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EFFICIENT SEMI-ANALYTICAL INTEGRATION OF VORTEX SHEET INFLUENCE IN 3D VORTEX METHOD

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Abstract. The original numerical scheme is developed for vortex sheet intensity computation for 3D incompressible flow simulation using meshless Lagrangian vortex methods. It is based on tangential components of the velocity boundary condition satisfaction on the body surface instead of widespread condition for normal components. For the body triangulated surface the corresponding integral equation is approximated by the system of linear algebraic equations, which dimension is doubled number of triangular panels. Vortex layer intensity on the panels assumed to be piecewise-constant.

The coefficients of the matrix are expressed through double integrals over the influence and control panels. When these panels have common edge or common vertex these integrals become improper. In order to compute them it is necessary to exclude the singularities, i.e., to split the integrals into regular and singular parts. Regular parts are expressed by smooth functions, so they can be integrated numerically with high precision by using Gaussian quadrature formulae. For singular parts exact analytical integration formulae are derived.

The developed approach allows to raise significantly the accuracy of vortex layer intensity computation in vortex method for flow simulation around arbitrary 3D bodies.

The test problem of flow simulation around the sphere is considered. The exact analytical solution is known for it, and the developed numerical scheme provides more accurate results in comparison with 'classical' 3D vortex method, especially when non-uniform unstructured triangular meshes are used for bodies surface representation. It allows to use arbitrary triangular mesh on body surface and to refine mesh near sharp edges, what is especially important for flow simulation around bodies with complicated geometry.

1 Introduction and problem statement

The problem of 3D incompressible flow simulation around immovable body is considered. The governing equations are Navier — Stokes equations

$$\nabla \cdot \vec{V} = 0, \quad \frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla)\vec{V} = \nu \nabla^2 \vec{V} - \frac{\nabla p}{\rho_\infty},$$

with boundary conditions

$$\lim_{r \to \infty} \vec{V} = \vec{V}_\infty, \quad \lim_{r \to \infty} p = p_\infty, \quad \vec{V}(\vec{r}, t)\Big|_{\vec{r} \in K} = \vec{0},$$

where $\vec{V}$ is flow velocity; p — pressure; $\rho_\infty = \text{const}$ — density; ν — kinematic viscosity coefficient; $\vec{V}_\infty$ and p_∞ are parameters of the incident flow; K is body surface.

The viscosity assumed to be small, so according to L. Prandtl's theory it is possible to take its influence into account only as a cause of vorticity generation on body surface. So, the flow can be considered inviscid, with vorticity flux from the surface.

The immovable body is simulated by the influence of vortex sheet with unknown intensity $\vec{\gamma}(\vec{r}, t)$, which is placed on the body surface, $\vec{r} \in K$. The vorticity flux can be simulated if this vortex sheet is free; it means that at every time step this sheet is split into separate vortex elements which form vortex wake around the body.

Vortex wake evolution can be simulated by using one of Lagrangian vortex element methods [1, 2].

Vorticity flux simulation is one of the most important problems is vortex sheet intensity computation. There are two fundamental approaches, which are based on elimination of the limit values of normal or tangential velocity components on the body surface [3]. These approaches can be called "N-scheme" and "T-scheme", respectively.

The accuracy of N-scheme, especially in FSI-applications, when the bodies are movable and deformable, sometimes is not enough for practice. In 2D-case T-scheme allows to obtain much more accurate results, but it requires the usage of more precise integration schemes [3]. Such schemes are constructed and investigated by authors for 2D-case [4].

The aim of the present research is development of the numerical algorithm for T-scheme in 3D case.

2 Integral equation for vortex sheet intensity

Due to the presence of vortex sheet on the body surface, velocity field, which can be expressed by using generalized Biot — Savart law, has jump discontinuity, and its limit value from body side is

$$\vec{V}_-(\vec{r},\,t) = \vec{V}_\infty + \frac{1}{4\pi}\int\limits_{S(t)} \frac{\vec{\Omega}(\vec{\xi},\,t)\times(\vec{r}-\vec{\xi})}{\left|\vec{r}-\vec{\xi}\right|^3}dS_\xi + \\ + \frac{1}{4\pi}\int\limits_{K} \frac{\vec{\gamma}(\vec{\xi},\,t)\times(\vec{r}-\vec{\xi})}{\left|\vec{r}-\vec{\xi}\right|^3}dS_\xi - \frac{\vec{\gamma}(\vec{r},\,t)\times\vec{n}(\vec{r})}{2}, \quad \vec{r}\in K.$$

Here $S(t)$ is vortex wake region; $\vec{\Omega}(\vec{\xi},\,t) = \operatorname{curl}\vec{V}(\vec{\xi},\,t)$ is vorticity distribution in $S(t)$, which assumed to be known; $\vec{n}(\vec{r})$ is unit outer normal vector to body surface K.

In order to satisfy the boundary condition on the body surface, vortex sheet intensity should satisfy the integral equation $\vec{V}_-(\vec{r},\,t) = \vec{0}$, $\vec{r}\in K$.

As it proved in [3], it is enough to satisfy this equation only for tangent component of the limit value of velocity field:

$$\vec{n}(\vec{r}) \times \left(\vec{V}_-(\vec{r},\,t)\times\vec{n}(\vec{r})\right) = \vec{0}.$$

It leads to the integral equation of the 2-nd kind

$$\frac{\vec{n}(\vec{r})}{4\pi} \times \left(\int\limits_{K} \frac{\vec{\gamma}(\vec{\xi},\,t)\times(\vec{r}-\vec{\xi})}{\left|\vec{r}-\vec{\xi}\right|^3}\times\vec{n}(\vec{r})dS_\xi\right) - \frac{\vec{\gamma}(\vec{r},\,t)\times\vec{n}(\vec{r})}{2} = \vec{f}(\vec{r},\,t), \quad \vec{r}\in K, \qquad (1)$$

where

$$\vec{f}(\vec{r},\,t) = -\vec{n}(\vec{r}) \times \left(\vec{V}_\infty + \frac{1}{4\pi}\int\limits_{S(t)} \frac{\vec{\Omega}(\vec{\xi},\,t)\times(\vec{r}-\vec{\xi})}{\left|\vec{r}-\vec{\xi}\right|^3}\,dS_\xi\right)\times\vec{n}(\vec{r})$$

is known vector function.

3 Integral equation discretization

In order to find approximate solution of integral equation (1), which kernel is unbounded when $|\vec{r}-\vec{\xi}\,| \to 0$, the following assumptions can be made:

1. Body surface is triangulated into N flat cells K_i with areas A_i and normal vectors $\vec{n}_i$, $i = 1,\dots,N$.
2. The unknown vortex sheet intensity on the i-th cell is constant vector $\vec{\gamma}_i$, $i = 1,\dots,N$, which lies in the plane of the i-th cell, i.e. $\vec{\gamma}_i\cdot\vec{n}_i = 0$.
3. The integral equation (1) is satisfied on average over the cells.

According to these assumptions the discrete analogue of equation (1) can be derived:

$$\frac{1}{4\pi A_i}\sum_{j=1}^{N}\int\limits_{K_i}\left(\int\limits_{K_j}\vec{n}_i \times \left(\frac{\vec{\gamma}_j \times (\vec{r}-\vec{\xi})}{\left|\vec{r}-\vec{\xi}\right|^3}\times \vec{n}_i\right)dS_\xi\right)dS_r - \frac{\vec{\gamma}_i \times \vec{n}_i}{2} = \\ = \frac{1}{A_i}\int\limits_{K_i}\vec{f}(\vec{r},\,t)dS_r, \quad i=1,\ldots,N. \quad (2)$$

To write down (2) in form of linear algebraic system we should choose local orthonormal basis on every cell $(\vec{\tau}_i^{(1)},\,\vec{\tau}_i^{(2)},\,\vec{n}_i)$, where tangent vectors $\vec{\tau}_i^{(1)},\,\vec{\tau}_i^{(2)}$ can be chosen arbitrary and $\vec{\tau}_i^{(1)} \times \vec{\tau}_i^{(2)} = \vec{n}_i$. So $\vec{\gamma}_i = \gamma_i^{(1)}\vec{\tau}_i^{(1)} + \gamma_i^{(2)}\vec{\tau}_i^{(2)}$ and we can project (2) for every i-th panel on directions $\vec{\tau}_i^{(1)}$ and $\vec{\tau}_i^{(2)}$:

$$\begin{aligned} &\frac{1}{4\pi A_i}\vec{\tau}_i^{(1)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\vec{\nu}_{ij}^{(1)} + \sum_{j=1}^{N}\gamma_j^{(2)}\vec{\nu}_{ij}^{(2)}\right) - \frac{\gamma_i^{(2)}}{2} = \frac{b_i^{(1)}}{A_i},\\ &\frac{1}{4\pi A_i}\vec{\tau}_i^{(2)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\vec{\nu}_{ij}^{(1)} + \sum_{j=1}^{N}\gamma_j^{(2)}\vec{\nu}_{ij}^{(2)}\right) + \frac{\gamma_i^{(1)}}{2} = \frac{b_i^{(2)}}{A_i}. \end{aligned} \quad (3)$$

Here

$$\vec{\nu}_{ij}^{(k)} = \int\limits_{K_i}\left(\int\limits_{K_j}\frac{\vec{\tau}_j^{(k)} \times (\vec{r}-\vec{\xi})}{\left|\vec{r}-\vec{\xi}\right|^3}dS_\xi\right)dS_r, \quad b_i^{(k)} = \int\limits_{K_i}\vec{\tau}_i^{(k)}\cdot\vec{f}(\vec{r},\,t)\,dS_r, \quad k=1,\,2;\; i,j=1,\,\ldots,\,N.$$

Algebraic system (3) has infinite set of solutions; in order to select the unique solution we should satisfy additional condition for total vorticity (integral from the vorticity over the body surface)

$$\int\limits_{K}\vec{\gamma}(\vec{r},\,t)dS_r = \vec{0},$$

which can be written down in the following form:

$$\sum_{i=1}^{N}A_i\left(\gamma_i^{(1)}\vec{\tau}_i^{(1)} + \gamma_i^{(2)}\vec{\tau}_i^{(2)}\right) = \vec{0}. \quad (4)$$

System (3)-(4) is overdetermined, it should be regularized, for example, by introducing the ‘regularization vector’ $\vec{R} = \left(R_1,\,R_2,\,R_3\right)^T$:

$$\begin{aligned}
&\frac{1}{4\pi A_i}\vec{\tau}_i^{(1)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\vec{\nu}_{ij}^{(1)}+\sum_{j=1}^{N}\gamma_j^{(2)}\vec{\nu}_{ij}^{(2)}\right)-\frac{\gamma_i^{(2)}}{2}+\vec{R}\cdot\vec{\tau}_i^{(2)}=\frac{b_i^{(1)}}{A_i},\\
&\frac{1}{4\pi A_i}\vec{\tau}_i^{(2)}\cdot\left(\sum_{j=1}^{N}\gamma_j^{(1)}\vec{\nu}_{ij}^{(1)}+\sum_{j=1}^{N}\gamma_j^{(2)}\vec{\nu}_{ij}^{(2)}\right)+\frac{\gamma_i^{(1)}}{2}+\vec{R}\cdot\vec{\tau}_i^{(1)}=\frac{b_i^{(2)}}{A_i},\\
&\sum_{j=1}^{N}A_j\left(\gamma_j^{(1)}\vec{\tau}_j^{(1)}+\gamma_j^{(2)}\vec{\tau}_j^{(2)}\right)=0,\qquad i=1,\,\ldots,\,N.
\end{aligned}\tag{5}$$

Numerical computations show that system (5) is well-conditioned; its dimension is $2N+3$.

4 Matrix coefficients calculation

The main problem for practical usage of the suggested approach is coefficients $\vec{\nu}_{ij}^{(k)}$ calculation for system (5):

$$\vec{\nu}_{ij}^{(k)}=\vec{\tau}_j^{(k)}\times\int\limits_{K_i}\left(\int\limits_{K_j}\frac{\vec{r}-\vec{\xi}}{\left|\vec{r}-\vec{\xi}\right|^3}dS_\xi\right)dS_r=\vec{\tau}_j^{(k)}\times\vec{I}_{ij},\quad k=1,\,2,\quad i,\,j=1,\,\ldots,\,N \tag{6}$$

Integral $\vec{I}_{ij}$ is calculated over triangular cells K_i and K_j, where i-th cell we call 'control', j-th cell — 'influence' cell.

4.1 The general approach

The inner integral in (6) over the influence cell K_j

$$\vec{J}_j(\vec{r})=\int\limits_{K_j}\frac{\vec{r}-\vec{\xi}}{\left|\vec{r}-\vec{\xi}\right|^3}dS_\xi \tag{7}$$

can be calculated exactly. There is well-known way for analytical calculation of integral (7) via considering of the integral from $\left|\vec{r}-\vec{\xi}\right|^{-1}$ with respect to $\vec{\xi}$ over the triangle K_j:

$$\vec{J}_j(\vec{r})=\int\limits_{K_j}\frac{\vec{r}-\vec{\xi}}{\left|\vec{r}-\vec{\xi}\right|^3}dS_\xi=\int\limits_{K_j}\nabla_\xi\frac{1}{\left|\vec{r}-\vec{\xi}\right|}dS_\xi=-\int\limits_{K_j}\nabla_r\frac{1}{\left|\vec{r}-\vec{\xi}\right|}dS_\xi=-\nabla_r\left(\int\limits_{K_j}\frac{1}{\left|\vec{r}-\vec{\xi}\right|}dS_\xi\right).$$

The last integral is very usual in potential theory, analytical expression for it can be found, for example, in [5]. However, that expression is cumbersome and it also should be differentiated with respect to components of vector $\vec{r}$.

Using computational software of symbolic mathematics *Wolfram Mathematica* and Handbook of integrals [6] it is possible to integrate (7) straightforwardly if vectors

$\vec{s}_k = \vec{r}_k^{(j)} - \vec{r}$, $k = 1, 2, 3$, are only known, where $\vec{r}$ is point for which integral (7) is calculated, $\vec{r}_k^{(j)}$ are vertices of K_j triangular cell. Denoting

$$\vec{e}_k^{(j)} = \frac{\vec{s}_{k+1} - \vec{s}_k}{\left|\vec{s}_{k+1} - \vec{s}_k\right|} = \frac{\vec{r}_{k+1}^{(j)} - \vec{r}_k^{(j)}}{\left|\vec{r}_{k+1}^{(j)} - \vec{r}_k^{(j)}\right|}, \qquad \vec{\sigma}_k = \frac{\vec{s}_k}{\left|\vec{s}_k\right|}, \quad k = 1, 2, 3,$$

and assuming all the indices to be calculated using a modulus of 3, we obtain

$$\vec{J}_j(\vec{r}) = \Theta_j \vec{n}_j + \vec{\Psi}_j \times \vec{n}_j, \quad j = 1, \ldots, N,$$

where

$$\vec{\Psi}_j = \sum_{k=1}^{3} \ln\left(\frac{\left|\vec{s}_k\right|}{\left|\vec{s}_{k+1}\right|} \frac{1 + \vec{e}_k^{(j)} \cdot \vec{\sigma}_k}{1 + \vec{e}_k^{(j)} \cdot \vec{\sigma}_{k+1}} \right) \vec{e}_k,$$

and Θ_j is solid angle subtended by triangular cell K_j which can be calculated, for example, by using the formula [7]

$$\Theta_j = 2 \arctan\left(\frac{\vec{s}_1 \vec{s}_2 \vec{s}_3}{\left|\vec{s}_1\right| \cdot \left|\vec{s}_2\right| \cdot \left|\vec{s}_3\right| + \left(\vec{s}_1 \cdot \vec{s}_2\right)\left|\vec{s}_3\right| + \left(\vec{s}_2 \cdot \vec{s}_3\right)\left|\vec{s}_1\right| + \left(\vec{s}_3 \cdot \vec{s}_1\right)\left|\vec{s}_2\right|} \right),$$

here $\vec{s}_1 \vec{s}_2 \vec{s}_3$ denotes the scalar triple product of the vectors.

The outer integral in (6)

$$\vec{I}_{ij} = \int\limits_{K_i} \vec{J}_j(\vec{r})\, dS_r \tag{8}$$

can't be simply expressed analytically in elementary functions, so the suitable way for its computation is Gaussian quadrature formula usage:

$$\vec{I}_{ij} = \int\limits_{K_i} \vec{J}_j(\vec{r})\, dS_r \approx A_i \sum_{p=1}^{N_{GP}} \omega_p \vec{J}_j(\vec{\eta}_p),$$

where N_{GP} is number of Gaussian points; ω_p are weight coefficients; $\vec{\eta}_p$ are positions of Gaussian points. Values of ω_p and $\vec{\eta}_p$ for different values of N_{GP} can be found, for example, in [8].

It works perfectly if influence and control cells are far one from the other; numerical experiments show that it is enough to use the corresponding quadratures with small number of points ($N_{GP} = 1 \ldots 4$).

However, for cells which have common edge or common vertex (such cells we call 'neighboring cells') the corresponding integral is improper, so Gaussian quadratures become unsuitable. Direct numerical computation of improper integral is non-trivial problem, so for such cases semi-analytical approach is developed.

If the cells have common edge or common vertex, we need to exclude the singularity from the $\vec{J}_j(\vec{r})$ and write it down as sum of two terms:

$$\vec{J}_j(\vec{r}) = \vec{J}_j^{\,\mathrm{reg}}(\vec{r}) + \vec{J}_j^{\,\mathrm{sing}}(\vec{r}).$$

where $\vec{J}_j^{\,\mathrm{reg}}(\vec{r})$, which has the form

$$\vec{J}_j^{\,\mathrm{reg}}(\vec{r}) = \big(\Theta_j(\vec{r}) - \Theta_j^{\mathrm{sing}}(\vec{r})\big)\vec{n}_j + \big(\vec{\Psi}_j(\vec{r}) - \vec{\Psi}_j^{\,\mathrm{sing}}(\vec{r})\big) \times \vec{n}_j,$$

has no singularities and can be easily integrated numerically with high accuracy by using Gaussian quadrature formulae

$$\int\limits_{K_i} \vec{J}_j^{\,\mathrm{reg}}(\vec{r})\, dS_r \approx A_i \sum_{p=1}^{N_{GP}} \omega_p \vec{J}_j^{\,\mathrm{reg}}(\vec{\eta}_p).$$

For the improper (singular) integral

$$\int\limits_{K_i} \vec{J}_j^{\,\mathrm{sing}}(\vec{r})\, dS_r = \Bigg(\int\limits_{K_i} \Theta_j^{\mathrm{sing}}(\vec{r})\, dS_r\Bigg)\vec{n}_j + \Bigg(\int\limits_{K_i} \vec{\Psi}_j^{\,\mathrm{sing}}(\vec{r}) dS_r\Bigg) \times \vec{n}_j$$

exact analytical formulae are derived, which are shown below.

4.2 Neighboring cells with common edge

If cells K_i and K_j have common edge with directing unit vector $\vec{e}_3$, as it is shown in fig. 1, singular terms have the following form (hereinafter upper index (j) in unit vectors $\vec{e}_1^{\,(j)}$, $\vec{e}_2^{\,(j)}$ and $\vec{e}_3^{\,(j)}$ is omitted):

$$\Theta_j^{\mathrm{sing}}(\vec{r}) = 2\Bigg(\arctan \frac{\vec{b}\,\vec{e}_2\,\vec{e}_3}{|\vec{b}\,|(1-\vec{e}_2\cdot\vec{e}_3) - \vec{b}\cdot(\vec{e}_2-\vec{e}_3)} - \arctan \frac{\vec{a}\,\vec{e}_1\,\vec{e}_3}{|\vec{a}\,|(1-\vec{e}_1\cdot\vec{e}_3) + \vec{a}\cdot(\vec{e}_1-\vec{e}_3)}\Bigg),$$

$$\vec{\Psi}_j^{\,\mathrm{sing}}(\vec{r}) = \vec{e}_3 \ln \frac{|\vec{a}\,|\cdot|\vec{c}\,| - \vec{a}\cdot\vec{c}}{|\vec{b}\,|\cdot|\vec{c}\,| - \vec{b}\cdot\vec{c}} + \vec{e}_1 \ln \frac{|\vec{a}\,| + \vec{a}\cdot\vec{e}_1}{|\vec{c}\,|} + \vec{e}_2 \ln \frac{|\vec{b}\,| - \vec{b}\cdot\vec{e}_2}{|\vec{c}\,|},$$

where

$$\vec{c} = \vec{r}_1^{\,(j)} - \vec{r}_3^{\,(j)}, \quad \vec{a} = \vec{r}_1^{\,(j)} - \vec{r}, \quad \vec{b} = \vec{r}_3^{\,(j)} - \vec{r},$$

Expression for Θ_j^{sing}, as well as all scalar multipliers of $\vec{\Psi}_j^{\,\mathrm{sing}}$ can be integrated analytically over the cell K_i, an finally we obtain:

$$\int\limits_{K_i} \Theta_j^{\mathrm{sing}}(\vec{r})\, dS_r = -2A_i\big(q_0(\xi,\alpha,\beta,\mu,\gamma,\lambda) + q_0(\xi,\beta,\alpha,\sigma,\delta,\theta)\big),$$

$$\int\limits_{K_i} \vec{\Psi}_j^{\,\mathrm{sing}}(\vec{r})\, dS_r = A_i\big(q_{12}(\xi,\alpha,\beta,\mu,\gamma,\lambda)\vec{e}_1 + q_{12}(\xi,\beta,\alpha,\sigma,\delta,\theta)\vec{e}_2 + q_3(\alpha,\beta)\vec{e}_3\big).$$

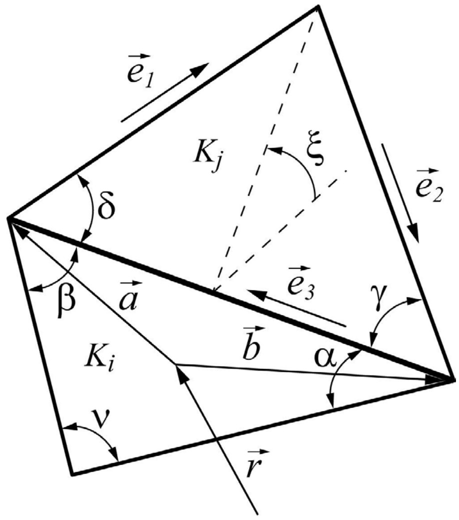

Figure 1: Cells K_i and K_j in case of having common edge

Here auxiliary functions q_0, q_{12} and q_3 have the form

$$q_0(\xi,\alpha,\beta,\mu,\gamma,\lambda) = \arctan\frac{\sin\xi\sin\alpha\sin\gamma}{1-\cos\alpha+\cos\gamma+\cos\lambda} + \\ + \frac{\sin\gamma\sin\nu}{\sin^2\mu\sin\alpha}\Big(\big(\cos\beta\sin\gamma-\sin\beta\cos\gamma\cos\xi\big)\arctan\frac{\sin\xi\sin\alpha\sin\gamma}{1+\cos\alpha-\cos\gamma+\cos\lambda} + \\ + \sin\xi\sin\beta\Big(\cos^2\frac{\mu}{2}\ln\frac{\cos\beta/2}{\sin\nu/2}+\sin^2\frac{\mu}{2}\ln\frac{\sin\beta/2}{\cos\nu/2}+\ln\frac{\cos\lambda/2}{\sin\gamma/2}\Big)\Big),$$

$$q_{12}(\xi,\alpha,\beta,\mu,\gamma,\lambda) = \\ = -\frac{3}{2}+\frac{1}{\sin\alpha\sin^2\mu}\Big(\ln(1+\cos\lambda)\sin\beta(\cos\nu+\cos\mu\cos\lambda) + \\ + \ln(1-\cos\gamma)\sin\nu(\cos\beta+\cos\gamma\cos\mu) + \\ + \ln\frac{\sin\beta}{\sin\nu}\sin\beta(1-\cos\mu)(\cos\nu-\cos\lambda) - \\ - \sin\nu\sin\beta\Big(-2\sin\xi\sin\gamma\arctan\frac{\sin\xi\sin\gamma\sin\alpha}{1+\cos\lambda-\cos\gamma+\cos\alpha} + \\ + (\sin\gamma\cos\beta\cos\xi-\sin\beta\cos\gamma)\ln\frac{1-\cos\nu}{1+\cos\beta}\Big)\Big),$$

$$q_3(\alpha,\beta) = \frac{\sin\nu}{\sin\beta}\ln\Big(\tan\frac{\alpha}{2}\tan\frac{\nu}{2}\Big)+\frac{\sin\nu}{\sin\alpha}\ln\Big(\tan\frac{\beta}{2}\tan\frac{\nu}{2}\Big)+\ln\Big(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\Big).$$

Here α and β are the angles of the triangle K_i, which adjoin the common edge of the cells K_i and K_j, $\nu = \pi-\alpha-\beta$; γ and δ are the angles of the triangle K_j, which adjoin the common edge; ξ is the angle between the planes of the cells K_i and K_j; angles σ, μ, λ and θ can be calculated by using formulae

$$\begin{aligned}
\sigma &= \pi - \arccos(\cos\alpha\cos\delta + \cos\xi\sin\alpha\sin\delta),\\
\mu &= \pi - \arccos(\cos\beta\cos\gamma + \cos\xi\sin\beta\sin\gamma),\\
\lambda &= \pi - \arccos(\cos\alpha\cos\gamma - \cos\xi\sin\alpha\sin\gamma),\\
\theta &= \pi - \arccos(\cos\beta\cos\delta - \cos\xi\sin\beta\sin\delta).
\end{aligned}$$

4.3 Neighboring cells with common vertex

If cells K_i and K_j have common vertex, for example as it is shown in fig. 2, the regular part $\vec{J}_j^{\,\mathrm{reg}}(\vec{r}\,)$ has the following form (the previous denotation is used):

$$\Theta_j^{\mathrm{sing}}(\vec{r}\,) = -2\Big(\arctan\frac{\vec{b}\,\vec{e}_{**}\,\vec{e}_0}{|\vec{b}\,|(1+\vec{e}_{**}\cdot\vec{e}_0)+\vec{b}\cdot(\vec{e}_{**}+\vec{e}_0)} - \arctan\frac{\vec{b}\,\vec{e}_{*}\,\vec{e}_0}{|\vec{b}\,|(1+\vec{e}_{*}\cdot\vec{e}_0)+\vec{b}\cdot(\vec{e}_{*}+\vec{e}_0)}\Big),$$

$$\vec{\Psi}_j^{\,\mathrm{sing}}(\vec{r}\,) = -\Big(\vec{e}_{**}\ln\frac{|\vec{b}\,|+\vec{b}\cdot\vec{e}_{**}}{\sqrt{A_j}} - \vec{e}_{*}\ln\frac{|\vec{b}\,|+\vec{b}\cdot\vec{e}_{*}}{\sqrt{A_j}}\Big),$$

where $\vec{b} = \vec{r}_c^{\,(j)} - \vec{r}$; $\vec{e}_0$ is unit vector of intersection line of the planes of the cells K_i and K_j; $\vec{e}_*$ and $\vec{e}_{**}$ are unit vectors of sides of cell K_j, as it is shown in fig. 2.

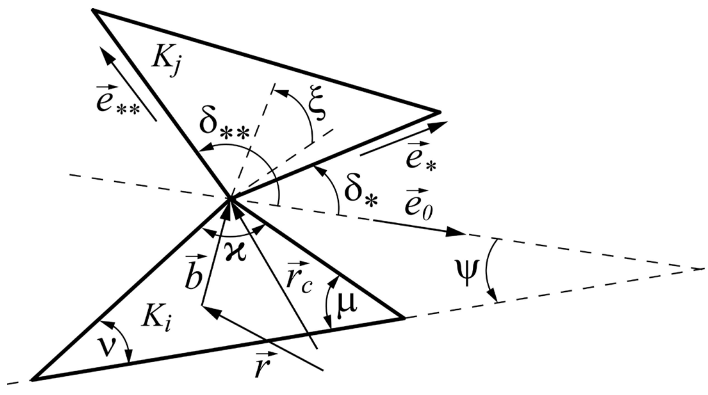

Figure 2: Cells K_i and K_j in case of having common vertex

These expressions also can be integrated analytically over the cell K_i:

$$\int\limits_{K_i} \Theta_j^{\mathrm{sing}}(\vec{r}\,)\,dS_r = -2A_i\big(q_4(\delta_{**}) - q_4(\delta_*)\big),$$

$$\int\limits_{K_i} \vec{\Psi}_j^{\,\mathrm{sing}}(\vec{r}\,)\,dS_r = -A_i\big(q_5(\delta_{**})\vec{e}_{**} - q_5(\delta_*)\vec{e}_*\big).$$

The auxiliary functions q_4 and q_5 are the following:

$$
\begin{aligned}
q_4(\delta) = \frac{1}{\sin\psi\sin\varkappa}\Bigg(&\sin\mu\sin(\nu+\psi)\arctan\frac{\sin\xi\sin\delta/2}{\cos\xi\sin\delta/2+\cos\delta/2\cot(\nu+\psi)/2} - \\
&- \sin\nu\sin(\mu-\psi)\arctan\frac{\sin\xi\sin\delta/2}{\cos\xi\sin\delta/2+\cos\delta/2\tan(\mu-\psi)/2} - \\
&- \frac{\sin\mu\sin\nu\sin\psi\sin\delta}{D}\Bigg(2(\cos\xi\cos\delta-\cot\psi\sin\delta)\omega + \\
&+ \sin\xi\Big(\ln\Big(\frac{1+\cos\lambda}{1-\cos\theta}\frac{\sin\nu}{\sin\mu}\Big)+\cos\sigma\ln\Big(\tan\frac{\nu}{2}\tan\frac{\mu}{2}\Big)\Big)\Bigg)\Bigg),
\end{aligned}
$$

$$
\begin{aligned}
q_5(\delta) = -\frac{3-\ln 2}{2} &+ \frac{\sin\mu\sin(\nu+\psi)\ln(1-\cos\theta)-\sin\nu\sin(\mu-\psi)\ln(1+\cos\lambda)}{\sin\varkappa\sin\psi} + \\
&+ \frac{1}{2}\ln\frac{\sin\nu\sin\mu}{\sin\varkappa} - \frac{\cos\nu\sin\mu}{\sin\varkappa}\ln\sin\nu - \frac{\cos\mu\sin\nu}{\sin\varkappa}\ln\sin\mu - \\
&- \frac{2\sin\nu\sin\mu}{D\sin\varkappa\sin\psi}\Bigg(\sin\delta\big(\sin\psi\cos\delta\cos\xi-\cos\psi\sin\delta\big)\ln\frac{1-\cos\theta}{1+\cos\lambda} - \\
&- \sin\psi\big(\sin\delta\cos\psi\cos\xi-\cos\delta\sin\psi\big)\ln\Big(\tan\frac{\nu}{2}\tan\frac{\mu}{2}\Big) + \\
+\sin\psi\Big(2\omega\sin\delta\sin\xi &- \frac{1}{2}\Big(\big(1-\sin^2\delta(1+\cos^2\xi)\big)\sin 2\psi-\sin 2\delta\cos 2\psi\cos\xi\Big)\ln\frac{\sin\nu}{\sin\mu}\Big)\Bigg).
\end{aligned}
$$

Here we denotes for simplicity

$$
\begin{aligned}
D &= 2\sin^2\psi+\sin^2\delta\big(\sin^2\xi+(1+\cos^2\xi)\cos 2\psi\big)-\sin 2\delta\sin 2\psi\cos\xi, \\
\omega &= \arctan\frac{\sin\xi\sin\delta\sin\varkappa/2}{\cos\xi\sin\delta\sin(\nu+\psi+\varkappa/2)+\cos\varkappa/2-\cos\delta\cos(\mu-\psi+\varkappa/2)}.
\end{aligned}
$$

Here $\varkappa$ is the angles of the triangle K_i, which adjoins the common vertex of the cells K_i and K_j; μ and ν are the other angles of cell K_i; ξ is the angle between the planes of the cells K_i and K_j; ψ is the angle between $\vec{e}_0$ and the side of triangle K_i, which is opposite to common vertex; δ_* and δ_{**} are the angles between $\vec{e}_0$ and vectors $\vec{e}_*$ and $\vec{e}_{**}$, respectively; angles σ, λ and θ can be calculated by using formulae

$$
\begin{aligned}
\sigma &= \pi - \arccos\big(\cos\psi\cos\delta+\cos\xi\sin\psi\sin\delta\big), \\
\lambda &= \pi - \arccos\big(\cos\delta\cos(\mu-\psi)-\cos\xi\sin\delta\sin(\mu-\psi)\big), \\
\theta &= \pi - \arccos\big(\cos\delta\cos(\nu+\psi)+\cos\xi\sin\delta\sin(\nu+\psi)\big).
\end{aligned}
$$

5 Numerical experiment

The developed semi-analytical numerical scheme makes it possible to use arbitrary triangular surface mesh even of very low quality. The corresponding linear system is being solved by using Gaussian elimination procedure. In the numerical examples shown below `Eigen` library, developed for C++, has been used [9]. In fig. 3 the results are shown for some test cases: flow around the sphere (with close to uniform mesh), flow around a weight (with mesh cells of very different size) and flow around a fish model (some mesh cells are very long and narrow). Red lines in the centers of cells shows the direction of vorticity in vortex sheet.

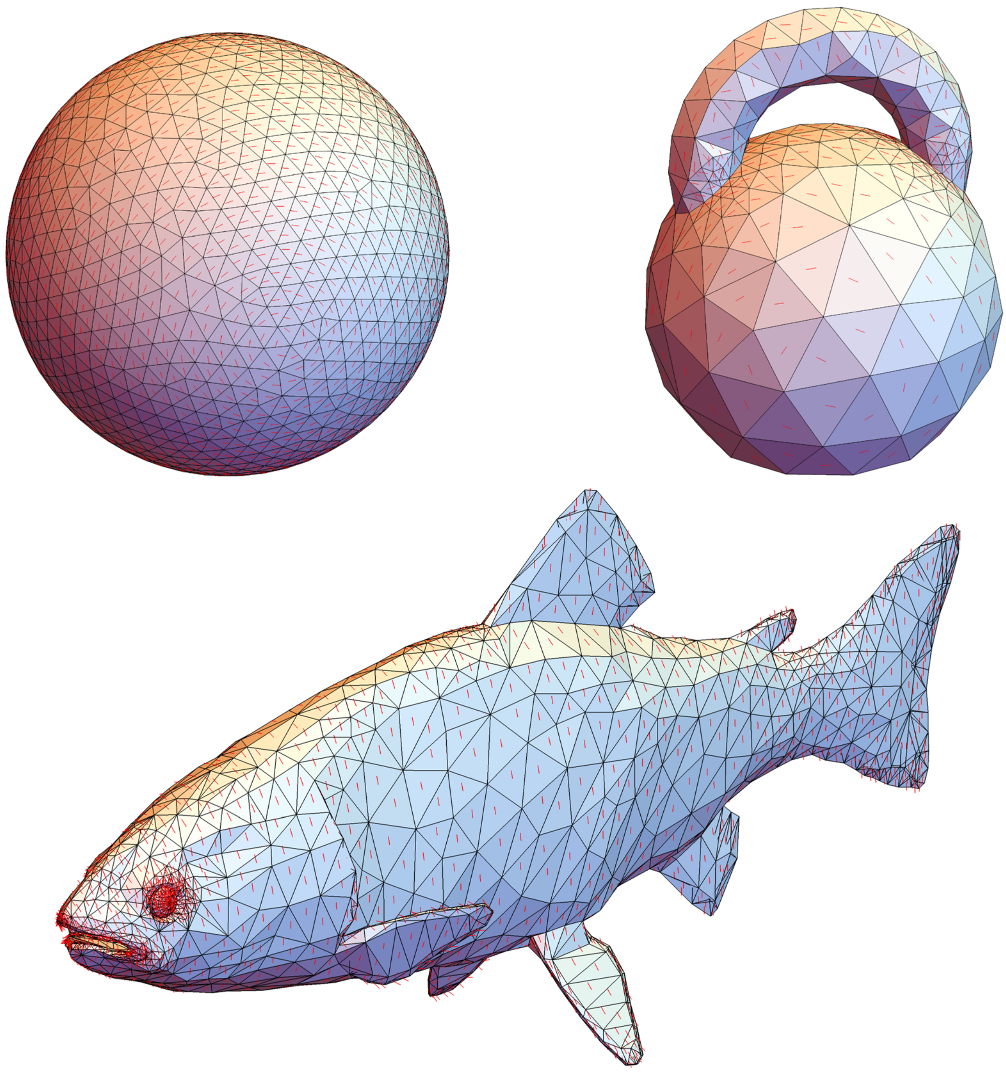

Figure 3: Vortex sheet on the sphere (number of panels $N = 2814$); on the weight ($N = 636$) and on the fish ($N = 3194$)

6 Conclusions

The derived formulae for $\vec{I}_{ij}$ makes it possible to construct numerical procedure for solving of the discrete analogue of the integral equation for vortex sheet intensity calculation in the framework of 'tangent' approach. It allows to use arbitrary triangular mesh on body surface and to refine mesh near sharp edges, that is especially important for flow around 3D wings simulation. Despite the fact that the dimension of the linear system in the developed numerical scheme is twice as large then in traditional implementations of vortex methods, its accuracy is much higher.

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