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PHASE PORTRAITS OF QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS HAVING AS SOLUTION SOME CLASSICAL PLANAR ALGEBRAIC CURVES OF DEGREE 4

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Abstract. We classify the phase portraits of quadratic polynomial differential systems having some relevant classic quartic algebraic curves as invariant algebraic curves, i.e. these curves are formed by solution curves of a quadratic polynomial differential system. We show the existence of 25 different global phase portraits in the Poincaré disc for such quadratic polynomial differential systems realizing exactly 14 different invariant algebraic curves of degree 4.

1. Introduction and statement of the main results

We call quadratic differential systems, simply *quadratic systems* or **(QS)**, the differential systems of the form

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real polynomials in the variables x and y, such that the $\max\{\deg(P), \deg(Q)\} = 2$. Here the dot denotes, as usual, differentiation with respect to the time t. To such a system one can always associate the quadratic vector field $\mathcal{X} = P(x, y)\partial/\partial x + Q(x, y)\partial/\partial y$.

If system (1) has an algebraic trajectory curve, which is defined by a zero set of a polynomial, $h(x, y) = 0$. Then it is clear that the derivative of h with respect to the time will not change along the curve $h = 0$, and by the Hilbert's Nullstellensatz (see for instance [5]) we have

$$\frac{dh}{dt} = \frac{\partial h}{\partial x}P + \frac{\partial h}{\partial y}Q = hk, \tag{2}$$

where k is a polynomial in x and y of degree at most 1, called the *cofactor* of the *invariant algebraic curve* $h(x, y) = 0$. For more details on the invariant algebraic curves of a polynomial differential system see Chapter 8 of [4].

Recently the quadratic systems have been intensely studied using algebraic, geometric, analytic and numerical tools. More than one thousand papers on these systems have been published, see for instance the books of Ye Yanquian et al. [11], Reyn [10], and Artés et al. [2] and the references quoted therein.

The main goal of this paper is to characterize the global phase portraits in the Poincaré disc of the quadratic systems having some relevant classical invariant

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algebraic curves of degree 4. More precisely, having the invariant algebraic curves from Table 1.

TABLE 1. Classical algebraic curves of degree 4 realizable by quadratic systems.

Name	Curve
Oblique Bifolium	$f_1(x,y) = -x^2(ax+by)+(x^2+y^2)^2$, $ab \neq 0$
Right Bifolium	$f_2(x,y) = -ax^3+(x^2+y^2)^2$, $a \neq 0$
Bow	$f_3(x,y) = x^4 - x^2y + y^3$,
Cardioid	$f_4(x,y) = (x^2+y^2-ax)^2 - a^2(x^2+y^2)$, $a \neq 0$
Campila	$f_5(x,y) = (x^2+y^2) - a^2x^4$, $a \neq 0$
Külp's Concoid	$f_6(x,y) = -a^2(a^2-x^2)+x^2y^2$, $a \neq 0$
Steiner's Curve	$f_7(x,y) = -27r^4 + 18r^2(x^2+y^2) + (x^2+y^2)^2 + 8rx(3y^2-x^2)$, $r \neq 0$
Simple Folium	$f_8(x,y) = -4rx^3+(x^2+y^2)^2$, $r \neq 0$
Montferrier's Lemniscate	$f_9(x,y) = x^2(x^2-a^2)+b^2y^2$, $ab \neq 0$
Pear Curve	$f_{10}(x,y) = r^4 - 2r^3y + (x-r)^2y^2$, $r \neq 0$
Besace	$f_{11}(x,y) = (x^2-by)^2 - a^2(x^2-y^2)$, $ab \neq 0$
Piriform	$f_{12}(x,y) = b^2y^2 - x^3(a-x)$, $ab \neq 0$
Ramphoid cusp	$f_{13}(x,y) = y^4 - 2axy^2 - 4ax^2y - ax^3 + a^2x^2$, $a \neq 0$
Limaçon of Pascal	$f_{14}(x,y) = (x^2+y^2-bax)^2 - a^2(x^2+y^2)$, $ab \neq 0$

Our first main result is the following.

Theorem 1. *The global phase portraits of the planar quadratic polynomial differential systems* (1)*, with the polynomials P and Q coprime, exhibiting an invariant algebraic curve of degree* 4 *of Table 1, are topologically equivalent to the phase portraits of the following systems:*

(*i*) *QS with the Oblique Bifolium invariant curve:*

$$\dot{x} = 3b^3x + 6abx^2 - 8(3a^2+2b^2)xy - 2aby^2,$$
$$\dot{y} = -ab^2x + 2b^3y + 2(3a^2+2b^2)x^2 + 8abxy - 6(3a^2+2b^2)y^2.$$

(*ii*) *QS with the Right Bifolium invariant curve:*

$$\dot{x} = -3ax/4 + 3x^2/4 - 4cxy - y^2/4, \quad \dot{y} = -9ay/16 + cx^2 + xy - 3cy^2.$$

(*iii*) *QS with Bow invariant curve:*

$$\dot{x} = x(2-9y), \quad \dot{y} = 2(x^2+y-6y^2).$$

(*iv*) *QS with the Cardioid invariant curve:*

$$\dot{x} = -2ax + aby + x^2 + 4bxy - 3y^2, \quad \dot{y} = -3bx^2 - 3ay + 4xy + by^2.$$

(*v*) *QS with the Campila invariant curve:*

$$\dot{x} = xy, \quad \dot{y} = x^2 + 2y^2.$$

(*vi*) *QS with the Külp's concoid invariant curve:*

$$\dot{x} = -xy, \quad \dot{y} = a^2 + y^2.$$

(*vii*) *QS with the Steiner's invariant curve:*

$$\dot{x} = 9r^2 + 6r(x - cy) - 3x^2 - 4cxy + y^2, \quad \dot{y} = 9cr^2 - 6r(cx + y) + cx^2 - 4xy - 3cy^2.$$

(*viii*) *QS with the Simple Folium invariant curve:*

$$\dot{x} = -12rx + 3x^2 + 4cxy - y^2, \quad \dot{y} = -cx^2 - 9ry + 4xy + 3cy^2.$$

(*ix*) *QS with the Montferrier's Lemniscate invariant curve:*

$$\dot{x} = b^2xy, \quad \dot{y} = -a^2x^2 + 2b^2y^2.$$

(*x*) *QS with the Pear invariant curve:*

$$\dot{x} = (x - r)(y - r), \quad \dot{y} = y(r - 2y).$$

(*xi*) *QS with the Besace invariant curve:*

$$\dot{x} = bx - xy, \quad \dot{y} = x^2 + by - 2y^2.$$

(*xii*) *QS with the Piriform invariant curve:*

$$\dot{x} = -\frac{a}{4}x + \frac{1}{4}x^2 - \frac{1}{16}acy + \frac{c}{4}xy, \quad \dot{y} = -\frac{3a^2c}{32b^2}x^2 - \frac{3a}{8}y + \frac{1}{2}xy + \frac{c}{2}y^2.$$

(*xiii*) *QS with the Ramphoid cusp invariant curve:*

$$\dot{x} = -5ax + (1/5)(c - 9)x^2 + (1/5)(2c - 3)xy + y^2,$$

$$\dot{y} = (a(34 - c)/20)x - 2ay + (3(c - 9)/20)xy + (1/4)cy^2.$$

(*xiv*) *QS with the Limaçon of Pascal invariant curve:*

$$\dot{x} = ay + 4bxy, \quad \dot{y} = a(-1 + b^2)x - 3bx^2 + by^2.$$

The first ten systems of Theorem 1 that have an invariant algebraic curve of degree four were already found in [6]. The last four systems of that theorem are new. Here we shall provide the global phase portraits in the Poincaré disc of all these systems, including the first ten systems whose phase portraits were not studied in [6]. More precisely

Theorem 2. *The phase portraits in the Poincaré disc of the fourteen systems of Theorem 1 are:*

- (a.1) 1 *for system* (*i*) *either when* $49b^2 - 144a^2 > 0$ *and* $6a^2 - b^2 \neq 0$, *or* $49b^2 - 144a^2 < 0$.
- (a.2) 2 *for system* (*i*) *when* $6a^2 - b^2 = 0$.
- (b) 3 *for system* (*ii*).
- (c) 4 *for system* (*iii*).
- (d.1) 5 *for system* (*iv*) *when* $b \in (5\sqrt{5/3}, +\infty)$.
- (d.2) 6 *for system* (*iv*) *when* $b \in [0, 5\sqrt{5/3})$. *This phase portrait is topologically equivalent to the phase portrait 13 but the invariant algebraic curves are different.*
- (d.3) 7 *for system* (*iv*) *when* $b = 5\sqrt{5/3}$.
- (e) 8 *for system* (*v*).
- (f) 9 *for system* (*vi*).

(g.1) 10 *for system* (*vii*) *when* $c \in (0, 1/\sqrt{3}) \cup (1/\sqrt{3}, +\infty)$.
(g.2) 11 *for system* (*vii*) *when* $c = 1/\sqrt{3}$.
(g.3) 12 *for system* (*vii*) *when* $c = 0$.
(h) 13 *for system* (*viii*).
(i) 14 *for systems* (*ix*).
(j) 15 *for system* (*x*).
(k) 16 *for system* (*xi*).
(l.1) 17 *for system* (*xii*) *when* $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 < 0$ *and* $c \neq 0$.
(l.2) 18 *for system* (*xii*) *when* $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 > 0$.
(l.3) 19 *for system* (*xii*) *when* $c = 0$.
(l.4) 20 *for system* (*xii*) *when* $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 = 0$.
(m.1) 21 *for system* (*xiii*) *when* $c = 9$.
(m.2) 22 *for system* (*xiii*) *when* $c < 27/8$.
(m.3) 23 *for system* (*xiii*) *when* $c = 27/8$.
(m.4) 24 *for system* (*xiii*) *when* $c > 27/8$ *and* $c \neq 9$.
(n.1) 25 *for system* (*xiv*) *when* $b = 1/2$.
(n.2) 26 *for system* (*xiv*) *when* $b \in (0, 1/2)$.
(n.3) 27 *for system* (*xiv*) *when* $b \in (1, \infty)$.
(n.4) 28 *for system* (*xiv*) *when* $b \in (1/2, 1)$.

2. Preliminaries and basic results

2.1. **Quadratic systems having a classical quartic invariant curve.** In this subsection we present the four new invariant algebraic curves of degree 4 detected for the quadratic systems which do not appear in [6].

The *Besace* curve $h(x, y) = (x^2 - by)^2 - a^2(x^2 - y^2) = 0$ is an invariant quartic algebraic curve with cofactor $2(b - 2y)$ for the QS (*xii*).

The *Pirifurm* curve $h(x, y) = b^2y^2 - x^3(a - x) = 0$ is an invariant algebraic curve with cofactor $(-3a)/4 + x + cy$ for the QS (*xiv*).

Ramphoid cusp curve $h(x, y) = y^4 - 2axy^2 - 4ax^2y - ax^3 + a^2x^2 = 0$ is an invariant algebraic curve with cofactor $-10a + \frac{3}{5}(-9 + c)x + cy$ for the QS (*xv*).

Limaçon of Pascal curve $h(x, y) = (x^2 + y^2 - bax)^2 - a^2(x^2 + y^2) = 0$ is an invariant algebraic curve with cofactor $4by$ for the QS (*xvi*).

2.2. **Poincaré compactification.** In this subsection we give some basic results which are necessary for studying the behavior of the trajectories of a planar polynomial differential system near infinity. Let $\mathcal{X}(x, y) = (P(x, y), Q(x, y))$ be a polynomial vector field of degree n. We consider the Poincaré sphere $\mathbb{S}^2 = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\}$. We identify the plane $\mathbb{R}^2$, where we have defined the polynomial vector field $\mathcal{X}$, with the tangent plane $T_{(0,0,1)}\mathbb{S}^2$ to the sphere $\mathbb{S}^2$ at the north pole $(0, 0, 1)$. We consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 \longrightarrow \mathbb{S}^2$ such that to each point of the plane $q \in T_{(0,0,1)}\mathbb{S}^2$, f associates the two intersection points of

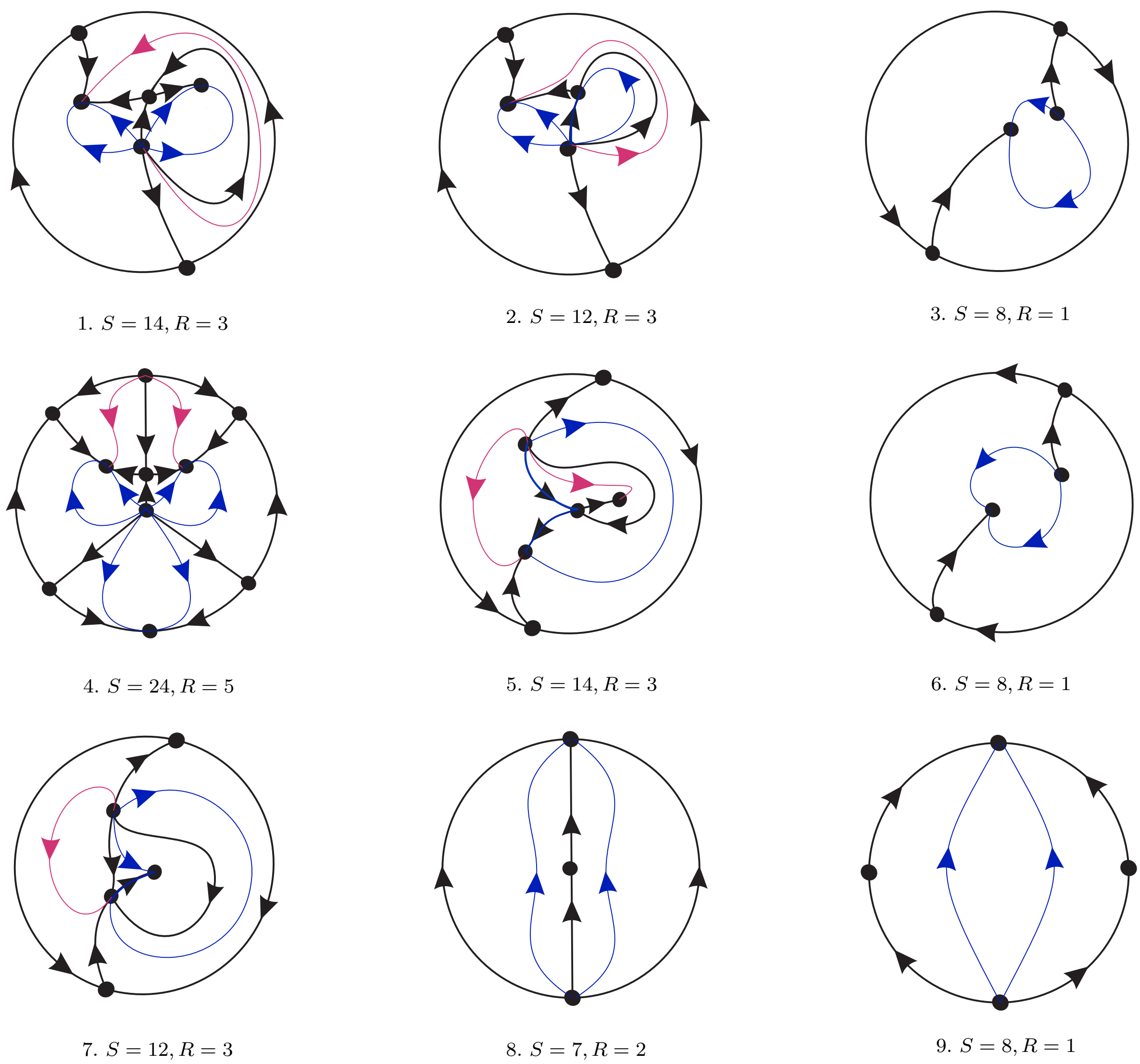
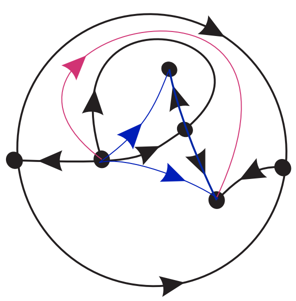
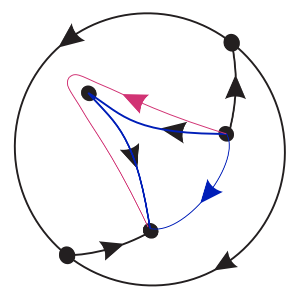


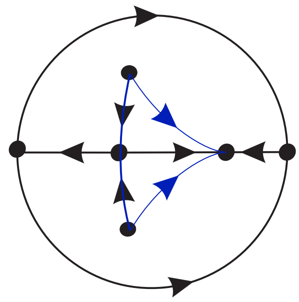
FIGURE 1. Phase portraits in the Poincaré disc. The invariant algebraic curves of degree 4 are drawn in blue color. An orbit inside a canonical region is drawn in red except if it is contained in the invariant algebraic curve. The separatrices are drawn in black except if the separatrix is contained in the invariant algebraic curve then it is of blue color but its arrow is black in order to indicate that is a separatrix. In each phase portrait S indicates the number of its separatrices and R the number of its canonical regions.



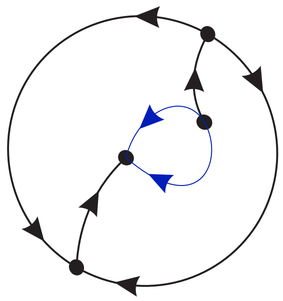
10. $S = 14, R = 3$



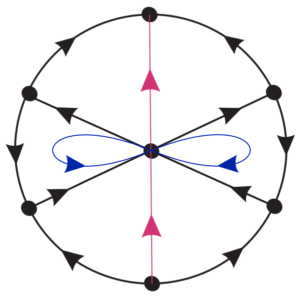
11. $S = 11, R = 2$



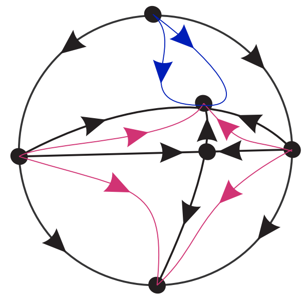
12. $S = 13, R = 2$



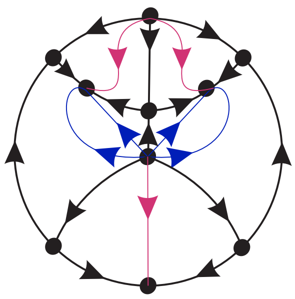
13. $S = 8, R = 1$



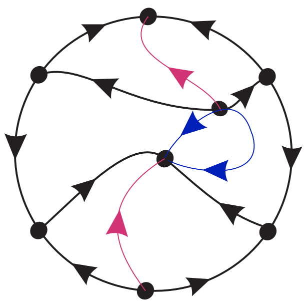
14. $S = 17, R = 4$



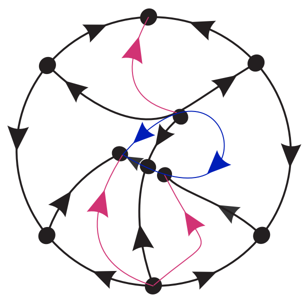
15. $S = 16, R = 5$



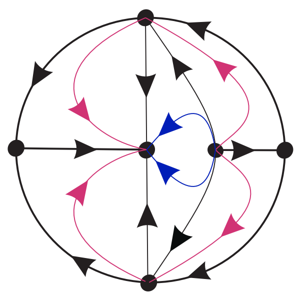
16. $S = 24, R = 5$



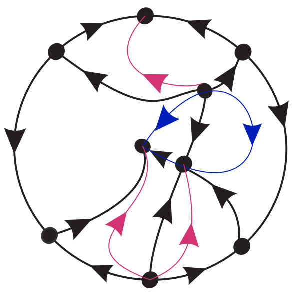
17. $S = 18, R = 3$



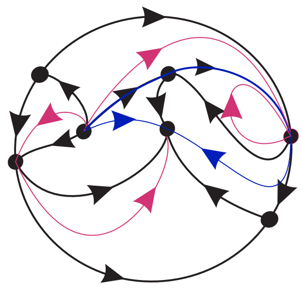
18. $S = 24, R = 5$



19. $S = 16, R = 5$

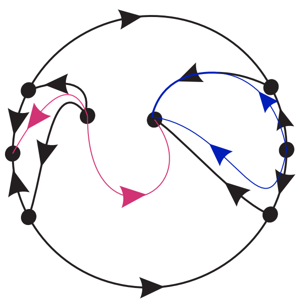


20. $S = 22, R = 5$

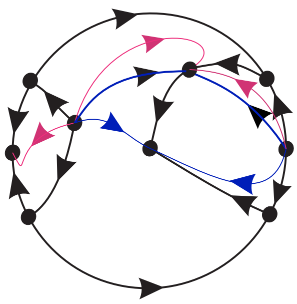


21. $S = 19, R = 6$

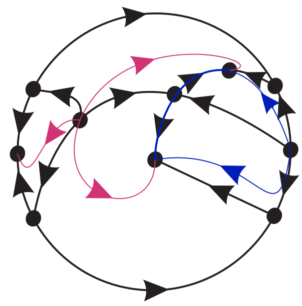
FIGURE 2. Continuation of Figure 1.



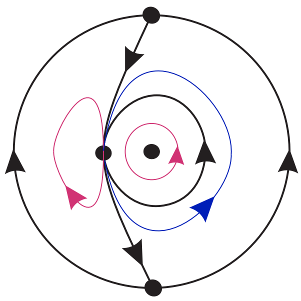
22. $S = 18, R = 3$



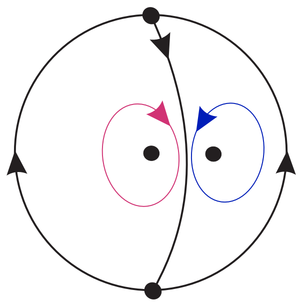
23. $S = 22, R = 5$



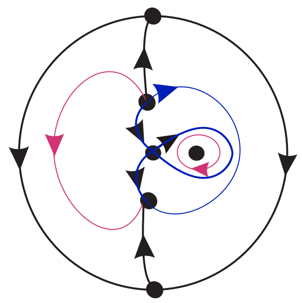
24. $S = 24, R = 5$



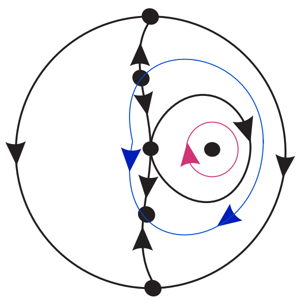
25. $S = 9, R = 3$



26. $S = 7, R = 2$



27. $S = 13, R = 3$



28. $S = 13, R = 3$

FIGURE 3. Continuation of Figure 1.

the straight line which connects the points q and $(0, 0, 0)$ with the sphere $\mathbb{S}^2$. The equator $\mathbb{S}^1 = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_3 = 0\}$ corresponds to the infinity points of the plane $\mathbb{R}^2 \equiv T_{(0,0,1)}\mathbb{S}^2$. In summary we get a vector field $\mathcal{X}'$ defined in $\mathbb{S}^2 \setminus \mathbb{S}^1$, which is formed by two symmetric copies of $\mathcal{X}$, one in the northern hemisphere and the other in the southern hemisphere. We extend it to a vector field $p(\mathcal{X})$ on $\mathbb{S}^2$ by scaling the vector field $\mathcal{X}$ by y_3^n. By studying the dynamics of $p(\mathcal{X})$ near $\mathbb{S}^1$ we get the dynamics of $\mathcal{X}$ close to infinity.

Since we need to do calculations on the Poincaré sphere we consider the local charts $U_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i > 0\}$, and $V_i = \{(y_1, y_2, y_3) \in \mathbb{S}^2 : y_i < 0\}$ for $i = 1, 2, 3$; with the associated diffeomorphisms $F_k : U_i \longrightarrow \mathbb{R}^2$ and $G_k : V_i \longrightarrow \mathbb{R}^2$ for $k = 1, 2, 3$ where $F_k(y_1, y_2, y_3) = -G_k(y_1, y_2, y_3) = (y_m/y_k, y_n/y_k)$ for $m < n$ and $m, n \neq k$. Let $z = (u, v)$ the value of $F_k(y_1, y_2, y_3)$ or $G_k(y_1, y_2, y_3)$ for any k, note that the coordinates (u, v) play different roles depending on the local chart that we are working. In the local charts U_1, U_2, V_1 and V_2 the points (u, v) corresponding to the infinity have its coordinate $v = 0$.

After a scaling of the independent variable in the local chart (U_1, F_1) the expression for $p(\mathcal{X})$ is

$$\dot{u} = v^n \left[-uP\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right) \right], \qquad \dot{v} = -v^{n+1} P\left(\frac{1}{v}, \frac{u}{v}\right);$$

in the local chart (U_2, F_2) the expression for $p(X)$ is

$$\dot{u} = v^n \left[P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right) \right], \qquad \dot{v} = -v^{n+1} Q\left(\frac{u}{v}, \frac{1}{v}\right);$$

and for the local chart (U_3, F_3) the expression for $p(X)$ is

$$\dot{u} = P(u, v), \qquad \dot{v} = Q(u, v).$$

Note that for studying the singular points at infinity we only need to study the infinite singular points of the chart U_1 and the origin of the chart U_2, because the singular points at infinity appear in pairs diametrally opposite.

For more details on the Poincaré compactification see Chapter 5 of [4].

2.3. **Singular points.** As usual we classify the singular points of a planar differential system in *hyperbolic*, the singular points such that their linear part of the differential system at them have eigenvalues with nonzero real part, see for instance Theorem 2.15 of [4] for the classification of their local phase portraits.

The *semi-hyperbolic* are the singular points having a unique eigenvalue equal to zero, their phase portraits are well known, see for instance Theorem 2.19 of [4].

The *nilpotent* singular points have both eigenvalues zero but their linear part is not identically zero. See for example Theorem 3.5 of [4] for the classification of their local phase portraits.

Finally the *linearly zero* singular points are the ones such that their linear part is identically zero, and their local phase portraits must be studied using the changes of variables called blow-up's, see for instance [1] or chapter 2 and 3 of [4].

2.4. **Phase portraits on the Poincaré disc.** In this subsection we shall see how to characterize the global phase portraits in the Poincaré disc of all the gradient quadratic polynomial differential systems.

A *separatrix* of $p(\mathcal{X})$ is an orbit which is either a singular point, or a limit cycle, or a trajectory which lies in the boundary of a hyperbolic sector at a singular point. Neumann [8] proved that the set formed by all separatrices of $p(\mathcal{X})$; denoted by $S(p(\mathcal{X}))$ is closed. We denote by S for the number of separatrices.

The open connected components of $\mathbb{D}^2 \setminus S(p(\mathcal{X}))$ are called *canonical regions* of $p(\mathcal{X})$: We define a *separatrix configuration* as a union of $S(p(\mathcal{X}))$ plus one solution chosen from each canonical region. Two separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are said to be *topologically equivalent* if there is an orientation preserving or reversing homeomorphism which maps the trajectories of $S(p(\mathcal{X}))$ into the trajectories of $S(p(\mathcal{Y}))$. The following result is due to Markus [7], Neumann [8] and Peixoto [9]. We denote by R for the number of canonical regions.

Theorem 3. *The phase portraits in the Poincaré disc of the two compactified polynomial differential systems $p(\mathcal{X})$ and $p(\mathcal{Y})$ are topologically equivalent if and only if their separatrix configurations $S(p(\mathcal{X}))$ and $S(p(\mathcal{Y}))$ are topologically equivalent.*

Due to this theorem in the phase portraits in the Poincaré disc of Figures 1,2 and 3 we plot at least one orbits in each canonical region, and there is more than one if the invariant algebraic curve has more than one orbit in the canonical region.

2.5. **Reduction of the parameters.** Each system which was given in Theorem 1, except systems (iii) and (v), is invariant by the symmetries mentioned below, so we only need to study their phase portraits for the values of the parameters indicated.

(a) System (i) is invariant under the changes $(x, y, t, a, b) \to (-x, y, t, -a, b)$ and $(x, y, t, a, b) \to (x, -y, -t, a, -b)$, then we study it for $a > 0$ and $b > 0$.

(b) System (ii) is invariant under the changes $(x, y, t, a, c) \to (-x, -y, -t, -a, c)$ and $(x, y, t, a, c) \to (x, -y, t, a, -c)$, then we study it for $a > 0$ and $c \geq 0$.

(c) System (iv) is invariant under the changes $(x, y, t, a, b) \to (-x, -y, -t, -a, b)$ and $(x, y, t, a, b) \to (x, -y, t, a, -b)$, then we study it for $a > 0$ and $b \geq 0$.

(d) System (vi) is invariant under the change $(x, y, t, a) \to (x, y, t, -a)$, then we study it for $a > 0$.

(e) System (vii) is invariant under the changes $(x, y, t, r, c) \to (-x, -y, -t, -r, c)$ and $(x, y, t, r, c) \to (x, -y, t, r, -c)$, then we study it for $r > 0$ and $c \geq 0$.

(g) System $(viii)$ is invariant under the changes $(x, y, t, r, c) \to (-x, -y, -t, -r, c)$ and $(x, y, t, r, c) \to (x, -y, t, r, -c)$, then we study it for $r > 0$ and $c \geq 0$.

(h) System (ix) is invariant under the changes $(x, y, t, a, b) \to (x, y, t, -a, b)$ and $(x, y, t, a, b) \to (x, y, t, a, -b)$, then we study it for $a > 0$ and $b > 0$.

(i) System (x) is invariant under the change $(x, y, t, r) \to (-x, -y, -t, -r)$, then we study it for $r > 0$.

(j) System (xii) is invariant under the change $(x, y, t, b) \to (-x, -y, -t, -b)$, then we study it for $b > 0$.

(k) System (xiv) is invariant under the changes $(x, y, t, a, c) \to (-x, y, -t, -a, -c)$ and $(x, y, t, a, c) \to (x, -y, t, a, -c)$, then we study it for $a > 0$ and $c \geq 0$.

(l) System (xv) is invariant under the change $(x, y, t, a, c) \to (-x, -y, -t, -a, c)$, then we study it for $a > 0$.

(m) System (xvi) is invariant under the changes $(x, y, t, a, b) \to (-x, y, t, -a, b)$ and $(x, y, t, a, b) \to (-x, y, -t, a, -b)$, then we study it for $a > 0$ and $b > 0$.

3. Finite and infinite singularities

For all quadratic system presented in Theorem 1 their singular points are characterized in the following result. We recall that we are going to take into consideration the sign of the parameters of the systems which are given previously. In what follows an *antisaddle* will be either a hyperbolic focus or node. In the next proposition the saddles, nodes and foci will be hyperbolic otherwise we will mention its nature, and the saddle–nodes will be semi-hyperbolic if we do not say the contrary.

Proposition 4. *The following statements hold for the quadratic systems of Theorem 1.*

(*i*) *System* (*i*) *in the local chart* U_1 *has one infinite saddle at* $(-(a^2+2b^2)/ab, 0)$, *and the origin of the local chart* U_2 *is not a singularity of this system.*

For the finite singular points, assume first that $6a^2 - b^2 \neq 0$, *then the system has four singularities, an unstable node at the origin of coordinates; the point* $(ab^2/6(a^2+b^2), b^3/6(a^2+b^2))$ *is either a stable focus if* $49b^2 - 144a^2 < 0$ *and* $6a^2 - b^2 > 0$, *or a stable node if* $49b^2 - 144a^2 \geq 0$ *and* $6a^2 - b^2 > 0$, *or a saddle if* $6a^2 - b^2 < 0$; *and the other two singular points are either nodes if* $6a^2 - b^2 < 0$, *or a node and a saddle if* $6a^2 - b^2 > 0$.

If $6a^2 - b^2 = 0$, *then the system has three singular points, two nodes, an unstable at the origin, a stable at* $(-(27a)/121, (18\sqrt{6}a)/121)$, *and a saddle-node at* $(a/7, (\sqrt{6}a)/7)$.

(*ii*) *System* (*ii*) *in the local chart* U_1 *has one infinite saddle at* $(-4c, 0)$, *and the origin of the local chart* U_2 *is not a singular point.*

This system has two finite nodes, a stable one at the origin and the second is unstable.

(*iii*) *System* (*iii*) *in the local chart* U_1 *has two infinite saddles at* $(\pm\sqrt{2/3}, 0)$, *and the origin of the local chart* U_2 *is an unstable node.*

The system has four finite singularities, an unstable node at the origin, two stable nodes at $(\pm\sqrt{2}/3\sqrt{3}, 2/9)$, *and a saddle at* $(0, 1/6)$.

(*iv*) *System* (*iv*) *in the local chart* U_1 *has one infinite saddle at* $(b, 0)$, *and the origin of the chart* U_2 *is not a singularity. For the finite singular points we have three cases.*

If $b \in [0, 5\sqrt{5/3})$, *the system has two singularities, a stable node at the origin and an unstable node.*

If $b = 5\sqrt{5/3}$, *in addition to the origin the system has an unstable node at* $(-3a/16, 3\sqrt{15}a/16)$, *and a saddle-node at* $(-7a/64, 3\sqrt{15}a/16)$.

If $b \in (5\sqrt{5/3}, +\infty)$, *the system has four singular points, the origin as in the previous case; two nodes and one saddle.*

(*v*) *System* (*v*) *has no infinite singularities in the local chart* U_1. *In the local chart* U_2 *the origin is a stable node.*

The system has one finite linearly zero singular point at the origin of coordinates and doing a blow-up, we know that its local phase portrait is formed by two hyperbolic sectors.

(*vi*) *System* (*vi*) *in the local chart* U_1 *has one infinite linearly zero singular point at the origin where the local phase portrait is formed by two hyperbolic sectors, and on the local chart* U_2 *the origin of the system is a stable node.*

The system has no finite singular points.

(vii) *System (vii) in the local chart* U_1 *has an infinite saddle at the point* $(c, 0)$, *and the origin of the local chart* U_2 *is not a singularity.*

If $c = 0$ *the system has four finite singular points, two unstable nodes at* $(-3r/2, \pm 3\sqrt{3}r/2)$, *a stable node at* $(3r, 0)$, *a saddle at* $(-r, 0)$.

If $c \in (0, 1/\sqrt{3}) \cup (1/\sqrt{3}, +\infty)$ *the system has four finite singular points, a stable node at* $(3r, 0)$, *a saddle at* $((-1-6c^2+3c^4)/(1+c^2)^2, 8cr/(1+c^2)^2)$, *an unstable node at* $(-3r/2, -3\sqrt{3}r/2)$ *and a node at* $(-3r/2, 3\sqrt{3}r/2)$ *which is unstable if* $c \in (0, 1/\sqrt{3})$ *and stable if* $c \in (1/\sqrt{3}, +\infty)$.

If $c = 1/\sqrt{3}$ *the system has in addition to the two nodes* $(3r, 0)$ *and* $(-3r/2, -3\sqrt{3}r/2)$, *the nilpotent singular point* $(-3r/2, 3\sqrt{3}r/2)$ *which is a cusp.*

($viii$) *System* ($viii$) *in the local chart* U_1 *has an infinite saddle at* $(c, 0)$, *and the origin of the chart* U_2 *is not a singularity.*

The system has two finite nodes, a stable one at the origin, and the second one is unstable.

(ix) *System* (ix) *in the local chart* U_1 *has two infinite saddles at* $(\pm a/b, 0)$; *and the origin of* U_2 *is a stable node.*

The system has one finite linearly zero singular point at the origin with two elliptic and two parabolic sectors.

(x) *System* (x) *in the local chart* U_1 *has one infinite linearly zero singular point at the origin with local phase portrait consists of two parabolic and two hyperbolic sectors, and the origin of the local chart* U_2 *is an unstable node.*

The system has two finite singular points, a saddle at $(r, 0)$ *and a stable node at* $(r, r/2)$.

(xi) *System* (xi) *in the local chart* U_1 *has two infinite saddles at* $(\pm 1, 0)$. *The origin of the local chart* U_2 *is an unstable node.*

The system has four finite singular points, an unstable node at the origin, two stable nodes at $(\pm b, b)$, *and a saddle at* $(0, b/2)$.

(xii) *Assume* $c > 0$. *System (xii) in the local chart* U_1 *has two infinite saddles at* $((-2b^2 \pm \sqrt{2}\sqrt{2b^4 + 3a^2b^2c^2})/(4b^2c), 0)$, *and the origin of the local chart* U_2 *is a stable node.*

If $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 < 0$, *the system has two finite nodes, a stable one at the origin, and the second is unstable.*

If $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 = 0$, *the system has two nodes as in the previous case, and the third singular point is a saddle-node.*

If $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 > 0$, *in addition to the two previous nodes the system has another node and one saddle.*

Assume $c = 0$. *Then in the local chart* U_1 *the origin is an infinite saddle, and the origin of the chart* U_2 *is a linearly zero singular point such that its local phase portrait consists of four parabolic and two hyperbolic sectors.*

The system has two finite nodes, a stable at the origin and an unstable at $(a, 0)$.

($xiii$) *Assume* $c \neq 9$. *System (xiii) in the local chart* U_1 *has three infinite singular points, a node at the origin of coordinates stable if* $c < 9$ *and unstable if* $c > 9$, *and two saddles at* $((12 - 3c \pm \sqrt{864 - 152c + 9c^2})/40, 0)$. *The origin on the chart* U_2 *is not a singularity.*

If $c < 27/8$ the system has two finite singular points, an unstable focus at $(-4a, 2a)$ and a stable node at the origin.

If $c = 27/8$ the system has three finite singularities, a stable node at the origin, a saddle-node at $(256a/81, 112a/27)$, and an unstable focus at $(-4a, 2a)$.

If $c > 27/8$ in addition to the stable node at $(0, 0)$ and to the unstable focus at $(-4a, 2a)$, the system has two more finite singularities, a node and a saddle.

Assume $c = 9$. Then the system in the local chart U_1 has two singular points, $(-3/4, 0)$ which is a saddle and a nilpotent singular point at the origin with one hyperbolic, one elliptic and two parabolic sectors.

For the finite singular points the system in addition to the two previous singular points at $(0, 0)$ and $(-4a, 2a)$, the system has a finite saddle at $(16a/81, 20a/27)$.

(*xiv*) *System (xiv) in the local chart U_1 has no infinite singular points, and the origin of the chart U_2 is a saddle.*

If $b \in (0, 1/2)$ the system has two finite singular points which are centers; the origin and the point $S = (a(b^2 - 1)/(3b), 0)$.

If $b = 1/2$ the system has two finite singular points, a center at the origin; and a nilpotent singular point at $(-a/2, 0)$, its local phase portrait consists of one hyperbolic, two parabolic and one elliptic sectors.

If $b \in (1/2, 1)$ the system has a center at the origin, a saddle at S, an unstable node at $(-a/(4b), a\sqrt{4b^2 - 1}/(4b))$, and a stable node at $(-a/(4b), -a\sqrt{4b^2 - 1}/(4b))$.

If $b \in (1, \infty)$ the system has a saddle at the origin, a center at S, and the other two singularities are as in the previous case.

Proof. **System (i)** in the local chart U_1 becomes

$$\begin{aligned}\dot{u} &= 6a^2(1 + u^2) + ab(2u + 2u^3 - bv) + b^2(4 + 4u^2 - buv),\\ \dot{v} &= v(24a^2u + 2ab(-3 + u^2) + b^2(16u - 3bv)).\end{aligned}$$

This system has one infinite hyperbolic singular point $q_1 = ((-3a^2 - 2b^2)/ab, 0)$ with eigenvalues $(-6(a^2 + b^2)(9a^2 + 4b^2))/(ab)$ and $(2(a^2 + b^2)(9a^2 + 4b^2))/(ab)$, then it is a saddle. On the local chart U_2 writes

$$\begin{aligned}\dot{u} &= -6a^2(u + u^3) + b^2u(-4 - 4u^2 + bv) + ab(-2 + u^2(-2 + bv)),\\ \dot{v} &= v(-6a^2(-3 + u^2) + abu(-8 + bv) - 2b^2(-6 + 2u^2 + bv)).\end{aligned}$$

It is clear that the origin is not a singular point of this system.

For the finite singular points the system has four hyperbolic singularities: $p_1 = (0, 0)$ with eigenvalues $2b^3$ and $3b^3$. Hence by using Theorem 2.15 of [4] we get that p_1 is an unstable node because $b > 0$; $p_2 = ((ab^2)/(6(a^2 + b^2)), b^3/(6(a^2 + b^2)))$ with eigenvalues

$$\frac{b^2\left(-5b \pm \sqrt{49b^2 - 144a^2}\right)}{6},$$

such that $\lambda_1.\lambda_2 = 2b^4(6a^2 - b^2)/3$.

Assume that $6a^2 - b^2 \neq 0$. If $6a^2 - b^2 < 0$ then p_2 is a hyperbolic saddle. If $6a^2 - b^2 > 0$ then p_2 is a stable node if $49b^2 - 144a^2 \geq 0$, and a stable focus if $49b^2 - 144a^2 < 0$. The two other singular points are $p_{3,4} = \Big(A_1/((a^2 + b^2)(9a^2 + 4b^2)^2), B_1/((a^2 + b^2)(9a^2 + 4b^2)^2)\Big)$ with $A_1 = b^4(-10a^3 - 4ab^2 \pm (7a^2 + 4b^2)\sqrt{7a^2 + 3b^2})$ and $B_1 = 3b^3(13a^4 + 15a^2b^2 + 4b^4 \pm a^3\sqrt{7a^2 + 3b^2})$. We can check that the expression of their eigenvalues are non-zero and real, but they are very big, and this make difficult to determine their local phase portraits. We may calculate their (topological) indices by using the Poincaré-Hopf Theorem, see for instance Theorem 6.30 of [4].

In the Poincaré sphere system (i) has ten isolated singular points, we denote by i_1', the index of the infinite singular point q_1 in the local chart U_1, and by i_1, i_2, i_3 and i_4, the indices of the finite singular points p_1, p_2, p_3 and p_4, respectively. It is well known that the index of a saddle is -1, and that the index of a node is 1, then $i_1' = i_2 = -1$ and $i_1 = 1$. The Poincaré-Hopf Theorem asserts that the sum of all the indices of the singular points of system (i) in the Poincaré sphere is equal to 2, therefore we have $2(i_1') + 2(i_1 + i_2 + i_3 + i_4) = 2$. In this equality we need to know the values of i_3 and i_4. We have two cases according with the index of p_2.

If p_2 is a saddle we get that $i_3 + i_4 = 2$, this implies that both p_3 and p_4 have index 1, then they are nodes or foci, but they cannot be foci because the points p_3 and p_4 are on the oblique bifolium invariant curve.

If p_2 is a node or focus we get that $i_3 + i_4 = 0$, this implies that one of them has index 1, then it is a node, and the other has index -1 and it is a saddle.

If $6a^2 - b^2 = 0$, since $b > 0$ the system has an unstable node at p_1, a stable node at $p_2 = (-(27a)/121, (18\sqrt{6}a)/121)$, and by using Theorem 2.19 of [4] for the semi-hyperbolic singular points we obtain that the third singular point $p_3 = (a/7, (\sqrt{6}a)/7)$ is a saddle-node.

System (ii) in the local chart U_1 writes

$$\begin{aligned} \dot{u} &= c(1 + u^2) + 1/16u(4 + 4u^2 + 3av), \\ \dot{v} &= 1/4v(-3 + 16cu + u^2 + 3av). \end{aligned} \tag{3}$$

This system has one hyperbolic singular point $q_1 = (-4c, 0)$, with eigenvalues verifying $\lambda_1.\lambda_2 = -(3/16)(1 + 16c^2)^2$, then it is a saddle. On the local chart U_2 becomes

$$\begin{aligned} \dot{u} &= 1/16(-4 - 4u^2 - 16c(u + u^3) - 3auv), \\ \dot{v} &= -uv - c(-3 + u^2)v + (9av^2)/16, \end{aligned} \tag{4}$$

therefore the origin of U_2 is not a singular point.

This system has the origin as a finite hyperbolic node with eigenvalues $-23a/32$ and $-18a/32$, then it is stable because $a > 0$; for the other three possible real singular points their y coordinates are given by the real solutions of the following cubic equation $432a^3c + 27a^2(7 + 256c^2)y + 512ac(5 + 72c^2)y^2 + 256(1 + 16c^2)^2y^3 = 0$. The number of real roots of this cubic is determined by its discriminant $\delta = -27648a^6(27 + 16c^2)^2(343 + 5184c^2)$. Since $\delta < 0$ we have that the cubic has only one real root. Then the system has one real singular point additional to the origin. By the Poincaré-Hopf Theorem its index is 1, since the two finite singular points are on the right folium invariant curve it is a node, which analyzing its eigenvalues is unstable.

System (iii) in the local chart U_1 becomes $\dot{u} = 2 - 3u^2$, $\dot{v} = (9u - 2v)v$. This system has two hyperbolic saddles, one at $(-\sqrt{2/3}, 0)$ with eigenvalues $2\sqrt{6}$ and $-3\sqrt{6}$, and the other at $(\sqrt{2/3}, 0)$ with eigenvalues $-2\sqrt{6}$ and $3\sqrt{6}$. In the local chart U_2 writes $\dot{u} = 3u - 2u^3$, $\dot{v} = -2v(-6 + u^2 + v)$, so the origin of this system is an unstable node with eigenvalues 3 and 12.

This system has four finite hyperbolic singular points, an unstable node at $(0, 0)$ with eigenvalues 2 and 2, two stable nodes at $(\pm\sqrt{2}/3\sqrt{3}, 2/9)$ with eigenvalues -2 and $-4/3$, and a saddle at $(0, 1/6)$ with eigenvalues -2 and $1/2$.

System (iv) in the local chart U_1 is

$$\begin{aligned} \dot{u} &= u(3 + 3u^2 - av) - b(3 + u^2(3 + av)), \\ \dot{v} &= v(-bu(4 + av) + (-1 + 3u^2 + 2av)). \end{aligned} \tag{5}$$

This system has only an infinite singular point, a hyperbolic saddle at $q_1 = (b, 0)$ with eigenvalues $-(1 + b^2)$ and $3(1 + b^2)$. On the local chart U_2 writes

$$\begin{aligned} \dot{u} &= b(3u + 3u^3 + av) + (-3 - 3u^2 + auv), \\ \dot{v} &= v(-b - 4u + 3bu^2 + 3av), \end{aligned} \tag{6}$$

therefore the origin of U_2 is not a singular point.

The system (iv) has the origin, denoted by p_1, as a hyperbolic stable node with eigenvalues $-3a$ and $-2a$ because $a > 0$. The y coordinates of the other three possible singular points are given by the solutions of the cubic equation $12a^3b + a^2(3b^4 - 66b^2 - 5)y + 64ab(b^2 + 1)y^2 - 16(b^2 + 1)^2y^3 = 0$. The number of real solutions of this cubic equation is determined by its discriminant $\delta = 64a^6(3b^2 - 125)(b^2 + 1)^3(3b^4 - 6b^2 - 1)^2$. We distinguish three cases and recall that we can assume that $b > 0$.

If $3b^2 - 125 < 0$, the cubic equation has one real solution, in addition to the stable node at origin, the system has an unstable node.

If $3b^2 - 125 = 0$, the cubic equation has two real roots one simple and one double. Then the system has three singular points, a hyperbolic stable node at the origin, an unstable node at $(-3a/16, 3\sqrt{5}a/16)$ with eigenvalues $10a$ and $12a$, and a semi-hyperbolic point at $(-7a/64, -\sqrt{15}a/64)$ with eigenvalues $-8a$ and 0. In order to obtain the local phase portrait at this finite semi-hyperbolic singular point we use Theorem 2.19 of [4], and we obtain that the origin is a saddle-node.

If $3b^2 - 125 > 0$, the cubic equation has three real solutions. Then three finite singular points for the system additionally to the origin. According to the Berlinskïi Theorem (see Theorem 7 of [3]), and since all the eigenvalues of the singular points are real, and due to the fact that the origin is a node; two of these three points are nodes or foci and the third one is a saddle, or two of them are saddles and the third one is a node or a focus. To know which one of these two cases hold we need to apply the Poincaré-Hopf Theorem. In the Poincaré sphere the compactified system (iv) has ten isolated singular points, the index of the infinite singular points q_1 is $i_1 = -1$, and we know also the index of the finite singular point p_1 which is $i_1 = 1$. We need to know the indices i_2, i_3 and i_4 of the three other finite singularities. Applying the Poincaré-Hopf Theorem we get the following equality: $2(i'_1)+2(i_1+i_2+i_3+i_4)=2$, then $i_2+i_3+i_4=1$, this implies that two of these singular points are nodes or foci and one is a saddle. But this system has no foci because the four finite singular points are on the cardioid invariant curve.

System (v) in the local chart U_1 becomes $\dot{u}=1+u^2$, $\dot{v}=-uv$. So the system has no infinite singularities in U_1. In the local chart U_2 it writes $\dot{u}=-u(1+u^2)$, $\dot{v}=-(2+u^2)v$. Therefore the origin of this system is a stable node with eigenvalues -2 and -1.

This system has only one finite linearly zero singular point at the origin of coordinates. We need to do a blow-up $y=zx$ for describing its local phase portrait. After eliminating the common factor x of $\dot{x}$ and $\dot{z}$, by doing the rescaling of the independent variable $ds=xdt$, we obtain the system $\dot{x}=xz$, $\dot{z}=1+z^2$. This system has no singular points. Going back through the two changes of variables and taking into account the flow of the system on the axes of coordinates, we obtain that the local phase portrait at the origin of system (v) is formed by two hyperbolic sectors.

System (vi) in the local chart U_1 writes $\dot{u}=2u^2+a^2v^2$, $\dot{v}=uv$. This system has only one infinite singular point, which is a linearly zero singular point at the origin. Doing the blow-up $v=wu$, and after eliminating the common factor u of $\dot{u}$ and $\dot{w}$ by doing the rescaling of the independent variable $ds=udt$, we obtain the differential system $\dot{u}=2u+a^2uw^2$, $\dot{w}=-w-a^2w^3$. The only singular point of this system with $u=0$ is $(0,0)$, with eigenvalues 2 and -1. Hence it is a saddle. Then going back through the two changes of variables, $ds=udt$ and $v=wu$, and taking into account the flow of the system on the axes, we obtain that the local phase portrait at the origin of U_1 is formed by two hyperbolic sectors. In the local chart U_2 the system becomes $\dot{u}=-u(2+a^2v^2)$, $\dot{v}=-v(1+a^2v^2)$. So the origin of this system is a stable node with eigenvalues, -2 and -1.

Since $a>0$ this system has no finite singular points.

System (vii) in the local chart U_1 is

$$\begin{aligned}\dot{u}&=-u(1+u^2+12rv+9r^2v^2)+c((1-3rv)^2+u^2(1+6rv)),\\ \dot{v}&=-v(-3+u^2+6rv+9r^2v^2-2cu(2+3rv)).\end{aligned}$$

This system has a unique infinite singular point, a hyperbolic saddle at $(c,0)$ with eigenvalues $-(1+c^2)$ and $3(1+c^2)$. In the local chart U_2 becomes

$$\begin{aligned}\dot{u}&=1+u^2+12ruv+9r^2v^2-c(u+u^3+6rv-6ru^2v+9r^2uv^2),\\ \dot{v}&=-v(-4u-6rv+c(-3+u^2-6ruv+9r^2v^2)).\end{aligned}$$

So its origin is not a singular point.

If $c=0$, this system has four finite hyperbolic singular points, a stable node at $(3r,0)$ with eigenvalues $-18r$ and $-12r$, two unstable nodes at $((-3r)/2,\pm(3r\sqrt{3})/2)$ with same eigenvalues $9r$ and $6r$, a saddle at $(-r,0)$ with eigenvalues $\lambda_1=-2r$ and $\lambda_2=12r$.

If $c\in(0,1/\sqrt{3})\cup(1/\sqrt{3},+\infty)$, this system has four finite hyperbolic singular points, a stable node at $(3r,0)$ with eigenvalues $-18r$ and $-12r$, another node at $((-3r)/2,(3r\sqrt{3})/2)$ with eigenvalues $9(1-\sqrt{3}c)r$ and $6(1-\sqrt{3}c)r$, then it is stable if $c\in(1/\sqrt{3},+\infty)$ and unstable if $c\in(0,1/\sqrt{3})$, a third unstable node at $((-3r)/2, (-3r\sqrt{3})/2)$ with eigenvalues $7(1+\sqrt{3}c)r$ and $8(1+\sqrt{3}c)r$ because $r > 0$, a saddle at $((-1-6c^2+3c^4)/(1+c^2)^2, (8cr)/(1+c^2)^2)$ with eigenvalues $\lambda_1.\lambda_2 = (-24(1-3c^2)^2r^2)/(1+c^2)^2$.

If $c = 1/\sqrt{3}$, we have the differential system

$$\begin{aligned}\dot{x} &= 9r^2 - 3x^2 - (4xy)/\sqrt{3} + y^2 + 6r(x - y/\sqrt{3}),\\ \dot{y} &= 3\sqrt{3}r^2 + x^2/\sqrt{3} - 4xy - \sqrt{3}y^2 - 6r((x/\sqrt{3}) + y).\end{aligned}$$

In addition to the hyperbolic node $(3r, 0)$ this system has another hyperbolic unstable node at $(((-3r)/2, (-3r\sqrt{3})/2)$ with eigenvalues $12r$ and $18r$, and the nilpotent singular point $((-3r)/2, (3r\sqrt{3})/2)$. In order to know the nature of this singular point.

First, we put these singular points at the origin of coordinates by performing the translation $x = x_1 - (3r)/2, y = x_2 + (3r\sqrt{3})/2$, and we get

$$\begin{aligned}\dot{x}_1 &= 3\sqrt{3}rx_2 + x_2^2 + 9rx_1 - (4x_2x_1)/\sqrt{3} - 3x_1^2,\\ \dot{x}_2 &= -9rx_2 - \sqrt{3}x_2^2 - 9\sqrt{3}rx_1 - 4x_2x_1 + x_1^2/\sqrt{3}.\end{aligned}$$

Second, we transform this system into its normal form by doing the change of variables $x_1 = z, x_2 = -\sqrt{3}z + w$, and we have

$$\begin{aligned}\dot{z} &= 3\sqrt{3}rw + w^2 + 4z^2 - (10/\sqrt{3})wz,\\ \dot{w} &= (16/\sqrt{3})z^2 - 8zw.\end{aligned}$$

By applying Theorem 3.5 of [4], we obtain that the origin is a cusp.

System (viii) in the local chart U_1 becomes

$$\dot{u} = u + u^3 - c(1+u^2) + 3ruv, \qquad \dot{v} = v(-3 - 4cu + u^2 + 12rv).$$

This system has one infinite hyperbolic saddle at $(c, 0)$ with eigenvalues $-3(1+c^2)$ and $(1+c^2)$. In the local chart U_2 writes

$$\dot{u} = -1 - u^2 + c(u + u^3) - 3ruv, \quad \dot{v} = v(-4u + c(-3 + u^2) + 9rv).$$

The origin of this system is not a singular point.

This system has a finite hyperbolic stable node at the origin with eigenvalues $-12r$ and $-9r$ because $r > 0$. The y coordinate of the other three possible singular points are given by the solution of the cubic equation $-432cr^3 + 27(7+16c^2)r^2y - 16c(10+9c^2)ry^2 + 16(1+c^2)^2y^3 = 0$. The number of real roots of this cubic are determined by its discriminant $\delta = -1728(27+c^2)^2(343+324c^2)r^6$. Since $\delta < 0$ the cubic equation has a unique real root. Then additional to the origin the system has a node. This follows using the Poincaré-Hopf Theorem and the fact that the singular points are on the simple folium invariant curve. Moreover that node is unstable.

System (ix) in the local chart U_1 becomes $\dot{u} = -a^2 + b^2u^2$, $\dot{v} = -b^2uv$. This system has two hyperbolic saddles, one at $(-a/b, 0)$ with eigenvalues $-2ab$ and ab, and the other at $(a/b, 0)$ with eigenvalues $-ab$ and $2ab$. In the local chart U_2 the system writes $\dot{u} = -b^2u + a^2u^3$, $\dot{v} = -2b^2v + a^2u^2v$. The origin of this system is a hyperbolic stable node with eigenvalues $-2b^2$ and $-b^2$.

This system has a unique finite singular point, which is a linearly zero singular point at the origin of coordinates. Doing the blow-up $y = zx$, and eliminating the common factor x of $\dot{x}$ and $\dot{z}$ by doing the rescaling of the independent variable $ds = xdt$, we obtain the system $\dot{x} = b^2xz$, $\dot{z} = -a^2 + b^2z^2$. This system has two singular points on $x = 0$, a stable node at $(0, -a/b)$ with eigenvalues $-2ab$ and $-ab$, and an unstable node at $(0, a/b)$ with eigenvalues $2ab$ and ab. Going back through the two changes of variables, $ds = xdt$ and $y = zx$, and taking into account the flow on the axes, we obtain that the local phase portrait at the origin of system (ix) is formed by two elliptic and two parabolic sectors.

System (x) in the local chart U_1 becomes $\dot{u} = u(rv(2 - rv) + u(-3 + rv))$, $\dot{v} = -v(-1 + rv)(-u + rv)$. This system has a unique infinite singular point at the origin, which is linearly zero. Doing the blow-up and the rescaling of the independent variable as in system (ix), we obtain that its local phase portrait consists of two parabolic and two hyperbolic sectors. In the local chart U_2 the system writes $\dot{u} = u(3 - 2av) + av(-1 + av)$, $\dot{v} = -v(-2 + av)$. The origin of this system is a hyperbolic unstable node with eigenvalues 2 and 3.

This system has two finite hyperbolic singular points, a saddle at $(r, 0)$ with eigenvalues r and $-r$; and a stable node at $(r, r/2)$ with eigenvalues $-r$ and $-r/2$.

System (xi) in the local chart U_1 system (xi) becomes $\dot{u} = 1 - u^2$, $\dot{v} = v(u - bv)$, this system has two saddles, one at $(1, 0)$ with eigenvalues 2 and -1, the second at $(-1, 0)$ with eigenvalues -2 and 1. In the local chart U_2 the system is given by $\dot{u} = u - u^3$, $\dot{v} = -v(-2 + u^2 + bv)$, the origin of this system is an unstable node with eigenvalues 1 and 2.

This system has four finite hyperbolic singular points, a saddle at $(0, b/2)$ with eigenvalues $-b$ and $b/2$; a node at the origin with eigenvalues b and b, then it is unstable, two another nodes at $(-b, b)$ and (b, b) with eigenvalues $-2b$ and $-b$, then they are stable.

System (xii) in the local chart U_1 becomes

$$\dot{u} = \frac{1}{32b^2}(-3a^2c + 8b^2u(1 + cu) + 2ab^2u(-2 + cu)v),$$
$$\dot{v} = \frac{1}{16}v(-4 + 4av + cu(-4 + av)).$$

Assume $c > 0$. This system has two hyperbolic saddles at $((-2b^2 - \sqrt{2}\sqrt{2b^4 + 3a^2b^2c^2})/(4b^2c), 0)$ with eigenvalues verifying $\lambda_1\lambda_2 = (-2b^2 - 3a^2c^2 + \sqrt{4b^4 + 6a^2b^2c^2})/(64b^2)$; and the second is $((-2b^2 + \sqrt{2}\sqrt{2b^4 + 3a^2b^2c^2})/(4b^2c), 0)$ with eigenvalues verifying $\lambda_1\lambda_2 = -(2b^2 + 3a^2c^2 + \sqrt{4b^4 + 6a^2b^2c^2})/(64b^2)$. In the chart U_2 the system becomes

$$\dot{u} = \frac{1}{32b^2}(3a^2cu^3 - 2b^2(4cu + 4u^2 + acv - 2auv)),$$
$$\dot{v} = \frac{1}{32b^2}v(3a^2cu^2 - 4b^2(4c + 4u - 3av)),$$

the origin of this system is a node with eigenvalues $-c/2$ and $-c/4$, then it is stable.

For the finite singularities the system has the origin as a hyperbolic node with eigenvalues $-3a/8$ and $-a/4$, then it is stable, the y coordinates of the other three possible finite singular points is given by the solution of the cubic equation $192a^3b^2c + (-256b^4 - 480a^2b^2c^2 + 3a^4c^4)y + 512ab^2c^3y^2 - 256b^2c^4y^3 = 0$. The numbers of real and complex roots of this cubic equation are determined by the discriminant $\delta = 1024b^2c^4(256b^4 + 3a^4c^4)^2(-256b^4 - 1184a^2b^2c^2 + 3a^4c^4)$. We distinguish three cases.

If $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 < 0$ with $0 < c < ((592 + 224\sqrt{7})b)/(a\sqrt{3})$, the cubic equation has one real solution. Then in addition to the origin the system has one real singular point which is an unstable node.

If $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 = 0$ with $c = ((592 + 224\sqrt{7})b)/(a\sqrt{3})$, the cubic equation has two real solutions, one simple and one double. And the system has a hyperbolic stable node at the origin, a hyperbolic node at $((1/2)(-2 + \sqrt{7})a, (3/(4b))\sqrt{-37 + 14\sqrt{7}a^2}))$ with eigenvalues $(7a)/8$ and $(1/4)(5+2\sqrt{7})a$, then it is unstable and a semi-hyperbolic point at $(a/(6+2\sqrt{7}), (-a^2/(8b))\sqrt{-37 + 14\sqrt{7}})$ with eigenvalues $(2 + \sqrt{7})a/4$ and 0. In order to obtain the local phase portrait at this finite semi-hyperbolic singular point we use Theorem 2.19 of [4], and we obtain that the point is a saddle-node.

If $-256b^4 - 1184a^2b^2c^2 + 3a^4c^4 > 0$ with $c > ((592 + 224\sqrt{7})b)/(a\sqrt{3})$ the cubic equation has three real simple solutions. Then four real singular points for the system. According to the Berlinskïi Theorem, and since all the eigenvalues of the singular points are real, and to the fact that the origin is a node; two of these three points are nodes and the third one is a saddle, or two of them are saddles and the third one is a node. To know which one of these two cases hold we need to apply Poincaré-Hopf Theorem.

In the Poincaré sphere the compactified system (xiv) has fourteen isolated singular points, the index of the two infinite singular points in the chart U_1 is -1, and the index of the origin of the chart U_2 is $+1$; and we know also the index of one finite singular point is 1. We need to know the indices i_2, i_3 and i_4 of the three other finite singularities. Applying Poincaré-Hopf Theorem, we get the following equality: $2(-1) + 2(-1) + 2(1) + 2(1) + 2(i_2 + i_3 + i_4) = 2$, then $i_2 + i_3 + i_4 = 1$. This implies that both of these three singular points are nodes and one is a saddle.

Now assume that $c = 0$. Then system (xiv) at infinity has a saddle at the origin of the chart U_1 with eigenvalues $-1/4$ and $1/4$, and the origin of the chart U_2 is a linearly zero singular point. Doing a blow-up, we obtain that its local phase portrait is formed by four parabolic and two hyperbolic sectors.

This system has a finite stable node at the origin with eigenvalues $(-3a)/8$ and $(-a/4)$, and an unstable node at $(a, 0)$ with eigenvalues $a/8$ and $a/4$.

System (xiii) Assume $c \neq 4, 9$. In the local chart U_1 the system becomes

$$\begin{aligned}
\dot{u} &= (1/20)(-3(-4 + c)u^2 - 20u^3 - a(-34 + c)v + u(9 - c + 60av)),\\
\dot{v} &= -(1/5)v(-9 + c - 3u + 2cu + 5u^2 - 25av).
\end{aligned}$$

This system has three finite hyperbolic singular points, the origin with eigenvalues $(9 - c)/5$ and $(9 - c)/20$, then it is a stable node if $c > 9$, an unstable node if $c < 9$, and the two singular points $(1/40(12 - 3c \pm \sqrt{864 - 152c + 9c^2}), 0)$ such that the product of its two eigenvalues is negative, so they are saddles. The origin of the local chart U_2 is not a singularity. For the finite singular points we distinguish three cases.

If $c < 27/8$ the system has two hyperbolic singularities; a node at the origin with eigenvalues $-5a$ and $-2a$, then it is stable; and a focus at $(-4a, 2a)$ with eigenvalues $-a/5((-29+c) \pm i(-4+c))$. Hence it is unstable.

If $c = 27/8$ the system has two hyperbolic singularities. In addition of the two previous finite singularities in the case 1, the system has a semi-hyperbolic singularity at $((256a)/81, (112a)/27)$ with eigenvalues $-((20a)/3)$ and 0. By applying Theorem 2.19 of [4], we get that this point is a saddle-node.

If $c > 27/8$ the system has four hyperbolic singularities, the node and the focus mentioned in case 1, and two other hyperbolic singular points whose expressions are big and we do not provide them here. We know that their eigenvalues are real but it is difficult to know their nature. We use Poincaré-Hopf Theorem, we get that their indices equal to 1 and -1, then one of them is a node and the second is a saddle.

Assume $c = 4$. System (xv) in the local chart U_1 has three hyperbolic singularities, an unstable node $(0,0)$ with eigenvalues $1/4$ and 1, a stable node $(-1/2, 0)$ with eigenvalues $5/4$ and $-1/2$, and a saddle $(1/2, 0)$ with eigenvalues $-1/2$ and $1/4$.

Then the system has four finite hyperbolic singularities, two stable nodes $(0,0)$ and $(16a, 12a)$ with eigenvalues $-10a$ and $-5a$, an unstable node at $(-4a, 2a)$ with eigenvalues $5a$ and $5a$, and a saddle $(a, 2a)$ with eigenvalues $-5a$ and $(5a)/4$.

Suppose now that $c = 9$. System (xv) in the local chart U_1 is

$$\begin{aligned} \dot{u} &= -((3u^2)/4) - u^3 + (5av)/4 + 3auv, \\ \dot{v} &= -v(3u + u^2 - 5av). \end{aligned} \tag{7}$$

This system has two singular points, $(-3/4, 0)$ with eigenvalues $-9/16$ and $27/16$, then it is a saddle. The seconde point is a nilpotent singularity at the origin. In order to obtain its local phase portrait we use Theorem 3.5 of [4], and we obtain that it consists of one hyperbolic, one elliptic and two parabolic sectors.

For the finite singular points in addition to the two previous singular points at $(0,0)$ and $(-4a, 2a)$, the system has a hyperbolic saddle at $(16a/81, 20a/27)$ with eigenvalues $-10a/3$ and $17a/9$.

System (xiv) in the local chart U_1 the system is

$$\dot{u} = -3b(1+u^2) + ab^2 v - a(1+u^2)v, \quad \dot{v} = -uv(4b + av),$$

this system has no infinite singular points. In the chart U_2 the system becomes

$$\begin{aligned} \dot{u} &= 3b(u + u^3) - ab^2u^2v + a(1+u^2)v, \\ \dot{v} &= -v(b - 3bu^2 - auv + ab^2uv), \end{aligned}$$

the origin of this system is a saddle with eigenvalues $-b$ and $3b$.

If $b \in (0, 1/2)$ the system has two finite singularities both are centers; one at the origin with eigenvalues $\pm ai\sqrt{1-b^2}$, and the other at $(a(b^2-1)/3b, 0)$ with eigenvalues $\pm a\sqrt{-1+5b^2-4b^4}/\sqrt{3}$.

If $b = 1/2$ the system has two finite singular points, a center at the origin with eigenvalues $\pm ai\sqrt{3}/2$, and a nilpotent singular point at $(-a/2, 0)$ with eigenvalues 0 and $ai\sqrt{3}/2$. By Theorem 3.5 of [4], and we obtain that it consist of one hyperbolic, two parabolic and one elliptic sectors.

If $b \in (1/2, \infty)$ the system has four finite singular points, the origin which is a center if $b \in (1/2, 1)$, and a saddle if $b \in (1, \infty)$; the point $(a(b^2-1)/3b, 0)$ with the same previous eigenvalues, which is a center if $b \in (1, \infty)$, and a saddle if $b \in (1/2, 1)$; the point $(-a/4b, -a\sqrt{4b^2-1}/4b)$ with eigenvalues $-a\sqrt{4b^2-1}/2$ and $-a\sqrt{4b^2-1}$, then it is a stable node; and the point $(-a/4b, a\sqrt{4b^2-1}/4b)$ with eigenvalues $\frac{1}{2}a\sqrt{4b^2-1}$ and $a\sqrt{4b^2-1}$, then it is an unstable node. □

4. Local and global phase portraits

System (i) can have two different phase portraits according to the value of $b^2 - 6a^2$.

If $b^2 \neq 6a^2$ and from statement (i) of Proposition 4 we obtain the local phase portrait of the finite and infinite singular points. Due to the fact that three of the finite singular points are on the Oblique Bifolium invariant curve of the system, we obtain some orbits on this invariant curves connecting those singular points, these connections vary if either $49b^2 - 144a^2 > 0$ and $6a^2 - b^2 > 0$, or $49b^2 - 144a^2 < 0$ (see local phase portraits 1 of Figure 4); or if $49b^2 - 144a^2 > 0$ and $6a^2 - b^2 < 0$ (see local phase portraits 1 of Figure 4). Since $\dot{x}_{|x=0} = -2aby^2 < 0$, the separatrices for which we do not know their α- or ω-limit can be easily determined from the mentioned figures, obtaining the global phase portrait 1 of Figure 1.

If $b^2 - 6a^2 = 0$ the system has two hyperbolic nodes and one saddle-node; the three finite singular points belong to the Oblique Bifolium invariant curve of the system, and by using the same arguments as in the previous case (see also local phase portraits 2 in Figure 4) we get the global phase portrait 2 in Figure 1.

System (ii) from statement (ii) of Proposition 4, we obtain that the system has two finite nodes which belong to the Right Bifolium invariant curve of the system, so they connect each one to the other. The system has only one infinite saddle in the local chart U_1. Since $\dot{x}_{|x=0} < 0$ and $\dot{y}_{|y=0} > 0$, we get that the α-limit of the infinite saddle in the local chart U_1 is the finite unstable node and the ω-limit of the infinite saddle in the local chart V_1 is the finite stable node (see the local phase portrait 3 of Figure 4). Hence the phase portrait 3 of Figure 1 is the global one of this system.

System (iii) according to statement (iii) of Proposition 4 we obtain the local phase portrait of this system, which contains three finite hyperbolic nodes belong to the Bow invariant curve, and one finite hyperbolic saddle. In the infinity the system has four saddles and two nodes. Since $x = 0$ is an invariant straight line of the system and the fact that the node at the origin and a saddle are localized on this line and by taking into account that $\dot{y}_{|y=0} > 0$, (see the local phase portrait 4 of Figure 4) it results the global phase portrait 4 of Figure 1.

System (iv) for this system we distinguish three different global phase portraits according to the sign of $3b^2 - 125$.

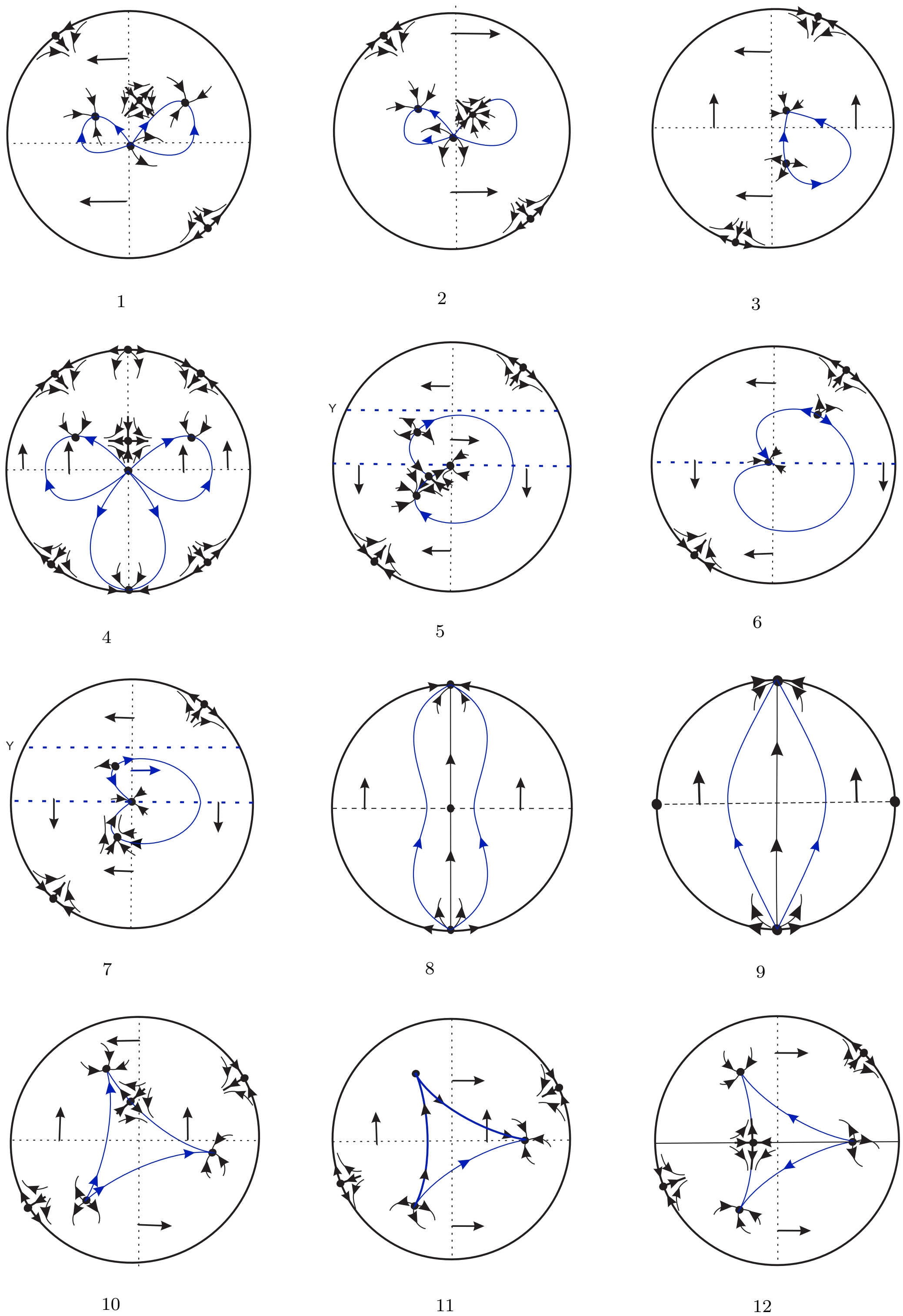


Figure 4. Local phase portraits at the singular points. The invariant algebraic curves of degree 4 are drawn in blue color.

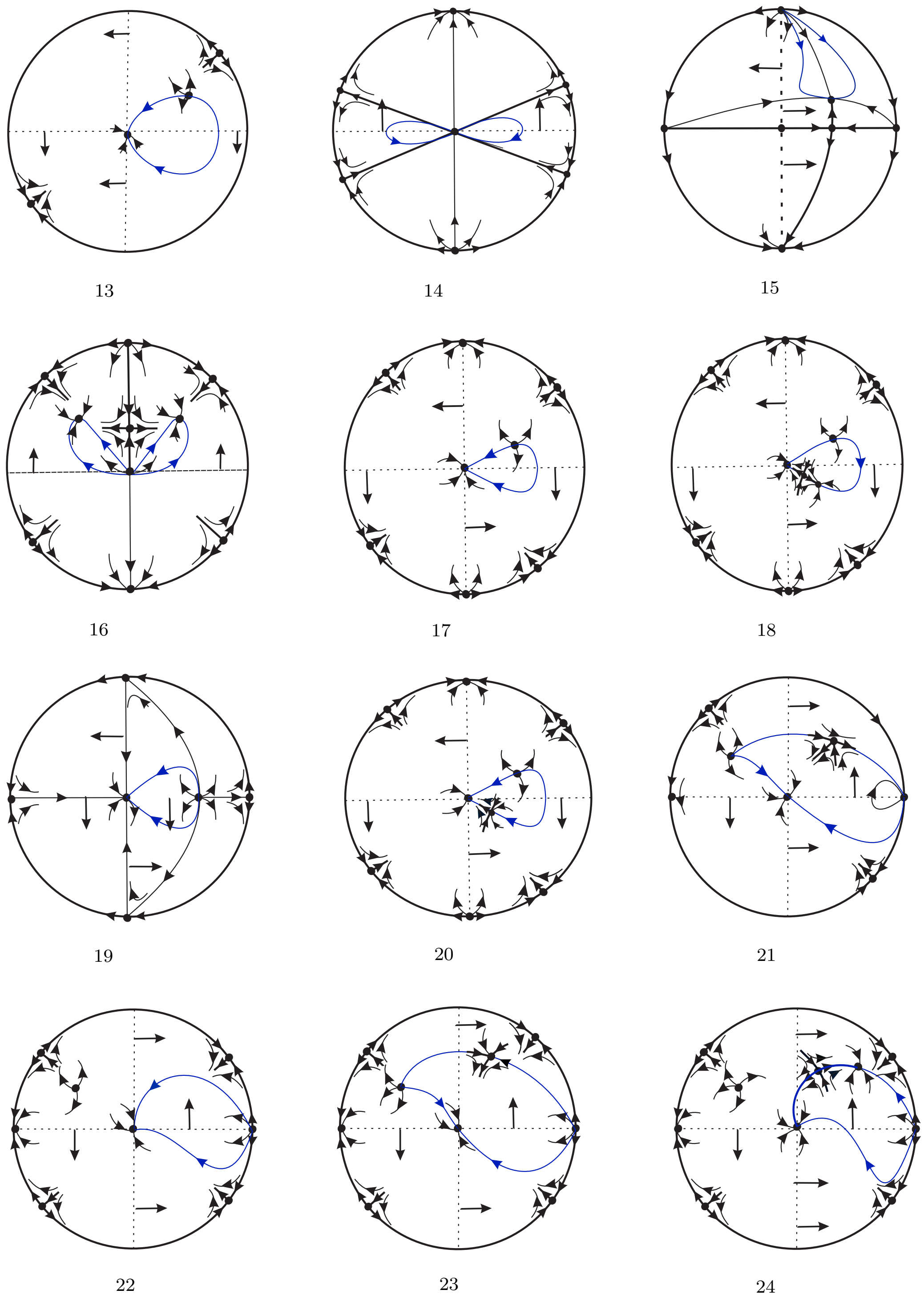
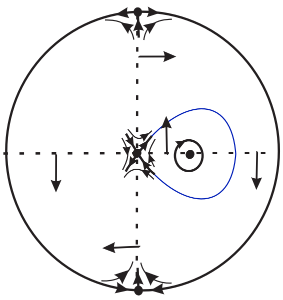


FIGURE 5. Continuation of Figure 4.



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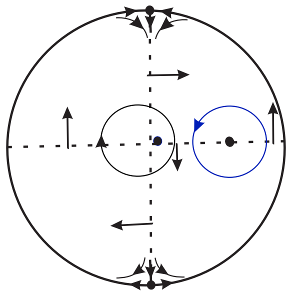

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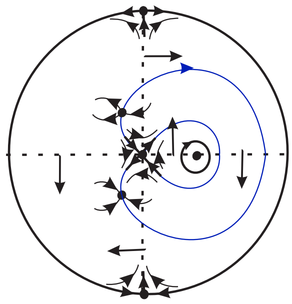
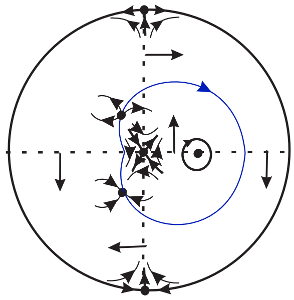


27



28

FIGURE 6. Continuation of Figure 4.

If $3b^2 - 125 > 0$ we use the same tools as in the previous case and according to the local phase portrait 5 of Figure 4, we get the global phase portrait 5 of Figure 1.

If $3b^2 - 125 < 0$ the system has the local phase portrait 6 of Figure 4, and the same configuration of equilibria than system (ii). So it has the global phase portrait 6 of Figure 1.

If $b = \sqrt{125}/\sqrt{3}$ from statement (iv) of Proposition 4 the system has two finite hyperbolic nodes and one semi-hyperbolic saddle-node. These three finite singular points belong to the Cardioid invariant curve of the system, and since the variartion of its vector field on the axes is given by $\dot{x}_{|x=0} = y((a\sqrt{125})/(3\sqrt{3}) - y)$, $\dot{y}_{|y=0} = -5\sqrt{15}x^2 < 0$ (see the local phase portrait 7 of Figure 4.), we get the global phase portrait 7 of Figure 1.

System (v) according to statement (v) of Proposition 4, system (v) has one linearly zero singular point at the origin, whose local phase portrait consists of two hyperbolic sectors. For the infinite ones it has one hyperbolic stable node at the origin of the local chart U_2. Knowing that both of Campila curve and the straight line $x = 0$ are invariant for the system, and the fact that $\dot{y}_{|y=0} > 0$,(see local phase portrait 8 of Figure 4), so it follows the global phase portrait 8 of Figure 1.

System (vi) from statement (vi) of Proposition 4 and the local phase portrait 9 of Figure 4, and since $\dot{y}_{|y=0} > 0$ it follows the phase portrait 9 of Figure 2.

System (vii) has two different global phase portraits.

If $c \neq 1/\sqrt{3}$ it has the same configuration of equilibria than systems (i) either when $49b^2 - 144a^2 > 0$ and $6a^2 - b^2 \neq 0$ or when $49b^2 - 144a^2 < 0$ and system (iv) when $3b^2 - 124k^2 > 0$. So it has the global phase portrait 10 of Figure 2.

If $c = 1/\sqrt{3}$ the system has a stable node $(3r, 0)$, an unstable node $(-3r/2, -3\sqrt{3}/2)$ and a cusp $(-3r/2, (-3\sqrt{3}/2)$. These three singularities are on Steiner's Invariant curve of the system. At infinity it has only one hyperbolic saddle at $(1/\sqrt{3}, 0)$ in U_1. Since $\dot{x}_{|x=0} = (y - \sqrt{3}r)^2 > 0$ and $\dot{y}_{|y=0} = (1/\sqrt{3})(x - 3r)^2 > 0$, see the local phase portrait 11 of Figure4, so we get that the global phase portrait of this case is 11 of Figure2.

If $c = 0$ the system has two unstable nodes at $(-3r/2, \pm 3\sqrt{3}r/2)$, a stable node at $(3r, 0)$, a saddle at $(-r, 0)$. These four singularities are on Steiner's Invariant curve of the system. At infinity it has only one hyperbolic saddle at $(1/\sqrt{3}, 0)$ in U_1. Since $\dot{x}_{|x=0} = y^2 + 9r^2 > 0$ and $\dot{y}_{|y=0} = 0$, see the local phase portrait 12 of Figure4, so we get that the global phase portrait of this case is 12 of Figure 2.

System $(viii)$ has the same configuration of equilibria than system (ii). Hence it has the global phase portrait 13 of Figure 2.

System (ix) The origin of this system is a linearly zero singular point whose local phase portrait consists of two elliptic and two parabolic sectors. Since the straight line $x = 0$ and the Montferrier's curve are invariant for the system intersecting at the origin, the fact that $\dot{y}_{|y=0} < 0$ and $\dot{y}_{|x=0} > 0$, and from the local phase portrait 14 of Figure 5, we obtain the global phase portrait 14 of Figure 2.

System (x) has two finite hyperbolic singular points, a stable node $n = (r, r/2)$ which belongs to the Pear invariant curve $H = 0$ of this system and a saddle $s = (r, 0)$ outside of $H = 0$. For the infinite ones it has only one linearly zero singular point at the origin of U_1, whose local phase portrait consists of two hyperbolic sectors separated by two parabolic sectors. Since each one of the straight lines $y = 0$, $x = r$ and $y = r/2$ are invariant for the system they contain in their intersection and according to the local phase portrait 15 of Figure 5. The global phase portrait of this system is 15 of Figure 2.

System (xi) has the same configuration of equilibria than system (iii), then it has the global phase portrait 16 of Figure 2.

System (xii) for this system we get four different global phase portraits according with the values of the parameter c.

If $c \in (0, b\sqrt{592 + 224\sqrt{7}}/(a\sqrt{3})$ the system has one stable node and an unstable node which belong to the Piriform invariant curve. At infinity the system has two hyperbolic saddles at U_1 and one hyperbolic stable node at the origin of U_2. Taking into account the following directions of the vector field of the system $\dot{y}_{|y=0} = -((3a^2c)/(32b^2))x^2 < 0$ and $\dot{x}_{|x=0} = -(1/16)acy$ (see local phase portrait 17 of Figure 5), we obtain the global phase portrait 17 of Figure 2.

If $c \in (b\sqrt{592+224\sqrt{7}}/(a\sqrt{3}), +\infty)$ from Proposition 4 this system has four hyperbolic finite singular points; three nodes and one saddle. In the chart U_1 it has two hyperbolic saddles and the origin of U_2 is a stable node. To know the $\alpha-$ and the ω- limit of the finite and infinite separatrices of the saddles we calculate the variation of the vector field of the system on the axes, we get $\dot{x}_{|x=0} = -(1/16)acy$, $\dot{y}_{|y=0} = (-3a^2c)/(32b^2)x^2 < 0$. From the local phase portrait 18 of Figure 5; we get that the global one is 18 of Figure 2.

If $c = 0$ the system has two finite hyperbolic nodes, the stable at the origin and the unstable at $(a, 0)$. These two points belong to the Piriform invariant curve of the system. For the infinite singularities the system has one hyperbolic saddle at the origin of the local chart U_1. The local phase portrait of the origin of U_2 is a linearly zero singular point that its local phase portrait consists of two hyperbolic and two parabolic sectors. Since $\dot{x}_{|y=0} = -(1/4)(a-x)x$ and that straight lines $x = 0$, $y = 0$ and $x = a$ are invariant for the system we can know the direction of the vector field on the axes, see the local phase portrait 19 of Figure 5 which forces the global phase portrait 19 of Figure 3.

If $c = b\sqrt{592+224\sqrt{7}}/(a\sqrt{3})$ from Proposition 4 we get the local phase portrait 20 of Figure 5 for the finite and infinite singular points of the system. Since $\dot{x}_{|x=0} = (-1/12)\sqrt{111+42\sqrt{7}}by$ and $\dot{y}_{|y=0} = (-1/8b)3\sqrt{111+42\sqrt{7}}ax^2 < 0$ we get that the global phase portrait in this case is of Figure 3.

System $(xiii)$ we get three different phase portraits.

If $c = 9$ we get the local phase portrait for the finite and the infinite singularities from Proposition 4. The three finite singular points are on the Ramphoid cusp the invariant curve of the system. Since $\dot{x}_{|x=0} = y^2 > 0$ and $\dot{y}_{|y=0} = (5ax)/4$ we get the local phase portrait 21 of Figure 5, so the global one is 21 of Figure 3.

If $c < 27/8$ the system has two finite hyperbolic singularities, a stable node at the origin and an unstable focus at $(-4a, 2a)$ and three infinite singularities, two saddles and one unstable node. By knowing that $\dot{x}_{|x=0} = y^2$ and $\dot{y}_{|y=0} = (a/20)(34-c)x$ we get the local phase portrait 22 of Figure 6, so the global one is 22 of Figure 3.

If $c = 27/8$ from Proposition 4 we get the local phase phase portrait for finite and infinite singularities. According the direction of the vector field of the system which gives by $\dot{x}_{|x=0} = y^2$ and $\dot{y}_{|y=0} = (49ax)/32$ we get the local phase portrait 23 of Figure 6, so the global one is 23 of Figure 3.

If $c > 27/8$ and $c \neq 9$ by using the same tools we get the global phase portrait 24 of Figure 3.

System (xiv) has three different global phase portraits and in all the cases it has one infinite hyperbolic saddle at the origin of U_2 and no equilibria in the chart U_1.

If $b = 1/2$ we get the local phase portrait from the statement (xvi) of Proposition 4. The two finite singular points of the system are in the Limaçon of Pascal the invariant curve of the system. The variation of the vector field on the axes gives by $\dot{x}_{|x=0} = ay$ and $\dot{y}_{|y=0} = (-3/4)x(a+2x)$ (see local phase portrait 25 of Figure 6). So we get the global phase portrait 25 of Figure 3.

If $b \in (0, 1/2)$ then for the finite singular points the system has two centers one at the origin and the second at $A = (a(b^2 - 1)/(3b), 0)$. In this case the origin belongs to the Limaçon of Pascal $H = 0$ the invariant curve of the system which is homeomorphic to a circle. The variation of the vector field on the axes gives by the two equations $\dot{x}_{|x=0} = ay$ and $\dot{y}_{|y=0} = x(a(-1+b^2) - 3bx)$ (see local phase portrait 26 of Figure 6), then we can conclude that the global phase portrait is 26 of Figure 3.

If $b \in (1, +\infty)$ then from statement (xvi) of Proposition 4 the system has a center, a hyperbolic saddle at the origin and two hyperbolic nodes symmetric with respect to the x-axis, one is stable and the other unstable. The three hyperbolic finite points are on the Limaçon of Pascal invariant curve, see local phase portrait 27 of Figure 6). As the previous cases using the vector field on the axes, we obtain the global phase portrait 27 of Figure 3.

If $b \in (1/2, 1)$ using Proposition 4 and working as in the previous case we obtain the global phase portrait 28 of Figure 3.

We note that the case $b = 1$ is not studied because in this case we have a cardioid which already has been considered.

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