DISCRETE LEAST SQUARES MESHLESS METHOD FOR MODELING 2D CRACK PROBLEMS

H. ARZANI[1], M. MOBARAKI[2] AND M. TORABI [3]

[1]Civil Engineering Department
Shahid Rajaee Teacher Training University
Lavizan,16788-15811 Tehran, Iran
E-mail:Arzani@iust.ac.ir

[2]MSC of
Civil Engineering Department
Shahid Rajaee Teacher Training University
Lavizan,16788-15811 Tehran, Iran
E-mail:muhammad_mobaraki@yahoo.com

[3] MSC of
Civil Engineering Department
Shahid Rajaee Teacher Training University
Lavizan,16788-15811 Tehran, Iran
E-mail:mehdi_torabi_gl@yahoo.com

Key words: Meshless method, DLS, Diffraction Method, Crack

Abstract. In this paper, Discrete Least Squares meshless method (DLSM) is developed to analyze cracked structures in an elastostatic problem. DLSM is a new really meshless method that does not use any mesh in computation. The method is based on the minimization of the least squares functional with respect to the nodal parameters. The least squares functional are formed as the weighted summation of the residual of the differential equation and its boundary conditions. In this method, the domain of problem is discretized by some nodes that are used to produce Moving Least Squares shape functions. This type of discretization eliminates the Finite Element Method shortcomings. In this study, diffraction method was used to produce continuous shape functions around the crack. In diffraction method, the domain of influence wrap around the crack tip and it results in continuous derivatives of shape functions. Finally, the DLSM high efficiency and accuracy is presented by comparing the analytical results with numerical ones.

1 INTRODUCTION

One of the most important advances in the field of numerical methods was the development of the Finite Element Method (FEM) in the 1950s. In FEM, a continuum with a complicated shape is divided into elements. The FEM is a robust and thoroughly developed method, and hence it is widely used in engineering fields due to its versatility for complex geometry and flexibility for many types of linear and non-linear problems. However, the FEM has the inherent shortcomings of numerical methods that rely on meshes or elements that are connected together by nodes in a properly predefined manner.

This method encounters some difficulties when dealing with problems involving moving boundaries, crack propagation or extremely large deformation due to their need for remeshing of the domain. This problem, to a lesser degree, is also faced when exploiting the most important advantage of the FEM over other mesh based numerical methods, namely adaptive refinement of the solution, where the mostly used method of h-refinement requires adding and/or repositioning of the nodal points.

For the elimination of at least part of the structure under consideration by constructing the approximation entirely in terms of nodes, a class of meshless has been developed. These include some important examples of these methods are the smoothed particle hydrodynamic (SPH) method , reproducing kernel particle method (RKPM), element free Galerkin (EFG) method, meshless local Petrov–Galerkin (MLPG) method, local boundary integral equation (LBIE) method, and hp-cloud method. As a consequence, they are totally free from the mesh entanglements and distortions commonly associated with the use of the finite element method in the computational simulation of large deformation problems.

Arzani and Afshar [1] proposed the discrete least squares method (DLSM) for solving Poisson’s equation. Firoozjaee and Afshar [2-3] developed this method by using collocation points to solve elliptic partial differential equations and studied the effect of the collocation points on the convergence and accuracy of the method. DLSM was later used by Naisipour et al. [4] to solve elasticity problems on irregular distribution of nodal points.

In this paper DLSM used for solving two-dimensional elasticity crack problems. In this method, the problem domain is discretized by distributed field nodes. The field nodes are used to construct the trial functions by employing the moving least-squares interpolant. This method is based on minimizing a least squares functional with respect to the nodal parameters.

In most of meshless methods using the smooth and high continuous approximation function, difficulties arise, in nonconvex bodies such as cracks. Therefore, meshless methods used some techniques in encountering these problems. Krysl and Belytschko [5] described some techniques so called visibility criterion that lead to discontinuous weight function, and consequently shape functions which are discontinuous within the domain. And later they used the two approaches so called diffraction and transparencymethods [6] that lead to continuous weight functions around the crack tip.

In this paper, diffraction method used for constructing continuous MLS approximation in DLS meshless method around the crack tip.In diffraction method, domain of influence wraps around concave boundary similar to the way light diffracts.

The purpose of this paper was using DLSM for fracture (mode I) analysis of crack in homogenous, isotropic, linear-elastic two-dimensional solid.

2 MOVING LEAST SQUARES APPROXIMATION

Among the available meshless approximation schemes, the moving least squares (MLS) method is generally considered to be one of the best methods to interpolate random data with a reasonable accuracy, because of its completeness, robustness and continuity. The MLS approximation can be constructed as:

$$u(x) \cong \phi(x_k) = \sum_{k=1}^{m} P_k(x)a(x) = P_k^T(x)a(x) \qquad \forall x \in \Omega_x \tag{1}$$

Where $\phi(x_k)$ denotes the approximate value of u(x) and $P_k^T(x) = [P_1(x), P_2(x), ..., P_m(x)]$ is the basis vector that built using Pascal's triangle and m is the number of monomials in the basis vector *p(x)*. For a 2D problem we can specify $P = [1, x, y, x^2, xy, y^2]$ for m=6. In the MLS approximation, the shape functions are obtained by minimizing a weighted residual J to determine the coefficients *a(x)* where

$$J(x) = \sum_{k=1}^{n} W(x - x_k)[u_k - P_k^T(x)a(x)]^2 \tag{2}$$

Where u_k is value of *u(x)* at x_k , *n* is the number of nodes in support at x_k . The weight function *W(x-x_k)* is usually built in such a way that it takes a unit value in the vicinity of the point k where the function and its derivatives are to be computed and vanishes outside a region surrounding the point x_k . In this research the cubic spline weight function is considered as follow:

$$W(d) = W(x - x_k) = \begin{cases} \frac{2}{3} - 4d^2 + 4d^3 & ; d \le \frac{1}{2} \\ \frac{4}{3} - 4d + 4d^2 - \frac{4}{3}d^3 & ; \frac{1}{2} < d \le 1 \\ 0 & ; d > 1 \end{cases} \tag{3}$$

Where $d = \frac{\|x_k - x\|}{d_{\max}}$ and $d_{\max}$ is the size of influence domain of point x_k . Minimization of equation (2) leads to

$$a(x) = B(x)A^{-1}(x)u_k \tag{4}$$

Where

$$A(x) = \sum_{k=1}^{m} W(x - x_k)P(x_k)P^T(x_k) \tag{5}$$

$$B^T(x) = [W(x - x_1)P(x_1), W(x - x_2)P(x_2), ..., W(x - x_m)P(x_m)] \tag{6}$$

Substituting equation (4) in equation (1) gives

$$\phi(x_k) = P^T(x)A^{-1}(x)B(x)u_k = \Phi(x_k)u_k \tag{7}$$

Where $\Phi(x_k)$ contains the MLS shape functions of nodes at point x_k .

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3 DISCRETE LEAST SQUARES MESHLESS METHOD

Consider the following (partial) differential equation

$$A(\phi) + F = 0 \qquad in\ \Omega \tag{8}$$

Subject to appropriate Drichlet and Neumann boundaries

$$B(\phi) - \bar{t} = 0 \qquad on\ \Gamma_t \tag{9}$$

$$\phi - \bar{\phi} = 0 \qquad on\ \Gamma_u \tag{10}$$

Where *A* and *B* are (partial) differential operators, and *F* represents external forces or source term on the problem domain. $\bar{\phi}$ and $\bar{t}$ are vector of prescribed displacements and tractions on the Dirichlet and Neumann boundaries, respectively, Γ_u and Γ_t are the displacement and traction boundaries, respectively. For plane stress, relations (8) to (10) can be written as follow:

$$A(\phi) + F = \begin{bmatrix} \frac{E}{2(1-v)} + \mu & 0 \\ 0 & \mu \end{bmatrix} \frac{\partial^2 \phi}{\partial x^2} + \begin{bmatrix} 0 & \frac{E}{2(1-v)} \\ \frac{E}{2(1-v)} & 0 \end{bmatrix} \frac{\partial^2 \phi}{\partial xy} + \begin{bmatrix} \mu & 0 \\ 0 & \frac{E}{2(1-v)} + \mu \end{bmatrix} \frac{\partial^2 \phi}{\partial y^2} + F = 0 \tag{11}$$

$$B(\phi) - \bar{t} = \begin{bmatrix} \left(\lambda - \frac{v\lambda}{(1-v)} + 2\mu \right) n_x & \mu n_y \\ \left(\lambda - \frac{v\lambda}{(1-v)} \right) n_y & \mu n_x \end{bmatrix} \frac{\partial \phi}{\partial x} + \begin{bmatrix} \mu n_y & \left(\lambda - \frac{v\lambda}{(1-v)} \right) n_x \\ \mu n_x & \left(\lambda - \frac{v\lambda}{(1-v)} + 2\mu \right) n_y \end{bmatrix} \frac{\partial \phi}{\partial y} - \begin{bmatrix} \bar{t}_x \\ \bar{t}_y \end{bmatrix} = 0 \tag{12}$$

$$\phi - \bar{\phi} = 0 \tag{13}$$

Where ϕ is the vector of unknowns that defined as $\phi = [u, v]^T$ and λ, μ are the Lame constants and shear modules, respectively, defined as:

$$\lambda = \frac{vE}{(1-2v)(1+v)} \qquad , \mu = \frac{E}{2(1+v)} \tag{14}$$

Upon discretization of the problem domain and its boundaries, using equation (7), the residual of partial differential equation at a typical nodal point *j* is defined as:

$$R_\Omega(x_j) = A(\phi(x_j)) + F(x_j) = \sum_{i=1}^{n} A(\Phi(x_j))\phi_i + F(x_j) \qquad , j = 1, 2, ..., N_\Omega \tag{15}$$

The residual of Neumann boundary condition at typical nodal point k on the Neumann boundary can also be written as:

$$R_t(x_j) = B(\phi(x_j)) - \bar{t}(x_j) = \sum_{i=1}^{n} B(\Phi(x_j))\phi_i - \bar{t}(x_j) \qquad , j = 1, 2, ..., N_t \tag{16}$$

And finally the residual of Drichlet boundary condition at nodes on the Drichlet boundary could be stated by

$$R_u(x_j) = \phi - \bar{\phi}(x_j) = \sum_{i=1}^{n} \Phi(x_j)\phi_i - \bar{\phi}(x_j) \qquad , j = 1,2,...,N_u \tag{17}$$

Where n is the total number of nodes, N_Ω is the internal nodes of domain of problem, N_t is the number of nodes on the Neumann boundary and N_u is the number of nodes on the Dirichlet boundary. The least squares functional of the residuals is defined as:

$$J = \sum_{j=1}^{N_\Omega} {R_\Omega}^2(x) + \alpha_t \sum_{j=1}^{N_t} {R_t}^2(x) + \alpha_u \sum_{j=1}^{N_u} {R_u}^2(x), \qquad N = N_\Omega + N_t + N_u \tag{18}$$

Where α_t and α_u are penalty coefficients for Neumann and Drichlet boundary conditions respectively.
Minimization of the functional with respect to nodal parameters leads to the following system of equations

$$K\phi = F \tag{19}$$

Where $\phi = [\phi_1, \phi_2, ..., \phi_N]^T$ is the vector of unknown's nodal parameters and K,F are the stiffness and right hand side matrices with typical components defined as:

$$K_{ij} = \sum_{k=1}^{N_\Omega} A\left(N_J(x_k)\right) A\left(N_i(x_k)\right) + \alpha_t \sum_{k=1}^{N_t} B\left(N_j(x_k)\right) B\left(N_i(x_k)\right) + \alpha_u \sum_{k=1}^{N_u} N_j(x_k) N_i(x_k) \tag{20}$$

$$F_j = \sum_{k=1}^{N_\Omega} A\left(N_J(x_k)\right) f(x_k) + \alpha_t \sum_{k=1}^{N_t} B\left(N_j(x_k)\right) \bar{t}(x_k) + \alpha_u \sum_{k=1}^{N_u} N_j(x_k) \phi(x_k) \tag{21}$$

Where K is stiffness matrix that is both square and symmetric. Therefore, the final system of equations can be solved directly via efficient solvers.

4 CONTINUOUS APPROXIMATION BY DIFFRACTION METHOD

In meshless methods, the displacement discontinuity due to a crack can be modeled by cutting the domain of influence. In this approach, discontinuity is defined by diffraction method. Organ et al. [6] have described a method for construction of approximations around the tip of a discontinuity. The method is called diffraction since the weight function encloses the crack surface similar to the way light diffracts around corners. When the discontinuity divides the domain of influence into two separate parts, the points that located behind the discontinuity are removed from influence domains. Consider the end of a line of discontinuity as shown in figure1. In this method the distance *S(x)* between the node point x_I and sampling point x is modified for all points x for which the line (x_I,x) intersects the crack line.

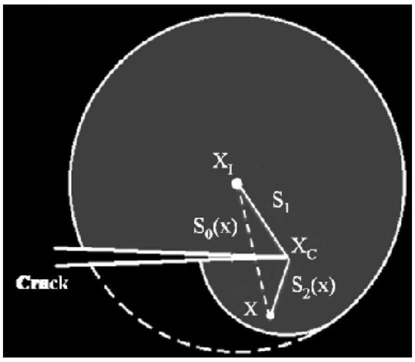


Figure 1: Diffraction method planning near the crack tip

The modified weight function distance S(x) is given by

$$S(x) = (\frac{S_1 + S_2(x)}{S_0(x)})^{\eta} S_0(x) \tag{22}$$

Where $s_1 = ||x_I - x_c||$, $s_2(x) = ||x - x_c||$, $s_0(x) = ||x - x_I||$, x_I is the node, x is the sampling point, x_c is the coordinate of the crack tip, and η is the diffraction method parameter. The exponent η is used to reduce the size of support behind the crack. Numerical experiments have shown $\eta = 1$ for problems with a normal nodal distributions and a linear basis to be reasonable choices.

The spatial derivatives of the weight function with the diffraction method are calculated by the chain rule.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial S(x)} \frac{\partial S(x)}{\partial x} \tag{23}$$

$$\frac{\partial S(x)}{\partial x} = \eta \left(\frac{S_1 + S_2(x)}{S_0(x)} \right)^{\eta - 1} \frac{\partial S_2}{\partial x} + (1 - \eta) \left(\frac{S_1 + S_2(x)}{S_0(x)} \right)^{\eta} \frac{\partial S_0}{\partial x} \tag{24}$$

$$\begin{aligned} \frac{\partial^2 S(x)}{\partial x^2} = \eta & \left[\frac{\partial S_2}{\partial x} (\eta - 1) (\frac{S_1 + S_2}{S_0})^{\eta - 2} \left(\frac{\partial S_2}{\partial x} S_0 - \frac{\partial S_0}{\partial x} (S_1 + S_2) \right) + \frac{\partial^2 S_2}{\partial x^2} (\frac{S_1 + S_2}{S_0})^{\eta - 1} \right] \\ & + (\eta - 1) \left[\frac{\partial S_0}{\partial x} \eta (\frac{S_1 + S_2}{S_0})^{\eta - 1} \left(\frac{\partial S_2}{\partial x} S_0 - \frac{\partial S_0}{\partial x} (S_1 + S_2) \right) + \frac{\partial^2 S_0}{\partial x^2} (\frac{S_1 + S_2}{S_0})^{\mu} \right] \end{aligned} \tag{25}$$

Where

$$\frac{\partial S_0}{\partial x} = \frac{x - x_I}{S_0}, \qquad \frac{\partial S_2}{\partial x} = \frac{x - x_C}{S_2} \tag{26}$$

5 NUMERICAL EXAMPLE

In this section the accuracy of the DLS method for solving crack problem by using diffraction method for the construction of weight functions near the crack tip is considered. Hence, example inclusive finite rectangular plates with edge crack and plate have been analyzed. The obtained results for this example is compared with existing analytical solution that have been published in the literature. In this approach, the purpose is considering the stress filed near the crack tip by using DLS method for mode-I deformation. A detailed expression for the stress and strain fields in a singularity dominated zone, near the tip of the crack, for mode-I is given in the following expressions.

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \cos\frac{\theta}{2} \begin{Bmatrix} 1-\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ 1+\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{3\theta}{2} \end{Bmatrix} \tag{27}$$

$$\begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \frac{K_I}{2\mu}\sqrt{\frac{r}{2\pi}} \begin{Bmatrix} \cos\frac{\theta}{2}[\kappa-1+2\sin^2\frac{\theta}{2}] \\ \sin\frac{\theta}{2}[\kappa+1-2\cos^2\frac{\theta}{2}] \end{Bmatrix} \tag{28}$$

whereK_I is the stress intensity factor, r and θ are polar coordinates with an origin at the crack tip, σ_{ij} are components of the stress tensor, *i,j=x,y*. The Kolovos constant κ is related to Poisson's ratio v by

$$\kappa = \begin{cases} 3-4v & \textit{for plane stress} \\ \frac{3-v}{1+v} & \textit{for plane strain} \end{cases} \tag{29}$$

5.1 Rectangular finite plate with edge crack

The first numerical example that is solved is a rectangular finite plate with edge crack under a distributed load (figure 2). The load is $\sigma=1kpa$, and the other parameters are *L=52mm*, *b=10mm*, *a=4mm*. The parameters taken in the computation are Young's modulus $E=2\times10^5$ *MPa* and Poisson ratio $v=0.25$ and plane stress conditions are assumed.

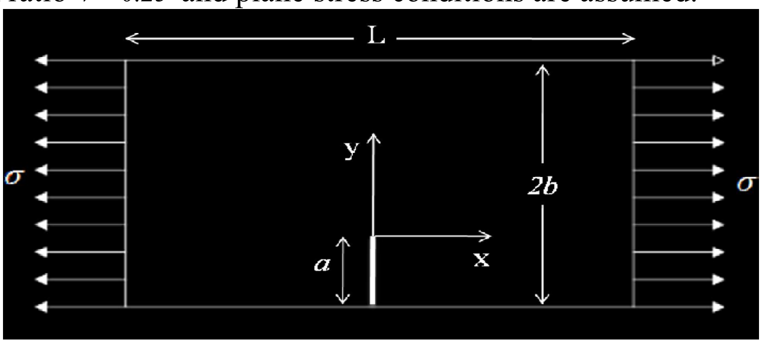


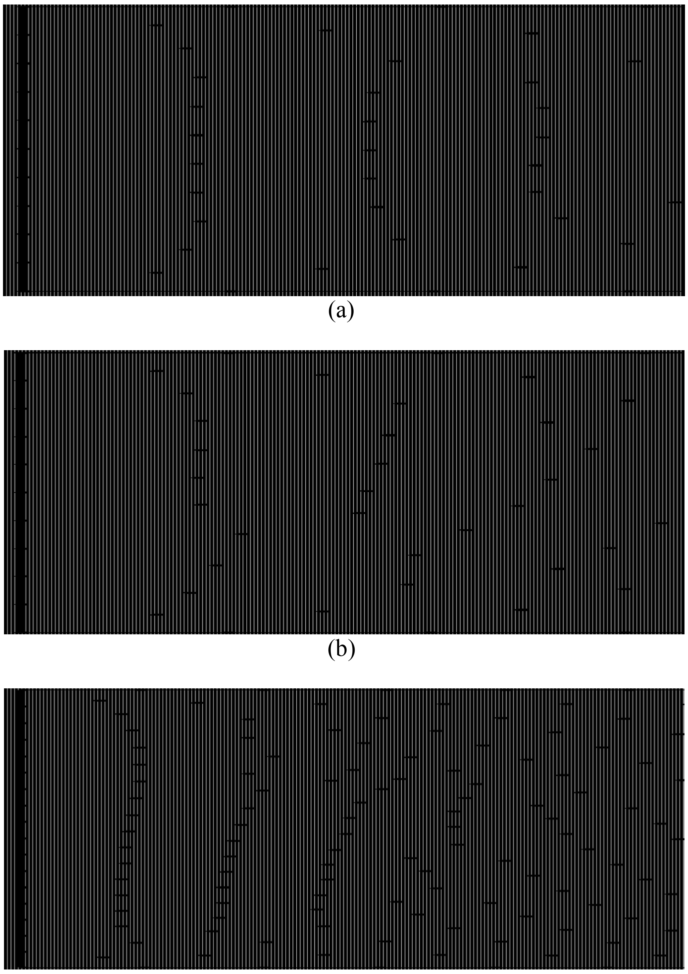
Figure 2: A rectangular plate with a edge crack under a distributed load

The boundary conditions are assigned as follow:

$$\begin{cases} on \;\; \text{x=26} \;\; : \quad \sigma_{xx} = 1kpa \\ on \;\; \text{x=-26} \;\; : \quad \sigma_{xx} = 1kpa \end{cases}$$

$$u_x(0,0)=0, \qquad u_x(0,16)=0, \qquad u_y(0,16)=0 \qquad (30)$$

The calculations are carried out using three different irregular nodal distributions, with 795, 1075, and 3166 nodes, see figure 3.



(a) (b) (c)

Figure 3: Three nodal distributions for modeling the plate with an edge crack: (a) 795 nodes; (b) 1075 nodes; (c) 3166 nodes.

The mode-I stress intensity factor K_I that used in equations (27) and (28), is calculate by

$$K_I = C\sigma\sqrt{\pi a} \qquad (31)$$

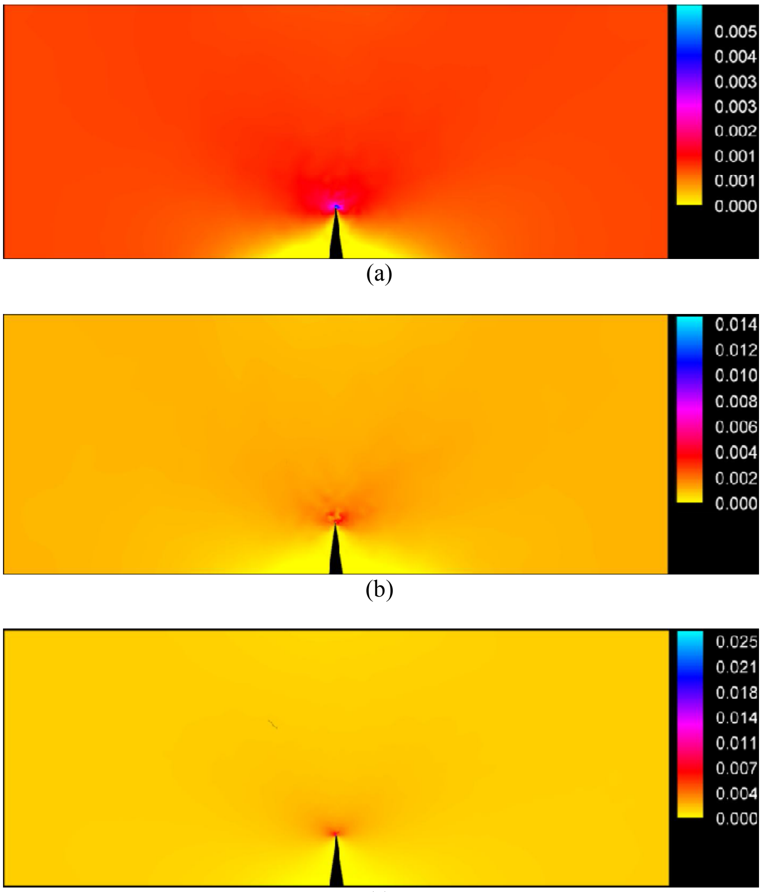
Where C is correction factor for the marginal effect that is given as following:

$$C = 1.12 - 0.231\left(\frac{a}{2b}\right) + 10.55\left(\frac{a}{2b}\right)^2 - 21.72\left(\frac{a}{2b}\right)^3 + 30.39\left(\frac{a}{2b}\right)^4 \tag{34}$$

Therefore, the exact value of stress intensity factor for this example is computed as the following value.

$$K_I = 4.85 kpa.(mm)^{1/2} \tag{33}$$

By solving system of equations (19), the stress field can be presented. The contours of stress field σ_{xx} for the three nodal configurations are shown in figure 4.



(c)

Figure 4: Contours of stress σ_{xx} for three nodal distributions (a) 795 nodes; (b) 1075 nodes; (c) 3166 nodes.

Also, by substituting K_I in relation (33) in equations (27) the stress field can be compute. In figure 5, stress field near crack tip for analytical and numerical solutions for three different irregular nodal distributions are compared.

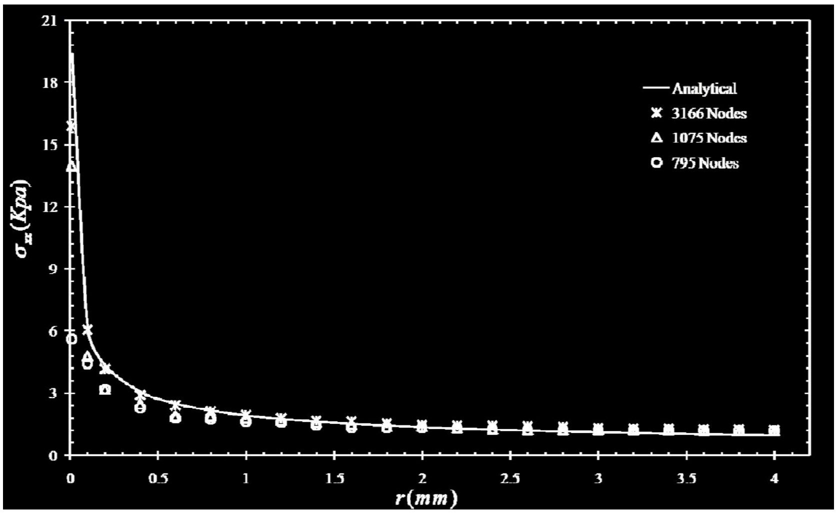


Figure 5: The stresses field σ_{xx} as a function of distance from the crack tip

Having the stress field near crack tip, numerical stress intensity factor using the following equation is obtained.

$$K_I = \lim_{|r| \to 0} \sigma_x \sqrt{2\pi r} \qquad (34)$$

Figure 6 presents the comparing of stress intensity factor near crack tip for analytical and numerical solutions for three different irregular nodal distributions.

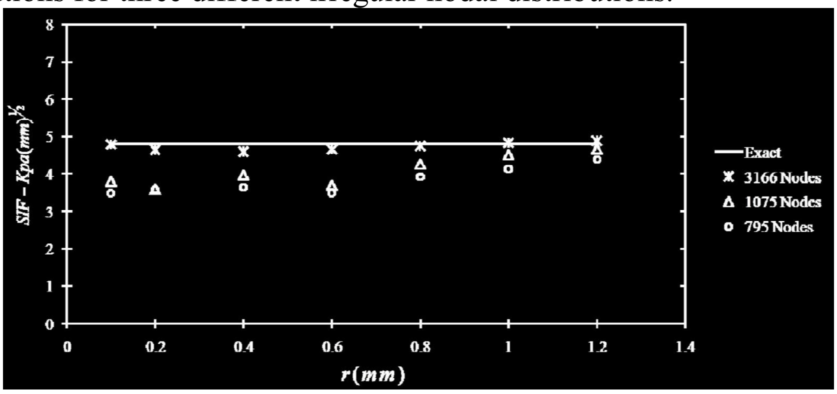


Figure 6: The Stress intensity factor as a function of distance from the crack tip

The convergence rate of the method for three nodal configurations is depicted in Figure 7. The results clearly show that a stable convergence rate is obtained for the present DLSM.

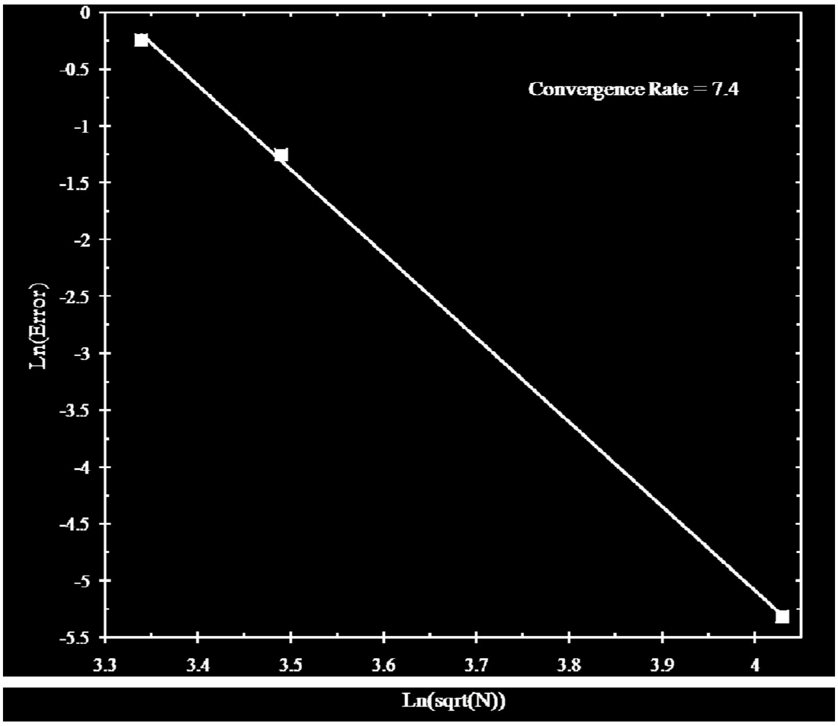


Figure 7: Rate of convergence for the plate with an edge crack

6 CONCLUSION

A true meshless approach, discrete least square (DLS) method, is extended in this paper, for solving two-dimensional crack problem. In the present DLSM, a structured mesh is not required and the problem domains is discretizated by distributed field nodes. The DLS method is based on the minimization of the least squares functional with respect to the nodal parameters. The least squares functional are formed as the weighted summation of the residual of the differential equation and its boundary conditions. The moving least-squares interpolant is employed to construct the trial functions. The displacement discontinuity due to a crack is modeled by the diffraction method. In diffraction method, if the domain of influence divided by discontinuity into two separate parts, the points that located behind the discontinuity are removed from influence domains. But near the crack tip, the domains of influence wrap around the crack tip. Since, some points at the back of the crack can be placed in domain of influence. In consequence, the continuous weight function and its ensued shape function have been created. The proposed method was applied to calculate stress-intensity factors and stress field in near crack tip in an example of two-dimensional cracked plate. Comparing the numerical results with available analytical solutions, revealed to high efficiency and excellent accuracy of presented method. The results clearly show that a stable convergence rate is obtained for the present DLSM method.

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