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SMOOTH PLASTICITY AND DAMAGE MODEL FOR THE MATERIAL POINT METHOD

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Abstract. In the Material Point Method (MPM) the structure is discretized into a set of material points that hold all the state variables of the system [1] such as stress, strain, velocities etc. A background grid is employed and the variables are mapped to the nodes of the grid. The conservation of momentum equations with energy and mass conservation considerations are solved at the grid nodes and the updated state variables are again mapped back to the material points updating their positions and velocities. The background grid is used only to solve the governing equations at the end of each computational step and then it is reset back to its original undeformed configuration. It is used only as a scratchpad for calculations and thus mesh distortion that constitutes a problem in Finite Element simulations is avoided. In this work the explicit formulation of the MPM is employed. According to the strain decomposition rule the strains are uncoupled into an elastic and an inelastic part. The constitutive law follows a Bouc-Wen [2] type formulation for smooth transition from the elastic to the inelastic regime. In the same manner the constitutive equations for elastoplasticity coupled with damage are smoothed according to Lemaitre's elastoplastic damage theory [3,4]. The above formulation is expressed and incorporated in the tangent modulus of elasticity as Heaviside type functions that control the inelastic behavior and damage. Results are presented that validate and verify the proposed formulation in the context of the Material Point Method.

1 INTRODUCTION

The Material Point Method is an extension of the Particle in Cell (PIC) method. It is a hybrid method in a sense that it is based both on a Lagrangian and a Eulerian description. In Lagrangian methods the computational grid is embedded and deformed with the material. On the contrary in Eulerian methods the computational grid is fixed and the material moves through the grid. Eulerian methods are more appropriate in problems in which the material becomes heavily distorted.

The MPM tracks the deformation history of the material points and drastically reduces the numerical dissipation that can be found in Eulerian methods. In MPM since the grid is fixed and the particles track the deformation, problems related to mesh distortion and element entanglement are alleviated. It has been used successfully in slope stability problems, sea ice dynamics [11], multiphase flows [9], hypervelocity problems [10] etc.

The method is considered a hybrid method since it takes advantage of both the Eulerian and Lagrangian description. At the beginning of each time step a Eulerian background grid is employed. The material is discretized into a number of material points that hold the properties and the state of the material (such as position, velocity, density, mass, stresses, strains etc.). The properties are then transferred to the background grid nodes where the governing equations are solved. The material points are then updated and the background grid is reset to its original form. Although the background grid nodes can be moved it is not necessary and in practice is often avoided. This happens mainly because in a structured grid the identification of the element that each material point lies in is straightforward and computationally inexpensive. This is in contrast with mesh-free methods like the Smoothed Particle Hydrodynamics where the nearest neighbor search takes a significant percentage of the computational time.

2 THE MATERIAL POINT METHOD

The material points hold all the properties: position, velocity, mass, stress, strain. The governing equations consist of conservation equations, the constitutive equation, kinematic condition as well as boundary conditions and initial values. These equations, in their general form, are presented below:

$$\begin{aligned}
&\text{Mass conservation:} && pJ = p_0 \\
&\text{Momentum conservation:} && \sigma \cdot \nabla + pb = p\dot{v} \\
&\text{Energy conservation:} && p\dot{e} = D : \sigma + ps + \nabla \cdot \left(k \nabla T\right) \\
&\text{Constitutive equation:} && \sigma = \sigma\left(D, \sigma, \text{etc}\right) \\
&\text{Rate of deformation:} && D = \frac{1}{2}\left(L + L^T\right)
\end{aligned} \tag{1}$$

where σ denotes the Cauchy stress, p the current density, b the body force per unit mass and $\dot{v}$ is the acceleration. As in the Finite Element Method the MPM also uses the weak form formulation. In the Material Point Method each particle represents a sub-domain of the whole domain Ω. Using the Dirac delta function the mass density can be expressed as a function of the material point positions and the material point masses as:

$$p = \sum_{i=1}^{N_p} m_p \delta\left(x - x_p\right) \tag{2}$$

The material gradients are calculated on a background computational grid in a similar manner to Finite Elements. The solution is approximated with the use of shape functions. This way the velocities and accelerations of the material points are expressed by the following relations, in terms of the grid nodal accelerations:

$$v\left(x_p\right)=\sum_{i=1}^{N} v_i N_i \quad , \quad \alpha\left(x_p\right)=\sum_{i=1}^{N} \frac{dv_i}{dt} N_i \tag{3}$$

In both relations the summation is over element nodes of the background grid that the material point resides in. In this work the shape functions used are cubic B-Splines [5]. They have been shown to reduce the quadrature errors and the grid crossing errors associated with discontinuous shape functions [6]. Applying the Galerkin method and using the previous relations in the momentum equation, integrals are replaced by sums and the momentum equation is stated as:

$$\sum_{i=1}^{N} m_i \frac{dv_i}{dt} = F_i^{\text{int}} + F_i^{ext} \tag{4}$$

where the internal and the external forces are defined as:

$$F_i^{\text{int}} = -\sum_{i=1}^{N_p} \frac{m_{p,i}}{p} \sigma_{p,i} \nabla N_i \quad , \quad F_i^{ext} = \tau_i + b_i \tag{5}$$

Subscript p,i denotes the material point i.

The main algorithm of the MPM is the following: firstly, the mass momentum and internal forces are mapped from the particles to the background grid nodes. The nodal force vector is computed and the nodal momenta are updated. Then information is transferred back to the material points by updating their velocities and positions. The last step based on the Modified Update Stress Last scheme of MPM is to recalculate the grid nodal momenta based on the new particle velocities and calculate the new stress based on the strain increments. In this last step for an elastic material the plane stress elasticity matrix is used. Based on the theory of plasticity and damage, the stress increments will be calculated from the strain increments times the tangent elasticity matrix.

$$\{\dot{\sigma}\} = [E_t]\{\dot{\varepsilon}\} \tag{6}$$

3 SMOOTH PLASTICITY AND DAMAGE

The elastoplastic model coupled with damage that is used in this work is Lemaitre's model [3-4]. It can simulate the evolution of internal damage as well as isotropic or kinematic hardening. The additive decomposition of the strain rates tensor is:

$$\{\dot{\varepsilon}\} = \{\dot{\varepsilon}^e\} + \{\dot{\varepsilon}^{pl}\} \tag{7}$$

while the yield function has the following form:

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$$\Phi = \frac{\sqrt{3J_2(s-\eta)}}{(1-D)} - \sigma_y \ , \ \sigma_y = {\sigma_y}^0 + H'\varepsilon_{ps} \tag{8}$$

where η is the back stress tensor and D is the damage parameter, which can take values from zero to one, σ_y^0 is the initial yield strength and H' is the hardening modulus. The coupled elastoplastic damage constitutive law is stated as:

$$\{\sigma\} = (1-D)[C]\{\varepsilon - \varepsilon_{pl}\} \tag{9}$$

Differentiating equation (9) and using the expression for the plastic strains based on the flow rule the following relation is derived:

$$\{\dot{\sigma}\} = (1-D)[C]\left\{\{\dot{\varepsilon}\} - \dot{\lambda}\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}\right\} - \dot{D}[C]\{\varepsilon - \varepsilon_{pl}\} \tag{10}$$

The evolution equation for the damage variable is given by [4]:

$$\dot{D} = \dot{\lambda}\frac{H_3}{1-D}\left(-\frac{Y}{r}\right)^s \tag{11}$$

where H_3 is a Heaviside function that holds a critical value (damage threshold) such as damage growth can start only at this critical value. Parameters r and s are material constants and are identified from experimental procedures, while Y is the thermodynamical force conjugate to the damage internal variable and is given by:

$$Y = -\frac{1}{2}\{\varepsilon^e\}^T[C]\{\varepsilon^e\} \tag{12}$$

Using equations (10) and (11) the following relation is obtained:

$$\{\dot{\sigma}\} = (1-D)[C]\{\dot{\varepsilon}\} - \dot{\lambda}\left\{(1-D)[C]\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\} + \frac{H_3}{1-D}\left(-\frac{Y}{r}\right)^s[C]\{\varepsilon - \varepsilon_{pl}\}\right\} \tag{13}$$

The consistency condition states that:

$$\dot{\Phi} = 0 \tag{14}$$

which leads to:

$$\begin{aligned}&\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}^{\mathrm{T}}\{\dot{\sigma}\} + \frac{\partial\Phi}{\partial D}\dot{D} - H'\dot{\lambda} = 0 \Rightarrow \\ &\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}^{\mathrm{T}}\{\dot{\sigma}\} + \frac{\partial\Phi}{\partial D}\left(\dot{\lambda}\frac{H_3}{1-D}\left(-\frac{Y}{r}\right)^s\right) - H'\dot{\lambda} = 0\end{aligned} \tag{15}$$

Substituting relation (13) into (15) and solving for the plastic multiplier the following relation is derived:

$$\dot{\lambda} = \lambda_1^D(1-D)\left\{\frac{\partial\Phi}{\partial\{\sigma\}}\right\}^{\mathrm{T}}[C]\{\dot{\varepsilon}\} \tag{16}$$

where λ_1^D is given by the following formula:

$$\lambda_1^D = \left\{ H' + (1-D) \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C] \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\} + \frac{H_3}{1-D} \left(-\frac{Y}{r} \right)^s \left(\left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C] \{ \varepsilon - \varepsilon_{pl} \} - \frac{\partial \Phi}{\partial D} \right) \right\}^{-1} \tag{17}$$

The previous equations hold when yielding has occurred. In order to generalize the plastic multiplier in the whole domain [7-8] and smooth the transition from the elastic to the inelastic regime two Heaviside type functions are used [2,12], thus making redundant the need for a piece wise approach for the domains of the Kunh-Tucker conditions:

$$H_1 = \left| \frac{\Phi}{\Phi_0} \right|^n \quad , \qquad H_2 = 0.5 + 0.5 sign\left(\{\varepsilon\}^T \{\dot{\sigma}\} \right) \tag{18}$$

This way the final expression for the plastic multiplier becomes:

$$\dot{\lambda} = H_1 H_2 \lambda_1^D (1-D) \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} [C] \{\dot{\varepsilon}\} \tag{19}$$

Plugging the plastic multiplier back to the strain rate equation (13) and after some algebraic manipulation the following relation is derived:

$$\{\dot{\sigma}\} = (1-D)[C] \left\{ [I] - H_1 H_2 \lambda_1^D (1-D) \left[\left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\} \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} + \frac{H_3}{(1-D)^2} \left(-\frac{Y}{r} \right)^s \{\varepsilon_{el}\} \left\{ \frac{\partial \Phi}{\partial \{\sigma\}} \right\}^{\mathrm{T}} \right] [C] \right\} \{\dot{\varepsilon}\} \tag{20}$$

This way the tangent constitutive matrix can be directly calculated for each material point by replacing the classic elasticity matrix in (6) thus facilitating the solution and neglecting the need for a demanding bookkeeping mechanism.

4 NUMERICAL EXAMPLES

4.1 Cantilever beam considering plasticity only

This example is about a cantilever beam subjected to a tip point load. Material is steel with *E=210GPa*, and yield strength of s_y*=240MPa*. The dimensions of the beam are *1m* by *0.1m*. In this analysis the beam was discretized with 360 material points and 1440 points. The force is applied gradually over time until its maximum value. The force-displacement diagram at the tip of the beam is presented in Figure 1 and is compared with results from Finite Element code. In addition, the Von Mises stresses are presented in Figure 2. Results are in good agreement and the hysteretic MPM model can accurately predict both the displacement history of the beam as well as its stress state.

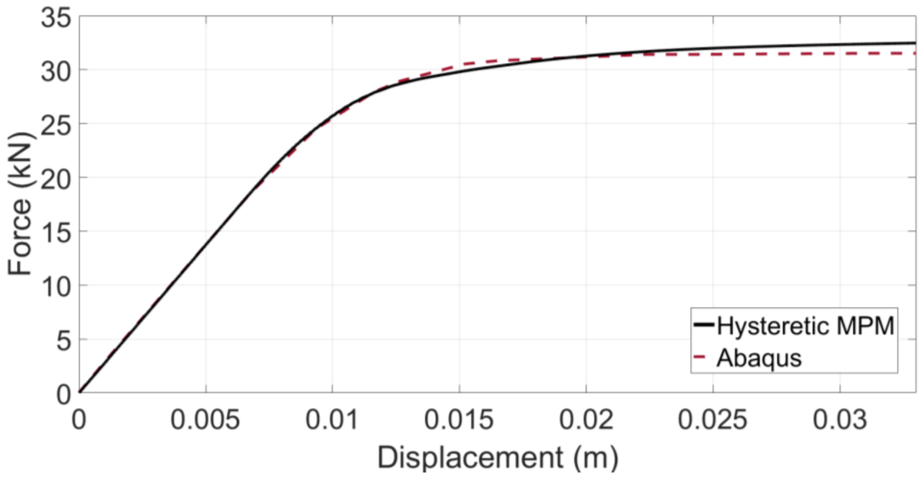

Figure 1: Beam force – displacement diagram.

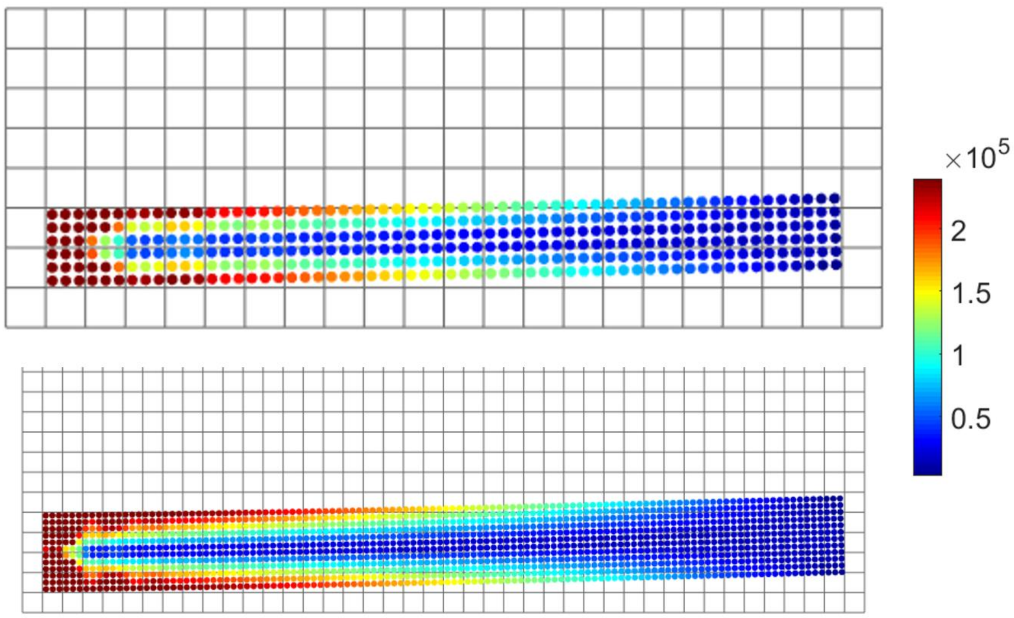

Figure 2: Von Mises stresses.

4.2 Cantilever beam considering plasticity and damage

For this example, the same cantilever beam is analyzed but this time damage is also considered. The beam is again discretized using 360 and 1440 material points. In both cases 9 points per element are used. In the first case the element size (of the background grid) is taken as 0.05m while for the second case 0.025m. Parameters regarding plasticity are kept the same. In Figure 3 the stress – strain diagram for the material point that lies on the left corner of the beam is presented. In Figure 4 the value of the damage index D across the whole beam is plotted for both discretizations. The maximum value of D is 1 and represents a failed material while when it is close to 0 the material is considered undamaged. As expected the damaged region is focused around the fixed end of the beam.

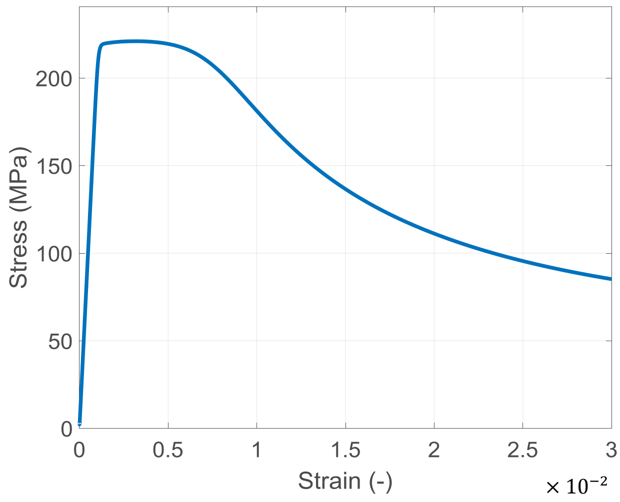

Figure 3: Von Mises Stress – Strain diagram.

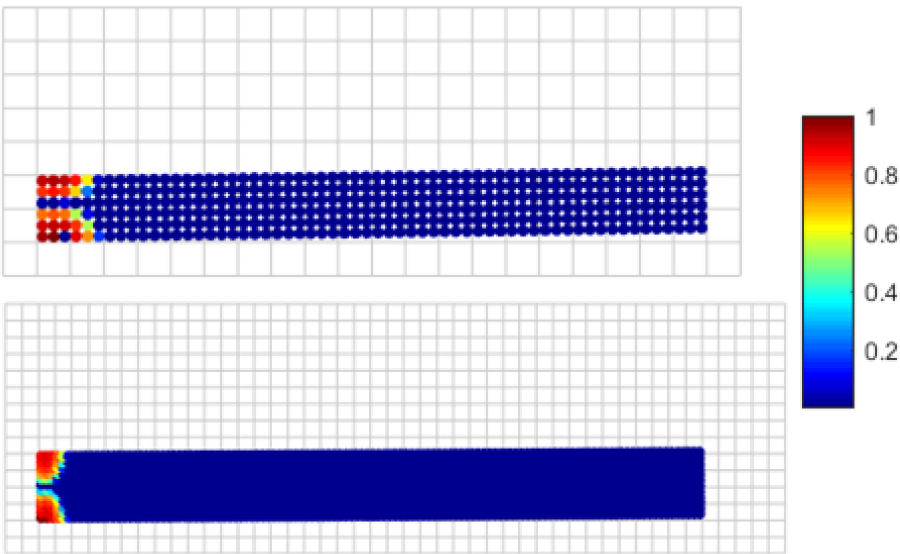

Figure 4: Damaged regions and damaged index.

5 CONCLUSIONS

- The Material Point Method is used in an explicit formulation scheme to model plasticity and damage under dynamic loading.
- The hysteretic - plasticity model for nonlinear analysis accounts for smooth transition from the elastic to the inelastic regime.

- The damage model is also smoothed.
- The plasticity and damage model have been incorporated into the MPM framework by modifying the tangent modulus of elasticity.
- Use of higher order cubic B-Splines effectively minimizes the grid crossing errors and improves the accuracy of the MPM method.
- Numerical examples are presented that verify the proposed model ability to simulate plastic and damage phenomena.

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