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MESHFREE METHOD FOR THE STOCHASTIC LANDAU-LIFSHITZ NAVIER-STOKES EQUATIONS

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Abstract. The current study aimed to develop a meshfree Lagrangian particle method for the Landau-Lifshitz Navier-Stokes (LLNS) equations. The LLNS equations incorporate thermal fluctuation into macroscopic hydrodynamics by addition of white noise fluxes whose magnitudes are set by a fluctuation-dissipation theorem. Moreover, the study focuses on capturing correct variance and correlation computed at equilibrium flows, which are compared with available theoretical values and found very good agreement.

1 INTRODUCTION

Physical quantities which describe a macroscopic system in equilibrium are seems to very near to their mean value. Nevertheless, due to microscopic fluctuation, random deviation from mean value though small, do occur. Thermal fluctuation is a source of noise in many system. Fluctuation plays a major role in phase transitions and chemical kinetics.

Investigation of thermal fluctuation in the motion of fluid becomes essential at micro and nano scale, because of the various applications of micro and nano scale flow, ranging from micro-engineering to molecular biology. Micro-machines have major impact on many disciplines (e.g. biology, medicine, optics, aerospace, and mechanical and electrical engineering) [1]. The study of fluctuation at micro and nano scale is particularly interesting when the fluid is under extreme conditions or near a hydrodynamic instability, e.g. the breakup of droplet in nanojet, fluid mixing in the Rayleigh-Taylor instability [8].

The validity of continuum approach has been identified with the validity of Navier-Stokes equations. This require the Knudsen number ($Kn = \lambda/L$)[1] should be very small

[1] λ denotes mean free path and L represents the characteristic length.

compared with unity because the presence of thermal fluctuation becomes significant for larger Knudsen number ($Kn \geq 0.01$), which corresponds to rarefied regime, resulted in failure of continuum hypothesis [2]. Then the flow is computed with the help of kinetic equations. LLNS PDE which is an extended form of Navier-Stokes equation trying to capture the thermal fluctuation as accurate as possible.

To describe the general theory of fluctuation in fluid dynamics is equivalent to setting up the "equation of motion" for fluctuating quantities. Landau and Lifshitz has done this work by introducing the appropriate additional terms in the general equation of fluid dynamics and gave an extended form of Navier-Stokes equations. The Landau-Lifshitz Navier-Stokes equations is written by the expression

$$\mathbf{U}_t + \nabla.\mathbf{F} = \nabla.\mathbf{D} + \nabla.\mathbf{S}, \tag{1}$$

where $\mathbf{U}$ stands for the vector of conserved quantities, density of mass, momentum and energy.

$$\mathbf{U} = \begin{pmatrix} \rho \\ \mathbf{J} \\ E \end{pmatrix}, \tag{2}$$

$\mathbf{F}$ denotes the hyperbolic flux and $\mathbf{D}$ denotes the diffusive flux of fluid dynamics equations. $\mathbf{F}$ and $\mathbf{D}$ are given by

$$\mathbf{F} = \begin{pmatrix} \rho\mathbf{v} \\ \rho\mathbf{v}.\mathbf{v} + P\mathbf{I} \\ \mathbf{v}E + P\mathbf{v} \end{pmatrix}, \tag{3}$$

$$\mathbf{D} = \begin{pmatrix} 0 \\ \tau \\ \tau.\mathbf{v} - \mathbf{q} \end{pmatrix}, \tag{4}$$

where $\mathbf{v}$ is the fluid velocity, P is the pressure and T denotes the temperature. $\tau = \eta\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T - \frac{2}{3}\mathbf{I}\nabla\cdot\mathbf{v}\right)$ is the stress tensor. $\mathbf{q} = -\kappa\nabla T$ denotes the heat flux. Here η and κ are the coefficients of viscosity and thermal conductivity, respectively. For the given expression of τ we have assumed the bulk viscosity is zero.

The given expression for τ and $\mathbf{q}$ relates these quantities to the velocity and temperature gradients respectively. But, in the presence of fluctuation there are also spontaneous local stresses and heat fluxes in the fluid, which are not related to velocity and temperature gradient. For these spontaneous local stresses tensor and heat fluxes, the LLNS equations introduce additional vector in fluid dynamics equations called stochastic flux

$$\mathbf{S} = \begin{pmatrix} 0 \\ S \\ H + \mathbf{v}\cdot S \end{pmatrix}, \tag{5}$$

where the stochastic stress tensor (SST) S and stochastic heat flux (SHF) H have zero mean and their covariances given by

$$Cov(S_{ij}(\mathbf{r},t),S_{kl}(\mathbf{r}^{'},t^{'})) = 2k_B\eta T\left(\delta^K_{ik}\delta^K_{jl}+\delta^K_{il}\delta^K_{jk}-\frac{2}{3}\delta^K_{ij}\delta^K_{kl}\right)\delta(\mathbf{r}-\mathbf{r}^{'})\delta(t-t^{'}), \tag{6}$$

$$Cov(H_i(\mathbf{r},t),H_j(\mathbf{r}^{'},t^{'})) = 2k_B\kappa T^2\delta^K_{ij}\delta(\mathbf{r}-\mathbf{r}^{'})\delta(t-t^{'}), \tag{7}$$

$$Cov(S_{ij}(\mathbf{r},t),H_k(\mathbf{r}^{'},t^{'})) = 0. \tag{8}$$

where k_B is the Boltzmann's constant. These stochastic properties for S and H have been derived by a variety of approaches. Originally, these properties have been derived for equilibrium fluctuation [5, 9, 10, 11] and later validity of LLNS equations for non-equilibrium systems has been shown [12].

In this work a meshfree numerical scheme has been developed for solving LLNS equations. For simplicity, we will deal with one-dimensional system. We will solve Lagrangian form of LLNS. The Lagrangian form of LLNS for $1D$ system in terms of primitive variables can be written as

$$\frac{D\rho}{Dt} = -\rho\frac{\partial u}{\partial x} \tag{9}$$

$$\rho\frac{Du}{Dt} = -\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left(\frac{4}{3}\eta\partial_x u\right)+\frac{\partial s}{\partial x} \tag{10}$$

$$c_v\rho\frac{DT}{Dt} = -P\frac{\partial u}{\partial x}+\frac{4}{3}\eta\left(\frac{\partial u}{\partial x}\right)^2+\frac{\partial}{\partial x}\left(\kappa\frac{\partial T}{\partial x}\right)+s\frac{\partial u}{\partial x}+\frac{\partial h}{\partial x}. \tag{11}$$

u is the fluid velocity in x-direction and T is the temperature. s and h represent SST and SHF in 1D respectively. Momentum $\mathbf{J} = \rho u$ and energy density $E = c_v\rho T + \frac{1}{2}\rho u^2$ expressed in terms of ρ, u, T. By D/Dt we denotes the Lagrangian derivative. We take the above system with equation of state $P = \rho RT$, where R is gas constant.

We will demonstrate our result for a mono-atomic, hard sphere gas for which $R = k_B/m$ and $c_v = \dfrac{R}{\gamma-1}$ where m is molecular mass and $\gamma(=\frac{5}{2})$ is the ratio of specific heat.

Now for covariances of stochastic fluxes in $1D$ system, from equations (6), (7) and (8)

$$\begin{aligned} Cov(s(x,t),s(x^{'},t^{'})) &= \frac{1}{\sigma^2}\int dy\int dy^{'}\int dz\int dz^{'}Cov(S_{xx}(\mathbf{r},t)S_{xx}(r^{'},t^{'})) \\ &= \frac{8k_B\eta T}{3\sigma}\delta(x-x^{'})\delta(t-t^{'}). \end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned} Cov(h(x,t),h(x^{'},t^{'})) &= \frac{1}{\sigma^2}\int dy\int dy^{'}\int dz\int dz^{'}Cov(H_x(\mathbf{r},t)H_x(r^{'},t^{'})) \\ &= \frac{2k_B\kappa T^2}{\sigma}\delta(x-x^{'})\delta(t-t^{'}). \end{aligned} \tag{13}$$

here σ represents the surface area of the system for yz - plane.

A number of numerical schemes has been developed for stochastic hydrodynamics equations. A stochastic lattice-Boltzmann model developed for simulating solid-fluid suspensions by Ladd [13]. In the following application of this approach for modelling Brownian motion of particles has been used by Sharma and Patankar [14], where they coupled the fluctuating hydrodynamics equations with the particle equations of motion which results in Brownian motion of particle.

Serrano and Español [15] have developed a thermodynamically consistent measoscopic fluid particle model by casting their model into GENERIC structure which allows to introduce thermal fluctuation. They describe a finite volume Lagrangian discretization of the continuum equations of hydrodynamics using Voronoi tessellation.

Garcia et al.[3] has been developed a simple finite difference scheme for the linearized LLNS equations. This scheme was successful, but has been designed for specific problems and cannot extended in general due to certain assumption. In the context of Adaptive Mesh and Algorithm Refinement hybrid method that couple continuum and particle algorithms based on finite difference scheme has been developed. Demonstration of diffusion equation [18], the "train model" [19] and the stochastic Burger's equation [20] has been done by similar kind of scheme.

In the later work of Garcia et al. [4] CFD based scheme for stochastic PDE has been developed. In this work they have been considered compressible flow and developed numerical scheme for LLNS equations. Numerical schemes demonstrated equilibrium flow and computed spatial and time correlation at equilibrium result has been compared with theoretical value and DSMC simulation. Effect of fluctuation on shock drift has been shown and result compare with DSMC simulation. The most successful scheme has been considered in this work is "Variance-preserving third order Runge-Kutta". The method is based on a third order, TVD Runge-Kutta temporal integrator (RK3) combine with a centered discretization of hyperbolic and diffusive fluxes. This scheme has also incorporate a specific interpolation for required accuracy in variance.

In this work we will present a mesh free method for LLNS equations. We will consider Finite Pointset Method (FPM) like smoothed particle hydrodynamics (SPH) based on least square approach [16] for numerical solution of fluctuating hydrodynamics. We will concentrate here to capture correct variance in equilibrium flow and compare the result with theoretical values. The concluding section will emphasize on the successful mesh free simulation of fluctuating hydrodynamics equations for compressible flow and discuss future work. Since, it is a mesh free method and the distribution of particles (moving grid) can be quite arbitrary, the method is suitable for complicated geometry and multiphase flows. The FPM is suitable to handle a wide range of dynamical fluid structure interaction.

2 Numerical Method

The $1D$ Navier-Stokes equations have already solved with least square SPH like FPM for compressible flow [16]. A hybrid method also has been developed for kinetic and con-

tinuum equations where the solution of kinetic equation has been done by DSMC and for continuum equations meshfree method used [17].

To extend the successful idea of meshfree method for $1D$ Navier-Stokes equations we will solve $1D$ LLNS equations with a meshfree Lagrangian method. Though, it is a bit difficult to develop a meshfree framework for stochastic partial differential equations (SPDE) due to already existed randomization in position of mesh particles, it has a number of advantage likewise the meshfree method is a more natural choice to study the dynamics of small particles at fluid interfaces, to generate a hybrid method for coupling of Boltzmann and fluctuating hydrodynamics equations for very small geometry. Earlier studies deal the coupling of DSMC for Boltzmann equation and finite volume method for the LLNS [21].

In this method the particle position are itself the grid and to take this consideration in account we will simulate an additional equation with (9 - 11). This equation will determine the particle position and given as

$$\frac{Dx}{Dt} = u. \tag{14}$$

Here u is the fluid velocity and x denote the position of particle in $1D$. To approximate spatial derivative at every grid point is equivalent to approximate the spatial derivative at every particle positions. For solving the Lagrangian LLNS system given by (9-11) together with (14) by FPM, first we fill the domain by particles. These particle moves with fluid velocity and then approximate spatial derivative in equation (9-11) at each particle position from its neighbouring particles. This reduce the given system of partial differential equations (PDE) to a system of ordinary differential equations (ODE) with respect to time per particle.

We will use MacCormack scheme [4] for reduced ODE on each particle. This MacCormack scheme is a variant of Lax-Wendroff. On applying the desired MacCormack's discretization of LLNS will be

Step 1

$$x_i^* = x_i^n + \triangle t u_i^n, \tag{15}$$

$$\rho_i^* = \rho_i^n - \triangle t \rho_i^n \left(\frac{\partial u}{\partial x}\right)_i^n, \tag{16}$$

$$u_i^* = u_i^n + \frac{\triangle t}{\rho_i^n}\left\{-\left(\frac{\partial P}{\partial x}\right)_i^n + \frac{4}{3}\eta_i^n\left(\frac{\partial^2 u}{\partial x^2}\right)_i^n + \frac{4}{3}\left(\frac{\partial \eta}{\partial x}\right)_i^n\left(\frac{\partial u}{\partial x}\right)_i^n + \left(\frac{\partial s}{\partial x}\right)_i^n\right\}, \tag{17}$$

$$T_i^* = T_i^n + \frac{\triangle t}{c_v \rho_i^n}\left\{-P_i^n\left(\frac{\partial u}{\partial x}\right)_i^n + \frac{4}{3}\eta_i^n\left(\left(\frac{\partial u}{\partial x}\right)^2\right)_i^n + \kappa_i^n\left(\frac{\partial^2 T}{\partial x^2}\right)_i^n + \left(\frac{\partial \kappa}{\partial x}\right)_i^n\left(\frac{\partial T}{\partial x}\right)_i^n + s_i^n\left(\frac{\partial u}{\partial x}\right)_i^n + \left(\frac{\partial h}{\partial x}\right)_i^n\right\}, \tag{18}$$

Step 2

$$x_i^{**} = x_i^* + \triangle t u_i^*, \tag{19}$$

$$\rho_i^{**} = \rho_i^* - \triangle t \rho_i^* \left(\frac{\partial u}{\partial x}\right)_i^*, \tag{20}$$

$$u_i^{**} = u_i^* + \frac{\triangle t}{\rho_i^*}\left\{-\left(\frac{\partial P}{\partial x}\right)_i^* + \frac{4}{3}\eta_i^*\left(\frac{\partial^2 u}{\partial x^2}\right)_i^* + \frac{4}{3}\left(\frac{\partial \eta}{\partial x}\right)_i^*\left(\frac{\partial u}{\partial x}\right)_i^* + \left(\frac{\partial s}{\partial x}\right)_i^*\right\}, \tag{21}$$

$$T_i^{**} = T_i^* + \frac{\triangle t}{c_v \rho_i^*}\left\{-P_i^*\left(\frac{\partial u}{\partial x}\right)_i^* + \frac{4}{3}\eta_i^*\left(\left(\frac{\partial u}{\partial x}\right)^2\right)_i^* + \kappa_i^*\left(\frac{\partial^2 T}{\partial x^2}\right)_i^* + \left(\frac{\partial \kappa}{\partial x}\right)_i^*\left(\frac{\partial T}{\partial x}\right)_i^* + s_i^*\left(\frac{\partial u}{\partial x}\right)_i^* + \left(\frac{\partial h}{\partial x}\right)_i^*\right\}, \tag{22}$$

Final Step

$$x_i^{n+1} = \frac{1}{2}\left(x_i^n + x_i^{**}\right), \tag{23}$$

$$\rho_i^{n+1} = \frac{1}{2}\left(\rho_i^n + \rho_i^{**}\right), \tag{24}$$

$$u_i^{n+1} = \frac{1}{2}\left(u_i^n + u_i^{**}\right), \tag{25}$$

$$T_i^{n+1} = \frac{1}{2}\left(T_i^n + T_i^{**}\right), \tag{26}$$

For each step given above P, η, κ will be computed by

$$P = \rho R T, \tag{27}$$

$$\eta = \frac{5}{16 d^2}\sqrt{\frac{m k_B}{\pi} T}, \tag{28}$$

$$\kappa = \frac{15 k_B \eta}{4m}, \tag{29}$$

Here d denotes the molecular diameter. $n = 0, 1, 2, \ldots$ represents the time step and $i = 1, 2, \ldots N$ goes for particles in domain.

The approximation for SST and SHF for each particle at any instant is computed as

$$s_i^n = \sqrt{\frac{8 k_B}{3 \triangle t V_c}\left(\eta_i^n T_i^n\right)}\ \Re_i^n, \tag{30}$$

$$h_i^n = \sqrt{\frac{2 k_B}{\triangle t V_c}\left(\kappa_i^n (T_i^n)^2\right)}\ \Re_i^n, \tag{31}$$

where V_c denotes the volume between two particle of spatial discretization and $\Re$ are independent, identically distributed (iid), Gaussian random value with zero mean and unit variance.

The stochastic fluxes required some extra care in two-step scheme. The s and h are independent, identically distributed (iid) Gaussian random variable with mean zero and variance σ^2 for $l = n, *$. Let denotes s and h in combined by $\boldsymbol{s}$. Substituting this in MacCormack scheme we find variance in the stochastic flux $\boldsymbol{s}$ on simulation particle as

$$\begin{aligned} Var\left(\left(\frac{1}{2}\boldsymbol{s}^n + \frac{1}{2}\boldsymbol{s}^*\right)^2\right) &= \left(\frac{1}{2}\right)^2 Var\left(\boldsymbol{s}^n\right) + \left(\frac{1}{2}\right)^2 Var\left(\boldsymbol{s}^*\right) \\ &= \frac{1}{2} Var\left(\boldsymbol{s}\right) \\ &= \frac{\sigma^2}{2}. \end{aligned} \tag{32}$$

We have neglected the multiplicity of noise by taking $Var\left(\boldsymbol{s}^n\right) = Var\left(\boldsymbol{s}^*\right)$.

The above equation means that because of the temporal averaging variance in the flux reduced to half of its original magnitude. So, to include this observation the correct stochastic flux for a two step scheme will be $\tilde{\boldsymbol{s}} = \sqrt{2}\boldsymbol{s}$ instead of $\boldsymbol{s}$.

Now we have to solve the equations (15 - 29) and for this the remaining task is to approximate the spatial derivatives on right hand side of the prescribed equations.

2.1 Meshfree approximation of spatial derivatives

In many practical applications the mesh plays a very important role in simulation and many solvers loose their accuracy if the mesh is poorly constructed. In some complicated geometry the mesh generation becomes a difficult task. A meshfree method does not require a regular grid and gives a very good approximation of spatial derivative even for randomly distributed grid point so overcome to mesh generation difficulties. Initially we will fill the domain with particles in a regular grid method but when they move with fluid velocity then their distribution become quite arbitrary after short time.

We will describe the least square approximation of spatial derivatives in $1D$. As mentioned earlier, in this method grid points are particle positions. Therefore, we have to approximate the derivative at every particle position. Let $f(t, x)$ be a scalar function at x and $f_i(t)$ its value at $x_i \in [0, L]$ for $i = 1, 2, 3, \ldots, N$ for any instant t. Approximation of spatial derivatives of $f(x)$ at x will be in terms of the values of $f(x)$ on a set of neighbouring points. For limiting number of neighbouring points of x we will consider a weight function $w = w(x_i - x; h)$ with small compact support, where h determines the size of support. The choice of weight function can be quite arbitrary but we will consider

a Gaussian weight function in the following form

$$w\left(x_i - x; h\right) = \begin{cases} \exp\left(-\alpha \dfrac{\|x_i - x\|^2}{h^2}\right), & \text{if } \dfrac{\|x_i - x\|}{h} \leq 1 \\ 0, & \text{else.} \end{cases} \tag{33}$$

with α a positive constant, we have taken the value of α is 6.25, h defines the neighbourhood radius for x. Let $P(x,h) = \{x_i : i = 1, 2, \ldots, n_x\}$ be the set of n_x neighbouring points of x in an interval of radius h. For consistency reason some obvious restriction required for h as there should be enough number of neighbouring particle for least square approximation. In general for $1D$, $h = 3dx$, where dx is the initial spacing of particles.

Derivatives of function can be computed easily and accurately by using Taylor's series expansion and the least square approximation. We will write Taylor's series expansion of $f(t,x)$ around x with unknown coefficients and then compute these coefficients by minimizing a weighted error over the neighbouring points.

Suppose we want to approximate the derivatives of a function $f(t,x)$ from its n_x neighbouring points sorted with respect to its distance from x. Consider Taylor's expansion of $f(t,x_i)$ around x

$$\begin{aligned} f\left(t,x_i\right) = f\left(t,x\right) + \left(f\left(t,x\right)\right)_x \left(x_i - x\right) \\ + \frac{1}{2}\left(f\left(t,x\right)\right)_{xx} \left(x_i - x\right)^2 + e_i \end{aligned} \tag{34}$$

where e_i is the error in Taylor's expansion at the point x_i. The unknowns f_x, f_{xx} are required derivatives computed by minimizing the error e_i for $i = 1, 2, 3, \ldots, n_x$. The above system can be written as

$$\vec{e} = M\vec{a} - \vec{b} \tag{35}$$

where,

$$M = \begin{pmatrix} x_1 - x & \frac{1}{2}\left(x_1 - x\right)^2 \\ x_2 - x & \frac{1}{2}\left(x_2 - x\right)^2 \\ \vdots & \vdots \\ x_{n_x} - x & \frac{1}{2}\left(x_{n_x} - x\right)^2 \end{pmatrix}, \tag{36}$$

$a = \left[f_x, f_{xx}\right]^T$, $b = \left[f_1 - f, f_2 - f, \ldots, f_{n_x} - f\right]^T$ and $e = \left[e_1, e_2, e_3, \ldots, e_{n_x}\right]^T$.

For $n_x > 2$ this system will be over-determined for two unknowns f_x and f_{xx}.

The unknowns $\vec{a}$ are obtained from a weighted least square method by minimizing the quadratic form

$$J = \sum_{i=1}^{n_x} w_i e_i^2 = \left(M\vec{a} - \vec{b}\right)^T W \left(M\vec{a} - \vec{b}\right) \tag{37}$$

where

$$W = \begin{pmatrix} w_1 & 0 & \dots & 0 \\ 0 & w_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & w_{n_x} \end{pmatrix}$$

Finally, the minimization of J gives

$$\vec{a} = \left(M^T W M\right)^{-1} \left(M^T W\right) \vec{b} \tag{38}$$

required derivatives of function $f(t,x)$ as a liner combination of discrete neighbour values f_i.

3 Numerical Results

3.1 Equilibrium

This section gives results of described method for equilibrium scenario. The physical domain has been chosen such that the fluctuation in system becomes significant. System's parameter for numerical simulation given in Table 1.

Table 1: System parameter in CGS units for simulation of dilute gas.

Molecular diameter (Argon)	3.66×10^{-8}	Molecular mass (Argon)	6.63×10^{-23}
Reference mass density	1.78×10^{-3}	Reference temperature	273.0
Sound speed	30781	Reference velocity	0.5×30781 or 0
System length	1.25×10^{-4}	Reference mean free path	6.26×10^{-6}
System volume	1.96×10^{-16}	Time step	1.0×10^{-13}

The initial spacing of particle will be $dx = L/N$, where N is the initially distributed particles in domain. Initially 40 number of particles has been considered in domain. This number will not fixed during simulation and there will be an updating in number of particles.

The stability condition is found to be consistent with that suggested in [4].

$$(\mid u \mid + c_s) \frac{\triangle t}{\triangle x} \leq 1 \tag{39}$$

$$max\left(\frac{4}{3}\frac{\bar{\eta}}{\bar{\rho}}, \frac{\bar{\kappa}}{\bar{\rho} c_v}\frac{\triangle t}{\triangle x^2}\right) \leq \frac{1}{2}. \tag{40}$$

where c_s is the sound speed; the bar indicates the reference value of quantities around which system fluctuate. For the given reference state in Table 1 and initial spacing of particle, the time step has been chosen $\triangle t = 10^{-13} s$.

3.2 Variance at Equilibrium

The first benchmark for given schemes is capturing correct variance of conserved variables for a system at equilibrium. For test problems, we consider a periodic domain with zero net flow and constant non-zero net flow. We take constant average density and temperature in both cases as given in Table 1. The variance computed from 10^7 samples. Due to moving grid we will calculate statistics in global sense as given below

$$mean\,(\rho) = \mathbf{E}(\rho) = \frac{1}{\sum_{n=1}^{N_s} M(n)} \left(\sum_{n=1}^{N_s} \sum_{i=1}^{M(n)} \rho_i^n \right), \tag{41}$$

$$\begin{aligned} Var\,(\rho) &= \mathbf{E}(\rho^2) - (\mathbf{E}(\rho))^2 \\ &= \frac{1}{\sum_{n=1}^{N_s} M(n)} \left(\sum_{n=1}^{N_s} \sum_{i=1}^{M(n)} (\rho_i^n)^2 \right) \\ &\quad - \left(\frac{1}{\sum_{n=1}^{N_s} M(n)} \left(\sum_{n=1}^{N_s} \sum_{i=1}^{M(n)} \rho_i^n \right) \right)^2, \end{aligned} \tag{42}$$

Here, N_s is the total number of samples and $M(n)$ is the number for particle at time $(nSkip + n)\triangle t$ ($nSkip$ is the number of initial time step for stabilizing the system). In the same way statistics for momentum and energy can be computed.

Tables 2 and 3 compare the theoretical variances have been computed in ([6], [3]) with measured variances in meshfree stochastic scheme.

Table 2: Variance in conserved quantities at equilibrium for zero net flow.

Variance of	**Exact value**	**Meshfree MacCormack**	**error**
Density (ρ)	2.35×10^{-8}	2.11×10^{-8}	-10.2%
Momentum ($\mathbf{J}$)	13.34	13.33	-0.07%
Energy ($\mathbf{E}$)	2.84×10^{10}	2.68×10^{10}	-5.6%

Table 3: Variance in conserved quantities at equilibrium for constant non-zero net flow.

Variance of	**Exact value**	**Meshfree MacCormack**	**error**
Density (ρ)	2.35×10^{-8}	2.12×10^{-8}	-9.7%
Momentum ($\mathbf{J}$)	18.91	19.01	$+0.05\%$
Energy ($\mathbf{E}$)	3.67×10^{10}	3.85×10^{10}	$+4.9\%$

4 Conclusion

Stochastic Partial differential equation were discussed for modelling fluctuation in compressible flow of gas dynamics. For numerical simulation of the SPDEs a meshfree discretization approach has been introduced. The results of the simulation shows that the Lagrangian particle scheme gives a good agreement with theoretical value of variances for conserved variables, which guarantee that the scheme is able to accurately represent fluctuation in equilibrium flow.

It has already been mentioned in earlier literature that the ability of continuum model to accurately capture the fluctuation is very much sensitive to the construction of numerical scheme. Therefore, to construct a meshfree frame work for LLNS model becomes much more complicated because of a quite arbitrary distribution of particle. Minor changes in implementation leads to significant changes in accuracy and behaviour. Future work can be extended to some other non-equilibrium scenario and higher dimension for which the stochastic stress fluxes are more complex .

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