\mathcal{I}_0, \mathcal{I}_1 \in \mathcal{U}$, then $\mathcal{I}_0 \cap \mathcal{I}_1 \in \mathcal{U}$;
(2) directness: for every $\mathcal{I}_o \subset \mathcal{I}_1$, where $\mathcal{I}_o$ belongs to $\mathcal{U}$, then $\mathcal{I}_1 \in \mathcal{U}$;
(3) maximality: for every $\mathcal{I}_0 \subset \mathcal{I}$, either $\mathcal{I}_0 \in \mathcal{U}$ or $\mathcal{I} \setminus \mathcal{I}_0 \in \mathcal{U}$.

An ultrafilter is *principal* if there exists $i \in \mathcal{I}$ such that the subsets of $\mathcal{I}$ that contains i are in the ultrafilter. Ultrafilters not of this form are called *nonprincipal* or *free*. It is easy to show that an ultrafilter is nonprincipal exactly when it contains no finite sets. An ultrafilter is *cofinal* when the index set is a directed set, and the sets $\{i \in \mathcal{I} : i \geq i_0\}$ are in $\mathcal{U}$ for every $i_0 \in \mathcal{I}$.

When dealing with directed sets with the property that there is no maximal element, every cofinal ultrafilter is nonprincipal. Moreover, when the ultrafilter is over $\mathbb{N}$, being cofinal is the same as being nonprincipal. If a directed set has a maximal element, then every cofinal ultrafilter is principal.

Definition 2.1. Let $\mathcal{U}$ be an ultrafilter over $\mathcal{I}$. Let (X, d) be a metric space and let $(a_i)_{i\in\mathcal{I}} \subset X$. We say that $(a_i)_{i\in\mathcal{I}}$ is convergent along $\mathcal{U}$, if there exists an element $a \in X$ such that, for every $\varepsilon > 0$, the set $\{i \in \mathcal{I} : d(a_i, a) < \varepsilon\}$ belongs to $\mathcal{U}$. The element a is called the $\mathcal{U}$-limit of $(a_i)_{i\in\mathcal{I}}$ and it is denoted by $\lim_{\mathcal{U}} a_i$.

2.1. **Ultraproducts of C^*-algebras.** Let $\mathcal{U}$ be an ultrafilter over a set $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. Denote by $\prod_{\mathcal{I}} \mathcal{A}$ the set $\{(a_i)_{i\in\mathcal{I}} : \sup_{i\in\mathcal{I}} \|a_i\| < \infty\}$ and let $\mathcal{N}_{\mathcal{U}}$ be the subspace generated by those $(a_i)_{i\in\mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ such that $\lim_{\mathcal{U}} \|a_i\| = 0$. Denote by $\mathcal{A}^{\mathcal{U}}$ the quotient $\prod_{\mathcal{I}} \mathcal{A}/\mathcal{N}_{\mathcal{U}}$. This is a vector space, and with the norm defined by $\|(a_i)_{i\in\mathcal{I}}\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|a_i\|$, and the involution defined by $(a_i)^*_{i\in\mathcal{I}} := (a_i^*)_{i\in\mathcal{I}}$, so $\mathcal{A}^{\mathcal{U}}$ becomes a C^*-algebra.

Remark 2.2. Let $((a_i^n)_{i\in\mathcal{I}})_{n\in\mathbb{N}} \subset \mathcal{A}^{\mathcal{U}}$ be a sequence that converges to $(a_i)_{i\in\mathcal{I}} \in \mathcal{A}^{\mathcal{U}}$. Then, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $\|(a_i^n)_{i\in\mathcal{I}} - (a_i)_{i\in\mathcal{I}}\|_{\mathcal{U}} < \varepsilon$. We claim that if $n \geq n_0$, then the set $\Omega_n(\varepsilon) := \{i \in \mathcal{I} : \|a_i^n - a_i\| < \varepsilon\}$ belongs to $\mathcal{U}$. To show this, set $\alpha_n := \|(a_i^n)_{i\in\mathcal{I}} - (a_i)_{i\in\mathcal{I}}\|_{\mathcal{U}}$. For every $\delta > 0$, we have $\{i \in \mathcal{I} : |\|a_i^n - a_i\| - \alpha_n| < \delta\} \in \mathcal{U}$. Taking $\delta = \varepsilon - \alpha_n > 0$ we get $\varepsilon - \alpha_n > |\|a_i^n - a_i\| - \alpha_n| \geq \|a_i^n - a_i\| - \alpha_n$. It follows that $\{i \in \mathcal{I} : |\|a_i^n - a_i\| - \alpha_n| < \delta\} \subset \Omega_n(\varepsilon)$. By directness, this implies that $\Omega_n(\varepsilon) \in \mathcal{U}$.

From now on we fix a faithful and non-degenerate representation of $\mathcal{A}$ on $B(\mathcal{H})$.

Lemma 2.3. *Let $\mathcal{U}$ be an ultrafilter defined over $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. For each $(a_i)_{i\in\mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$, there exists a unique element $a_{\mathcal{U}\text{-}WOT} \in B(\mathcal{H})$ such that for every $\xi, \eta \in \mathcal{H}$, it holds that $\langle a_{\mathcal{U}\text{-}WOT}\xi, \eta\rangle = \lim_{\mathcal{U}} \langle a_i\xi, \eta\rangle$. The operator $a_{\mathcal{U}\text{-}WOT}$ is called the $\mathcal{U}$-WOT-limit of $(a_i)_{i\in\mathcal{I}}$.*

Proof. Let $(a_i)_{i\in\mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ and let $\xi, \eta \in \mathcal{H}$. Then $(\langle a_i\xi, \eta\rangle)_{i\in\mathcal{I}} \subset \mathbb{C}$ is bounded, hence it has a $\mathcal{U}$-limit, which is denoted by $b_{\xi,\eta}$. It is easy to see that the map

$(\xi,\eta)\mapsto b_{\xi,\eta}$ is a bounded sesquilinear form on $\mathcal{H}\times\mathcal{H}$. Take $a_{\mathcal{U}\text{-}WOT}\in B(\mathcal{H})$ the unique operator associated to it. □

Definition 2.4. Let $\mathcal{U}$ be an ultrafilter defined over $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. An element $(a_i)_{i\in\mathcal{I}}\in\prod_{\mathcal{I}}\mathcal{A}$ is $\mathcal{U}$-*strict* convergent if there exists an operator $a_{\mathcal{U}}\in B(\mathcal{H})$ such that, for every $x\in\mathcal{A}$, and every $\varepsilon>0$ we have $\{i\in\mathcal{I}:\|a_ix-a_{\mathcal{U}}x\|<\varepsilon,\|xa_i-xa_{\mathcal{U}}\|<\varepsilon\}\in\mathcal{U}$. The operator $a_{\mathcal{U}}$ is called the $\mathcal{U}$-*strict limit* of $(a_i)_{i\in\mathcal{I}}$. Observe that $a_{\mathcal{U}}x$ and $xa_{\mathcal{U}}$ are elements of $\mathcal{A}$ for every $x\in\mathcal{A}$.

In what follows, it will be convenient to have the following notation at hand.

Notation 2.5. Let $(a_i)_{i\in\mathcal{I}},(b_i)_{i\in\mathcal{I}}\in\Pi_{\mathcal{I}}\mathcal{A}$ that are $\mathcal{U}$-strict convergent to $a_{\mathcal{U}}$ and $b_{\mathcal{U}}$, respectively. For every $x\in\mathcal{A}$ and every $\varepsilon>0$, put

$$A_x(\varepsilon):=\{i\in\mathcal{I}:\|x(a_i-a_{\mathcal{U}})\|,\|(a_i-a_{\mathcal{U}})x\|<\varepsilon\}\in\mathcal{U},$$
$$B_x(\varepsilon):=\{i\in\mathcal{I}:\|x(b_i-b_{\mathcal{U}})\|,\|(b_i-b_{\mathcal{U}})x\|<\varepsilon\}\in\mathcal{U}.$$

Proposition 2.6. *Let $\mathcal{U}$ be an ultrafilter defined over $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. If $(a_i)_{i\in\mathcal{I}},(b_i)_{i\in\mathcal{I}}\in\Pi_{\mathcal{I}}\mathcal{A}$ define the same element in $\mathcal{A}^{\mathcal{U}}$, then*

(1) *if $(a_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-WOT convergent to $a_{\mathcal{U}\text{-}WOT}$, then $(b_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-WOT convergent to $a_{\mathcal{U}\text{-}WOT}$;*
(2) *if $(a_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $a_{\mathcal{U}}$, then $(b_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $a_{\mathcal{U}}$.*

Proof. To prove (1), take $\xi,\eta\in\mathcal{H}$ of norm 1, let $\varepsilon>0$, and take i in the set $\{i\in\mathcal{I}:\|a_i-b_i\|<\frac{\varepsilon}{2}\}\cap\{i\in\mathcal{I}:|\langle a_{\mathcal{U}\text{-}WOT}\xi,\eta\rangle-\langle a_i\xi,\eta\rangle|<\frac{\varepsilon}{2}\}\in\mathcal{U}$. Then

$$|\langle a_{\mathcal{U}\text{-}WOT}\xi,\eta\rangle-\langle b_i\xi,\eta\rangle|\le|\langle(a_{\mathcal{U}\text{-}WOT}-a_i)\xi,\eta\rangle|+|\langle(a_i-b_i)\xi,\eta\rangle|<\varepsilon.$$

It follows that the set $\{i\in\mathcal{I}:|\langle a_{\mathcal{U}\text{-}WOT}\xi,\eta\rangle-\langle b_i\xi,\eta\rangle|<\varepsilon\}$ belongs to $\mathcal{U}$.

To prove (2), take $\varepsilon>0$, $x\in\mathcal{A}$ and $i\in\{i\in\mathcal{I}:\|a_i-b_i\|<\frac{\varepsilon}{2\|x\|}\}\cap A_x(\frac{\varepsilon}{2})\in\mathcal{U}$. Then

$$\|x(b_i-a_{\mathcal{U}})\|\le\|x(b_i-a_i)\|+\|x(a_i-a_{\mathcal{U}})\|<\varepsilon,$$
$$\|(b_i-a_{\mathcal{U}})x\|\le\|(b_i-a_i)x\|+\|(a_i-a_{\mathcal{U}})x\|<\varepsilon.$$

It follows that the set $\{i\in\mathcal{I}:\|x(b_i-a_{\mathcal{U}})\|,\|(b_i-a_{\mathcal{U}})x\|<\varepsilon\}$ belongs to $\mathcal{U}$. □

Proposition 2.7. *Let $\mathcal{U}$ be an ultrafilter defined over $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. The set*

$$\mathcal{A}^{s\mathcal{U}}:=\{(a_i)_{i\in\mathcal{I}}\in\mathcal{A}^{\mathcal{U}}:\text{there exists } a_{\mathcal{U}}\in B(\mathcal{H}):(a_i)_{i\in\mathcal{I}}\text{ is }\mathcal{U}\text{-strict convergent to }a_{\mathcal{U}}\}$$

is a C^*-algebra.

Proof. Let $(a_i)_{\in\mathcal{I}},(b_i)_{i\in\mathcal{I}}\in\mathcal{A}^{s\mathcal{U}}$ that are $\mathcal{U}$-strict convergent to $a_{\mathcal{U}}$ and $b_{\mathcal{U}}$, respectively, let $\lambda\neq0$ be a complex number, and let $x\in\mathcal{A}$. By following Notation 2.5, if $i\in A_x\left(\frac{\varepsilon}{2}\right)\cap B_x\left(\frac{\varepsilon}{2|\lambda|}\right)\in\mathcal{U}$, then

$$\|(a_i+\lambda b_i-a_{\mathcal{U}}-\lambda b_{\mathcal{U}})x\|\le\|(a_i-a_{\mathcal{U}})x\|+|\lambda|\|(b_i-b_{\mathcal{U}})x\|<\varepsilon,$$

and

$$\|x(a_i+\lambda b_i-a_{\mathcal{U}}-\lambda b_{\mathcal{U}})\|\le\|x(a_i-a_{\mathcal{U}})\|+|\lambda|\|x(b_i-b_{\mathcal{U}})\|<\varepsilon.$$

It follows that

$$A_x\left(\frac{\varepsilon}{2}\right)\cap B_x\left(\frac{\varepsilon}{2|\lambda|}\right)\subset\{i\in\mathcal{I}:\|(a_i+\lambda b_i-a_{\mathcal{U}}-\lambda b_{\mathcal{U}})x\|<\varepsilon,\|x(a_i+\lambda b_i-a_{\mathcal{U}}-\lambda b_{\mathcal{U}})\|<\varepsilon\},$$

so $(a_i+\lambda b_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $a_{\mathcal{U}}+\lambda b_{\mathcal{U}}$.

It is clear that $\mathcal{A}^{s\mathcal{U}}$ is $*$-closed. To show that $\mathcal{A}^{s\mathcal{U}}$ is closed under taking products, set $M=\sup_{i\in\mathcal{I}}\{\|a_i\|,\|b_i\|\}$ and take $i\in A_x(\frac{\varepsilon}{2M})\cap A_{b_{\mathcal{U}}x}(\frac{\varepsilon}{2})\cap B_x(\frac{\varepsilon}{2M})\cap B_{xa_{\mathcal{U}}}(\frac{\varepsilon}{2})\in\mathcal{U}$. Then

$$\|(a_ib_i-a_{\mathcal{U}}b_{\mathcal{U}})x\|\leq\|(a_ib_i-a_ib_{\mathcal{U}})x\|+\|(a_ib_{\mathcal{U}}-a_{\mathcal{U}}b_{\mathcal{U}})x\|\leq\varepsilon$$

and

$$\|x(a_ib_i-a_{\mathcal{U}}b_{\mathcal{U}})\|\leq\|x(a_ib_i-a_{\mathcal{U}}b_i)\|+\|x(a_{\mathcal{U}}b_i-a_{\mathcal{U}}b_{\mathcal{U}})\|\leq\varepsilon,$$

which means that $(a_ib_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $a_{\mathcal{U}}b_{\mathcal{U}}$.

It is left to show that $\mathcal{A}^{s\mathcal{U}}$ is norm closed. Let $((a_i^n)_{i\in\mathcal{I}})_{n\in\mathbb{N}}$ be a sequence in $\mathcal{A}^{s\mathcal{U}}$ that converges to $(\alpha_i)_{i\in\mathcal{I}}$ in $\mathcal{A}^{\mathcal{U}}$. We need to see that $(\alpha_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent.

As a first step, we will show that, for a fixed element $x\in\mathcal{A}$, $(\alpha_ix)_{i\in\mathcal{I}}$ and $(x\alpha_i)_{i\in\mathcal{I}}$ have $\mathcal{U}$-limit in $\mathcal{A}$ (in the sense of Definition 2.1).

Let $x\in\mathcal{A}$ be fixed and $x\neq 0$. For each $n\in\mathbb{N}$ let $a_{\mathcal{U}}^n$ be the $\mathcal{U}$-strict limit of $(a_i^n)_{i\in\mathcal{I}}\in\mathcal{A}^{s\mathcal{U}}$. We proceed to show that $(a_{\mathcal{U}}^nx)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{A}$. By Remark 2.2, for every $\varepsilon>0$, there exists $n_0\in\mathbb{N}$ such that, for all $n\geq n_0$, the sets $\Omega_n(\frac{\varepsilon}{4\|x\|})$ are elements of $\mathcal{U}$. It follows that the set

$$\{i\in\mathcal{I}:\|a_i^nx-a_{\mathcal{U}}^nx\|<\frac{\varepsilon}{4}\}\cap\{i\in\mathcal{I}:\|a_i^mx-a_{\mathcal{U}}^mx\|<\frac{\varepsilon}{4}\}\cap\Omega_n\left(\frac{\varepsilon}{4\|x\|}\right)\cap\Omega_m\left(\frac{\varepsilon}{4\|x\|}\right)$$

is an element of $\mathcal{U}$ for all $n,m\geq n_0$. Take i in this set. Then

$$\begin{aligned}\|a_{\mathcal{U}}^nx-a_{\mathcal{U}}^mx\|&\leq\|a_{\mathcal{U}}^nx-a_i^nx\|+\|a_i^nx-a_i^mx\|+\|a_i^mx-a_{\mathcal{U}}^mx\|\\&\leq\frac{\varepsilon}{4}+\|a_i^n-\alpha_i\|\|x\|+\|\alpha_i-a_i^m\|\|x\|+\frac{\varepsilon}{4}<\varepsilon.\end{aligned}$$

Let $\rho(x):=\lim_{n\in\mathbb{N}}a_{\mathcal{U}}^nx\in\mathcal{A}$. We will show that $\lim_{\mathcal{U}}\alpha_ix=\rho(x)$, that is, $\{i\in\mathcal{I}:\|\rho(x)-\alpha_ix\|<\varepsilon\}\in\mathcal{U}$ for each $\varepsilon>0$. Let $n\in\mathbb{N}$ large enough such that

$$\|\rho(x)-a_{\mathcal{U}}^nx\|<\frac{\varepsilon}{3}\quad\text{and}\quad\Omega_n\left(\frac{\varepsilon}{3\|x\|}\right)\in\mathcal{U}.$$

For such $n\in\mathbb{N}$, take $i\in\{i\in\mathcal{I}:\|a_{\mathcal{U}}^nx-a_i^nx\|<\frac{\varepsilon}{3}\}\cap\Omega_n(\frac{\varepsilon}{3\|x\|})\in\mathcal{U}$. Then

$$\|\rho(x)-\alpha_ix\|\leq\|\rho(x)-a_{\mathcal{U}}^nx\|+\|a_{\mathcal{U}}^nx-a_i^nx\|+\|a_i^nx-\alpha_ix\|\leq\varepsilon.$$

Repeating this with $(x\alpha_i)_{i\in\mathcal{I}}$ concludes the first step.

Let $\alpha_{\mathcal{U}\text{-}WOT}\in B(\mathcal{H})$ be the $\mathcal{U}$-WOT-limit of $(\alpha_i)_{i\in\mathcal{I}}$. We will show that $\alpha_{\mathcal{U}\text{-}WOT}x=\rho(x)$. Take $\eta,\xi\in\mathcal{H}$ of norm 1, $\varepsilon>0$, and

$$i\in\{i\in\mathcal{I}:|\langle(\alpha_i-\alpha_{\mathcal{U}\text{-}WOT})x\xi,\eta\rangle|<\frac{\varepsilon}{2}\}\cap\{i\in\mathcal{I}:\|\rho(x)-\alpha_ix\|<\frac{\varepsilon}{2}\}\in\mathcal{U}.$$

We then have that

$$|\langle(\rho(x)-\alpha_{\mathcal{U}\text{-}WOT}x)\xi,\eta\rangle|\leq|\langle(\rho(x)-\alpha_ix)\xi,\eta\rangle|+|\langle(\alpha_ix-\alpha_{\mathcal{U}\text{-}WOT}x)\xi,\eta\rangle|<\varepsilon,$$

which implies that $\alpha_{\mathcal{U}\text{-}WOT}x = \rho(x) = \lim_{\mathcal{U}} \alpha_i x$. Therefore, for all $x \in \mathcal{A}$ and $\varepsilon > 0$, we have $\{i \in \mathcal{I} : \|\alpha_{\mathcal{U}\text{-}WOT}x - \alpha_i x\| < \varepsilon\} \in \mathcal{U}$. In a similar manner, one shows that $\{i \in \mathcal{I} : \|x\alpha_{\mathcal{U}\text{-}WOT} - x\alpha_i\| < \varepsilon\} \in \mathcal{U}$. It follows that $(\alpha_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $\alpha_{\mathcal{U}\text{-}WOT}$. □

Proposition 2.8. *Let $\mathcal{U}$ be an ultrafilter defined over $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. The set*

$$J := \{(a_i)_{i\in\mathcal{I}} \in \mathcal{A}^{s\mathcal{U}} : a_i \text{ is } \mathcal{U}\text{-strict convergent to } 0\}$$

is an ideal of $\mathcal{A}^{s\mathcal{U}}$.

Proof. We only have to show that J is norm closed. Consider $((a_i^n)_{i\in\mathcal{I}})_{n\in\mathbb{N}} \subset J$ a sequence that converges in norm to $(\alpha_i)_{i\in\mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$. Let $\alpha_{\mathcal{U}}$ be the $\mathcal{U}$-strict limit of $(\alpha_i)_{i\in\mathcal{I}}$. Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ large enough such that $\Omega_n\left(\frac{\varepsilon}{3\|x\|}\right) \in \mathcal{U}$, and $i \in \{i \in \mathcal{I} : \|(\alpha_i - \alpha_{\mathcal{U}})x\| < \frac{\varepsilon}{3}\} \cap \{i \in \mathcal{I} : \|a_i^n x\| < \frac{\varepsilon}{3}\} \cap \Omega_n\left(\frac{\varepsilon}{3\|x\|}\right) \in \mathcal{U}$. We then have that $\|\alpha_{\mathcal{U}}x\| \leq \|(\alpha_{\mathcal{U}} - \alpha_i)x\| + \|\alpha_i x\| \leq \frac{\varepsilon}{3} + \|(\alpha_i - a_i^n)x\| + \|a_i^n x\| \leq \varepsilon$. Since the action of $\mathcal{A}$ on $\mathcal{H}$ is nondegenerate, $\alpha_{\mathcal{U}} = 0$. □

There exists a natural embedding of $\mathcal{A}$ in $\mathcal{A}^{s\mathcal{U}}$, via the constant sequences $a \mapsto (a)_{i\in\mathcal{I}}$. It is clear that this element is $\mathcal{U}$-strict convergent to a. Moreover, since the representation of $\mathcal{A}$ in $B(\mathcal{H})$ is faithful and nondegenerate, it follows that there exists a natural embedding of $\mathcal{A}$ in $\mathcal{A}^{s\mathcal{U}}/J$.

Recall that an ideal I in a C^*-algebra $\mathcal{A}$ is *essential* if $I \cap K$ is nontrivial for every ideal $K \neq \{0\}$, or equivalently, $aI = 0$ implies $a = 0$.

Lemma 2.9. *Let $\mathcal{U}$ be an ultrafilter defined over $\mathcal{I}$ and let $\mathcal{A}$ be a C^*-algebra. Consider the C^*-algebra $\mathcal{A}^{s\mathcal{U}}/J$. The image of $\mathcal{A}$ in $\mathcal{A}^{s\mathcal{U}}/J$ is an essential ideal.*

Proof. Take $(b_i)_{i\in\mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$, let $b_{\mathcal{U}}$ be its $\mathcal{U}$-strict limit, and take $a \in \mathcal{A}$. Then $(b_i a)_{i\in\mathcal{I}}$ and $(b_{\mathcal{U}}a)_{i\in\mathcal{I}}$ are both $\mathcal{U}$-strict convergent to $b_{\mathcal{U}}a$. It follows that $(b_i a)_{i\in\mathcal{I}}$ and $(b_{\mathcal{U}}a)_{i\in\mathcal{I}}$ are equal in $\mathcal{A}^{s\mathcal{U}}/J$. Analogously, $(ab_i)_{i\in\mathcal{I}}$ is equal to $(ab_{\mathcal{U}})_{i\in\mathcal{I}}$ in $\mathcal{A}^{s\mathcal{U}}/J$.

Suppose that $J' \subset \mathcal{A}^{s\mathcal{U}}/J$ is an ideal such that $J' \cap \mathcal{A} = \{0\}$. If $(b_i)_{i\in\mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$ projects to J', then $(b_i x)_{i\in\mathcal{I}} \in J$,for each $x \in \mathcal{A}$. Let $b_{\mathcal{U}}$ be the $\mathcal{U}$-strict limit of $(b_i)_{i\in\mathcal{I}}$. Hence $(b_i x)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $b_{\mathcal{U}}x$. Then $b_{\mathcal{U}}x = 0$ for all $x \in \mathcal{A}$. It follows that $b_{\mathcal{U}} = 0$ and then $J' = \{0\}$. □

2.2. **Ultrafilters and approximate units.** Every C^*-algebra $\mathcal{A}$ has an approximate unit, namely, there exist a directed set $\mathcal{I}$ and a net $(e_i)_{i\in\mathcal{I}} \subset \mathcal{A}$ such that for every $x \in \mathcal{A}$, the nets $(xe_i)_{i\in\mathcal{I}}$ and $(e_i x)_{i\in\mathcal{I}}$ converge to x (see [6, Chapter I.4]). Approximate units can be taken to be positive and uniformly bounded, in which case they are elements of $\prod_{\mathcal{I}} \mathcal{A}$. In what follows, we will focus in the case where the ultrafilters are defined over this directed set $\mathcal{I}$. Observe that for a cofinal ultrafilter $\mathcal{U}$, the sets $\{i \in \mathcal{I} : \|xe_i - x\| < \varepsilon\}$ and $\{i \in \mathcal{I} : \|e_i x - x\| < \varepsilon\}$ belong to $\mathcal{U}$, for every $x \in \mathcal{A}$ and for every $\varepsilon > 0$. Moreover, when $\mathcal{A}$ is unital, $\mathcal{I}$ can be taken to be the set with one element $\{1_{\mathcal{A}}\}$ and the approximate unit to be equal to $\{1_{\mathcal{A}}\}$. In this case, the only ultrafilter is the set $\{1_{\mathcal{A}}\}$, which is cofinal.

Theorem 2.10. *Let $\mathcal{A}$ be a C^*-algebra and let $(e_i)_{i\in\mathcal{I}} \in \prod_{i\in\mathcal{I}} \mathcal{A}$ be an approximate unit for $\mathcal{A}$. Let $\mathcal{U}$ be a cofinal ultrafilter over $\mathcal{I}$. Then the C^*-algebra $\mathcal{A}^{s\mathcal{U}}/J$ is the multiplier algebra of $\mathcal{A}$.*

Proof. We saw in Lemma 2.9 that $\mathcal{A}$ is an essential ideal in $\mathcal{A}^{s\mathcal{U}}/J$. We are left to show that for any C^*-algebra $\mathcal{B}$ containing $\mathcal{A}$ as an ideal, there is a unique C^*-homomorphism $\varphi : \mathcal{B} \to \mathcal{A}^{s\mathcal{U}}/J$ such that $\varphi(a) = a$.
To this end, let $b \in \mathcal{B}$ and consider $\psi : \mathcal{B} \to \mathcal{A}^{s\mathcal{U}}$ defined by $\psi(b) = (be_i)_{i\in\mathcal{I}}$. To see that ψ is well defined, let $\varepsilon > 0$ and let $x \in \mathcal{A}$. Let $i_0 \in \mathcal{I}$ such that if $i \geq i_0$, then $\|xbe_i - xb\| < \varepsilon$. Let $i_1 \in \mathcal{I}$ such that if $i \geq i_1$, then $\|e_i x - x\| < \dfrac{\varepsilon}{\|b\|}$. By the cofinality of $\mathcal{U}$ one obtains that $\{i \in \mathcal{I} : \|be_i x - bx\| < \varepsilon, \|xbe_i - xb\| < \varepsilon\} \in \mathcal{U}$. Observe that since $b \notin B(\mathcal{H})$, the last line does not imply that $(be_i)_{i\in\mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$. We must "represent" b in $B(\mathcal{H})$. To this end, let $b_{\mathcal{U}\text{-}WOT} \in B(\mathcal{H})$ be the $\mathcal{U}$-WOT-limit of $(be_i)_{i\in\mathcal{I}} \in \mathcal{A}^{\mathcal{U}}$, an argument similar to one given in the proof of Proposition 2.7, shows that $b_{\mathcal{U}\text{-}WOT}x = bx$ and $xb_{\mathcal{U}\text{-}WOT} = xb$. Thus $(be_i)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $b_{\mathcal{U}\text{-}WOT}$. The same procedure shows that $(e_i b)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to $b_{\mathcal{U}\text{-}WOT}$.

Call π the quotient projection to $\mathcal{A}^{s\mathcal{U}}/J$, and let $\varphi = \pi \circ \psi$. It is clear that φ is linear and bounded. Since $\psi(b^*) = (b^* e_i)_{i\in\mathcal{I}}$ and $\psi(b)^* = (e_i b^*)_{i\in\mathcal{I}}$ and they are both $\mathcal{U}$-strict convergent to $(b^*)_{\mathcal{U}\text{-}WOT}$, hence $\psi(b^*) - \psi(b)^*$ is an element of J, so φ is a $*$-preserving homomorphism.

To see that φ is multiplicative, fix $b, b' \in \mathcal{B}$ of norm 1 and take $x \in \mathcal{A}$, $\varepsilon > 0$, $M = \sup_{i\in\mathcal{I}}\{\|e_i\|\}$ and i in the set

$$\{i \in \mathcal{I} : \|e_i x - x\| < \frac{\varepsilon}{3M}\} \cap \{i \in \mathcal{I} : \|x - e_i x\| < \frac{\varepsilon}{3M}\} \cap \{i \in \mathcal{I} : \|e_i b' x - b' x\| < \frac{\varepsilon}{3M}\},$$

which is an element of $\mathcal{U}$. Then

$$\begin{aligned}\|(be_i b' e_i - bb' e_i)x\| &\leq \|e_i b' e_i x - e_i b' x\| + \|e_i b' x - b' x\| + \|b' x - b' e_i x\| \\ &\leq \|e_i b'\|\|e_i x - x\| + \|e_i b' x - b' x\| + \|b'\|\|x - e_i x\| < \varepsilon.\end{aligned}$$

Take $i \in \{i \in \mathcal{I} : \|xbe_i - xb\| < \dfrac{\varepsilon}{M}\} \in \mathcal{U}$. Then $\|x(be_i b' e_i - bb' e_i)\| \leq \|xbe_i - xb\|\|b' e_i\| < \varepsilon$. It follows that $\psi(b)\psi(b') - \psi(bb') = (be_i b' e_i - bb' e_i)_{i\in\mathcal{I}}$ is an element of J.

Since $(ae_i - a)_{i\in\mathcal{I}}$ is $\mathcal{U}$-strict convergent to 0, for all $a \in \mathcal{A}$, so $\varphi(a) = a$ in $\mathcal{A}^{s\mathcal{U}}/J$.

Suppose that there exists another C^*-homomorphism $\varphi' : \mathcal{B} \to \mathcal{A}^{s\mathcal{U}}/J$ such that $\varphi'(a) = a$ for all $a \in \mathcal{A}$. Then

$$\varphi'(b)a = \varphi'(b)\varphi'(a) = ba = \varphi(b)\varphi(a) = \varphi(b)a.$$

By Lemma 2.9, $\varphi(b) = \varphi'(b)$. □

Ultraproducts provide a new point of view for dealing with multiplier algebras. For instance, the identification of $\mathcal{M}(\mathcal{A})$ with $\mathcal{A}^{s\mathcal{U}}/J$ yields an easy proof of the next characterization of multipliers, without using double centralizers.

Corollary 2.11. *$\mathcal{M}(\mathcal{A})$ is isomorphic to $\mathcal{M} := \{m \in B(\mathcal{H}) :$ for all $a \in \mathcal{A}, am \in \mathcal{A}, ma \in \mathcal{A}\}$. In particular, $\mathcal{M}(\mathcal{A})$ is unital.*

Proof. Consider $\varphi : \mathcal{A}^{s\mathcal{U}} \to \mathcal{M}$ defined by $\varphi((a_i)_{i\in\mathcal{I}}) = \lim_{\mathcal{U}\text{-strict}} a_i$. This map is well defined, it is a C^*-homomorphism (Proposition 2.7), and $\ker(\varphi) = J$. To show that φ is surjective, let $m \in \mathcal{M}$. Then $am \in \mathcal{A}$ and $ma \in \mathcal{A}$ for every $a \in \mathcal{A}$. Hence, for all $\varepsilon > 0$, the sets $\{i \in \mathcal{I} : \|a(me_i - m)\| < \varepsilon\}$ and $\{i \in \mathcal{I} : \|(me_i - m)a\| < \varepsilon\}$ are elements of $\mathcal{U}$. Then $(me_i)_{i\in\mathcal{I}} \in \mathcal{A}^{\mathcal{U}}$ is $\mathcal{U}$-strict convergent to m.

Taking $m = 1$, it follows that the image of $(e_i)_{i\in\mathcal{I}}$ in $\mathcal{A}^{s\mathcal{U}}/J$ is the unit of $\mathcal{A}^{s\mathcal{U}}/J$. □

For a second application, observe that every C^*-homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ defines a natural C^*-homomorphism $\phi' : \mathcal{A}^{\mathcal{U}} \to \mathcal{B}^{\mathcal{U}}$. When ϕ is surjective, a proof similar to one given in Proposition 2.7 shows that $\phi'(\mathcal{A}^{s\mathcal{U}}) \subset \mathcal{B}^{s\mathcal{U}}$. This together with Theorem 2.10 immediately gives the following known result.

Proposition 2.12. *Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a surjective homomorphism. The natural extension $\phi' : \mathcal{A}^{\mathcal{U}} \to \mathcal{B}^{\mathcal{U}}$ induces the following commutative diagram:*

$$\begin{array}{ccccccc}
\mathcal{A} & \longrightarrow & \mathcal{A}^{s\mathcal{U}} & \longrightarrow & \mathcal{M}(\mathcal{A}) & \longrightarrow & \mathcal{M}(\mathcal{A})/\mathcal{A} \\
\Big\downarrow{\scriptstyle \phi} & & \Big\downarrow{\scriptstyle \phi'} & & \Big\downarrow{\scriptstyle \phi''} & & \Big\downarrow{\scriptstyle \phi'''} \\
\mathcal{B} & \longrightarrow & \mathcal{B}^{s\mathcal{U}} & \longrightarrow & \mathcal{M}(\mathcal{B}) & \longrightarrow & \mathcal{M}(\mathcal{B})/\mathcal{B}.
\end{array}$$

3. THE CASE OF COMMUTATIVE AND SEPARABLE C^*-ALGEBRAS

Recall that when a C^*-algebra $\mathcal{A}$ is separable, it is σ-unital, namely, there exists a countable approximate unit (see [6, Chapter I.4]). That entails that the index set $\mathcal{I}$ of the previous section can be taken to be equal to $\mathbb{N}$, in which case nonprincipal ultrafilters are cofinal and $\prod_{\mathcal{I}} \mathcal{A}$ is $\ell^\infty(\mathcal{A})$.

In [1], the authors built the multiplier algebra for commutative and separable C^*-algebras using ultraproducts of C^*-algebras. More precisely they considered $\mathcal{A} = C_0(X)$ where X is a second countable, locally compact topological space and took $\mathcal{U}$ a nonprincipal ultrafilter defined over $\mathbb{N}$ to construct the multiplier algebra of $\mathcal{A}$, $C_b(X)$, identifying it with a quotient of a sub-C^*-algebra of $\prod_{\mathcal{I}} \mathcal{A}$. For that, the authors usee the key fact that the hypothesis on X entails the existence of a proper metric compatible [8]. In what follows, we will then identify the second countable, locally compact topological space X with the metric space (X, d), where d is a proper metric on X. The closed ball of radius $r > 0$ centered at a fixed base-point $o \in X$, will be denoted by $B_o(r)$. The main technical tool of [1] is the following definition.

Definition 3.1. [1, Section 3] Let (X, d) be as in the preceding discussion. Let $\mathcal{A} = C_0(X)$ and let $\mathcal{U}$ be a nonprincipal ultrafilter over $\mathbb{N}$. For $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{A})$, we say that $(f_n)_{n\in\mathbb{N}}$ is *$\mathcal{U}$-equicontinuous on bounded sets* if, for every $r, \varepsilon > 0$, there is $\delta > 0$ such that the set $\{n \in \mathbb{N} :$ for all $s, t \in B_o(r)$ with $d(s,t) < \delta \implies |f_n(s) - f_n(t)| < \varepsilon\}$ belongs to $\mathcal{U}$.

Given $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{A})$, and a fixed $x \in X$, the $\mathcal{U}$-limit of the sequence $(f_n(x))_{n\in\mathbb{N}}$ is well defined. We denote $f_{\mathcal{U}} : X \to \mathbb{C}$, $f_{\mathcal{U}}(x) := \lim_{\mathcal{U}}(f_n(x))$. The following fact was observed in [1].

Lemma 3.2. *If $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{A})$ is $\mathcal{U}$-equicontinuous on bounded sets, then $f_{\mathcal{U}}$ is uniformly continuous on bounded sets.*

The next proposition shows that our notion of $\mathcal{U}$-strict convergence coincides with the notion of $\mathcal{U}$-equicontinuity on bounded sets in the case of $\mathcal{A} = C_0(X)$, where X is a locally compact, second countable, topological space. This entails that the work done here in section 2 is indeed a generalization of the work done in [1, Section 3].

Proposition 3.3. *Take $(f_n) \in \ell^\infty(\mathcal{A})$ and let $f_{\mathcal{U}}(x) = \lim_{\mathcal{U}}(f_n(x))$. The following two conditions are equivalent:*

(1) *The sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{U}$-equicontinuous on bounded sets.*
(2) *The sequence $(f_n)_{n\in\mathbb{N}}$ is $\mathcal{U}$-strict convergent to $f_{\mathcal{U}}$.*

Proof. To show that (1) implies (2), let $(f_n)_{n\in\mathbb{N}} \in \ell^\infty(\mathcal{A})$ be $\mathcal{U}$-equicontinuous on bounded sets. Let $\varepsilon > 0$ and let $g \in C_0(X)$. Set $M = \sup_{n\in\mathbb{N}}\{\|f_n\|, \|g\|\}$, and take $K \subset X$ a compact set such that $|g(x)| < \dfrac{\varepsilon}{2M}$ if $x \notin K$. There exists δ_1 such that for $x, y \in K$ with $d(x, y) < \delta_1$, $\{n \in \mathbb{N} : |f_n(x) - f_n(y)| \leq \dfrac{\varepsilon}{3M}\} \in \mathcal{U}$. By Lemma 3.2, there exists $\delta_2 > 0$ such that $|f_{\mathcal{U}}(x) - f_{\mathcal{U}}(y)| < \dfrac{\varepsilon}{3}$, for $x, y \in K$ with $d(x, y) < \delta_2$. Take $\delta = \min\{\delta_1, \delta_2\}$ and cover K with a finite number of balls $B_{x_j}(\delta)$ of radius δ centered at x_j, $j = 1, \dots, m$. Since $f_{\mathcal{U}}(x_j) = \lim_{\mathcal{U}} f_n(x_j)$, it follows that the sets $A_j = \left\{n \in \mathbb{N} : |f_n(x_j) - f_{\mathcal{U}}(x_j)| < \dfrac{\varepsilon}{3M}\right\}$ belong to $\mathcal{U}$. Therefore if $n \in \bigcap_{j=1}^m A_j \in \mathcal{U}$, and $x \in K$, taking x_j with $d(x, x_j) < \delta$ we get

$$|(f_n(x) - f_{\mathcal{U}}(x))g(x)| \leq |f_n(x) - f_n(x_j)|\|g\| + |f_n(x_j) - f_{\mathcal{U}}(x_j)|\|g\| + |f_{\mathcal{U}}(x_j) - f_{\mathcal{U}}(x)|\|g\| < \varepsilon.$$

On the other hand, if $x \notin K$, then $|(f_n - f_{\mathcal{U}})g(x)| < \varepsilon$. This shows that $\{n \in \mathbb{N} : \|f_n g - f_{\mathcal{U}} g\| < \varepsilon\} \in \mathcal{U}$.

To show the converse, take $g \in C_0(X)$ such that $g = 1$ in $B_o(r)$. By hypothesis, for all $\varepsilon > 0$ the sets $\{n \in \mathbb{N} : \|(f_n - f_{\mathcal{U}})g\| < \varepsilon\}$ are in $\mathcal{U}$ and are infinite. So we can build a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $(f_{n_k})_{k\in\mathbb{N}}$ is uniformly convergent to $f_{\mathcal{U}}$ in $B_0(r)$. By hypothesis, closed balls are compact, so $f_{\mathcal{U}}$ is uniformly continuous on $B_o(r)$. Let $\delta > 0$ such that $d(x, y) < \delta$ implies $|f_{\mathcal{U}}(y) - f_{\mathcal{U}}(x)| < \varepsilon$ in $B_o(r)$. Take $n \in \{n \in \mathbb{N} : \|(f_n - f_{\mathcal{U}})g\| < \dfrac{\varepsilon}{3}\} \in \mathcal{U}$. Then for $x, y \in B_o(r)$ such that $d(x, y) < \delta$ we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_{\mathcal{U}}(x)| + |f_{\mathcal{U}}(x) - f_{\mathcal{U}}(y)| + |f_n(y) - f_{\mathcal{U}}(y)| \leq \varepsilon,$$

therefore $\{n \in \mathbb{N} : \|(f_n - f_{\mathcal{U}})g\| < \frac{\varepsilon}{3}\}$ is a subset of $\{n \in \mathbb{N} :$ for all $x, y \in B_o(r)$ with $d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \varepsilon\}$. This shows that the last set belongs to $\mathcal{U}$. ∎

4. An application to boundary amenability of groups

Let Γ be a countable group. Let $\mathcal{A}$ be a unital C^*-algebra endowed with a Γ-action by $*$-automorphisms. Let $C_c(\Gamma, \mathcal{A})$ be the space of finitely supported functions from Γ to $\mathcal{A}$. This is a $*$-algebra with the product given by

$$T * S(\gamma) = \sum_{\gamma_1\gamma_2=\gamma} T(\gamma_1)(\gamma_1 \cdot S(\gamma_2))$$

and the involution given by

$$T^*(\gamma) = \gamma \cdot T(\gamma^{-1})^*.$$

The space $C_c(\Gamma, \mathcal{A})$ has a pre-Hilbert $\mathcal{A}$-module structure via the inner product $\langle T, S\rangle_{\mathcal{A}} := \sum_{\gamma\in\Gamma} T(\gamma)^*S(\gamma)$, and the corresponding norm $\|T\|_{\mathcal{A}} := \langle T, T\rangle^{1/2}$.

Definition 4.1. A group Γ is boundary amenable (or exact) if there exist a compact space Y and a sequence $(S_i)_{i\in\mathbb{N}} \subset C_c(\Gamma, C(Y))$ such that

(1) S_i is positive, that is, $S_i(\gamma) \geq 0$ for all $\gamma \in \Gamma$;
(2) $\sum_{\gamma\in\Gamma} S_i^2(\gamma) = 1$;
(3) for every $\gamma_1 \in \Gamma$, it follows that $\|S_i - \gamma_1 * S_i\|$ goes to 0 when i goes to infinity.

In [1, Section 4], the authors applied their ultraproduct construction of the multiplier algebra of a separable C^* algebra to show that groups acting properly and transitively on a locally finite tree are boundary amenable. The assumption that the action is proper and transitive implies that the vertex stabilizers are all isomorphic finite groups of cardinal m. In the proof given in [1], this constant m contains all the information that is needed about the stabilizers. Since it is known that groups acting on a locally finite tree with exact stabilizers are boundary amenable (see, for instance,[2, 3, 5, 7]), it is interesting to see if the strategy developed in [1] can be extended to prove this more general result. We partially achieve this in the proof of the next theorem.

Theorem 4.2. *If a countable group Γ acts transitively on a locally finite tree $\mathcal{T}$ with boundary amenable stabilizers, then Γ is boundary amenable.*

Proof. Fix a vertex $o \in \mathcal{T}$ as a base-point. We will denote with $B(i)$ the closed ball of radius r centered in o. The geodesic that connects o with t will be denoted by $[o, t]$. In what follows $Stab\{o\}$ will be denoted with Λ.

Since the action of Γ on $\mathcal{T}$ is transitive, we choose a cross section for it, namely, for each $v \in \mathcal{T}$ we choose $g_v \in \Gamma$ such that $g_v o = v$.

Set $\left(\Gamma_i\right)_{i\in\mathbb{N}}$ an increasing sequence of finite subsets of Γ that covers Γ and let

$$\Lambda_i := \bigcup_{v\in B(i)} \bigcup_{w\in B(i)} \left(g_v^{-1}\Gamma_i g_w \cap \Lambda\right).$$

Since $\mathcal{T}$ is locally finite, Λ_i is finite.

Since Λ is boundary amenable, there exist a compact set Y and a sequence $(S_i)_{i\in\mathbb{N}} \subset C_c((\Lambda, C(Y))$ satisfying the three conditions of Definition 4.1. The compact space Y will be taken to be a Γ-space (see [3, pg. 178]). Moreover, we can assume that S_i is defined over all Γ by extending it by 0 in $\Gamma \setminus \Lambda$. For every $i \in \mathbb{N}$, let

$$\kappa(i) := \min\{k \in \mathbb{N} : ||S_l - \lambda * S_l\| < \frac{1}{i} \text{ for all } l \geq k \text{ and for all } \lambda \in \Lambda_i\}.$$

Consider the C^*-algebra $\mathcal{A} := C_0(\mathcal{T} \times Y)$. Since Y is compact and $\mathcal{T}$ is countable, $\mathcal{A}$ is a σ-unital C^*-algebra, hence we are in the case discussed at the beginning of Section 3. Let $\mathcal{U}$ be a non principal ultrafilter over $\mathbb{N}$ and consider the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$, which we identify with $\mathcal{A}^{s\mathcal{U}}/J/\mathcal{A}$. This is a unital C^*-algebra.

For each $i \in \mathbb{N}$, set $T_i : \Gamma \to \mathcal{A}^{s\mathcal{U}}$, where $T_i(\gamma) = (T_i^n(\gamma))_{n\in\mathbb{N}}$ is defined by

$$T_i^n(\gamma)(t, y) = \frac{1}{\sqrt{\Big|[o, t] \cap B(i)\Big|}} \chi_{B(n)}(t) \sum_{v\in B(i)} \chi_{[o,t]}(v) S_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y),$$

where $\Big|[o, t] \cap B(i)\Big|$ denotes the number of points in the geodesic $[o, t]$ that lie inside the ball $B(i)$. Note that $(T_i^n(\gamma))_{n\in\mathbb{N}}$ is $\mathcal{U}$-strict convergent to

$$T_i = T_i(\gamma)(t, y) = \frac{1}{\sqrt{\Big|[o, t] \cap B(i)\Big|}} \sum_{v\in B(i)} \chi_{[o,t]}(v) S_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y)$$

We will denote by T_i its projection to $\mathcal{A}^{s\mathcal{U}}/J/\mathcal{A}$. We remark that the definition of T_i is inspired by the definition of $\mu_{x,y}$ given by Ozawa in [5].

Claim: The sequence $(T_i)_{i\in\mathbb{N}} \subseteq C(\Gamma; \mathcal{A}^{s\mathcal{U}}/J/\mathcal{A})$ satisfies the conditions of Definition 4.1.

To show this, first observe that if $\Omega_{\kappa(i)} \subseteq \Lambda$ denotes the support of $S_{\kappa(i)}$, then the support of T_i is contained in $\bigcup_{v\in B(i)} g_v\Omega_{\kappa(i)}$. Since $\mathcal{T}$ is locally finite, this is a finite set.

To show that condition (1) holds true, it is enough to notice that T_i is sum and product of positive functions.

To show that condition (2) holds true, note that for each fixed $\gamma \in \Gamma$, there exists only one g_v such that $g_v^{-1}\gamma \in \Lambda$. Then there exists at most one g_v such that $S_{\kappa(i)}(g_v^{-1}\gamma)$ is nonzero. Then

$$\begin{aligned}
\sum_{\gamma\in\Gamma}(T_i)^2(t, y) &= \sum_{\gamma\in\Gamma} \frac{1}{\Big|[o, t] \cap B(i)\Big|} \sum_{v\in B(i)} \chi_{[o,t]}(v) S^2_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y) \\
&= \frac{1}{\Big|[o, t] \cap B(i)\Big|} \sum_{v\in B(i)} \chi_{[o,t]}(v) \sum_{\gamma\in\Gamma} S^2_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y) \\
&= \frac{1}{\Big|[o, t] \cap B(i)\Big|} \sum_{v\in B(i)} \chi_{[o,t]}(v) \sum_{\lambda\in\Lambda} S^2_{\kappa(i)}(\lambda)(g_v^{-1}y) \\
&= \frac{1}{\Big|[o, t] \cap B(i)\Big|} \sum_{v\in B(i)} \chi_{[o,t]}(v) = 1.
\end{aligned}$$

It remains to show that condition (3) holds true. To this end, fix $\gamma_1 \in \Gamma$. Observe that if $(t, y) \in \mathcal{T} \times Y$, then $\gamma_1 * T_i(\gamma)(t, y) = T_i(\gamma_1^{-1}\gamma)(\gamma_1^{-1}t, \gamma_1^{-1}y)$.

Hence

$$\begin{aligned}\|T_i - \gamma_1 * T_i\|^2_{C_c(\Gamma,\mathcal{A}^s\mathcal{U}/J/\mathcal{A})} &= \|\sum_{\gamma\in\Gamma}(T_i(\gamma) - \gamma_1 * T_i(\gamma))^2\|_{\mathcal{A}^s\mathcal{U}/J\big/\mathcal{A}}\\ &= \inf_{a\in\mathcal{A}} \|\sum_{\gamma\in\Gamma}(T_i(\gamma) - \gamma_1 * T_i(\gamma))^2 - a\|_{\mathcal{A}^s\mathcal{U}/J}\\ &= \inf_{a\in\mathcal{A}} \|2 - 2\sum_{\gamma\in\Gamma} T_i(\gamma)(\gamma_1 * T_i(\gamma)) - a\|_{\mathcal{A}^s\mathcal{U}/J} \qquad (4.1)\end{aligned}$$

If we set $\theta(i,t) := \Big|[o,t]\cap B(i)\Big|^{-1/2}\Big|[o,\gamma_1^{-1}t]\cap B(i)\Big|^{-1/2}$, then

$$\begin{aligned}&\sum_{\gamma\in\Gamma} T_i(\gamma)(t,y)T_i(\gamma_1^{-1}\gamma)(\gamma_1^{-1}t,\gamma_1^{-1}y)\\ &= \theta(i,t)\sum_{\substack{v\in B(i)\\ w\in B(i)}} \chi_{[o,t]}(v)\chi_{[o,\gamma_1^{-1}t]}(w)\sum_{\gamma\in\Gamma} S_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y)S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}\gamma)(g_w^{-1}\gamma_1^{-1}y)\\ &= \theta(i,t)\sum_{\substack{v\in B(i)\\ w\in B(i)}} \chi_{[o,t]}(v)\chi_{[o,\gamma_1^{-1}t]}(w)\sum_{\lambda\in\Lambda} S_{\kappa(i)}(\lambda)(z)S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_v z),\end{aligned} \qquad (4.2)$$

where the last equality follows from the changes of variables $\lambda := g_v^{-1}\gamma$ and $z := g_v^{-1}y$, and by recalling that $S_{\kappa(i)}(\lambda) = 0$ if $\lambda \notin \Lambda$.

In order to get that $S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda) \neq 0$, it is necessary that $g_w^{-1}\gamma_1^{-1}g_v$ belongs to Λ. Note that for a fixed v, there exists only one g_w such that $g_w^{-1}\gamma_1^{-1}g_v \in \Lambda$. For this g_w, we have that $w = \gamma_1^{-1}v$ and $\chi_{[o,\gamma_1^{-1}t]}(w) = \chi_{[o,\gamma_1^{-1}t]}(\gamma_1^{-1}v) = \chi_{[\gamma_1 o,t]}(v)$. Therefore, (4.2) is equal to the following sum, only depending on v.

$$\theta(i,t)\sum_{v\in B(i)\cap\gamma_1 B(i)} \chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)\sum_{\lambda\in\Lambda} S_{\kappa(i)}(\lambda)(z)S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_v z).$$

Replacing this in (4.1), we get that $\|T_i - \gamma_1 * T_i\|^2_{C_c(\Gamma,\mathcal{A}^s\mathcal{U}/J/\mathcal{A})}$ is equal to

$$\begin{aligned}&\inf_{a\in\mathcal{A}}\sup_{(t,y)\in T\times Y}\Bigg\{\Big|2 - a(t,y) -\\ &2\theta(i,t)\sum_{v\in B(i)\cap\gamma_1 B(i)} \chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)\sum_{\lambda\in\Lambda} S_{\kappa(i)}(\lambda)(z)S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_v z)\Big|\Bigg\}.\end{aligned} \qquad (4.3)$$

By the triangle inequality, to estimate (4.3), it is enough to estimate the following two quantities

$$2\theta(i,t)\sum_{v\in B(i)\cap\gamma_1 B(i)} \chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)\Big|1 - \sum_{\lambda\in\Lambda} S_{\kappa(i)}(\lambda)(z)S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_v z)\Big| \qquad (4.4)$$

and

$$\Big|2-2\theta(i,t)\sum_{v\in B(i)\cap\gamma_1 B(i)}\chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)-a(t,y)\Big| \tag{4.5}$$

In order to bound (4.4), first note that

$$2\Big|1-\sum_{\lambda\in\Lambda}S_{\kappa(i)}(\lambda)(z)S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_v z)\Big| = \|S_{\kappa(i)}-(g_v^{-1}\gamma_1 g_w)*S_{\kappa(i)}\|^2.$$

Then note that there exists $r\in\mathbb{N}$ such that $\gamma_1\in\Gamma_i$ for all $i\geq r$. It follows that for all $i\geq r$, $g_v^{-1}\gamma_1 g_w\in\Lambda_i$, whenever $v,w\in B_i$. Then, by the definition of $\kappa(i)$, it follows that

$$\|S_{\kappa(i)}-(g_v^{-1}\gamma_1 g_w)*S_{\kappa(i)}\|^2\leq\frac{1}{i},\ \text{ for all } i\geq r,\ \text{ whenever } v\in B(i)\cap\gamma_1 B(i).$$

Then, for all $i\geq r$, (4.4) is bounded by

$$\begin{aligned}
&\frac{1}{i}\theta(i,t)\sum_{v\in B(i)\cap\gamma_1 B(i)}\chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)\\
&\qquad\leq\frac{1}{i}\theta(i,t)\Big(\sum_{v\in B(i)\cap\gamma_1 B(i)}\chi_{[o,t]}(v)\Big)^{1/2}\Big(\sum_{v\in B(i)\cap\gamma_1 B(i)}\chi_{[\gamma_1 o,t]}(v)\Big)^{1/2}\\
&\qquad\leq\frac{1}{i}\theta(i,t)\Big|[o,t]\cap B(i)\Big|^{1/2}\Big|[\gamma_1 o,t]\cap\gamma_1 B(i)\Big|^{1/2}\\
&\qquad=\frac{1}{i}\theta(i,t)\Big|[o,t]\cap B(i)\Big|^{1/2}\Big|[o,\gamma_1^{-1}t]\cap B(i)\Big|^{1/2}\\
&\qquad=\frac{1}{i}.
\end{aligned}$$

In order to bound (4.5), for each $i\in\mathbb{N}$, we choose

$$a(t,y)=a(t):=\left(2-2\theta(i,t)\sum_{v\in B(i)\cap\gamma_1 B(i)}\chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)\right)\chi_{B(i)\cup\gamma_1 B(i)}(t).$$

This choice of $a\in\mathcal{A}$ left us to bound (4.5) only when $t\notin B(i)\cup\gamma_1 B(i)$. In this case $\theta(i,t)=\frac{1}{i}$. Moreover, if $i\geq d(o,\gamma_1 o)$ then

$$\Big|[o,t]\cap[\gamma_1 o,t]\cap B(i)\cap\gamma_1 B(i)\Big|\geq i-d(o,\gamma_1 o).$$

It follows that in (4.5) we have the bound

$$\begin{aligned}
2-\frac{2}{i}\sum_{v\in B(i)\cap\gamma_1 B(i)}\chi_{[o,t]}(v)\chi_{[\gamma_1 o,t]}(v)&=2-\frac{2}{i}\Big|[o,t]\cap[\gamma_1 o,t]\cap B(i)\cap\gamma_1 B(i)\Big|\\
&\leq 2-\frac{2}{i}(i-d(o,\gamma_1 o))\\
&=\frac{d(o,\gamma_1 o)}{i}.
\end{aligned}$$

All this combined entails that if $\varepsilon>0$, and $i\geq\max\{r;d(o,\gamma_1 o),2/\varepsilon;2d(o,\gamma_1 o)/\varepsilon\}$, then $\|T_i-\gamma_1*T_i\|^2_{C_c(\Gamma,\mathcal{A}^s\mathcal{U}/J/\mathcal{A})}<\varepsilon$. $\square$

Acknowledgments. F. Poggi was supported in part by a CONICET Doctoral Fellowship. R. Sasyk was supported in part through the grant PIP-CONICET 11220130100073CO. We thank Prof. Isaac Goldbring for his comments on the first version of the article that helped us improve the exposition. We thank the referee for his detailed reading of the manuscript and for prompting us to add a section on boundary amenability of groups.

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