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SOLUTION OF THE STATIONARY STOKES AND NAVIER-STOKES EQUATIONS USING THE MODIFIED FINITE PARTICLE METHOD IN THE FRAMEWORK OF A LEAST SQUARES RESIDUAL METHOD

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Abstract. The present work is concerned with the solution of stationary Stokes and Navier-Stokes flows using the Modified Finite Particle Method for spatial derivative approximations and the Least Square Residual Method for the solution of the linear system deriving from the collocation procedure. The combination of such approaches permits to easily handle the numerical difficulty of the inf-sup conditions, without distinguishing between the discretizations of velocity and pressure fields.

The obtained results, both in the cases of linear and non-linear flows, show the robustness of the proposed algorithm

1 Introduction

The numerical simulation of incompressible fluid flows represents a challenge for many numerical methods due to the *inf-sup* condition [1], which imposes limitations in the discretization of velocity and pressure fields, and therefore in numerical methods some attention is needed in field discretization.

Among the existing numerical methods an important role has been played in recent years by meshless methods, which present many potentiality with respect to traditional mesh-based or grid-based methods. The main characteristic of meshless methods is, in fact, that nodes are not "rigidly" connected to each other, and therefore they can easily model large deformation and fluid-dynamics problems.

In the context of meshless methods the Radial Basis Functions (RBF) have been widely investigated from a theoretical point of view [2, 3] and applied to function approximation problems [4] and partial differential problems [5], in particular wave propagation [6] and fluid dynamics [7].

RBF collocation has also been used in combination with a Least Square Residual Method (LSRM), an algorithm for the solution of linear systems based on error minimization. Such an algorithm is used when the number of equations overcomes the number of unknowns. In the field of collocation methods for the solution of partial differential problems, such a methodology is needed when the number of collocation points is higher then the number of nodal unknowns. In [8] it is shown that the combination of a collocation method with LSRM corresponds to the weak formulation with numerical quadrature. In [9] the use of LSRM is extended to linear incompressible elasticity, and it is shown that the different equations have to be properly weighted in order to balance the different components of the error. An important characteristic of the combination between Radial Basis Functions and Least Square Residual Method is that in this way pressure does not undergo any spurious oscillation, even approximating velocity and pressure fields using the same discrete differential operators.

In the present paper we combine the Least Square Residual Method with the Modified Finite Particle Method (MFPM), a meshless method for spatial derivative approximation first introduced in [10] and applied to 1D elasticity static and dynamic problems, to 2D elliptic problems [11], to 2D elasticity and plasticity [12]. In [13] a novel formulation is introduced and applied to 2D dynamics and to 3D statics.

The present paper is organized as follows: in Section 2 we present a version of the Modified Finite Particle Method where collocation points are decoupled from points where variables are evaluated; then in Section 3 we recall the governing equations for incompressible fluid flows and present a solution procedure using the Least Square Residual Method. In Section 4 we solve the problem of a flow in a quarter of annulus under internal body loads; in Section 5 we recall the stationary Navier-Stokes equations and introduce a numerical procedure for handling the non-linearity, and then, in Section 6, we solve the famous benchmark of the lid-driven cavity problem. Finally, in Section 7, we draw some conclusions.

2 Modified Finite Particle Method

The Modified Finite Particle Method is a numerical procedure for function and derivative approximation introduced in [10]. Here we introduce a novel formulation, where equations are collocated in a set of points (collocation points) and variables are evaluated in a different set of points (control points).

The node distributions considered in the MFPM are referred to as $\mathbf{x} = [x \quad y]^T$ and $\boldsymbol{\xi} = [\xi \quad \eta]$; the first one is the set of collocation points, placed on the physical domain; the second one is the set of control points, that is, the points where variables are computed. In general, the node distribution $\boldsymbol{\xi}$ has no physical evidence, and therefore nodes can be placed in any convenient way, i.e., on a Cartesian equispaced grid. We remark that this choice does not affect the characteristic of the MFPM of being a meshless method, since collocation nodes can assume any position within the physical domain.

In a first stage, in order to show the numerical approximation procedure of a function $u = u(\mathbf{x})$ and its derivatives, we assume to know the nodal values of u in the control points $\boldsymbol{\xi}$. The first step of the numerical method is the evaluation of the Taylor series expansion of the function $u(\mathbf{x})$ in a control point $\boldsymbol{\xi}_j$. The Taylor series is centered in a collocation point $\mathbf{x}_i$ and is expanded until second order

$$\begin{aligned} u(\boldsymbol{\xi}_j) \ =& u(\mathbf{x}_i) + D_x u(\mathbf{x}_i)(\xi_j - x_i) + D_y u(\mathbf{x}_i)(\eta_j - y_i) + \frac{1}{2} D^2_{xx} u(\mathbf{x}_i)(\xi_j - x_i)^2 \\ &+ \frac{1}{2} D^2_{yy} u(\mathbf{x}_i)(\eta_j - y_i)^2 + D^2_{xy} u(\mathbf{x}_i)(\xi_j - x_i)(\eta_j - y_i) \end{aligned} \tag{1}$$

Equation (1) contains 6 unknown terms (function and derivative values in the collocation point $\mathbf{x}_i$) and hence 6 equations are needed to compute their value. Therefore Equation (1) is evaluated in N_i control nodes surrounding $\mathbf{x}_i$, that constitute a subset referred as X_i. We then introduce 6 *projection functions* $W^i_\alpha = W_\alpha(\boldsymbol{\xi} - \mathbf{x_i})$ and multiply the evaluations of Equation (1) in the nodes of X_i by the evaluations of W^i_α in the same points, and sum all terms, obtaining

$$\begin{aligned} & u_i \sum_j W^{ij}_\alpha + D_x u_i \sum_j (\xi_j - x_i) W^{ij}_\alpha + D_y u_i \sum_j (\eta_j - y_i) W^{ij}_\alpha + \\ & + \frac{1}{2} D^2_{xx} u_i \sum_j (\xi_j - x_i)^2 W^{ij}_\alpha + \frac{1}{2} D^2_{yy} u_i \sum_j (\eta_j - y_i)^2 W^{ij}_\alpha + \\ & + D^2_{xy} u_i \sum_j (\xi_j - x_i)(\eta_j - y_i) W^{ij}_\alpha = \sum_j u(\boldsymbol{\xi}_j) W^{ij}_\alpha \qquad \alpha = 1, ..., 6 \end{aligned} \tag{2}$$

where $W^{ij}_\alpha = W_\alpha(\boldsymbol{\xi}_j - \mathbf{x_i})$. Equation (2) can be rearranged in matrix form as

$$\mathbf{A}^i \begin{pmatrix} u(\mathbf{x}_i) \\ D_x u(\mathbf{x}_i) \\ D_y u(\mathbf{x}_i) \\ D^2_{xx} u(\mathbf{x}_i) \\ D^2_{yy} u(\mathbf{x}_i) \\ D^2_{xy} u(\mathbf{x}_i) \end{pmatrix} = \begin{pmatrix} \sum_j u(\boldsymbol{\xi}_j) W^{ij}_1 \\ \sum_j u(\boldsymbol{\xi}_j) W^{ij}_2 \\ \sum_j u(\boldsymbol{\xi}_j) W^{ij}_3 \\ \sum_j u(\boldsymbol{\xi}_j) W^{ij}_4 \\ \sum_j u(\boldsymbol{\xi}_j) W^{ij}_5 \\ \sum_j u(\boldsymbol{\xi}_j) W^{ij}_6 \end{pmatrix} \tag{3}$$

Equation (3) is then rewritten in the form

$$\mathbf{A}^i \mathbf{D}(u_i) = \mathbf{C}^i \mathbf{u} \tag{4}$$

where

$$\mathbf{C}^i = [\mathbf{W}^{i1} \quad | \quad \mathbf{W}^{i2} \quad | \quad ... \quad | \quad \mathbf{W}^{iN_i}] \tag{5}$$

and

$$\mathbf{W}^{ij} = [W^{ij}_1 \quad | \quad W^{ij}_2 \quad | \quad ... \quad | \quad W^{ij}_6]^T \tag{6}$$

The vector $\mathbf{u}$ collects the known nodal values in the node set $\boldsymbol{\xi}$. Then, by inverting (4), we obtain

$$\mathbf{D}(u_i) = \mathbf{E}^i \mathbf{C}^i \mathbf{u} \tag{7}$$

where $\mathbf{E}^i = (\mathbf{A}^i)^{-1}$, and finally

$$\mathbf{D}(u_i) = \mathbb{D}^i \mathbf{u} \tag{8}$$

with

$$\mathbb{D}^i = \mathbf{E}^i \mathbf{C}^i \tag{9}$$

The 6xN_i operator $\mathbb{D}^i$, applied to $\mathbf{u}$, gives back a 6x1 vector collecting all the approximations of functions and derivatives of $u(\mathbf{x})$ in the collocation point $\mathbf{x}_i$. We are interested in building 6 linear operators ($\mathbb{I}$, $\mathbf{D}_x$, $\mathbf{D}_y$, $\mathbf{D}_{xx}$, $\mathbf{D}_{yy}$, $\mathbf{D}_{xy}$) that, applied to the vector $\mathbf{u}$, return the evaluation of function and derivatives in all collocation points $\mathbf{x}$. These operators are simply built collecting, for each i, the correct row of $\mathbb{D}^i$, where the correct row is identified through Equation (3).

For example, in order to build the linear operator $\mathbf{D}_x$ (the discrete counterpart of $\partial/\partial x$), we simply consider, for each i, the 2-nd row of $\mathbb{D}^i$. The final form of $\mathbf{D}_x$ is then

$$\mathbf{D}_x = \begin{bmatrix} \mathbb{D}_2^1 \\ \mathbb{D}_2^2 \\ \dots \\ \mathbb{D}_2^N \end{bmatrix} \tag{10}$$

where $\mathbb{D}_\alpha^i$ is the α-th row of $\mathbb{D}^i$.

3 Governing equations for incompressible flows

The governing equations of incompressible fluid flows are the well known Navier-Stokes equations

$$\begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{b} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{11}$$

where the first equation is the dynamic equilibrium equation and the second equation is the incompressibility constraint. The variable ρ is the fluid density, $\mathbf{u}$ is the velocity field, p is the pressure field, μ is the dynamic viscosity, $\mathbf{b}$ is the vector of internal loads.

Navier-Stokes equations are non-linear equations, due to the presence of the convective term $\mathbf{u} \cdot \nabla \mathbf{u}$. Nevertheless, when inertial forces are negligible with respect to viscous forces, Navier-Stokes Equations can be simplified in the form

$$\begin{cases} \rho \dfrac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu \Delta \mathbf{u} + \mathbf{b} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \tag{12}$$

known as Stokes equations. Systems (11) and (12) have to be completed with suitable boundary conditions, that can be imposed on velocity or on the outward stress.

In the present work we restrict to stationary flows, that is, $\partial \mathbf{u}/\partial t = 0$. Moreover, the work is divided in two parts: in the first part we concentrate on Stokes Equations, in order to study how the Modified Finite Particle Method, in combination with the Least Square Residual Method, deals with the inf-sup condition; in the second part we focus on the solution of the complete Navier-Stokes equations, and show a numerical procedure to handle the non linear term.

3.1 Solution of the Stokes equations

In the spirit of collocation methods the steady Stokes equations are discretized using the Modified Finite Particle Method. The discrete linear system of equations is

$$\begin{bmatrix} \mu\Delta & 0 & -\mathbf{D}_x \\ 0 & \mu\Delta & -\mathbf{D}_y \\ \mathbf{D}_x & \mathbf{D}_y & 0 \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}}_x \\ \hat{\mathbf{f}}_y \\ \mathbf{0} \end{pmatrix} \tag{13}$$

where $\Delta = \mathbf{D}_{xx}+\mathbf{D}_{yy}$ is the discrete Laplace operator, $\hat{\mathbf{u}}$, $\hat{\mathbf{v}}$, and $\hat{\mathbf{p}}$ are the nodal unknowns associated to the velocity components u and v and to the pressure p, and $\hat{\mathbf{f}}_x$ and $\hat{\mathbf{f}}_y$ are the components of the internal loads at collocation points.

The required boundary conditions are imposed as linear algebraic equations on boundary collocation points. No boundary conditions are required on the incompressibility constraint, since incompressibility is imposed both on the interior and on the boundary of the domain [14]. We remark that this is not common in collocation methods, especially with collocated grids, where particular boundary condition for the incompressibility constraint are imposed [15, 16].

When collocation and control points coincide, the values of the control unknowns can be simply retrieved by inverting system (13). Unfortunately this algorithm is not valid with Stokes equations, due to the non-compliance of the discrete system (13) with the discrete inf-sup condition.

In order to overcome such a difficulty, we recur to a Least Square Residual Method, that is, we use a number of collocation points higher than the number of control nodes, and therefore System (13) is a rectangular, overdetermined system of equations, which can be solved through an error minimization. The global squared error, defined as

$$E = \|e\|^2 = \sum_i (K_{ij}d_j - f_i)^2 = (\mathbf{Kd} - \mathbf{f})^T(\mathbf{Kd} - \mathbf{f}) \tag{14}$$

can be rewritten as the sum of four contributions: (i) the error E_{eq} associated to equilibrium equations; (ii) the error E_{inc} associated to the discretized incompressibility constraints; (iii) the error E_{dir} associated to Dirichlet boundary conditions; (iv) the error E_{neum} associated to Neumann boundary conditions. The total error is therefore written as

$$E = E_{eq} + E_{inc} + E_{dir} + E_{neum} \tag{15}$$

As suggested in [9] the components of such an error have to be properly weighted, since they contribute differently to the global error. For this reason we introduce a different definition of the error, referred as E_w, with expression

$$E_w = E_{eq} + \alpha_{inc} E_{inc} + \alpha_{dir} E_{dir} + \alpha_{neum} E_{neum} \tag{16}$$

where α_{inc}, α_{dir}, α_{neum} are weights associated to discrete incompressibility conditions, Dirichlet boundary conditions, and Neumann boundary conditions respectively.

Equation (16) can be rewritten as

$$E_w = \|\mathbf{K}_{eq}\mathbf{d} - f_{eq}\|^2 + \alpha_{inc}\|\mathbf{K}_{inc}\mathbf{d}\|^2 + \alpha_{dir}\|\mathbf{K}_{dir}\mathbf{d} - \bar{\mathbf{u}}\|^2 + \alpha_{neum}\|\mathbf{K}_{neum}\mathbf{d} - \bar{\mathbf{t}}\|^2 \tag{17}$$

where $\mathbf{K}_{eq}$, $\mathbf{K}_{inc}$, $\mathbf{K}_{dir}$, and $\mathbf{K}_{neum}$ are the rows of the stiffness matrix $\mathbf{K}$ associated to the equilibrium equations, incompressibility constraints, Dirichlet boundary conditions and Neumann boundary conditions. The vector $\bar{\mathbf{u}}$ is the known nodal velocity at Dirichlet boundary, $\bar{\mathbf{t}}$ is the known nodal outward stress at Neumann boundary.

The error (17) can be finally rewritten in compact form as

$$E_w = (\mathbf{Kd} - \mathbf{f})^T \mathbb{A} (\mathbf{Kd} - \mathbf{f}) \tag{18}$$

where $\mathbb{A}$ is a diagonal matrix collecting the weights α associated to the different equations of system (13). Finally we minimize the error E_w with respect to the nodal unknowns $\mathbf{d}$, obtaining the linear system

$$\mathbf{K}^T \mathbb{A} \mathbf{Kd} = \mathbf{K}^T \mathbb{A} \mathbf{f} \tag{19}$$

that is a linear, symmetric system of equations, and can be solved using dedicated algorithms for symmetric matrix inversion. Nodal unknowns are therefore computed inverting System (19).

3.2 Choice of the weights

The choice of the weights to be assigned to the equations of system (13) is an important topic for the application of LSRM, since a wrong definition of weights may prevent the convergence of the numerical method.

The rigorous analysis conducted by [9] which takes in account the particular choice of the shape functions (in this case, Radial Basis Functions are used) leads to the choice of the following weights

$$\alpha_{inc} = (\mu N_s)^2 \qquad \alpha_{dir} = (\mu N_s)^2 \qquad \alpha_{neum} = 1 \tag{20}$$

In the present work, we prefer a different approach in order to define the set of weights to adopt: in fact we base our analysis on the consideration that different equations have different scales, in particular:

1. The equation of equilibrium has the dimensions of $\mu \partial u / \partial x^2$
2. The equation of incompressibility has the dimensions of $\partial u / \partial x$

3. The Dirichlet boundary conditions have the dimensions of u
4. The Neumann boundary conditions have the dimensions of $\mu \partial u / \partial x$

The second derivatives scale with $1/h^2$, where h is the distance between the control nodes; the first derivatives scale with $1/h$. The distance between particles is related to the total number of control nodes N_s, in particular we can assume $h \simeq 1/\sqrt{N_s}$. With these considerations, we can write the correct scaling of each equation in the form:

1. Equilibrium: $o(\mu/h^2) = o(\mu N_s)$
2. Incompressibility constraint: $o(1/h) = o(\sqrt{N_s})$
3. Dirichlet boundary conditions: $o(1)$
4. Neumann boundary conditions: $o(\mu/h) = o(\mu\sqrt{N_s})$

In order to balance the squared error in Equation (16), all components are requested to have at least the same dimensions, that are the ones of the squared equation of equilibrium, $(\mu N_s)^2$. The other weights, following this principle, are:

$$\alpha_{inc} = C_1 N_s \qquad \alpha_{dir} = C_2 N_s^2 \qquad \alpha_{neum} = C_3 N_s \tag{21}$$

where C_1, C_2, and C_3 are constants that, in a first approximation, we can consider unitary.

4 Solution of the Stokes problem on a quarter of annulus under body loads

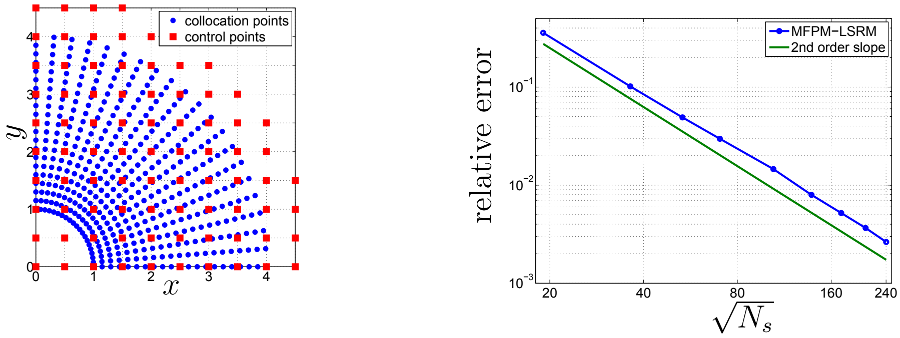
In the following we solve the problem of a Stokes flow in a quarter of annulus, clamped an all its boundary, under a polynomial body load. The problem has been studied in [17] using a stream line formulation and isogeometric analysis for the spatial discretization, exploiting the high regularity of isogeometric shape-functions, and also the possibility of reproducing exactly the geometry of the domain. The analytical solution of this problem is

$$\begin{cases} u = 10^{-6}x^2y^4(x^2+y^2-16)(x^2+y^2-1)(5x^4+18x^2y^2-85x^2+13y^4+80-153y^2) \\ v = -2\cdot 10^{-6}xy^5(x^2+y^2-16)(x^2+y^2-1)(5x^4-51x^2+6x^2y^2-17y^2+16+y^4) \end{cases} \tag{22}$$

The internal body loads are obtained using the manufactured solution (22).

In Figure 1(a) we show an example of distribution of collocation points and field nodes. In particular we underline the fact that collocation points are distributed in the physical domain, whereas control points are distributed on a regular, Cartesian grid.

For this problem the expected second-order accuracy is achieved, as shown in Figure 1(b)we remark that on the horizontal axis of Figure 1(b) we plot the square root of the number of control nodes used for the approximation, since they are directly proportional to the needed computational effort.



(a) Distribution of 441 collocation points and 83 control nodes

(b) Convergence diagram of the error using MFPM and LSRM

Figure 1: Quarter of annulus under body loads

A numerical test has been performed in order to understand what is the best ratio between the number of field nodes and the number of collocation points. In particular, for each amount N_s of control nodes, different analyses have been performed varying the number of collocation nodes. The results are shown in Figure 2 where we notice that, as expected, on the right side of the diagram (that is, when the number of collocation points equals the number of field nodes) the error is great because of the pressure instability. However, for most of the field node densities, a minimum of the error can be appreciated around the value of $\sqrt{N_s/N_c} = 0.5$.

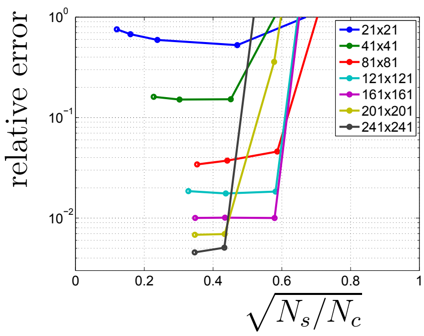


Figure 2: Quarter of annulus under body loads: relative error versus the ratio between the number of field nodes and collocation points

5 Stationary Navier-Stokes Equations

In the present section we recall the stationary Navier-Stokes Equations, obtained from Equation (11) cancelling the term $\rho\partial\mathbf{u}/\partial t$.

$$\begin{cases} \rho\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\mu\Delta\mathbf{u}+\mathbf{b} \\ \nabla\cdot\mathbf{u}=0 \end{cases} \tag{23}$$

The main difficulty of Equations (23), in addition to the need of satisfying the inf-sup condition, is the handling of the non linear term, which complicates the solution algorithm. A strategy for its solution is the linearization according to the Newton-Raphson algorithm. Unfortunately, despite the fast rate of convergence of this method, convergence is achieved when suitable initial guess solution is given. A guess solution for the Newton-Raphson algorithm is suitable when it is comprised in a bubble with radius is proportional to $1/Re$, where Re is the Reynold numbers, defined as the ratio between inertial and viscous forces. This means that the solution of the complete Navier-Stokes equations is more and more difficult when approaching problems with high Reynolds numbers.

A strategy to reduce this difficulty is the use of the Picard linearization, which consists in a "partial" linearization of the convective terms. This strategy exhibits a sub-optimal rate of convergence to the solution, the bubble for the initial guess solution is larger and it is more easily found (see [18] for details).

In both cases of Picard or Newton-Raphson linearization, the discrete problem can be rewritten in the form

$$\mathbf{K}^k\Delta\hat{\mathbf{d}}^k=\mathbf{R}^k \tag{24}$$

where $\mathbf{K}^k$ is the tangent (in the case of Newton Raphson) or secant (in the case of Picard) stiffness matrix, that is suitably assembled using the MFPM discrete differential operators; $\Delta\hat{\mathbf{d}}^k=[\Delta\hat{\mathbf{u}}\quad\Delta\hat{\mathbf{v}}\quad\Delta\hat{\mathbf{p}}]^T$ is the vector of the nodal increment, and $\mathbf{R}^k$ is the residual of the discretized Navier-Stokes system. In all cases, the superscript k reminds that we are performing an iterative process, and therefore k is the iteration number.

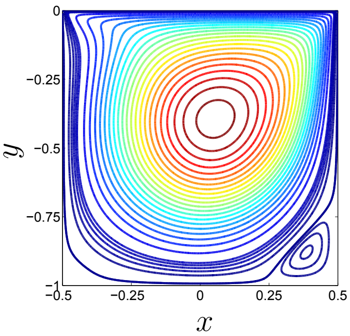
We again discretize equations using the MFPM and adopting $N_c>N_S$, therefore at each iteration we obtain a rectangular, overdetermined system, that again can be solved using the LSRM procedure. Therefore, at each iteration a weighted error is computed, with expression

$$\|e^2\|=(\mathbf{K}^k\Delta\hat{\mathbf{d}}^k-\mathbf{R}^k)^T\mathbb{A}^k(\mathbf{K}^k\Delta\hat{\mathbf{d}}^k-\mathbf{R}^k) \tag{25}$$

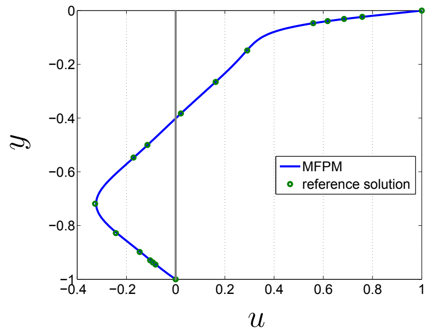
and then we minimize with respect to the control variable increment $\Delta\hat{\mathbf{d}}^k$

$$(\mathbf{K}^k)^T\mathbb{A}^k\mathbf{K}^k\Delta\hat{\mathbf{d}}^k=(\mathbf{K}^k)^T\mathbb{A}^k\mathbf{R}^k \tag{26}$$

The main difference with the linear case is that now we are weighting residual equations, so the dimensions of each equation is not depending on the differential operators, or on



(a) Streamlines solution using MFPM and LSRM



(b) Velocity profile in the x-direction along the axis $x = 0$ and comparison with the reference solution by [19].

Figure 3: Lid-driven cavity problem ($Re = 400$)

the material properties; therefore we propose, as weights:

$$\alpha_i^k = \left(\frac{3N_i}{\sum_{j=1}^{N_i} K_{ij}^k} \right)^2 \tag{27}$$

in order all equations to be of the same order of magnitude. N_i is the number of supporting nodes of the collocation points to which the i-th row of $\mathbf{K}^k$ is referred.

Once the weights have been established, we only have to solve Equation (26), update the solution, and then repeat until the norm of the weighted residual is less then a pre-established tolerance.

6 Solution of the Navier-Stokes Equations on the lid-driven cavity flow

In the following we apply what we have discussed in the previous section to the problem of the lid-driven cavity flow. The domain of the problem is a square of side $L = 1m$. On all boundary, Dirichlet boundary condition are assigned, in particular all sides have no velocity, while the top side is given a tangential velocity $U = 1m/s$.

We solved this problem using $\mu = 1/400 kg/m\, s$ and $\rho = 1kg/m^3$, obtaining a Reynolds number $Re = \rho LU/\mu = 400$. The results, in terms of stream-lines, are shown in Figure 3(a). In absence of analytical solution, we compare the results obtained in the present work with those published in [19] for the same Reynolds number in terms of horizontal velocity profile at the middle vertical axis (Figure 3(b)) and notice a substantial agreement between our results and the reference solution.

7 Conclusion

In the present work, we introduce an extension of the Modified Finite Particle Method that allows the decoupling between collocation points and control nodes. With this modification, the method is suitable for a Least Squares Residual Method. The present formulation, in combination with the Least Square Residual Method, shows some advantages with respect to other collocation methods: the matrix to invert in order to find the discrete solution of partial differential problems is symmetric, allowing the use of appropriate and faster solvers; the quality of the solution is minimally dependent on the distribution of collocation points, and also extremely random node distributions are permitted. This characteristic, in particular, permits the solution also on more complicated geometries, such as a quarter of annulus, also showing a correct second-order accuracy, when proper weights are assigned to the different equations. A discussion about weights is conducted, as well as the best possible ratio to impose between the number of collocation points and control nodes.

In the final part of the paper we attack the solution of non-linear Navier-Stokes equations and extend the use of LSRM also to non-linear problems. The comparison between the solution found in the present paper and the one given in the reference literature show good agreement, confirming the robustness of the present methodology.

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