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HIGH-ORDER NUMERICAL SCHEME FOR VORTEX SHEET APPROXIMATION IN VORTEX METHODS FOR 2D FLOW SIMULATION

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Abstract. The problem of high-order numerical schemes developing is considered for solving of boundary integral equation for vortex sheet intensity, which arises in 2D meshless vortex particle methods. Curvilinearity of the airfoil surface line is taken into account: its shape is approximated by Hermite cubic spline interpolation instead of rectilinear panels which are usually used in vortex methods. Coefficients of system of linear algebraic equations are represented by define integrals of known functions. For some coefficients (which can not be calculated numerically) approximate analytic formulae are obtained, the other coefficients can be computed using 4 point Gaussian integration. The developed numerical scheme provides 4-th order of accuracy for average values of vortex sheet intensities over the panels.

1 INTRODUCTION

Vortex method [1, 2, 3, 4] is well-known meshless lagrangian particle method, which allows to solve efficiently number of actual engineering problems. The main idea of vortex method is that vorticity is considered as a primary computed variable. The airfoil influence is replaced by a vortex sheet placed on the airfoil surface line. Its intensity can be found from boundary condition on the airfoil surface line. The modification of vortex method is used which corresponds to the integral equation of the 2-nd kind with bounded kernel (for smooth airfoils) [5, 6]. When solving this equation in vortex methods airfoil is usually approximated with a polygon consisting of N rectilinear segments (panels) and solution of boundary integral equation is assumed to be piecewise-constant or piecewise-linear function. Numerical experiments show that in some cases such approach doesn't allow to obtain high order of accuracy, this fact has a negative effect on the whole problem solution, especially for significantly nonuniform mashes of the airfoil surface line. This problem can be solved by improving of airfoil surface line discretization, i.e. by taking into account its curvilinearity.

The aim of this paper is to develop new numerical scheme for boundary integral equation approximation, which provides high order of accuracy.

2 The governing equations

The problem of two-dimensional incompressible flow simulation around rigid immovable airfoil is considered. In vortex method at every time step two sub-problems should be solved:

- vortex sheet intensity computation on the surface line of the airfoil;
- vorticity motion simulation in the flow.

The airfoil in the flow can be replaced with vortex sheet on its surface line, because the vortex sheet with appropriate intensity distribution influences the flow exactly in the same way as the airfoil itself [8, 9].

So the unknown intensity distribution $\gamma(\boldsymbol{r})$ can be found from the solution of some boundary integral equation which corresponds to no-slip (for viscous flow) or no-through (for inviscid media) boundary condition.

Two different approaches can be distinguished to derivation of such boundary integral equation [5], and that one which is based on the equality to zero of the tangent component of velocity limit value on the surface line seems to be most efficient. It leads to the Fredholm-type boundary integral equation of the second kind:

$$\frac{1}{2\pi}\oint_K Q_\tau(\boldsymbol{r},\boldsymbol{\xi})\gamma(\boldsymbol{\xi})dl_\xi - \frac{\gamma(\boldsymbol{r})}{2} = f_\tau(\boldsymbol{r}), \quad \boldsymbol{r}\in K. \tag{1}$$

Its kernel is uniformly bounded function for C^2-smooth curves (*i.e.*, when parametric equations of the surface line belong to C^2 class with respect to arc length) and it has the following form:

$$Q_\tau(\boldsymbol{r},\boldsymbol{\xi}) = \frac{\boldsymbol{k}\times(\boldsymbol{r}-\boldsymbol{\xi})}{|\boldsymbol{r}-\boldsymbol{\xi}|^2}\cdot\boldsymbol{\tau}(\boldsymbol{r}),$$

where $\boldsymbol{\tau}(\boldsymbol{r})$ is tangent unit vector.

It should be noted, that the alternative approach which is based on zero value of the normal components of flow velocity on the surface line [5], leads to singular integral equation of the 1-st kind with Hilbert-type kernel, so special numerical procedures should be applied in order to compute the principle value of the corresponding integral [7].

Right-hand side $f_\tau(\boldsymbol{r})$ of the equation (1) in the considered case is known function, which depends on the shape of the airfoil, incident flow velocity and vorticity distribution in the flow. In the most general case it also depends on the surface line velocity, which is unknown function in coupled hydroelastic problems, when the surface line of the airfoil moves under hydrodynamic loads. If the motion and/or deformation low of the airfoil is known, the right-hand side $f_\tau(\boldsymbol{r})$ again becomes known function.

In order to take into account non-zero velocity of the airfoil surface line, the attached source sheet and attached vortex sheets should be introduced [2, 10]. All the approaches and formulae being discussed hereinafter can be easily generalized for such case. The arbitrary vortity distribution in the flow domain also can be considered, but for simplicity we assume that the is no vorticity in the flow domain at all, in this case the right-hand side $f_\tau(\boldsymbol{r})$ has the simplest form:

$$f_\tau(\boldsymbol{r}) = -\boldsymbol{V}_\infty \cdot \boldsymbol{\tau}(\boldsymbol{r}), \tag{2}$$

where $\boldsymbol{V}_\infty$ is incident flow velocity.

The equation (1) has infinite set of solutions, which follows from the fact that the corresponding integral operator has the eigenvalue $\lambda = 0$. In particular, for circular airfoil $\lambda = 0$ corresponds to the eigenfunction which is equal to constant.

In order to select the unique solution, which is physically plausible, the most convenient way is to add an extra equation for the total value of vorticity, which the vortex sheet contains, *i.e.*, the integral of the solution should be equal to the given value:

$$\oint_K \gamma(\boldsymbol{\xi}) dl_\xi = \Gamma, \tag{3}$$

The value Γ is usually known from the physical sense of the considered problem, in particular, for unsteady flow simulation its value follows from the Helmholtz theorems and their analogues for viscous flows.

3 Numerical schemes on the surface line of the airfoil

3.1 Airfoil discretization by using rectilinear panels

The simplest way to approximate the shape of the airfoil is to replace it with polygon. The legs of the polygon are usually called "panels". Vortex sheet intensity distribution over the panels is normally assumed to be piecewise-constant or piecewise-linear.

The most accurate schemes with piecewise-constant vorticity distribution provide the 2-nd order of accuracy for average values of solution (vortex sheet intensity) over the panels, and the 1-st order of accuracy in L_1 norm. The schemes with piecewise-linear vorticity distribution just a little more accurate with respect to average values of solution over the panels, but they provide the 2-nd order of accuracy in L_1 norm.

It should be noted, that the accuracy becomes much lower when the lengthes of neighboring panels differ significantly. From the mathematical point of view it can be explained by the following way: instead of smooth part of the curve in numerical simulation we simulate, in fact, flow around the airfoil with angle points (fig. 1). In proximity to such point the kernel of the integral equation (1) becomes unbounded, and the exact solution for such airfoil shape can has "unphysical" weak singularity in angle points.

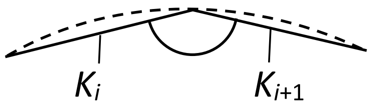

Figure 1: The part of airfoil surface and two rectilinear panels

In case of uniform splitting of the surface line of the airfoil into panels the mentioned singularities of the kernel, which arise at the endings of the panels, compensate each other. However, if neighboring panels lengthes differ appreciably, these singularities influences are significant and it leads to the errors of the coefficients computation of the resulting system of linear algebraic equations.

So, when the discretization of the surface line of the airfoil is uniform or close to uniform, the most accurate results for equations (1), (3) numerical solution, as the rule, can be obtained when the following approach is being used, which is similar to Discontinuous Galerkin (DG) [11] method. The approximate numerical solution is assumed to be linear combination of basis functions (constant and linear ones; it is useful to provide zero average value for linear basis function, which means that is orthogonal to the constant basis function). The unknown coefficients are found from solution of the linear system which corresponds to the residual orthogonality to basis functions. Naturally, when the described approach is being implemented, the numerical solution has discontinuities (jumps) between the panels.

3.2 Airfoil discretization taking into account curvilinearity of the panels

In [12] numerical scheme is developed which permits to take into account explicitly the curvilinearity of the surface line of the airfoil approximate it by smooth curve of spline-type or Hermite spline type. It provides the 4-th order of accuracy for the solution average values over the curvilinear panels, but its computational cost is extremely high, so it hardly can be implemented in practical applications.

In the present paper simplified numerical scheme is considered which provides the same (4-th) order of accuracy for approximate solution of boundary integral equation.

Let's denote the vertices of the airfoil as $\boldsymbol{C}_i$, $i = 1, \ldots, N$, rectilinear lines C_iC_{i+1} we call "chords" of the airfoil. As in [12], the surface line of the airfoil over the i-th chord is being approximated by smooth curve, that has the following parametric equation:

$$\boldsymbol{r}_i(t) = \boldsymbol{C}_i + L_i^0 \left(t\, \boldsymbol{\tau}_i^0 + p_i(t)\, \boldsymbol{n}_i^0 \right), \quad t \in [0;\, 1]. \tag{4}$$

Parameter values $t = 0$ and $t = 1$ correspond to the beginning and ending of the panel; L_i^0 is length of the i-th chord; $\boldsymbol{\tau}_i^0$ and $\boldsymbol{n}_i^0$ are tangent and normal unit vectors for the i-th chord. Function $p_i(t)$ is cubic polynomial:

$$p_i(t) = \alpha_i\, t(t-1) + \beta_i\, t(t-1/2)(t-1), \tag{5}$$

where the coefficients α_i and β_i can be found from the following condition: the directions of tangent vector should be the same for exact shape of the airfoil and its approximate representation (fig. 2). They can be written down in explicit form.

$$\alpha_i = -\frac{1}{2}(\tan\varphi_i + \tan\psi_i), \quad \beta_i = (\tan\varphi_i - \tan\psi_i).$$

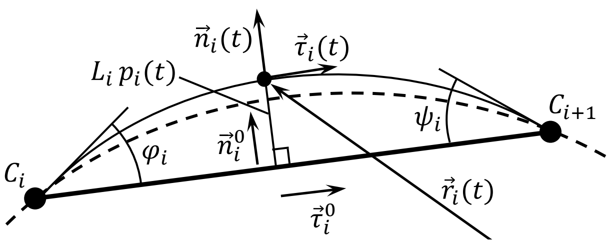

Figure 2: Curvilinear panel and the i-th chord of the airfoil

It should be noted, that if the curvilinear part of the airfoil surface line is represented over the i-th panel as smooth curve, that belongs to C^4 class, the error of the airfoil shape approximation with Hermite spline has order of $O(L_i^4)$.

3.3 Integral equation approximation

In order to construct numerical scheme for equations (1), (3), the principal ideas of Discontinuous Galerkin (DG) method seems to be useful. We introduce constant and linear basis functions $\varphi_0^i(t)$ and $\varphi_1^i(t)$, $i = 1, \ldots, N$, each of these functions differs from zero only on the i-th panel:

$$\varphi_0^i(t) \equiv 1, \qquad \varphi_1^i(t) = t - \frac{1}{2}, \qquad t \in [0;\, 1].$$

The solution over the every panel of the airfoil is considered to be linear distribution with respect to parameter t:

$$\gamma\big(\boldsymbol{r}_i(t)\big) = \gamma_i\varphi_0^i(t) + \delta_i\varphi_1^i(t), \qquad t \in [0;\, 1], \qquad i = 1, \ldots, N.$$

The unknown coefficients γ_i and δ_i can be found from orthogonality condition. The residual of the equation (1) on the i-th panel, which has the following form:

$$z_i(t) = \sum_{j=1}^{N}\left(\gamma_j \int_0^1 Q_\tau\big(\boldsymbol{r}_i(t),\, \boldsymbol{r}_j(\xi)\big)\,\varphi_0^j(\xi)\, J_j(\xi) d\xi \;+\right.$$
$$\left. +\, \delta_j \int_0^1 Q_\tau\big(\boldsymbol{r}_i(t),\, \boldsymbol{r}_j(\xi)\big)\varphi_1^j(\xi)\, J_j(\xi)\, d\xi\right) -$$
$$- \frac{\gamma_i\varphi_0^j(t) + \delta_i\varphi_1^j(t)}{2} - f_\tau\big(\boldsymbol{r}_i(t)\big), \quad i = 1, \ldots, N,$$

should be orthogonal to projection functions, which are chosen to be equal to basis functions $\varphi_0^i(t)$ and $\varphi_1^i(t)$.

Here

$$J_j(\xi) = \left\| \frac{d\boldsymbol{r}_j(\xi)}{d\xi} \right\| = L_j^0 \sqrt{1 + \left(\alpha_j (2\xi - 1) + \beta_j \left(3\xi^2 - 3\xi + \frac{1}{2} \right) \right)^2}$$

is the Jacobian, which is calculated according to the approximation (4), (5).

Discrete analogue of the equation (3) has the following form:

$$\sum_{j=1}^{N} \Bigg(\gamma_j \underbrace{\int_0^1 \varphi_0^j(\xi)\, J_j(\xi) d\xi}_{L_j} + \delta_j \underbrace{\int_0^1 \varphi_1^j(\xi)\, J_j(\xi) d\xi}_{L_j^*} \Bigg) = \Gamma.$$

As the result, we obtain linear system of algebraic equations, which is overdetermined. In order to regularize it, an additional variable R is introduced nearly in the same way as in [7]. The resulting system, being split into blocks, has the form

$$\underbrace{\begin{pmatrix} A^{00} + D^{00} & A^{01} + D^{01} & I_N \\ A^{10} + D^{10} & A^{11} + D^{11} & O_N \\ L & L^* & 0 \end{pmatrix}}_{M} \begin{pmatrix} \gamma \\ \delta \\ R \end{pmatrix} = \begin{pmatrix} b^0 \\ b^1 \\ \Gamma \end{pmatrix}.$$

Here blocks A^{pq} are $N \times N$ square matrices; D^{pq} are diagonal matrices, $p, q = 0, 1$; L and L^* are rows consist of panel lengthes L_i and coefficients L_i^*, respectively; I_N is column consists of ones; O_N is column consists of zeros; γ and δ are columns of unknown coefficients; b^0 and b^1 are the columns which form the right-hand side of the system.

The coefficients of the matrices A^{pq} and D^{pq} should be computed by using the following formulae:

$$A_{ij}^{pq} = \frac{1}{2\pi} \int_0^1 \left(\int_0^1 Q_\tau(\boldsymbol{r}_i(t),\, \boldsymbol{r}_j(\xi)) \varphi_q^j(\xi) J_j(\xi) d\xi \right) \varphi_p^i(t) J_i(t) dt,$$

$$D^{pq} = \operatorname{diag} \left\{ -\frac{1}{2\pi} \int_0^1 J_i(t) \left(\varphi_1^i(t) \right)^{p+q} dt \right\}, \quad p,\, q = 0,\, 1, \quad i,\, j = 1,\, \ldots,\, N.$$

The easiest way for non-diagonal coefficients computation for matrices A^{pq}, which provides the necessary accuracy, is numerical integration using Gaussian quadrature formulae with 4 integration points.

Note, that the computational complexity of the procedure of the integrals numerical calculation can be reduced significantly. In order to take into account the mutual influence of the panels which are sufficiently far one from the other, number of Gaussian points can be reduced. However, the investigation of this case in detail requires special research.

At the same time the accuracy of the suggested method can be not enough for the airfoils with sharp edges, when the neighboring panels (in proximity to sharp edge) are very close one to the other. In such cases much higher accuracy of numerical integration is required; this problem is not considered in the framework of this paper either.

For diagonal elements of the matrices A^{pq} the error of numerical integration is also unacceptably high, but for them the following approximate analytical formulae are derived:

$$A_{ii}^{00} \approx \left(\frac{\alpha_i}{2\pi} - \frac{\alpha_i^3}{12\pi}\right)L_i, \qquad A_{ii}^{01} \approx \left(\frac{\beta_i}{24\pi} - \frac{\alpha_i^2\beta_i}{144\pi}\right)L_i,$$
$$A_{ii}^{10} \approx \left(\frac{\beta_i}{12\pi} - \frac{23\alpha_i^2\beta_i}{720\pi}\right)L_i, \quad A_{ii}^{11} \approx -\frac{\alpha_i^3}{144\pi}L_i.$$

The coefficients of the matrices D^{pq} also can be computed approximately by using the following formulae:

$$D_{ii}^{00} \approx -\left(\frac{1}{2} - \frac{\beta_i^2}{80}\right)L_i, \quad D_{ii}^{01} = D_{ii}^{10} \approx -\frac{\alpha_i\beta_i}{60}L_i, \quad D_{ii}^{11} \approx -\left(\frac{1}{24} + \frac{\alpha_i^2}{180}\right)L_i.$$

Lengthes of the panels L_i and coefficients L_i^* can be computed analytically with high precision as

$$L_i \approx L_i^0\left(1 + \frac{\alpha_i^2}{6} + \frac{\beta_i^2}{40}\right), \qquad L_i^* \approx L_i^0\alpha_i\beta_i\left(\frac{1}{30} - \frac{\alpha_i^2}{70}\right).$$

Right-hand side vector coefficients are computed by using formulae

$$b_i^p = \frac{1}{L_i}\int_0^1 f_\tau\big(r_i(t)\big)\,\varphi_p^i(t)J_i(t)\,dt, \quad p = 0,\,1, \;\; i = 1,\,\ldots,\,N.$$

3.4 Accuracy estimation

For some simple particular cases, for example, for the problem of flow simulation around elliptical airfoil [13], it is possible to derive exact analytical solution $\gamma^*(s)$ for vortex sheet intensity by using conformal mappings theory (here s is a parameter which parameterizes the exact shape of the surface line of the airfoil).

If the linear algebraic system is solved and values of the variables γ_i and δ_i, $i = 1,\,\ldots,\,N$, are found, approximate vortex sheet intensity distribution is known. Introducing for it the same parametrization as for exact solution and comparing them in L_1 norm,

$$\|\Delta\gamma\|_{L_1} = \oint_K |\gamma(s) - \gamma^*(s)|\,dl_s,$$

one can found that the error is proportional to panel length in square, *i.e.*, the developed numerical scheme provides the second order of accuracy.

However, in vortex methods sometimes it is much more important to provide high accuracy for average values of vortex sheet intensity over the panels:

$$\tilde{\gamma}_i = \frac{\Gamma_i}{L_i} = \frac{1}{L_i}\int_0^1 \left(\gamma_i \varphi_0^i(t) + \delta_i \varphi_1^i(t)\right) J_i(t)dt, \quad i = 1, \ldots, N.$$

So the error for such values is calculated as the following:

$$\|\Delta\gamma\|_{C^h} = \max_{i=1,\ldots,N} |\tilde{\gamma}_i - \tilde{\gamma}_i^*|,$$

where

$$\tilde{\gamma}_i^* = \frac{1}{L_i}\int_0^1 \gamma^*(s)dl_s, \quad i = 1, \ldots, N.$$

is the exact average values of vortex sheet intensity over the panels.

4 Numerical experiment

In the fig. 3 the dependencies of errors $\|\Delta\gamma\|_{C^h}$ on number of panels are shown for flow simulation around elliptical airfoils with different ratios of semiaxes with equal panel lengthes. It is seen that the developed numerical scheme provides 4-th order of accuracy.

The error for total quantities of vorticity over the panels, consequently, has the 5-th order of accuracy.

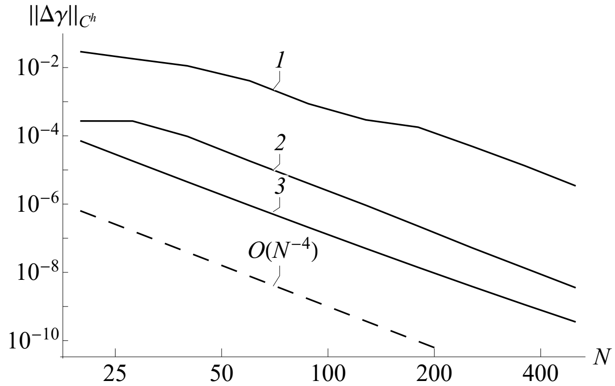

Figure 3: Dependencies of errors of average values of vortex sheet intensity computation over the panels on number of panels for elliptical airfoils with different ratios of semiaxes: *1* — 1 : 5; *2* — 1 : 2; *3* — 1 : 1; dashed line corresponds to the 4-th order of accuracy (in logarithmical scale)

5 Conclusions

New numerical scheme is developed for solving of boundary integral equation, which arises in flow simulation around airfoils using vortex methods. This scheme makes it possible to take into account the curvilinearity of the airfoil surface explicitly by using

Hermite spline interpolation. For numerical solution of integral equation the approach based on the ideas of discontinuous Galerkin method is used. The numerical solution is represented by piecewise-linear function. Such approach provides 2-nd order of accuracy in norm L_1 and 4-th order — for average values for vortex sheet intensities over the panels.

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