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SUBLOADING-DAMAGE MODEL AND ITS EXTENSION TO UNILATERAL DAMAGE EFFECT

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Abstract. The elastoplastic constitutive equation with the damage is formulated incorporating the subloading surface model. Further, it is extended to describe the unilateral damage phenomenon by formulating the actual Young's modulus tensor as the function of the signs of the principal actual damaged stresses and the damage variable. Here, we may perform the ordinary deformation analysis simply in the virtual undamaged state.

1 INTRODUCTION

The past elastoplastic-damage models [1] [2] [3] [4] have been formulated in the current damaged configuration, so that several complicated modifications of evolution rules of internal variables by incorporating the damage effect are required. In addition, they are incapable of describing cyclic loading behavior.

Further, when the material undergoes the damage, the microdefects may be partly closed leading to the reduction of the damage effect in the plane subject to the compressional normal stress in the most materials. This is more often the case for very brittle materials. The partial closure of microcracks revives the effective area which can carry the load in compression and thus the stiffness may then be partially or fully recovered in compression. It is called the *unilateral microdefect closure effect* or simply *unilateral damaged effect* by Ladeveze and Lemaitre [5]. The constitutive relations for the unilateral damage effect have been formulated in the current damaged configuration, introducing various transformation tensor of the actual damaged stress tensor to the virtual undamaged stress tensor [2] [5] [6].

The elastoplastic-damage model capable of describing the cyclic loading behavior is formulated by incorporating the subloading surface model [7] [8] [9] in this article. It is formulated in the virtual undamaged configuration, so that the elastoplastic constitutive equation in the ordinary subloading surface model itself is inherited to this model without any modification. Further, it is extended to describe the unilateral damage phenomenon by formulating the actual Young's modulus as the function of the signs of the principal actual damaged stresses and the damage variable, in which the complicated transformation tensor is not required. Here, it is noticeable that the simple deformation analysis by the ordinary constitutive equation without the influence of the damage can be performed in the virtual undamaged configuration, provided that the actual damaged stress tensor is calculated from the virtual undamaged stress tensor.

2 HYPERELASTIC EQUATION

The infinitesimal strain $\boldsymbol{\varepsilon}$ is additively decomposed into the elastic strain $\boldsymbol{\varepsilon}^e$ and the plastic strain $\boldsymbol{\varepsilon}^p$ as follows:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p \tag{1}$$

In what follows, we adopt the *hypothesis of strain equivalence* [10] insisting that the strain and its elastic and plastic parts in the virtual undamaged configuration are equivalent to those in the actual damaged configuration. It is based on the fact that the cracks possess infinitesimal thicknesses and various directions and thus it is assumed that the shape and the volume of the material in the actual configuration does not change from them in the virtual undamaged configuration. Mechanical quantities in the virtual undamaged configuration are specified by the symbols added the wave under them, i.e. $\underset{\sim}{(\ )}$.

The relation of the virtual Cauchy stress $\underset{\sim}{\boldsymbol{\sigma}}$ in the undamaged configuration and the elastic strain $\boldsymbol{\varepsilon}^e$ are given by the Helmholtz free energy function $\underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)$ and the Gibbs' free energy $\underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}})$ as follows:

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial \underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e}, \quad \boldsymbol{\varepsilon}^e = \frac{\partial \underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}})}{\partial \underset{\sim}{\boldsymbol{\sigma}}} \tag{2}$$

Adopting the simplest functions in the quadratic forms

$$\underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \underset{\sim}{\mathbf{E}} : \boldsymbol{\varepsilon}^e \left(= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e\right), \quad \underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}}) = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} \left(= \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e = \underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)\right) \tag{3}$$

it follows that

$$\underset{\sim}{\boldsymbol{\sigma}} = \frac{\partial \underset{\sim}{\psi}(\boldsymbol{\varepsilon}^e)}{\partial \boldsymbol{\varepsilon}^e} = \underset{\sim}{\mathbf{E}} : \boldsymbol{\varepsilon}^e, \quad \boldsymbol{\varepsilon}^e = \frac{\partial \underset{\sim}{\phi}(\underset{\sim}{\boldsymbol{\sigma}})}{\partial \underset{\sim}{\boldsymbol{\sigma}}} = \underset{\sim}{\mathbf{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} \tag{4}$$

where $\underset{\sim}{\mathbf{E}}$ is the fourth-order elastic modulus tensor. If $\underset{\sim}{\mathbf{E}}$ is the constant tensor, we have the rate linear relations:

$$\underset{\sim}{\overset{\bullet}{\boldsymbol{\sigma}}} = \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}}{}^e, \quad \overset{\bullet}{\boldsymbol{\varepsilon}}{}^e = \underset{\sim}{\mathbf{E}}^{-1} : \underset{\sim}{\overset{\bullet}{\boldsymbol{\sigma}}} \tag{5}$$

The elastic stiffness modulus tensor $\underset{\sim}{\mathbf{E}}$ is given for the Hooke's law as follows:

$$\begin{cases} \underset{\sim}{E}_{ijkl} = \dfrac{\underset{\sim}{E}}{1+\nu}\left[\dfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \dfrac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \\ \underset{\sim}{E}^{-1}_{ijkl} = \dfrac{1}{\underset{\sim}{E}}\left[\dfrac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl}\right] \end{cases} \tag{6}$$

where $\underset{\sim}{E}$ is the virtual Young's modulus in the virtual undamaged configuration and ν is the Poisson's ratio which is assumed to be constant independently of the damage. Equations (2)-(4) are expressed for Eq. (6) as follows:

$$\begin{cases} \underset{\sim}{\psi}(\varepsilon^e_{ij}) = \dfrac{1}{2}\dfrac{\underset{\sim}{E}}{1+\nu}\left[\varepsilon^e_{ij}\varepsilon^e_{ij} + \dfrac{\nu}{1-2\nu}(\varepsilon^e_{kk})^2\right] \\ \underset{\sim}{\phi}(\underset{\sim}{\sigma}_{ij}) = \dfrac{1}{2\underset{\sim}{E}}[(1+\nu)\underset{\sim}{\sigma}_{ij}\underset{\sim}{\sigma}_{ij} - \nu(\underset{\sim}{\sigma}_{kk})^2] \end{cases} \tag{7}$$

$$\underset{\sim}{\sigma}_{ij} = \frac{\underset{\sim}{E}}{1+\nu}\left(\varepsilon^e_{ij} + \frac{\nu}{1-2\nu}\varepsilon^e_{kk}\delta_{ij}\right), \quad \varepsilon^e_{ij} = \frac{1}{\underset{\sim}{E}}[(1+\nu)\underset{\sim}{\sigma}_{ij} - \nu\underset{\sim}{\sigma}_{kk}\delta_{ij}] \tag{8}$$

Analogously, let the following quadratic free energy functions be adopted by taking into account of the fact that the actual elastic modulus tensor is influenced by the damage variable D $(0 \le D \le 1)$.

$$\begin{cases} \psi(\boldsymbol{\varepsilon}^e,\ D) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{E}(D) : \boldsymbol{\varepsilon}^e \left(= \frac{1}{2}\boldsymbol{\sigma}(D) : \boldsymbol{\varepsilon}^e\right) \\ \phi(\boldsymbol{\sigma}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{E}^{-1}(D) : \boldsymbol{\sigma} \end{cases} \tag{9}$$

from which one has

$$\boldsymbol{\sigma} = \frac{\partial\psi(\boldsymbol{\varepsilon}^e,\ D)}{\partial\boldsymbol{\varepsilon}^e},\quad \boldsymbol{\varepsilon}^e = \frac{\partial\phi(\boldsymbol{\sigma},\ D)}{\partial\boldsymbol{\sigma}} \tag{10}$$

$$\boldsymbol{\sigma} = \mathbf{E}(D) : \boldsymbol{\varepsilon}^e,\ \boldsymbol{\varepsilon}^e = \mathbf{E}^{-1}(D) : \boldsymbol{\sigma} \tag{11}$$

Further, assume the following Hooke's type elastic modulus tensor with the damage effect, provided that the Poisson's ratio is not influenced by the damage.

$$\begin{cases} E_{ijkl}(D) = \frac{E(D)}{1+\nu}\left[\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \frac{\nu}{1-2\nu}\delta_{ij}\delta_{kl}\right] \\ E^{-1}_{ijkl}(D) = \frac{1}{E(D)}\left[\frac{1}{2}(1+\nu)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \nu\delta_{ij}\delta_{kl}\right] \end{cases} \tag{12}$$

$$\begin{cases} \psi(\varepsilon^e_{ij},\ D) = \frac{1}{2}\frac{E(D)}{1+\nu}\left[\varepsilon^e_{ij}\varepsilon^e_{ij} + \frac{\nu}{1-2\nu}(\varepsilon^e_{kk})^2\right] \\ \phi(\sigma_{ij},\ D) = \frac{1}{2E(D)}[(1+\nu)\sigma_{ij}\sigma_{ij} - \nu(\sigma_{kk})^2] \end{cases} \tag{13}$$

$$\sigma_{ij} = \frac{E(D)}{1+\nu}\left(\varepsilon^e_{ij} + \frac{\nu}{1-2\nu}\varepsilon^e_{kk}\delta_{ij}\right),\quad \varepsilon^e_{ij} = \frac{1}{E(D)}[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}] \tag{14}$$

The damaged virtual stress is related to the undamaged virtual stress from Eqs. (4) and (11) as follows:

$$\boldsymbol{\sigma} = \mathbf{E}(D) : \underset{\sim}{\mathbf{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}},\ \underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\mathbf{E}} : \mathbf{E}^{-1}(D) : \boldsymbol{\sigma} \tag{15}$$

3 SUBLOADING SURFACE MODEL IN VIRTUAL UNDAMAGED CONFIGUTRATION

The elastoplastic constitutive equation in the virtual undamaged configuration will be formulated based on the concept of subloading surface in this section (cf. [9]).

3.1 Normal-yield and subloading surfaces

The normal-yield surface with the isotropic and the kinematic hardening is described as

$$\underset{\sim}{f}(\hat{\boldsymbol{\sigma}}) = \underset{\sim}{F}(\underset{\sim}{H}) \tag{16}$$

where

$$\hat{\underset{\sim}{\boldsymbol{\sigma}}} \equiv \underset{\sim}{\boldsymbol{\sigma}} - \underset{\sim}{\boldsymbol{\alpha}} \tag{17}$$

The subloading surface for the normal-yield sur-face in Eq. (16) is given as follows.

$$\underset{\sim}{f}(\bar{\boldsymbol{\sigma}}) = \underset{\sim}{R}\underset{\sim}{F}(\underset{\sim}{H}) \tag{18}$$

where $\underset{\sim}{R}$ $(0 \le \underset{\sim}{R} \le 1)$ is the *normal-yield ratio* designating the ratio of the size of the subloading surface to that of the normal-yield surface and

$$\bar{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \bar{\boldsymbol{\alpha}} \tag{19}$$

where $\bar{\boldsymbol{\alpha}}$ stands for the conjugate (similar) point to the variable $\boldsymbol{\alpha}$ in the normal-yield surface. Here, $f(\hat{\boldsymbol{\sigma}})$ is chosen to be the homogeneous function of $\hat{\boldsymbol{\sigma}}$ in degree-one.

By letting $\mathbf{c}$ denote the center of similarity of the normal-yield and the subloading surfaces, i.e. the similarity-center, which is called *elastic-core* since the most elastic deformation behavior is induced when the stress lies on it fulfilling $R=0$ as will be explained later, the following relation holds.

$$\mathbf{c} - \bar{\boldsymbol{\alpha}} = R(\mathbf{c} - \boldsymbol{\alpha}) \tag{20}$$

from which one has

$$\bar{\boldsymbol{\alpha}} = \mathbf{c} - R\hat{\mathbf{c}}, \quad \bar{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}} + R\hat{\mathbf{c}} \tag{21}$$

where

$$\hat{\mathbf{c}} \equiv \mathbf{c} - \boldsymbol{\alpha}, \quad \tilde{\boldsymbol{\sigma}} \equiv \boldsymbol{\sigma} - \mathbf{c} \tag{22}$$

3.2 Plastic flow rule and evolution equations of internal variables

Adopt the associated flow rule for the subloading surface:

$$\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p = \overset{\bullet}{\lambda}\,\bar{\mathbf{n}} \quad (\overset{\bullet}{\lambda} = \|\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p\| > 0) \tag{23}$$

where

$$\bar{\mathbf{n}} \equiv \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \Big/ \left\| \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \boldsymbol{\sigma}} \right\| \quad (\|\bar{\mathbf{n}}\| = 1) \tag{24}$$

The rate of the isotropic hardening variable is described as

$$\overset{\bullet}{H}(\boldsymbol{\sigma}, H; \overset{\bullet}{\boldsymbol{\varepsilon}}{}^p) = f_{Hn}(\boldsymbol{\sigma}, H; \bar{\mathbf{n}})\,\overset{\bullet}{\lambda} \tag{25}$$

and the rate of the kinematic hardening variable is described as follows:

$$\overset{\bullet}{\boldsymbol{\alpha}} = c_k\Big(\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p - \frac{1}{b_k F}\|\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p\|\boldsymbol{\alpha}\Big) = \overset{\bullet}{\lambda}\,\bar{\mathbf{f}}_{kn}, \quad \bar{\mathbf{f}}_{kn} \equiv c_k\Big(\bar{\mathbf{n}} - \frac{1}{b_k F}\|\bar{\mathbf{n}}\|\boldsymbol{\alpha}\Big) \tag{26}$$

where c_k and b_k are the material constants.

The evolution rule of the normal-yield ratio is given by

$$\overset{\bullet}{R} = U(R)\|\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p\| \quad \text{for } \overset{\bullet}{\boldsymbol{\varepsilon}}{}^p \neq \mathbf{0} \tag{27}$$

where $U(R)$ is the monotonically-decreasing function of normal-yield ratio which is given explicitly as

$$U(R) = u\cot\Big(\frac{\pi}{2}\frac{\langle R - R_e\rangle}{1 - R_e}\Big) \tag{28}$$

where $\langle\ \rangle$ is the Macaulay's bracket and u is the material parameters and $R_e (<1)$ is the material constant denoting the value of R below which only elastic deformation is induced practically.

3.3 Evolution rule of elastic-core

Let the following *elastic-core surface* be introduced, which always passes through the elastic-core $\mathbf{c}$ and maintains a similarity to the normal-yield surface with respect to the kinematic-hardening variable $\boldsymbol{\alpha}$.

$$f(\hat{\mathbf{c}}) = \Re_c F(H), \text{ i.e. } \Re_c = f(\hat{\mathbf{c}}) / F(H) \tag{29}$$

$\Re_c$ designates the ratio of the size of the elastic-core surface to the normal-yield surface so that let it be called the *elastic-core yield ratio*. Then, let it be postulated that the elastic-core can never reach the normal-yield surface designating the fully-plastic stress state so that the elastic-core does not go over the following *limit elastic-core surface.*

$$f(\hat{\mathbf{c}}) = \chi F(H) \tag{30}$$

where $\chi\,(<1)$ is the material constant designating the limit value of the variable $\Re_c$.

The following evolution rule of the elastic-core is assumed [11].

$$\overset{\bullet}{\mathbf{c}} = c\|\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p\|\left(\frac{\chi}{R}\bar{\boldsymbol{\sigma}} - \hat{\mathbf{c}}\right) + \overset{\bullet}{\boldsymbol{\alpha}} + \frac{\overset{\bullet}{F}}{F}\hat{\mathbf{c}} = \overset{\bullet}{\lambda}\bar{\mathbf{f}}_{cn} + \overset{\bullet}{\lambda}\bar{\mathbf{f}}_{kn} + \overset{\bullet}{\lambda}\frac{F' f_{Hn}}{F}\hat{\mathbf{c}} \tag{31}$$

where c is the material constant and

$$\bar{\mathbf{f}}_{cn} \equiv c\left(\frac{\chi}{R}\bar{\boldsymbol{\sigma}} - \hat{\mathbf{c}}\right) \tag{32}$$

The time-differentiation of Eq. (21) leads to

$$\overset{\bullet}{\bar{\boldsymbol{\alpha}}} = R\overset{\bullet}{\boldsymbol{\alpha}} + (1-R)\overset{\bullet}{\mathbf{c}} - \overset{\bullet}{R}\hat{\mathbf{c}} \tag{33}$$

Substituting Eq. (31) into Eq. (33), one obtains

$$\overset{\bullet}{\bar{\boldsymbol{\alpha}}} = R\overset{\bullet}{\boldsymbol{\alpha}} + (1-R)\left[c\|\overset{\bullet}{\boldsymbol{\varepsilon}}{}^p\|\left(\frac{\chi}{R}\bar{\boldsymbol{\sigma}} - \hat{\mathbf{c}}\right) + \overset{\bullet}{\boldsymbol{\alpha}} + \frac{\overset{\bullet}{F}}{F}\hat{\mathbf{c}}\right] - \overset{\bullet}{R}\hat{\mathbf{c}} \tag{34}$$

3.4 Plastic strain rate

The time derivative of Eq. (18) leads to the consistency condition for the subloading surface:

$$\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \overset{\bullet}{\boldsymbol{\sigma}} - \frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \overset{\bullet}{\bar{\boldsymbol{\alpha}}} - R\overset{\bullet}{F} - \overset{\bullet}{R}F = 0 \tag{35}$$

Here, one has

$$\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \bar{\boldsymbol{\sigma}} = f(\bar{\boldsymbol{\sigma}}) = RF \tag{36}$$

based on the homogeneous function $f(\bar{\boldsymbol{\sigma}})$ of $\bar{\boldsymbol{\sigma}}$ in degree-one by the Euler's theorem. Then, it follows that

$$\bar{\mathbf{n}} : \bar{\boldsymbol{\sigma}} = \frac{\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}} : \bar{\boldsymbol{\sigma}}}{\left\|\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}}\right\|} = \frac{RF}{\left\|\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}}\right\|}, \quad \frac{1}{\left\|\frac{\partial f(\bar{\boldsymbol{\sigma}})}{\partial \bar{\boldsymbol{\sigma}}}\right\|} = \frac{\bar{\mathbf{n}} : \bar{\boldsymbol{\sigma}}}{RF} \tag{37}$$

The substitution of Eq. (37) into Eq. (35) leads to

$$\bar{\mathbf{n}} : \overset{\bullet}{\boldsymbol{\sigma}} - \bar{\mathbf{n}} : \left[\left(\frac{\overset{\bullet}{F}}{F} + \frac{\overset{\bullet}{R}}{R}\right)\bar{\boldsymbol{\sigma}} + \overset{\bullet}{\bar{\boldsymbol{\alpha}}}\right] = 0 \tag{38}$$

The substitution of Eq. (34) into Eq. (38) leads to

$$\underset{\sim}{\bar{\mathbf{n}}} : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}} - \underset{\sim}{\bar{\mathbf{n}}} : \Big[\frac{\overset{\bullet}{\underset{\sim}{F}}}{\underset{\sim}{F}} \hat{\boldsymbol{\sigma}} + \frac{\overset{\bullet}{\underset{\sim}{R}}}{\underset{\sim}{R}} \tilde{\underset{\sim}{\boldsymbol{\sigma}}} + (1-\underset{\sim}{R})\, c \Big(\frac{\chi}{\underset{\sim}{R}} \bar{\underset{\sim}{\boldsymbol{\sigma}}} - \underset{\sim}{\mathbf{c}} \Big) \| \overset{\bullet}{\boldsymbol{\varepsilon}}^p \| + \overset{\bullet}{\underset{\sim}{\boldsymbol{\alpha}}} \Big] = 0 \tag{39}$$

The substitutions of Eqs. (23), (25), (26), (27) and (31) into Eq. (39) leads to

$$\underset{\sim}{\bar{\mathbf{n}}} : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}} - \underset{\sim}{\bar{\mathbf{n}}} : \Big[\frac{\underset{\sim}{F'}}{\underset{\sim}{F}} \overset{\bullet}{\underset{\sim}{\lambda}} \underset{\sim}{f_{Hn}} \hat{\boldsymbol{\sigma}} + \overset{\bullet}{\underset{\sim}{\lambda}} \underset{\sim}{\bar{\mathbf{f}}_{kn}} + (1-\underset{\sim}{R}) \overset{\bullet}{\lambda} \underset{\sim}{\bar{\mathbf{f}}_{cn}} + \frac{\underset{\sim}{U}}{\underset{\sim}{R}} \overset{\bullet}{\lambda} \tilde{\underset{\sim}{\boldsymbol{\sigma}}} \Big] = 0 \tag{40}$$

from which the plastic multiplier $\overset{\bullet}{\underset{\sim}{\lambda}}$ and the plastic strain rate $\overset{\bullet}{\boldsymbol{\varepsilon}}^p$ are given as follows:

$$\overset{\bullet}{\underset{\sim}{\lambda}} = \frac{\underset{\sim}{\bar{\mathbf{n}}} : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}^p}, \quad \overset{\bullet}{\boldsymbol{\varepsilon}}^p = \frac{\underset{\sim}{\bar{\mathbf{n}}} : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}^p} \underset{\sim}{\bar{\mathbf{n}}} \tag{41}$$

where

$$\underset{\sim}{\bar{M}}^p = \underset{\sim}{\bar{\mathbf{n}}} : \Big[\frac{\underset{\sim}{F'}}{\underset{\sim}{F}} \underset{\sim}{f_{Hn}} \hat{\boldsymbol{\sigma}} + \underset{\sim}{\bar{\mathbf{f}}_{kn}} + (1-\underset{\sim}{R}) \underset{\sim}{\bar{\mathbf{f}}_{cn}} + \frac{\underset{\sim}{U}}{\underset{\sim}{R}} \tilde{\underset{\sim}{\boldsymbol{\sigma}}} \Big] \tag{42}$$

3.5 Stain rate vs. stress rate relations

The strain rate is given by substituting Eqs. (5) and $(41)_2$ into Eq. (1) as follows:

$$\overset{\bullet}{\boldsymbol{\varepsilon}} = \underset{\sim}{\mathbf{E}}^{-1} : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}} + \frac{\underset{\sim}{\bar{\mathbf{n}}} : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}^p} \underset{\sim}{\bar{\mathbf{n}}} = \Big(\underset{\sim}{\mathbf{E}}^{-1} + \frac{\underset{\sim}{\bar{\mathbf{n}}} \otimes \underset{\sim}{\bar{\mathbf{n}}}}{\underset{\sim}{\bar{M}}^p} \Big) : \overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}} \tag{43}$$

from which the magnitude of plastic strain rate described in terms of the strain rate, denoted by $\overset{\bullet}{\underset{\sim}{\Lambda}}$ instead of $\overset{\bullet}{\underset{\sim}{\lambda}}$, in the flow rule of Eq. (23) is given as follows:

$$\overset{\bullet}{\underset{\sim}{\Lambda}} = \frac{\underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}}}{\underset{\sim}{\bar{M}}^p + \underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \underset{\sim}{\bar{\mathbf{n}}}}, \quad \overset{\bullet}{\boldsymbol{\varepsilon}}^p = \overset{\bullet}{\underset{\sim}{\Lambda}} \underset{\sim}{\bar{\mathbf{n}}} = \frac{\underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}}}{\underset{\sim}{\bar{M}}^p + \underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \underset{\sim}{\bar{\mathbf{n}}}} \underset{\sim}{\bar{\mathbf{n}}} \tag{44}$$

The stress rate is given by the strain rate as follows:

$$\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}} = \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}} - \frac{\underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}}}{\underset{\sim}{\bar{M}}^p + \underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \underset{\sim}{\bar{\mathbf{n}}}} \underset{\sim}{\mathbf{E}} : \underset{\sim}{\bar{\mathbf{n}}} = \Big(\underset{\sim}{\mathbf{E}} - \frac{\underset{\sim}{\mathbf{E}} : \underset{\sim}{\bar{\mathbf{n}}} \otimes \underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}}}{\underset{\sim}{\bar{M}}^p + \underset{\sim}{\mathbf{n}} : \underset{\sim}{\mathbf{E}} : \underset{\sim}{\bar{\mathbf{n}}}} \Big) : \overset{\bullet}{\boldsymbol{\varepsilon}} \tag{45}$$

The loading criterion is given as follows [9]:

$$\begin{cases} \overset{\bullet}{\boldsymbol{\varepsilon}}^p \neq \mathbf{0} & \text{for } \underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}} > 0 \\ \overset{\bullet}{\boldsymbol{\varepsilon}}^p = \mathbf{0} & \text{for } \underset{\sim}{\bar{\mathbf{n}}} : \underset{\sim}{\mathbf{E}} : \overset{\bullet}{\boldsymbol{\varepsilon}} \le 0 \end{cases} \tag{46}$$

3.6 Improvement of inverse-reloading responses

The material parameter $\underset{\sim}{u}$ is extended in order to improve the description of the inverse-reloading behavior as follows:

$$\underset{\sim}{u} = \bar{u} \exp(u_c \underset{\sim}{\mathcal{R}_c} \underset{\sim}{C_\sigma}) \tag{47}$$

where $\bar{u}$ and u_c is the material constant and

$$\underset{\sim}{C_\sigma} \equiv \underset{\sim}{\hat{\mathbf{n}}_c} : \underset{\sim}{\bar{\mathbf{n}}} \quad (-1 \le \underset{\sim}{C_\sigma} \le 1) \tag{48}$$

with

$$\underset{\sim}{\hat{\mathbf{n}}_c} \equiv \frac{\partial f(\underset{\sim}{\hat{\mathbf{c}}})}{\partial \underset{\sim}{\mathbf{c}}} \Big/ \Big\| \frac{\partial f(\underset{\sim}{\hat{\mathbf{c}}})}{\partial \underset{\sim}{\mathbf{c}}} \Big\| \quad (\| \underset{\sim}{\hat{\mathbf{n}}_c} \| = 1) \tag{49}$$

4 EVOLUTION OF DAMAGE VARIABLE

The continuum damage variable D is interpreted as an indirect measure of density of microvoids and microcracks [12] and its evolution rule was given as follows [13]:

$$\dot{D} = \left(\frac{Y}{\zeta}\right)^a \frac{H(\varepsilon^{dp} - \varepsilon_D^{dp})}{1-D}\dot{\varepsilon}^p \tag{50}$$

where ζ and a are the material constants, and ε_D^p is the threshold value of the accumulation of the deviatoric plastic strain rate, i.e. $\varepsilon^{dp} \equiv \int \|\dot{\boldsymbol{\varepsilon}}^p\| dt$. Y is the virtual undamaged strain energy function given by

$$Y = \frac{1}{2}\boldsymbol{\varepsilon}^e : \underset{\sim}{\mathbf{E}} : \boldsymbol{\varepsilon}^e = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \boldsymbol{\varepsilon}^e = \frac{1}{2}\underset{\sim}{\boldsymbol{\sigma}} : \underset{\sim}{\mathbf{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} \tag{51}$$

5 BILATERAL DAMAGE EFFECT

The elastic modulus tensor and its inverse in the virtual undamaged configuration in Eq. (6) are expressed in the matrix form as

$$\underset{\sim}{\mathbf{E}} = \frac{\underset{\sim}{E}}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ & 1-\nu & \nu & 0 & 0 & 0 \\ & & 1-\nu & 0 & 0 & 0 \\ & & & 1-2\nu & 0 & 0 \\ & \text{Sym.} & & & 1-2\nu & 0 \\ & & & & & 1-2\nu \end{bmatrix} \tag{52}$$

and

$$\underset{\sim}{\mathbf{E}}^{-1} = \frac{1}{\underset{\sim}{E}}\begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ & 1 & -\nu & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & \text{Sym.} & & 1+\nu & 0 & 0 \\ & & & & 1+\nu & 0 \\ & & & & & 1+\nu \end{bmatrix} \tag{53}$$

Here, let the damaged Young's modulus in Eq. (12) be given by

$$E(D) = (1-D)\underset{\sim}{E} \tag{54}$$

leading to

$$\mathbf{E} = (1-D)\underset{\sim}{\mathbf{E}}\,, \quad \mathbf{E}^{-1} = \frac{1}{1-D}\underset{\sim}{\mathbf{E}}^{-1} \tag{55}$$

for which the damaged current stress is related to the undamaged virtual stress by substituting Eq. (54) into Eq. (15) as follows:

$$\boldsymbol{\sigma} = (1-D)\underset{\sim}{\boldsymbol{\sigma}}\,, \quad \underset{\sim}{\boldsymbol{\sigma}} = \frac{1}{1-D}\boldsymbol{\sigma} \tag{56}$$

The virtual undamaged strain energy function Y is given by substituting Eqs. (55) and (56) into Eq. (51) as

$$Y = \frac{1}{2}\boldsymbol{\varepsilon}^e : \underset{\sim}{\mathbf{E}} : \boldsymbol{\varepsilon}^e$$

$$= \frac{1}{2(1-D)} \boldsymbol{\varepsilon}^e : \mathbf{E} : \boldsymbol{\varepsilon}^e = \frac{1}{2(1-D)} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}^e = \frac{1}{2(1-D)} \boldsymbol{\sigma} : \mathbf{E}^{-1} : \boldsymbol{\sigma} = \frac{1}{2(1-D)^2} \boldsymbol{\sigma} : \underset{\sim}{\mathbf{E}}^{-1} : \boldsymbol{\sigma} \tag{57}$$

5 UNILATERAL DAMAGE EFFECT

Let the principal actual damaged axial stress σ_p (p=1, 2, 3) be given by the principal elastic axial strain ε_p^e in the uniaxial loading state as follows:

$$\begin{cases} \sigma_p = E_p(D)\varepsilon_p^e \\ \varepsilon_p^e = E_p^{-1}(D)\sigma_p \end{cases} \quad \text{under } \sigma_q = 0 \ (q \neq p) \tag{58}$$

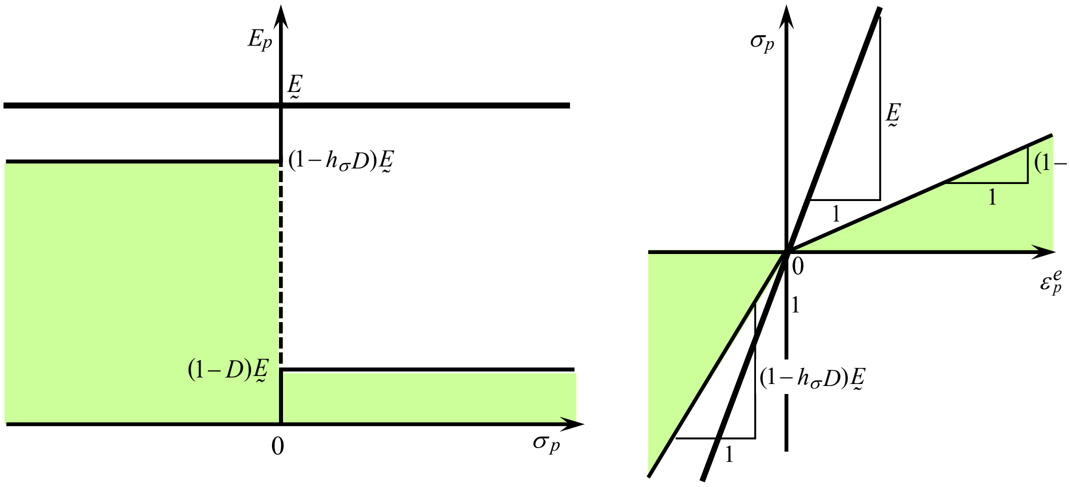
Let the damaged Young's modulus $E_p(D)$ in Eq. (58) be given as follows:

$$E_p = (1 - H_{\sigma p} D)\underset{\sim}{E} \tag{59}$$

$$= \begin{cases} (1-D)\underset{\sim}{E} & \text{for } \sigma_p \geq 0 \\ (1-h_\sigma D)\underset{\sim}{E} & \text{for } \sigma_p < 0 \end{cases}$$

$$H_{\sigma p} \equiv H(\sigma_p) + [1 - H(\sigma_p)]h_\sigma \tag{60}$$

$H(s)$ is the Heaviside's step function, i.e. $H(s)=1$ for $s \geq 0$ and $H(s)=0$ for $s<0$ (s: arbitrary scalar), and h_σ $(0 \leq h_\sigma \leq 1)$ is the material constant. Equation (59) is shown in Fig. 1.



(a) Relation of principal actual Young's modulus vs. principal actual damaged stress.

(a) Relation of principal actual damage stress vs. principal elastic strain

Fig. 1. Actual damaged Young's modulus and actual damaged stress in uniaxial loading for unilateral damage phenomenon.

Extending the inverse elastic modulus tensor to the unilateral damage by adopting the damaged Young's modulus in Eq. (59) on the premise that the elastic strain is not influenced by the lateral stresses, let the inverse elastic modulus tensor be given by the matrix form in the coordinate system with the base $\{\overline{\mathbf{e}}_P\}$ in the principal stress directions as follows:

$$\mathbf{E}^{-1} = \frac{1}{\underset{\sim}{E}}\begin{bmatrix} \Gamma_1 & -\nu & -\nu & 0 & 0 & 0 \\ & \Gamma_2 & -\nu & 0 & 0 & 0 \\ & & \Gamma_3 & 0 & 0 & 0 \\ & \text{Sym.} & & 1+\nu & 0 & 0 \\ & & & & 1+\nu & 0 \\ & & & & & 1+\nu \end{bmatrix} \tag{61}$$

$$\mathbf{E} = \frac{\underset{\sim}{E}}{\Pi}\begin{bmatrix} \Gamma_{23}-\nu^2 & \nu\Gamma_3+\nu^2 & \nu\Gamma_2+\nu^2 & 0 & 0 & 0 \\ & \Gamma_{13}-\nu^2 & \nu\Gamma_1+\nu^2 & 0 & 0 & 0 \\ & & \Gamma_{12}-\nu^2 & 0 & 0 & 0 \\ & \text{Sym.} & & \dfrac{\Pi}{1+\nu} & 0 & 0 \\ & & & & \dfrac{\Pi}{1+\nu} & 0 \\ & & & & & \dfrac{\Pi}{1+\nu} \end{bmatrix} \tag{62}$$

where

$$\Gamma_P(D) \equiv \frac{\underset{\sim}{E}}{E_P} = \frac{\underset{\sim}{E}}{(1-H_{\sigma P}D)\underset{\sim}{E}} = \frac{1}{1-H_{\sigma P}D} \tag{63}$$

$$\Gamma_{PQ} \equiv \Gamma_P\Gamma_Q = \frac{\underset{\sim}{E}^2}{E_PE_Q},\;\; \Gamma_{PQR} \equiv \Gamma_P\Gamma_Q\Gamma_R = \frac{\underset{\sim}{E}^3}{E_PE_QE_R} \tag{64}$$

$$\Pi = \Gamma_{123} - \nu^2(\Gamma_1+\Gamma_2+\Gamma_3) - 2\nu^3 \tag{65}$$

The actual damaged stress is given by the virtual undamaged stress as

$$\boldsymbol{\sigma} = \mathbf{E} : \boldsymbol{\varepsilon}^e = \mathbf{E} : \underset{\sim}{\mathbf{E}}^{-1} : \underset{\sim}{\boldsymbol{\sigma}} = \mathbb{I} : \underset{\sim}{\boldsymbol{\sigma}} \tag{66}$$

where

$$\mathbb{I} \equiv \mathbf{E} : \underset{\sim}{\mathbf{E}}^{-1} = \frac{1}{\Pi}\begin{bmatrix} \Gamma_{23}-\nu^2(\Gamma_2+\Gamma_3+1+2\nu) & -\nu(\Gamma_{23}+\nu\Gamma_2-\Gamma_3-\nu) & -\nu(\Gamma_{23}-\Gamma_2+\nu\Gamma_3-\nu) & 0 & 0 & 0 \\ -\nu(\Gamma_{31}-\Gamma_3+\nu\Gamma_1-\nu) & \Gamma_{31}-\nu^2(\Gamma_3+\Gamma_1+1+2\nu) & -\nu(\Gamma_{31}+\nu\Gamma_3-\Gamma_1-\nu) & 0 & 0 & 0 \\ -\nu(\Gamma_{12}+\nu\Gamma_1-\Gamma_2-\nu) & -\nu(\Gamma_{12}-\Gamma_1+\nu\Gamma_2-\nu) & \Gamma_{12}-\nu^2(\Gamma_3+\Gamma_1+1+2\nu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \Pi & 0 & 0 \\ 0 & 0 & 0 & 0 & \Pi & 0 \\ 0 & 0 & 0 & 0 & 0 & \Pi \end{bmatrix} (\neq \mathbb{I}^T) \tag{67}$$

The relation $\boldsymbol{\sigma}$ to $\underset{\sim}{\boldsymbol{\sigma}}$ is expressed by the components in the fixed coordinate system as follows:

$$\sigma_{ij} = Q_{Pi}Q_{Qj}Q_{Ra}Q_{Sb}\mathbb{I}_{PQRS}\underset{\sim}{\sigma}_{ab} \tag{68}$$

where

$$Q_{Ai} \equiv \bar{\mathbf{e}}_A \bullet \mathbf{e}_i \tag{69}$$

noting

$$\sigma_{ij} = \mathbf{e}_i \bullet (\mathbb{I}_{PQRS}\; \bar{\mathbf{e}}_P \otimes \bar{\mathbf{e}}_Q \otimes \bar{\mathbf{e}}_R \otimes \bar{\mathbf{e}}_S) : (\underset{\sim}{\sigma}_{ab}\mathbf{e}_a \otimes \mathbf{e}_b)\mathbf{e}_j$$

Inversely, the virtual undamaged stress is given by the actual damaged stress as follows:

$$\underset{\sim}{\boldsymbol{\sigma}} = \underset{\sim}{\mathbf{E}} : \boldsymbol{\varepsilon}^e = \underset{\sim}{\mathbf{E}} : \mathbf{E}^{-1} : \boldsymbol{\sigma} = \mathfrak{I} : \boldsymbol{\sigma} \tag{70}$$

where

$$\mathfrak{I} \equiv \underset{\sim}{\mathbf{E}} : \mathbf{E}^{-1} = \begin{bmatrix} \frac{(1-\nu)\Gamma_1 - 2\nu^2}{(1+\nu)(1-2\nu)} & \frac{\nu(\Gamma_2 - 1)}{(1+\nu)(1-2\nu)} & \frac{\nu(\Gamma_3 - 1)}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu(\Gamma_1 - 1)}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)\Gamma_2 - 2\nu^2}{(1+\nu)(1-2\nu)} & \frac{\nu(\Gamma_3 - 1)}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ \frac{\nu(\Gamma_1 - 1)}{(1+\nu)(1-2\nu)} & \frac{\nu(\Gamma_2 - 1)}{(1+\nu)(1-2\nu)} & \frac{(1-\nu)\Gamma_3 - 2\nu^2}{(1+\nu)(1-2\nu)} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} (\neq \mathfrak{I}^T) \tag{71}$$

noting $\mathfrak{I}$ is the non-symmetric tensor but the tensor $\mathfrak{I}:\mathbf{t}$ is the symmetric tensor for an arbitrary symmetric second-order tensor $\mathbf{t}$.

Equation (70) is described in the component form as follows:

$$\underset{\sim}{\sigma}_{ij} = Q_{Pi} Q_{Qj} Q_{Ra} Q_{Sb} \mathcal{I}_{PQRS} \sigma_{ab} \tag{72}$$

because of

$$\underset{\sim}{\sigma}_{ij} = \mathbf{e}_i \bullet (\mathcal{I}_{PQRS}\ \bar{\mathbf{e}}_P \otimes \bar{\mathbf{e}}_Q \otimes \bar{\mathbf{e}}_R \otimes \bar{\mathbf{e}}_S) : (\sigma_{ab} \mathbf{e}_a \otimes \mathbf{e}_b) \mathbf{e}_j$$

Noting

$$\dot{E}_p^{-1} = \frac{H_{\sigma p}}{E_p^2} \underset{\sim}{E} \dot{D} \quad \left(E_p^{-1} = \frac{1}{(1 - H_{\sigma p} D) \underset{\sim}{E}} \right) \tag{73}$$

one has

$$\dot{\mathbf{E}}^{-1} = \begin{bmatrix} \frac{H_{\sigma 1}}{E_1^2} & 0 & 0 & 0 & 0 & 0 \\ & \frac{H_{\sigma 2}}{E_2^2} & 0 & 0 & 0 & 0 \\ & & \frac{H_{\sigma 3}}{E_3^2} & 0 & 0 & 0 \\ & \text{Sym.} & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \underset{\sim}{E} \dot{D}, \tag{74}$$

It follows from Eq. (52) and (74) that

$$\underset{\sim}{\mathbf{E}} : \dot{\mathbf{E}}^{-1} = \bar{\mathfrak{I}} \dot{D} \tag{75}$$

where

$$\bar{\mathfrak{I}} \equiv \frac{1-\nu}{(1+\nu)(1-2\nu)} \begin{bmatrix} \frac{H_{\sigma 1}}{E_1^2} & 0 & 0 & 0 & 0 & 0 \\ & \frac{H_{\sigma 2}}{E_2^2} & 0 & 0 & 0 & 0 \\ & & \frac{H_{\sigma 3}}{E_3^2} & 0 & 0 & 0 \\ & \text{Sym.} & & 0 & 0 & 0 \\ & & & & 0 & 0 \\ & & & & & 0 \end{bmatrix} \tag{76}$$

It is required to calculate the virtual undamaged stress rate from the actual damaged stress rate at the boundary where the stress and its rate are given, while the virtual undamaged stress is calculated from the actual stress by Eq. (70). The virtual undamaged stress rate is given from the actual stress rate by Eqs. (70) and (75) as follows:

$$\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}\ (=\underset{\sim}{\mathbf{E}}:\overset{\bullet}{\underset{\sim}{\boldsymbol{\varepsilon}}}{}^{e}=\underset{\sim}{\mathbf{E}}:(\mathbf{E}^{-1}:\boldsymbol{\sigma})^{\bullet})=\underset{\sim}{\mathbf{E}}:\mathbf{E}^{-1}:\overset{\bullet}{\boldsymbol{\sigma}}+\underset{\sim}{\mathbf{E}}:\overset{\bullet}{\mathbf{E}}{}^{-1}:\boldsymbol{\sigma}=\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}+\overset{\bullet}{D}\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma} \tag{77}$$

or in the component description as follows:

$$\overset{\bullet}{\underset{\sim}{\sigma}}_{ij}=\mathcal{J}_{PQRS}Q_{Pi}Q_{Qj}Q_{Ra}Q_{Sb}\overset{\bullet}{\sigma}_{ab}+\overset{\bullet}{D}Q_{Pi}Q_{Qj}Q_{Ra}Q_{Sb}\bar{\mathcal{J}}_{PQRS}\sigma_{ab} \tag{78}$$

noting

$$\begin{aligned}\overset{\bullet}{\underset{\sim}{\sigma}}_{ij}&=\mathbf{e}_i\bullet[(\mathcal{J}_{PQRS}\,\bar{\mathbf{e}}_P\otimes\bar{\mathbf{e}}_Q\otimes\bar{\mathbf{e}}_R\otimes\bar{\mathbf{e}}_S):(\sigma_{ab}\mathbf{e}_a\otimes\mathbf{e}_b)^{\bullet}\\&\qquad+\overset{\bullet}{D}(\bar{\mathcal{J}}_{PQRS}\,\bar{\mathbf{e}}_P\otimes\bar{\mathbf{e}}_Q\otimes\bar{\mathbf{e}}_R\otimes\bar{\mathbf{e}}_S):(\sigma_{ab}\mathbf{e}_a\otimes\mathbf{e}_b)]\,\mathbf{e}_j\\&=\mathbf{e}_i\bullet[(\mathcal{J}_{PQRS}\,\bar{\mathbf{e}}_P\otimes\bar{\mathbf{e}}_Q\otimes\bar{\mathbf{e}}_R\otimes\bar{\mathbf{e}}_S):(\overset{\bullet}{\sigma}_{ab}\mathbf{e}_a\otimes\mathbf{e}_b)]\,\mathbf{e}_j+\overset{\bullet}{D}Q_{Pi}Q_{Qj}Q_{Ra}Q_{Sb}\bar{\mathcal{J}}_{PQRS}\sigma_{ab}\end{aligned}$$

However, note that the rate of the damage variable is involved in Eq. (77). Equation will be represented fully in terms of the actual stress and its rate in the following.

Equation (77) is rewritten by substituting Eq. (50) of the damage variable with Eq. (41) of the plastic strain rate as follows:

$$\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}=\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}+\left(\frac{Y}{\zeta}\right)^{a}\frac{H(\varepsilon^{dp}-\varepsilon^{dp}_{D})}{1-D}\frac{\underset{\sim}{\bar{\mathbf{n}}}:\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}{}^{p}}\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma} \tag{79}$$

from which one has

$$\frac{\underset{\sim}{\bar{\mathbf{n}}}:\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}{}^{p}}\underset{\sim}{\bar{M}}{}^{p}=\underset{\sim}{\bar{\mathbf{n}}}:\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}+\left(\frac{Y}{\zeta}\right)^{a}\frac{H(\varepsilon^{dp}-\varepsilon^{dp}_{D})}{1-D}\frac{\underset{\sim}{\bar{\mathbf{n}}}:\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}{}^{p}}\underset{\sim}{\bar{\mathbf{n}}}:\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma} \tag{80}$$

leading to

$$\frac{\underset{\sim}{\bar{\mathbf{n}}}:\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}}{\underset{\sim}{\bar{M}}{}^{p}}=\frac{\underset{\sim}{\bar{\mathbf{n}}}:\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}}{\underset{\sim}{\bar{M}}{}^{p}-\left(\frac{Y}{\zeta}\right)^{a}\frac{H(\varepsilon^{dp}-\varepsilon^{dp}_{D})}{1-D}\underset{\sim}{\bar{\mathbf{n}}}:\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma}} \tag{81}$$

The damage variable is rewritten by substituting Eq. (81) as follows:

$$\overset{\bullet}{D}=\frac{\underset{\sim}{\bar{\mathbf{n}}}:\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}}{\underset{\sim}{\bar{M}}{}^{p}\frac{1-D}{H(\varepsilon^{dp}-\varepsilon^{dp}_{D})}\left(\frac{\zeta}{Y}\right)^{a}-\underset{\sim}{\bar{\mathbf{n}}}:\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma}} \tag{82}$$

Substituting Eq. (82), the rate of the virtual undamaged stress is described by the actual damaged stress and its rate as follows:

$$\overset{\bullet}{\underset{\sim}{\boldsymbol{\sigma}}}=\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}+\frac{\underset{\sim}{\bar{\mathbf{n}}}:\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}}}{\underset{\sim}{\bar{M}}{}^{p}\frac{1-D}{H(\varepsilon^{dp}-\varepsilon^{dp}_{D})}\left(\frac{\zeta}{Y}\right)^{a}-\underset{\sim}{\bar{\mathbf{n}}}:\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma}}\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma} \tag{83}$$

where

$$\begin{cases}(\bar{\boldsymbol{\Im}}:\boldsymbol{\sigma})_{ij}=Q_{Pi}Q_{Qj}Q_{Ra}Q_{Sb}\bar{\mathcal{J}}_{PQRS}\sigma_{ab}\\(\boldsymbol{\Im}:\overset{\bullet}{\boldsymbol{\sigma}})_{ij}=Q_{Pi}Q_{Qj}Q_{Ra}Q_{Sb}\mathcal{J}_{PQRS}\,\overset{\bullet}{\sigma}_{ab}\end{cases} \tag{84}$$

Consider the deformation in the uniaxial loading process ($\sigma_2 = \sigma_3 = 0$) in which the principal directions are fixed. It follows from Eq. (70) with Eq. (71) that

$$\underset{\sim}{\sigma}_1 = \frac{(1-\nu)\Gamma_1 - 2\nu^2}{(1+\nu)(1-2\nu)}\sigma_1 \tag{85}$$

from which we have

$$\sigma_1 = \frac{(1+\nu)(1-2\nu)}{(1-\nu)\Gamma_1 - 2\nu^2}\underset{\sim}{\sigma}_1 \tag{86}$$

Further, it follows from Eq. (77) with Eqs. (71) and (76) that

$$\begin{cases} \overset{\bullet}{\underset{\sim}{\sigma}}_1 = \dfrac{(1-\nu)\Gamma_1 - 2\nu^2}{(1+\nu)(1-2\nu)}\overset{\bullet}{\sigma}_1 + \dfrac{1-\nu}{(1+\nu)(1-2\nu)}\overset{\bullet}{D}\dfrac{H_{\sigma 1}}{E_1^2}\sigma_1 \\ \overset{\bullet}{\underset{\sim}{\sigma}}_2 = \overset{\bullet}{\underset{\sim}{\sigma}}_3 = \overset{\bullet}{\underset{\sim}{\sigma}}_4 = \overset{\bullet}{\underset{\sim}{\sigma}}_5 = \overset{\bullet}{\underset{\sim}{\sigma}}_6 = 0 \end{cases} \tag{87}$$

where

$$\overset{\bullet}{D} = \frac{\dfrac{(1-\nu)\Gamma_1 - 2\nu^2}{(1+\nu)(1-2\nu)}\bar{n}_1\overset{\bullet}{\sigma}_1}{\underset{\sim}{\bar{M}}^p \dfrac{1-D}{H(\varepsilon^{dp} - \varepsilon_D^{dp})}\left(\dfrac{\underset{\sim}{\zeta}}{Y}\right)^a - \dfrac{(1-\nu)\Gamma_1 - 2\nu^2}{(1+\nu)(1-2\nu)}\bar{n}_1\sigma_1} \tag{88}$$

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