DEGENERACY LOCI OF TWISTED DIFFERENTIAL FORMS AND LINEAR LINE COMPLEXES

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ABSTRACT. We prove that the Hilbert scheme of degeneracy loci of pairs of global sections of $\Omega_{\mathbb{P}^{n-1}}(2)$, the twisted cotangent bundle on $\mathbb{P}^{n-1}$, is unirational and dominated by the Grassmannian of lines in the projective space of skew-symmetric forms over a vector space of dimension n. We provide a constructive method to find the fibers of the dominant map. In classical terminology, this amounts to giving a method to realize all the pencils of linear line complexes having a prescribed set of centers. In particular, we show that the previous map is birational when $n = 4$.

1. INTRODUCTION

Degeneracy loci of morphisms of the form $\phi : \mathcal{O}_{\mathbb{P}^{n-1}}^m \longrightarrow \Omega_{\mathbb{P}^{n-1}}(2)$ arise naturally in algebraic geometry and have been extensively studied. In classical terminology, a general degeneracy locus of this form is the set of centers of complexes belonging to a general linear system of dimension $m - 1$ of linear line complexes in $\mathbb{P}^{n-1}$, i.e. hyperplane sections of the Grassmannian $\mathbf{Gr}(1, \mathbb{P}^{n-1})$ of lines in $\mathbb{P}^{n-1} = \mathbf{P}(V)$ embedded in $\mathbf{P}(\Lambda^2 V)$ via the Plücker map (cfr. Sect. 2.2). This nice geometric interpretation, together with the fact that many classical algebraic varieties arise this way, motivated the interest of many classical algebraic geometers as Castelnuovo, Fano, and Palatini. Nonetheless, these loci have been recently studied by several authors, for example Chang and Ottaviani. A more detailed historical account and a few classical examples can be found, for instance, in [BM01, FF10].

In general, in order to parameterize the possible degeneracy loci X_ϕ as ϕ varies, it is useful to take a modern approach and introduce $\mathcal{H}$ as the union of the irreducible components, in the Hilbert scheme of subschemes of $\mathbb{P}^{n-1}$, containing X_ϕ for general ϕ's.

Let $\mathbf{P}(V)$ be the projectivization of an n-dimensional vector space V. Relying on a nice interpretation due to Ottaviani ([Ott92, §3.2]), we can identify a morphism of the form above with a skew-symmetric matrix of linear forms in m variables, or with an m-tuple of elements in $\Lambda^2 V$; the natural GL_m-action does not modify its degeneracy locus, so we get the rational map

$$(1) \qquad \rho : \mathbf{Gr}(m, \Lambda^2 V) \dashrightarrow \mathcal{H}$$

sending ϕ to X_ϕ.

The behavior of the map ρ is fully understood for most of the values of (m, n). In [Tan] it is shown that it is birational whenever $4 \leq m < n - 1$ or $(m, n) = (3, 5)$; if $(m, n) = (3, 6)$, ρ is dominant but $4 : 1$, while for $m = 3$ and $n > 6$ it is generically

injective but not dominant. Partial results in this line of thought were already provided in [BM01, FM02, FF10].

We are interested here in the case $m = 2$. Our aim is to give a description of the behavior of the map ρ, as n varies. The case $n = 6$ has already been tackled in [BM01], where ρ was proved to be dominant and its fibers were described and shown to be two-dimensional. Our main result is the following

Theorem. *Let $n \in \mathbb{N}$ such that $n \geq 4$, V be a vector space of dimension n and let*

$$\rho : \ \boldsymbol{Gr}(2, \Lambda^2 V) -- \!\!\succ \mathcal{H}$$

be the rational morphism introduced in (1), sending the class of a morphism $\phi :$ $\mathcal{O}^2_{\mathbf{P}(V)} \longrightarrow \Omega_{\mathbf{P}(V)}(2)$ to its degeneracy locus X_ϕ, considered as a point in the Hilbert scheme. Then ρ is dominant. Moreover

- i. *if n is even, the general element of $\mathcal{H}$ is the union of $\frac{n}{2}$ lines spanning the whole $\mathbf{P}(V)$. The general fiber is the Grassmannian $\boldsymbol{Gr}(1, \sigma)$ of lines lying on a suitable $(\frac{n-2}{2})$-dimensional projective space σ. In particular, ρ is birational for $n = 4$;*
- ii. *if n is odd, the general element of $\mathcal{H}$ is the image in $\mathbf{P}(V)$ of a map*

$$\mathbb{P}^1 \xrightarrow{\ [f_1 : \ldots : f_n]\ } \mathbf{P}(V),$$

where $f_1, \ldots, f_n$ are forms of degree $\frac{n-1}{2}$ spanning the whole vector space $\mathrm{H}^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\frac{n-1}{2}))$. The general fiber of ρ has dimension $\frac{n^2-3n}{2}$.

When n is even, the general X_ϕ is union of $\frac{n}{2}$ lines. A crucial step in proving the first part of the statement is the reinterpretation of a morphism ϕ as a pencil ℓ_ϕ of linear line complexes. The $\frac{n}{2}$ lines are the set of centers of the linear line complexes belonging to ℓ_ϕ; using a similar approach to the one adopted in [BM01], we are able to generalize the existing result to all even values of n.

When n is odd, the general X_ϕ is a rational curve, image of the map $[f_1 : \ldots : f_n]$ as in the statement. A priori, the forms $f_1, \ldots, f_n$ have to be the Pfaffians of order $n - 1$ of a linear $n \times n$ skew-symmetric matrix, but we will show that a general n-tuple of forms of degree $\frac{n-1}{2}$ arises as such.

As pointed out later, an explicit construction of the preimages of a given degeneracy locus is possible. For the special case $n = 4$, this amounts to the following: it is possible to construct the unique pencil of linear line complexes having two given skew lines as set of centers.

These results complete the general picture, describing the behavior of the map ρ for arbitrary values of m. For $m > 3$, such description had already been given in [Tan]; the above theorem, together with the previous contributions, shows that the unique pairs (m, n) with $2 \leq m < n - 1$ for which ρ is not generically injective are $(2, n)$ with $n \geq 5$ and $(3, 6)$, while the unique pairs for which ρ is not dominant are $(3, n)$ with $n \geq 7$.

We remark that the results contained in this paper hold over non-necessarily algebraically closed fields of arbitrary characteristic.

2. Notation and preliminaries

2.1. **Notation and geometric interpretation.** Let $\mathbf{k}$ be any field and let $n \in \mathbb{N}$ such that $n \geq 4$. We will denote by U, V two $\mathbf{k}$-vector spaces of dimensions 2,

n respectively; by $\mathbf{P}(U)$ and $\mathbf{P}(V)$ we will mean the projective spaces of their 1-quotients, i.e. $\mathrm{H}^0(\mathbf{P}(U), \mathcal{O}_{\mathbf{P}(U)}(1)) \cong U$. We set $\{y_0, y_1\}$ and $\{x_0, \ldots, x_{n-1}\}$ to be the bases of U and V respectively.

Let $\Omega_{\mathbf{P}(V)}$ be the rank $n-1$ vector bundle of differential forms on $\mathbf{P}(V)$ and let $\phi : U \otimes \mathcal{O}_{\mathbf{P}(V)} \to \Omega_{\mathbf{P}(V)}(2)$ be a general morphism. We can define $X = X_\phi$ to be the degeneracy locus associated to ϕ, i.e. the scheme cut out by the maximal minors of the matrix locally representing ϕ.

A different interpretation of X, due to Ottaviani [Ott92, §3.2], allows ϕ to be interpreted as an $(n \times n)$ skew-symmetric matrix $N_\phi = y_0 N_0 + y_1 N_1$ of linear forms in y_0, y_1. A point p in V represents a point in X_ϕ if and only if $p \in \ker(b_0 N_0 + b_1 N_1)$ for some $(b_0, b_1) \neq (0, 0)$. Thus, the geometry of X_ϕ strongly depends on the parity of n: when n is even, it is a scroll over the hypersurface cut out by the Pfaffian of N_ϕ, i.e. a union of $\frac{n}{2}$ lines in $\mathbf{P}(V)$; when n is odd, it is the blow-up of $\mathbb{P}^1$ along the set of points described by the n Pfaffians of order $n-1$ of N_ϕ, which is empty for a general ϕ. For more details about this geometric interpretation, we refer to [Tan, §2].

2.2. Linear line complexes.

Another interpretation (cfr. [BM01]) of a morphism $\phi : \mathcal{O}_{\mathbf{P}(V)}^2 \longrightarrow \Omega_{\mathbf{P}(V)}(2)$ is the following: let $\mathbb{G} := \mathbf{Gr}(1, \mathbf{P}(V))$ be the Grassmannian of lines in $\mathbf{P}(V)$, embedded in $\mathbf{P}(\Lambda^2 V)$ via the Plücker map. The dual space $\mathbf{P}(\Lambda^2 V^*)$ parameterizes hyperplane sections of $\mathbb{G}$, or, in classical terminology, *linear line complexes* in $\mathbf{P}(V)$.

An element $A \in \Lambda^2 V$, up to constants, may be regarded as an element of $\mathbf{P}(\Lambda^2 V^*)$, hence giving rise to a linear line complex Γ. A point $p \in \mathbf{P}(V)$ is called a *center* of Γ if all the lines through p belong to Γ; the space $\mathrm{Deg}(A) := \mathbf{P}(\ker(A))$, called the *singular space* of Γ, turns out to be the set of centers of Γ.

Let $\ell \in \mathbb{G}$ and $\mathbb{T}_\ell \mathbb{G}$ the corresponding tangent space. The hyperplane $\mathrm{V}(A)$ in $\mathbf{P}(\Lambda^2 V)$ contains $\mathbb{T}_\ell \mathbb{G}$, i.e. A belongs to the dual variety $\check{\mathbb{G}}$, if and only if $\mathrm{Deg}(A) \supseteq \ell$. We distinguish the following two cases:

- if n is even, then a general linear line complex Γ has no center, as a general skew-symmetric matrix of even order has maximal rank. Linear line complexes Γ corresponding to the points of $\check{\mathbb{G}}$ have at least a line as center and will be called *special*;
- if n is odd, then a general linear line complex Γ has a point as center. Γ will be said to be *special* if it corresponds to a point of $\check{\mathbb{G}}$: in this case, the center of Γ contains (at least) a $\mathbb{P}^2$.

We can therefore interpret the degeneracy locus of a general morphism $\phi : \mathcal{O}_{\mathbf{P}(V)}^2 \longrightarrow \Omega_{\mathbf{P}(V)}(2)$ as

$$\mathrm{Deg}(N_\phi) := \bigcup_{A \in N_\phi} \mathrm{Deg}(A),$$

where the skew-symmetric matrix $N_\phi = y_0 N_0 + y_1 N_1$ is regarded as a line in $\mathbf{P}(\Lambda^2 V^*)$. Special complexes on N_ϕ are parameterized by the intersection $N_\phi \cap \check{\mathbb{G}}$.

3. THE BEHAVIOR OF ρ: EVEN CASE

Within this section, we assume n even. We will show that the map ρ

$$\rho : \mathbf{Gr}(2, \Lambda^2 V) \dashrightarrow \mathcal{H}$$

defined in (1) is dominant. This leads naturally to asking what is the preimage of a general point X_ϕ in $\mathcal{H}$, i.e. which lines $N \subset \mathbf{P}(\Lambda^2 V^*)$ have $\mathrm{Deg}(N) = X_\phi$. We will give a geometric description of the fibers, providing a constructive procedure to realize the elements of a general preimage.

Recall that a linear line complex $A \in \mathbf{P}(\Lambda^2 V^*)$ is special if its center contains a $\mathbb{P}^1$. We distinguish special complexes *of the first type*, whose center is exactly a line, from special complexes *of the second type*, whose center is at least a $\mathbb{P}^3$.

A general matrix $N = N_\phi$ with linear forms in $\mathbf{k}[y_0, y_1]$ as entries has corank two in $\frac{n}{2}$ distinct points of $\mathbb{P}^1$, corresponding to the roots of $\mathrm{Pf}(N)$; N has maximal rank in any other point. This means that the general line of complexes $N \subset \mathbf{P}(\Lambda^2 V^*)$ does not contain any special complex of the second type, and that $\mathrm{Deg}(N)$ is the union of $\frac{n}{2}$ lines $\{\ell_1, \ldots, \ell_{\frac{n}{2}}\}$.

We claim that these lines are general, in the sense that their span is the whole $\mathbf{P}(V)$. It is clear that this condition is open, so it is sufficient to exhibit a matrix N satisfying it. We will do more, showing constructively in Proposition 2 that any set of lines $\{\ell_1, \ldots, \ell_{\frac{n}{2}}\}$ spanning $\mathbf{P}(V)$ is $\mathrm{Deg}(N)$ for some pencil of complexes N.

Let us examine the Gauss map

$$\zeta : \ \check{\mathbb{G}} \dashrightarrow \mathbb{G}$$

which sends a special complex of the first type A to the point in $\mathbb{G}$ corresponding to the line $\mathrm{Deg}(A)$. Fixing a line $\ell \in \mathbb{G}$, the fiber $\zeta^{-1}(\ell)$ is a linear space. A complex $A \in \mathbb{G}$ has ℓ as center if and only if the hyperplane $\mathrm{V}(A)$ in $\mathbf{P}(\Lambda^2 V)$ contains $\mathbb{T}_\ell \mathbb{G}$, so the space of such A's has dimension

$$\dim(\zeta^{-1}(\ell)) = \dim(\mathbf{P}(\Lambda^2 V)) - \dim(\mathbb{G}) - 1 = \frac{1}{2}(n-1)(n-4).$$

We observe that, given a linear space S of dimension $n-3$ in $\mathbf{P}(V)$, it is uniquely determined a complex H, up to constants, such that the center of H is S. Indeed, all the lines in $\mathbf{P}(V)$ intersecting S have to be contained in $\Gamma = \mathrm{V}(H) \cap \mathbb{G}$, but this is a linear condition in the Plücker coordinates.

Fixing a set of lines $\{\ell_1, \ldots, \ell_{\frac{n}{2}}\}$ spanning $\mathbf{P}(V)$, for any j such that $1 \le j \le \frac{n}{2}$ we will denote by $H_j \in \mathbf{P}(\Lambda^2 V^*)$ the unique complex having $<\ell_i>_{i \ne j}$ as center. We will denote by F_i the $(\frac{n}{2} - 2)$-dimensional linear span $<H_j>_{j \ne i} \subset \mathbf{P}(\Lambda^2 V^*)$; a point in F_i is a complex having at least ℓ_i as center.

Remark 1. Let $\ell_1, \ldots, \ell_{\frac{n}{2}}$ be lines spanning $\mathbf{P}(V)$. Up to a projectivity of $\mathbf{P}(V)$, we may assume that

$$\forall i, \quad \ell_i = \bigcap_{j \notin \{2i-2, 2i-1\}} \mathrm{V}(x_j^*),$$

being $\mathrm{V}(x_j^*)$ the hyperplane in $\mathbf{P}(V)$ whose points have x_j coordinate zero. With this choice, H_i is the complex represented by a skew-symmetric matrix with (j, k)-th entry

$$\begin{cases} \alpha & \text{if } j+1 = 2i = k \text{ for some } i \\ -\alpha & \text{if } k+1 = 2i = j \text{ for some } i \\ 0 & \text{otherwise} \end{cases}$$

for some $\alpha \in \mathbf{k} \setminus \{0\}$. The elements of $\zeta^{-1}(\ell_i)$ have zero entries in the $(2i-1)$-th, $(2i)$-th rows and columns.

Proposition 2. Let $\{\ell_1, \ldots, \ell_{\frac{n}{2}}\}$ be a set of lines spanning $\mathbf{P}(V)$. Let σ be the $(\frac{n-2}{2})$-dimensional linear space $<H_i>_{1 \le i \le \frac{n}{2}} \subset \mathbf{P}(\Lambda^2 V^*)$. Then, for a line $N \subset \mathbf{P}(\Lambda^2 V^*)$, the following are equivalent:

    i. $N \subseteq \sigma$ and $N \cap F_i \cap F_j = \emptyset$ for any $1 \leq i < j \leq \frac{n}{2}$;

    ii. N does not contain special complexes of the second type and $\mathrm{Deg}(N) = \{\ell_1, \ldots, \ell_{\frac{n}{2}}\}$.

Proof. i. $\Rightarrow$ ii. Let $\mathbb{P}_i := \zeta^{-1}(\ell_i)$. Let $N \subseteq \sigma$. For any j, the linear space F_j has dimension $\frac{n}{2} - 2$, so any line $N \subseteq \sigma$ intersects it; hence, $\mathrm{Deg}(N) \supseteq \{\ell_1, \ldots, \ell_{\frac{n}{2}}\}$. Moreover, with the choice of coordinates of Remark 1, points of σ are represented by skew-symmetric matrices with (j, k)-th entry

$$\begin{cases} \alpha_i & \text{if } \exists i \text{ such that } j+1 = 2i = k \\ -\alpha_i & \text{if } \exists i \text{ such that } k+1 = 2i = j \\ 0 & \text{otherwise} \end{cases}$$

for some sequence (α_i) in $\mathbf{k}$. It is easy to check that a point of σ
    – is a special complex if and only if $\alpha_j = 0$ for some j;
    – is special of the second type if and only if $\alpha_j = \alpha_k = 0$ for some $j \neq k$, i.e. if and only if it lies on $F_j \cap F_k$.

If $N \cap F_j \cap F_k = \emptyset$ for any $j < k$, then N contains $\frac{n}{2}$ special complexes of the first type and no special complexes of the second type, hence the conclusion.

ii. $\Rightarrow$ i. Let N be as in ii.. For any i define $R_i := N \cap \zeta^{-1}(\ell_i)$, which is non-empty by hypothesis. We have $R_i \neq R_j$ for any $i \neq j$, as otherwise N would contain a point having $<\ell_i, \ell_j>$ as center. We claim that $N \subseteq \sigma$, for which it is sufficient to show that R_1 and R_2 are both contained in σ.

    Let us choose the coordinates as in Remark 1. For any $i \neq j$ we have $N = <R_i, R_j>$, so

$$(2) \qquad R_1 \cup R_2 \subset N = \bigcap_{i<j} <R_i, R_j>.$$

Complexes in $R_i \subset \zeta^{-1}(\ell_i)$ have zero i-th and $(i+1)$-th rows and columns, hence the entries $A_{k,l}$ of a complex A in $<R_i, R_j>$ are zero at least when

$$(k, l) \in (\{2i-1, 2i\} \times \{2j-1, 2j\}) \cup (\{2j-1, 2j\} \times \{2i-1, 2i\}).$$

From (2), we deduce that any complex A in $R_1 \cup R_2$ has non-zero entries $A_{k,l}$ only if $\exists i$ such that $k+1 = 2i = l$ or $l+1 = 2i = k$, hence it belongs to $\sigma = <H_i>_{1 \leq i \leq \frac{n}{2}}$.

    This is enough to show that $N \subseteq \sigma$. If $N \cap F_i \cap F_j \neq \emptyset$ for some $i \neq j$, then $\mathrm{Deg}(N)$ would contain $<\ell_i, \ell_j>$, hence a contradiction. $\qquad \square$

Proposition 3. *If $m = 2$, $n \geq 4$ and n is even, then ρ is dominant. The general element of $\mathcal{H}$ is the union of $\frac{n}{2}$ lines spanning $\mathbf{P}(V)$. The general fiber has dimension $n-4$; its general element is a general line $N \subset \mathbf{P}(\Lambda^2 V^*)$ lying on σ, as in Proposition 2. In particular, ρ is birational if $(m, n) = (2, 4)$.*

Proof. We can define a rational map

$$\xi: \ \mathbf{Gr}(1, \mathbf{P}(V))^{\frac{n}{2}} \dashrightarrow \mathcal{H},$$

defined outside the closed subset corresponding to sets of $\frac{n}{2}$ lines non-spanning $\mathbf{P}(V)$, sending $\frac{n}{2}$ lines to the corresponding point in $\mathcal{H}$. It is finite and its image $\mathrm{Im}(\xi)$ is irreducible.

The general morphism ϕ has $\frac{n}{2}$ lines (ℓ_i) spanning $\mathbf{P}(V)$ as degeneracy locus, so $\operatorname{Im}(\xi) = \operatorname{Im}(\rho)$. It remains to show that

$$\dim(\operatorname{Im}(\xi)) = \dim \mathcal{H}, \tag{3}$$

so that $\overline{\operatorname{Im}(\rho)}$ is the unique irreducible component of $\mathcal{H}$, hence ρ is dominant. On the one hand, $\dim(\operatorname{Im}(\xi)) = \dim(\mathbf{Gr}(1, \mathbf{P}(V))^{\frac{n}{2}}) = n^2 - 2n$. On the other hand, let Y be the union of $\frac{n}{2}$ skew lines. From

$$0 \longrightarrow \mathcal{T}_Y \longrightarrow \mathcal{T}_{\mathbf{P}(V)}\big|_Y \longrightarrow \mathcal{N}_{Y/\mathbf{P}(V)} \longrightarrow 0$$

and Euler sequence restricted to Y, we get

$$\mathrm{h}^0(\mathcal{N}_{Y/\mathbf{P}(V)}) = \frac{n}{2}(2n - 1 - \chi(\mathcal{T}_Y)) \quad \text{and} \quad \mathrm{h}^1(\mathcal{N}_{Y/\mathbf{P}(V)}) = 0.$$

Riemann-Roch Theorem yields $\chi(\mathcal{T}_Y) = 3$, so equality (3) holds.

Finally, fixing a point $\cup_i \ell_i$ in $\mathcal{H}$, the general elements of its preimage via ρ is a general line $N \subseteq \sigma$, as showed in Proposition 2. In particular, the space of such lines has dimension

$$\dim \mathbf{Gr}(1, \sigma) = n - 4.$$

When $n = 4$, σ is a line and the unique preimage is σ itself. $\qquad\square$

We remark here that the fibers of ρ can be explicitly constructed. By means of a projectivity we can send any set of $\frac{n}{2}$ general lines to the lines chosen in Remark 1; then, we just need to apply to any line lying on σ as in Proposition 2 the same projectivity backwards.

4. The behavior of ρ: odd case

Let n be odd from now on. The general degeneracy locus X_ϕ is easy to describe: similarly to the case $m = 3$ in [Tan, §6], the elements in $\operatorname{Im}(\rho)$ are the images of maps

$$\mathbf{P}(U) \xrightarrow{\;[f_1:\dots:f_n]\;} \mathbf{P}(V), \tag{4}$$

where $f_1, \dots, f_n$ are forms of degree $\frac{n-1}{2}$ in $\mathbf{k}[y_0, y_1]$, Pfaffians of a general $n \times n$ skew-symmetric matrix N with entries in $\mathbf{k}[y_0, y_1]_1$. These forms are general, in the sense that they generate the whole vector space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$.

Lemma 4. For a general $n \times n$ skew-symmetric matrix N with entries in $\mathbf{k}[y_0, y_1]_1$, its Pfaffians of order $n - 1$ span the whole $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$.

Proof. For the $(n-1) \times (n-1)$ Pfaffians of a general N, not to span $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$ is a closed condition, so it is sufficient to exhibit, for any odd k, a matrix N_k not satisfying it. For this sake, we consider the $k \times k$ matrix

$$N_k = \begin{pmatrix} 0 & y_0 & & & & & \\ -y_0 & 0 & y_1 & & & & \\ & -y_1 & 0 & y_0 & & & \\ & & -y_0 & 0 & y_1 & & \\ & & & \ddots & \ddots & & \\ & & & & & 0 & y_1 \\ & & & & & -y_1 & 0 \end{pmatrix}.$$

If we denote by $\mathrm{Pf}_i(N_k)$ the $(k-1) \times (k-1)$ Pfaffian obtained from N_k by deleting the i-th row and column, it is easy to check that

$$\mathrm{Pf}_{2i+1}(N_k) = y_0^i y_1^{\frac{k-1}{2}-i} \qquad \text{for any } 0 \leq i \leq \frac{k-1}{2},$$

$$\mathrm{Pf}_{2i}(N_k) = 0 \qquad \text{for any } 1 \leq i \leq \frac{k-1}{2},$$

and this concludes the proof. $\qquad\square$

Remark 5. As a consequence of the previous lemma, every sequence of general forms $f_1, \ldots, f_n$ of degree $\frac{n-1}{2}$ corresponds to the sequence of Pfaffians of a suitable skew-symmetric matrix. Indeed, these forms can be expressed as linear combination of the Pfaffians of N_k above, giving rise to $\beta \in \mathrm{PGL}(V)$ such that the diagram

$$\begin{array}{ccc}
 & V_1 & \\
\overset{[f_1:\ldots:f_n]}{\nearrow} & & \overset{\beta}{\searrow} \\
\mathbf{P}(U) \xrightarrow[{[\mathrm{Pf}_1(N_k):\ldots:\mathrm{Pf}_k(N_k)]}]{\sim} & & V_2
\end{array}$$

commutes. This produces an automorphism of $\mathbf{P}(U)$, hence a change of basis of $\mathbf{k}[y_0, y_1]_1$. In terms of this new basis, N_k has the desired Pfaffians $f_1, \ldots, f_n$.

Proposition 6. If $m = 2$, $n \geq 5$ and n is odd, then ρ is dominant. The general element of $\mathcal{H}$ is the image in $\mathbf{P}(V)$ of a map
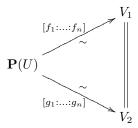
$$\mathbf{P}(U) \xrightarrow{[f_1:\ldots:f_n]} \mathbf{P}(V),$$

where $f_1, \ldots, f_n$ are forms of degree $\frac{n-1}{2}$ spanning the whole space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$. The general fiber of ρ has dimension $\frac{n^2-3n}{2}$.

Proof. Let $r = \dim(\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}) - 1 = \frac{n-1}{2}$. We can define a rational map

$$\xi : \mathbb{A}^{(r+1)n} \dashrightarrow \mathcal{H}$$

sending an n-tuple of forms $f_1, \ldots, f_n$ to the image of the map (4). It is defined on the n-tuples which span the whole linear space $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$; its image $\mathrm{Im}(\xi)$ is irreducible and its dimension is easily computable. Indeed, on the one hand there is a natural GL_2-action on $\mathbf{k}[y_0, y_1]_1$, acting as a change of basis on U; this induces an action on $\mathbf{k}[y_0, y_1]_{\frac{n-1}{2}}$ and therefore on $\mathbb{A}^{(r+1)n}$, and one can see that ξ factors through this action. On the other hand, take two points V_1, V_2 in $\mathrm{Im}(\xi)$ such that $V_1 = V_2$. By the commutativity of the diagram

$$\begin{array}{ccc}
 & & V_1 \\
 & \overset{[f_1:\ldots:f_n]}{\nearrow} & \| \\
\mathbf{P}(U) & & \| \\
 & \overset{[g_1:\ldots:g_n]}{\searrow} & \| \\
 & & V_2
\end{array}$$

we get an automorphism of $\mathbf{P}(U)$, i.e. the two maps $[f_1 : \ldots : f_n]$ and $[g_1 : \ldots : g_n]$ belong to the same class modulo GL_2. Hence

$$\dim(\mathrm{Im}(\xi)) = \dim(\mathbb{A}^{(r+1)n}) - \dim(\mathrm{GL}_2) = \frac{n^2+n-8}{2}.$$

Since n general forms are obtained as the Pfaffians of a suitable matrix N by Remark 5, $\mathrm{Im}(\rho) = \mathrm{Im}(\xi)$. If we prove that

$$\dim(\mathrm{Im}(\xi)) = \dim\mathcal{H},$$

then $\overline{\mathrm{Im}(\rho)}$ is the unique irreducible component of $\mathcal{H}$, hence ρ is dominant. This can be done again as in the proof of Proposition 3.

The dimension of the fibers is finally

$$\dim\mathbf{Gr}(2, \Lambda^2 V) - \dim\mathcal{H} = \frac{n^2 - 3n}{2}. \qquad \square$$

As in the even case, also for odd values of n it is possible to construct explicitly the fibers of ρ. Up to a projectivity, we can assume that the degeneracy locus is the image of the Pfaffians of the matrix N_k from Lemma 4. We can find by linear algebra all the skew-symmetric matrices having those as Pfaffians and apply the same projectivity backwards to get the elements of the desired fiber.

References

[BM01] D. Bazan and E. Mezzetti. On the construction of some Buchsbaum varieties and the Hilbert scheme of elliptic scrolls in $\mathbb{P}^5$. *Geom. Dedicata*, 86(1-3):191–204, 2001.

[FF10] D. Faenzi and M. L. Fania. Skew-symmetric matrices and Palatini scrolls. *Math. Ann.*, 347(4):859–883, 2010.

[FM02] M. L. Fania and E. Mezzetti. On the Hilbert scheme of Palatini threefolds. *Adv. Geom.*, 2(4):371–389, 2002.

[Ott92] G. Ottaviani. On 3-folds in $\mathbf{P}^5$ which are scrolls. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):451–471, 1992.

[Tan] F. Tanturri. On the Hilbert scheme of degeneracy loci of twisted differential forms. To appear in *Trans. Amer. Math. Soc.*.

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