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MODEL OF THE TETHERED SPACE SYSTEM IN VICINITY OF ELLIPSOIDAL ASTEROID AND ITS APPROXIMATIONS

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Abstract. While planning missions in vicinity of an asteroid/comet body one has to take into account several dynamical problems to overcome and among them are: (a) irregular distribution of the body internal masses; (b) too weak gravity acceleration near the body surface. In case of (a) one offers to apply approximate models of gravity. As an example we consider the case of a triaxial ellipsoid. For the problem (b) we apply docking procedures with help of anchor and a connecting tether.

For computing the force field of gravity being generated by the ellipsoid of three axes one has to calculate several values of elliptic integrals at each instant of the simulation process. For this we apply original algorithm interpreting elliptic integrals as a state variables in additional to dynamics system of ODEs. To resolve the problem (b) we use so-called hybrid automata to build up the tethered interconnection between a spacecraft and the asteroid.

Ellipsoidal asteroid performs free rotary motions about its mass center thus performing the Euler case of the rigid body rotary motion. The spacecraft moves under the force of

gravity from the asteroid and under the tether tension, in case of the constraint being imposed. So we have so-called restricted dynamical model because the asteroid does not "feel" any force from the spacecraft.

In addition to the hybrid automata dynamical model including impacts on constraint we also consider approximations of this model being really regularizations of the impact process. All these models are analysed and compared numerically.

1 INTRODUCTION

Pendulum system under consideration may be regarded as a delivery facility to transport payloads to/from the spacecraft from/to the asteroid surface, some kind of space elevator. This idea is close to that one, belonging to Tsiolkovsky and known as a space elevator located nearby the Earth. Engineering development of this construction concerns the pioneer work [1], see also [2]. Close idea of the Moon elevator probably arises due to works of F. Zahnder [3]. This latter idea was rediscovered in [4, 5, 6].

Specific property of the problem under analysis consists of typically irregular shape and mass distribution of asteroids. Another property relates to asteroid's motion as the Euler top, which is not a permanent rotation in general situations.

Studying of pendulums concerning the dynamical problems of orbital mechanics goes back to probably pioneer work [7]. Existence and conditions of stability for orbital pendulums in different formulations were investigated in [8, 9, 10], and also in papers [11, 12, 13, 14, 15, 16, 17, 18]. Investigation of space tethered systems is tightly interconnected with activity of Beletsky and his pupils and colleagues: [19, 20, 21, 22, 23, 24]. Dynamics of tethered systems attached to an asteroid is studied intensively in [25, 26, 27, 28, 29]). Orbital motion of compound satellites was studied in [30, 31]. The model useful for simulation the contact between particular spacecraft of the tethered system and the asteroid surface see in [32].

2 PROBLEM FORMULATION

When planning missions in vicinity of minor celestial body of the Solar system one has to take into account at least two serious difficulties to overcome: (a) internal masses distribution inside asteroid/comet is as a rule irregular; (b) gravity of such a body is too weak such that it makes practically impossible the traditional "landing" of the spacecraft onto the celestial object surface. For solving two problems mentioned we apply proper methodics. In case of (a) one offers to apply approximate models of gravity. As an example we consider gravity of the asteroid having a shape of triaxial ellipsoid. To overcome problems in case of (b) one can apply docking procedures with help of anchor and cable connection.

One encounters frequently celestial bodies of ellipsoidal shape or close to it in Solar system. For computing the force field of gravity being generated by such a body one has to calculate elliptic integrals on each step of the integration process for the dynamical model under construction. As it turned out one can use special additional system of ODEs for this problem efficient solution. For solving the problem (b) one uses so-called hybrid automata to build up the dynamical model of a spacecraft. There are exactly two states of this automata: free flight of the spacecraft: cable is slack; flight along the constraint: cable is tensed.

Ellipsoidal asteroid performs free rotary motions about its mass center. The spacecraft moves under the force of gravity from the asteroid and under the tension, in case of the constraint being imposed. Dynamical model under analysis can be classified as a restricted one. Indeed, spacecraft does not influence the motion of the asteroid.

Besides the hybrid automata dynamical model, which includes impacts on constraint, approximations of this model are also under consideration. These approximations are reduced to replacement of "exact" model of impact by its different schemes of regularization. Some of these models are analysed and compared numerically.

Thus first of all consider a tethered system (TS), consisting of a weightless tether of length ℓ, an asteroid $\mathcal{A}$, and a particle Q of mass m, see Figure 1. Suppose the tether connects a point P fixed on the asteroid surface Σ and a particle Q.

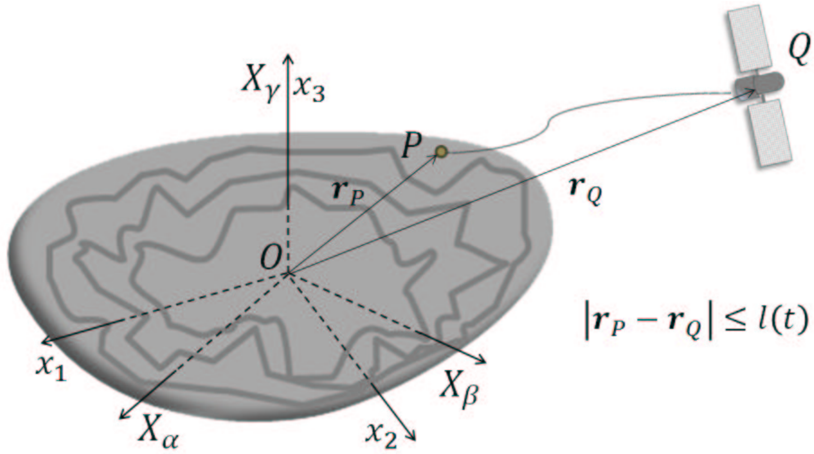

Figure 1: Mechanical system.

Let $OX_\alpha X_\beta X_\gamma$ be an absolute frame of reference (ARF). Assume that its origin coincides with the asteroid center of mass. Also let $Ox_1x_2x_3$ be the mobile reference system (MRF), connected with the asteroid. Its axes are assumed to be directed along its principal central axes of inertia. Unit vectors of the ARF base with respect to MRF can be represented in the following way

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T, \quad \boldsymbol{\beta} = (\beta_1, \beta_2, \beta_3)^T, \quad \boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)^T. \tag{1}$$

Using components of these vectors one can compose an orthogonal matrix

$$\mathbf{S} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}, \tag{2}$$

allowing to implement transfers from MRF to ARF and back: if a vector $\overrightarrow{OQ}$ is described as

$$\mathbf{X} = (X_\alpha, X_\beta, X_\gamma)^T, \quad \mathbf{x} = (x_1, x_2, x_3)^T$$

in ARF and MRF respectively, then

$$\mathbf{X} = \mathbf{S}\mathbf{x}. \tag{3}$$

The angular velocity

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)^T \tag{4}$$

given by its projections onto the axes of the MRF satisfies a matrix equation

$$\dot{\mathbf{S}} = \mathbf{S} \cdot \hat{\boldsymbol{\omega}}, \quad \hat{\boldsymbol{\omega}} = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \tag{5}$$

that can be treated as a matrix form of the Poisson equations describing variations of the body orientation. The same equations in the vector form read

$$\dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}. \tag{6}$$

If in the ARF velocity and acceleration of the particle Q are represented as

$$\mathbf{V} = \left(\dot{X}_\alpha, \dot{X}_\beta, \dot{X}_\gamma\right)^T \quad \text{and} \quad \mathbf{A} = \left(\ddot{X}_\alpha, \ddot{X}_\beta, \ddot{X}_\gamma\right)^T \tag{7}$$

respectively, then, as is known from kinematics, in the MRF velocity and acceleration are given as

$$\mathbf{v} = \mathbf{S}^T \cdot \mathbf{V} = \boldsymbol{\omega} \times \mathbf{x} + \dot{\mathbf{x}}, \tag{8}$$

$$\mathbf{a} = \mathbf{S}^T \cdot \mathbf{A} = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) + \dot{\boldsymbol{\omega}} \times \mathbf{x} + 2\boldsymbol{\omega} \times \dot{\mathbf{x}} + \ddot{\mathbf{x}}. \tag{9}$$

respectively.

Since the point of fixation P given by the vector $\overrightarrow{OP}$ with

$$\mathbf{P} = (P_\alpha, P_\beta, P_\gamma)^T, \quad \mathbf{p} = (p_1, p_2, p_3)^T$$

is fixed in asteroid, then its velocity and acceleration are

$$\mathbf{V}_P = \mathbf{S}(\boldsymbol{\omega} \times \mathbf{p}), \quad \mathbf{A}_P = \mathbf{S}\left(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}) + \dot{\boldsymbol{\omega}} \times \mathbf{p}\right) \tag{10}$$

respectively in the ARF. They also read

$$\mathbf{v}_P = \boldsymbol{\omega} \times \mathbf{p}, \quad \mathbf{a}_P = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{p}) + \dot{\boldsymbol{\omega}} \times \mathbf{p} \tag{11}$$

respectively in the MRF.

3 DYNAMICS

Let the pendulum be hanged at the point P of the asteroid surface with a massless inextensible tether of the length ℓ: $|PQ| \le \ell$. The mass m of the pendulum is supposed being concentrated at its endpoint Q. The asteroid is assumed performing rotation about its masscenter as the Euler top. One also assumes that the pendulum motion does not affect the asteroid rotation. The tether length we in general assume as known predefined function of time. Then equations of the particle Q motion w. r. t. MRF read

$$m\mathbf{a} = \mathbf{f}_\star + \lambda_\star \frac{\partial \varphi}{\partial \mathbf{x}}, \tag{12}$$

where $\mathbf{f}_\star$ and $\lambda_\star$ are the active force and the Lagrange multiplier respectively. The latter one is determined from the equation of the exerted constraint:

$$\varphi(x, p, \ell) = \frac{1}{2}\left[(\mathbf{x} - \mathbf{p}, \mathbf{x} - \mathbf{p}) - \ell^2\right] \equiv 0 \tag{13}$$

and two its time derivatives

$$(\dot{\mathbf{x}}, \mathbf{x} - \mathbf{p}) - \dot{\ell}\ell \equiv 0,\ (\ddot{\mathbf{x}}, \mathbf{x} - \mathbf{p}) - (\dot{\mathbf{x}}, \dot{\mathbf{x}}) - \ddot{\ell}\ell - \dot{\ell}^2 = 0. \tag{14}$$

Since the point P is assumed to be fixed in the MRF the equalities $\dot{\mathbf{p}} = 0$, $\ddot{\mathbf{p}} = 0$ are fulfilled.

Denoting $f_\star = mf$, $\lambda_\star = m\lambda$ and using identity (9) we find out from (12) that

$$(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) + \dot{\boldsymbol{\omega}} \times \mathbf{x} + 2\boldsymbol{\omega} \times \dot{\mathbf{x}} + \ddot{\mathbf{x}}) = \mathbf{f} + \lambda \frac{\partial \varphi}{\partial \mathbf{x}} \tag{15}$$

or in the equivalent form:

$$\ddot{\mathbf{x}} = ((\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega} + \mathbf{x} \times \dot{\boldsymbol{\omega}} + 2\dot{\mathbf{x}} \times \boldsymbol{\omega}) + \mathbf{f} + \lambda \frac{\partial \varphi}{\partial \mathbf{x}}. \tag{16}$$

Substituting of acceleration $\ddot{\mathbf{x}}$ from (16) into second identity (14) we have

$$((\boldsymbol{\omega} \times \mathbf{x}) \times \boldsymbol{\omega} + \mathbf{x} \times \dot{\boldsymbol{\omega}} + 2\dot{\mathbf{x}} \times \boldsymbol{\omega} + \mathbf{f} + \lambda(\mathbf{x} - \mathbf{p}), \mathbf{x} - \mathbf{p}) - (\dot{\mathbf{x}}, \dot{\mathbf{x}}) - \ddot{\ell}\ell - \dot{\ell}^2 = 0,$$

and one obtains the Lagrange multiplier

$$\lambda = \left[(\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{x}) + \dot{\boldsymbol{\omega}} \times \mathbf{x} + 2\boldsymbol{\omega} \times \dot{\mathbf{x}} - \mathbf{f}, \mathbf{x} - \mathbf{p}) + (\dot{\mathbf{x}}, \dot{\mathbf{x}}) + \ddot{\ell}\ell + \dot{\ell}^2\right] / \ell^2. \tag{17}$$

We assume that active forces are potential, and potential energy has the form $U_\star = mU(\mathbf{x})$. Then

$$\mathbf{f} = -\frac{\partial U}{\partial \mathbf{x}}. \tag{18}$$

Explicit expression of force (18) will be used below in equations (16). However it is impossible to wright down an exact expression for potential energy for an arbitrary celestial body, because of the absence of exact data about its mass distribution. Nevertheless if the point Q is located far enough from the asteroid then we can assume that

$$U = -GM\left[\frac{1}{r} + \frac{I_1 + I_2 + I_3}{2r^3} - \frac{3(\mathbf{I}\mathbf{x}, \mathbf{x})}{r^3} + \dots\right] \tag{19}$$

where $r = (\mathbf{x}, \mathbf{x})^{1/2}$, and $\mathbf{I} = \operatorname{diag}(I_1, I_2, I_3)$ is asteroid's central tensor of inertia, divided by its mass and represented in asteroid's principal central axes of inertia. In the further course we assume that selection of physical dimensions is such that $GM = 1$.

R e m a r k 1 *Approximate expression (19) for the potential is derived, as usual, from the exact one:*

$$U(\mathbf{x}) = -\frac{1}{m}\iiint\limits_{\mathcal{A}} \rho u(\mathbf{x}, \mathbf{y}) d\mathbf{y} \tag{20}$$

via a standard expansion of the integrand multiplier

$$u(\mathbf{x}, \mathbf{y}) = (\mathbf{x} - \mathbf{y}, \mathbf{x} - \mathbf{y})^{-1/2} \tag{21}$$

with respect to the parameter $\varepsilon = (-2(\mathbf{x}, \mathbf{y}) + (\mathbf{y}, \mathbf{y}))/(\mathbf{x}, \mathbf{x})$. *However, the potential can be also expanded with respect to* $\varepsilon_\star = -2(\mathbf{x}, \mathbf{y})/((\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}))$. *This parameter has no singularity at the origin. This representation has been applied in [33] for gravitating rings and in [34] for tetrahedral bodies*

$$u(\mathbf{x}, \mathbf{y}) = \frac{1}{((\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}))^{1/2}} \frac{1}{(1 + \varepsilon)^{1/2}} = u_0 + u_1 + \dots = \tag{22}$$

$$= \frac{1}{((\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y}))^{1/2}}\left(1 - \frac{1}{2}\varepsilon_\star + \frac{3}{8}\varepsilon_\star^2 - \frac{5}{16}\varepsilon_\star^3 + \dots\right).$$

In contrast to the standard expansion, integration of terms u_n, *which are not homogeneous function of the vector* $\mathbf{y} = (y_1, y_2, y_3)^T$ *components usually is non-trivial. Properties of this approximation for the potential, in particular, its behaviour in vicinity of the asteroid surface were not deeply investigated.*

Equations (17), (19) are suitable to describe motion of a massive particle in Euler case of asteroid's rotation as well as in particular cases of precessions and permanent rotations. In the latter case

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{e}_\psi, \tag{23}$$

where $\dot{\psi} = \mathrm{const}$ is the magnitude of angular velocity which is invariable in the body coordinate system, $\mathbf{e}_\psi$ is the unit vector which is directed along axis of rotation.

Regular precessions exist in the case of the body $\mathcal{A}$ dynamical symmetry. If $\mathbf{f}_1$, $\mathbf{f}_2$, $\mathbf{f}_3$ are unit vectors directed along its principal central axes of inertia, and $\mathbf{f}_3$ is directed along the axis of symmetry, then

$$\boldsymbol{\omega} = \dot{\psi}\mathbf{e}_\psi + \dot{\varphi}\mathbf{e}_\varphi. \tag{24}$$

Here and below the constant quantities $\dot{\psi}$ and $\dot{\varphi}$ denote magnitudes of angular velocities of precession and proper rotation respectively; θ is a constant angle of nutation;

$$\mathbf{e}_\psi = \cos\theta\mathbf{f}_3 + \sin\theta\left(\cos(\dot{\varphi}t)\mathbf{f}_1 + \sin(\dot{\varphi}t)\mathbf{f}_2\right)$$

is a unit vector of the axis of precession, fixed in the absolute space; $\mathbf{e}_\varphi = \mathbf{f}_3$.

Remark 2 If the value λ, determined by (17) is negative on the motions under consideration, then the constraint is tensed, and it can be implemented using the tether. Otherwise for the constraint implementation one needs a weightless rod. If the value λ can change its sign then releasing the constraint as well as its tension are possible.

It is natural to expect that the tether model is applicable if its length is large enough and its free endpoint Q is located in the region where a centrifugal force dominates over the force of gravity. At the same time, it is natural that the rod model ought be applied near the body surface, where force of gravity dominates.

4 MODEL OF A SPACECRAFT TETHERED WITH ASTEROID

The tether, implementing a unilateral constraint and connecting an asteroid surface and a massive particle, can be in a stretched or slackened state. We build up a computer model of the mechanical system consisting of a free rigid body and a massive particle, connected by a massless inextensible tether. The problem is considered within a restricted formulation: a particle, the spacecraft, does not affect dynamics of a rigid body, asteroid. At the same time, the particle moves under gravitational attraction of the asteroid and simultaneously under reaction of the tether. The unilateral constraint allows impacts satisfying, for example, the Newton model.

Description of the typical mission in vicinity of the asteroid including slack mode of the tether connecting the asteroid surface and the spacecraft, its decreasing oscillations with impacts, final landing on the constraint and the limit pendulum-like motions of the tethered system we undertook in [29].

Let us apply the model described above. Let the asteroid be a homogeneous triaxial ellipsoid with semi-axes $a_1 > a_2 > a_3$, described by

$$\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$$

in the MRF. Assume that ellipsoid's center of mass is fixed, and the ellipsoid itself rotates uniformly about its axis of inertia, corresponding to the maximal moment of inertia. Without loss of generality this is the axis OX_γ of the ARF, coincident to the axis Ox_3

of the MRF. In case of Euler such solution is possible. Suppose the point of tether's attachment P is located on the ellipsoid surface at the point of intersection with the axis of inertia, corresponding to the minimal moment of inertia. Without loss of generality this is the axis Ox_1 of the MRF.

Model of the tether as a unilateral constraint was implemented as a hybrid automata with a properly defined automata state variable. This variable takes two possible values: (a) "constraint is in a slackened state"; i. e. the spacecraft freely flies in field of the asteroid gravity; (b) "constraint is in a stretched state", i. e. the spacecraft moves as a spherical pendulum also in field of asteroid gravity.

In [29] we also demonstrated so-called "landing-on-the-constraint" process with repetitive impacts and final change motion to constrained mode of spherical pendulum. Really hybrid automata is one of possible implementations for unilateral constraint which in turn is sufficiently rough model for the tethered system of two bodies. The cable is assumed weighless and inextensible in the simplest case and without any resistance to bending. Thus when the distance between endpoints is less than the cable length then the tension force is equal to zero. The force interaction takes place only if the unilateral constraint arises with impacts or impactless. Evidently the case of the impact is the most general one. Here we use the simplest case of Newtonian impact model.

Approximations are known [35] for systems with impacts having potential energy of large value. Elastic and viscous components are included in frame of Newtonian impact model. We performed comparison of pure impact model with linear elastic and viscous models during the process of "landing on constraint" as an example. The first model performs multiple impacts and soon starts its limit pendulum-like motions. The second, linear, model performs its decreasing oscillations in much more slow pace.

At the same time if we consider the non-linear model of repulsing force instead of linear model then this force supposed to act inside the thin, of thickness $\varepsilon \ll 1$, layer. This layer simulates the boundary compliance for the constraint under analysis. The force can have the following expression

$$F_{\text{elast}} = \begin{cases} c \cdot \tan\left(\dfrac{\pi}{2}\dfrac{\Delta\ell}{\varepsilon}\right) & \text{for} \quad \Delta\ell \geq 0 \\ 0 & \text{for} \quad \Delta\ell < 0 \end{cases}$$

where c is the stiffness coefficient, $\Delta\ell$ is the deviation from the constraint virtual boundary such that for $\Delta\ell < 0$ the constraint/contact are absent and for $\Delta\ell \geq 0$ the constraint is simulated by the elastic resistance force being increased fast up to infinity.

Numeric results are shown in Figure 2 where two variables $\Delta\ell$ for two models under verification, one with impacts and nonlinear with compliance, are compared in the process of landing on the constraint. One can see easily that non-linear model represented above gives an approximation of much more better quality while the landing process.

Let us consider in more detail the motivation of the last model use. The complicated dynamical picture can arise in case of the account for the interconnecting tether mass.

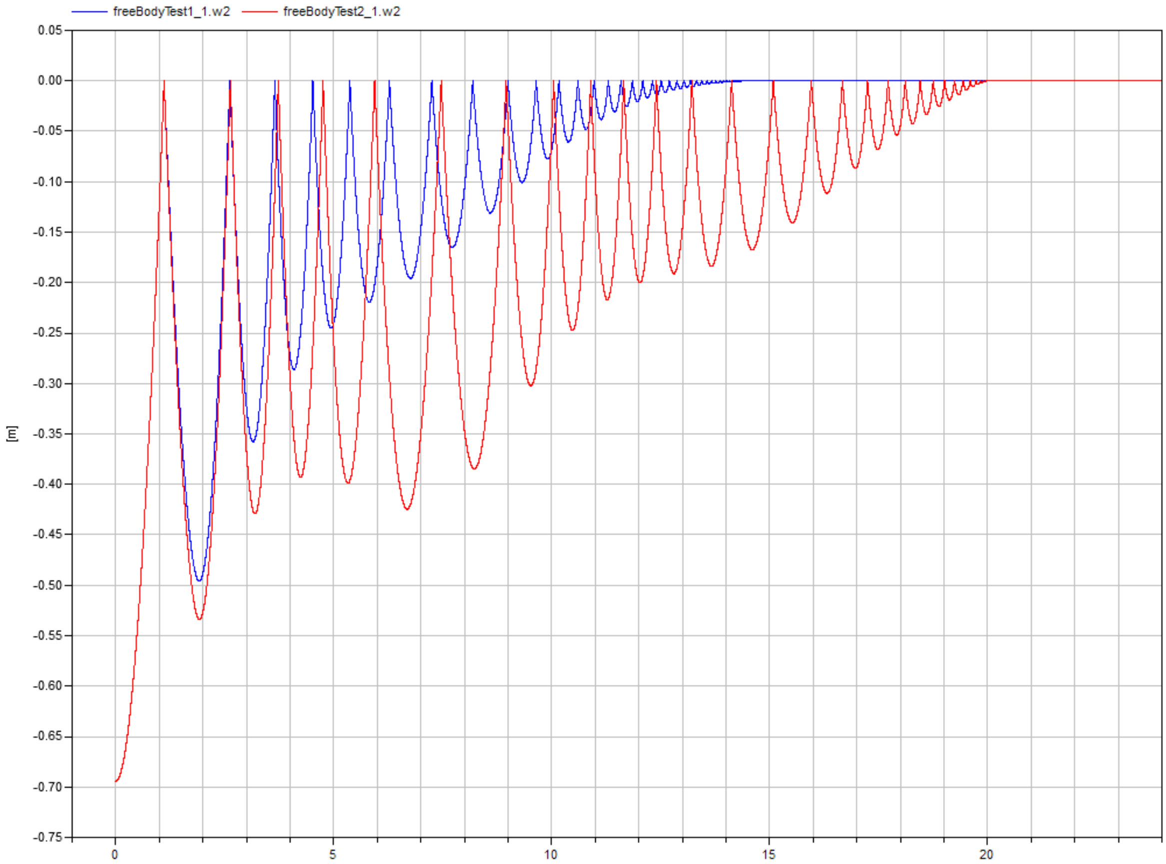

Figure 2: Comparison of two models

Let for definiteness and simplicity we have the tether lumped model as a chain sufficiently long and consisting of the discrete elements which are particles of sufficiently small masses being interconnected by weighless and inextensible cable segments. In such the model if each the segment is simulated as a hybrid automata then it is easy to see that combining interactions via impacts is transformed to the problem of large computational complexity.

To resolve this problem we can undertake the replacement of inextensible segments by ones of the model described above with sufficiently small compliance. Computational experiments show that inextensible and compliant segments demonstrate dynamical closeness. As a result we can conclude that application of lumped model with compliance can be useful for efficient approximation of the tethered system dynamics with massive interconnected cables. For this we also have to remark that real cables are sooner compliant than inextensible.

5 CONCLUSION

As a result we can make some numerical observations:

–Lumped model with inextensible elements with impacts approximates source problem having massive tethers;

–Lumped model with compliance approximates model of the hybrid automata with impacts;

–The latter model can be used for approximating dynamical problems with tethers having non-zero masses.

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