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# NONLINEAR STABILITY OF THE MPM METHOD

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**Abstract.** The Material Point Method (MPM) has been very successful in providing solutions to many challenging problems involving large deformations. The nonlinear nature of MPM makes it necessary to use a full nonlinear stability analysis to determine a stable timestep. The stability analysis of Spigler and Vianello is adapted to MPM and used to derive a stable timestep bound for a model problem. This bound is contrasted against a traditional CFL bound.

## 1 Introduction

The Material Point Method (MPM) may be viewed as being a solid mechanics method that is derived from the fluid implicit particle, FLIP and PIC methods and which has had considerable success on large deformation problems. Despite this success many theoretical issues to do with MPM remain unresolved. One such issue is the stability of the method, given its nonlinear nature. Currently either a fourier-based analysis e.g. [3, 10] or energy-conservation approach e.g. [1] is taken. However Wallstedt and Guilkey [14] rightly point out that the nonlinear nature of the MPM scheme makes classic linear stability analysis inappropriate. Similarly while energy conservation is of great importance it does not necessarily imply stability [11]. One way to start to address this is to note that the standard time integration methods used in MPM corresponds to the use of the semi-implicit Euler method, or symplectic Euler-A [9]. There is convergence and stability analysis of this method in [12] and this analysis is sufficiently general to be applied to the MPM, providing that care is taken with the nonlinear nature of MPM. The intention here is to use this approach to shed some light on the nonlinear stability of MPM by considering a one dimensional model problem as an ordinary differential equations system in the values at particles and nodes. While this does not address the well-known issues to do with ringing that we have previously considered [7, 3], the aim is to consider how to bound the timestep when nonlinearity is taken into account. Consequently Section 2 described the MPM method and the model problem used, while Section 3 provides the theoretical framework for the stability analysis in Section 4 which is summarized in Section 5.

## 2 A Simplified Form of MPM Method for Analysis

The description of MPM used here follows [7] in that the model problem used here is a pair of equations connecting velocity $v$, displacement $u$ and density $\rho$ (here assumed constant):

$$\frac{Du}{Dt} = v, \tag{1}$$

$$\rho \frac{Dv}{Dt} = \frac{\partial \sigma}{\partial x} + b, \tag{2}$$

with a linear stress model $\sigma = E\frac{\partial u}{\partial x}$ for which Young's modulus, $E$, is constant, a body force $b$ and with appropriate boundary and initial conditions. For convenience a mesh of equally spaced $N+1$ fixed nodes $X_i$ with intervals $I_i = [X_i, X_{i+1}]$, on on the interval $[a,b]$ is used where

$$a = X_0 < X_1 < ... < X_N = b, \tag{3}$$

$$h = X_i - X_{i-1}. \tag{4}$$

Further it is assumed that there are $m$ particles between each pair of nodes, situated at $x_p^n$ points where at each time step, $t^n = \delta t * n$, where $n$ is the nth time step, and the computed solution at the $p$th particles will be written as $u_p^n = u(x_p^n, t^n)$. Suppose that the particles in interval $i$ lie between $X_i$ and $X_{i+1}$ and have positions $x_{im+j}, j = 1,..,m$. The calculation of the internal forces in MPM at the nodes requires the calculation of the volume integral of the divergence of the stress, [14], which is written as

$$f_i^{int} = -\frac{1}{h}\sum_p D_{ip}^* \sigma_p V_p. \tag{5}$$

The coefficients $DS_{ip}^*$ may be chosen to reproduce derivatives of constant and linear functions exactly, [7], in a similar way to that used in other particle methods e.g. [4]. A further simplification is to assume uniform particle masses and that the initial volume of the particles is uniform for the $m$ particles in an interval. The particle volumes are defined using the absolute value of the deformation gradient, $|F_p^n|$, and the initial particle volume,

$$V_p^n = |F_p^n|\frac{h}{m}, \; where \; F_p^0 = 1. \tag{6}$$

From (5) the acceleration equation in MPM method after cancelling $h$ and using constant density is:

$$a_i^{n+1} = \frac{-1}{m}\left( \sum_{x_p \in I_{i-1}} D_{ip}^{n*} \sigma_p^n |F_p^n| + \sum_{x_p \in I_i} D_{ip}^{n*} \sigma_p^n |F_p^n| \right) \tag{7}$$

In the case of GIMP, [2], two extra terms involving particles $X_p \in I_{i-1}$ and $x_p \in I_{i+2}$ are also needed. The equation to update velocity at the nodes, as denoted by $v_i^n$ is then given by

$$v_i^{n+1} = v_i^n + dt a_i^{n+1}. \tag{8}$$

Using linear interpolation gives the equation for the update of the particle velocity:

$$v_p^{n+1} = v_p^n + dt[\lambda_{ip} a_i^{n+1} + (1-\lambda_{ip}) a_{i+1}^{n+1}], x_p \in I_i \tag{9}$$

where $\lambda_{ip} = \frac{x_p - X_i}{h}, x_p \in I_i$. The use of GIMP basis functions would give rise to an extended stencil involving $a_{i-1}^{n+1}$ and $a_{i+2}^{n+1}$. The equation for the particle position update is

$$x_p^{n+1} = x_p^n + v_p^{n+1} dt. \tag{10}$$

The immediate use of the updated velocity $v_p^{n+1}$ in this and subsequent equations is the Symplectic Euler Method. The update of the deformation gradients and stresses is given using their linear spatial derivative defined by :

$$\frac{\partial v^{n+1}}{\partial x}(x_p) = \frac{(v_{i+1}^{n+1} - v_i^{n+1})}{h}, x_p \in I_i. \tag{11}$$

The displacement is updated using

$$F_p^{n+1} = F_p^n + \frac{\partial v^{n+1}}{\partial x}(x_p) F_p^n dt, x_p \in I_i. \tag{12}$$

While stress is updated using the appropriate constitutive model and Young's Modulus, $E$,

$$\sigma_p^{n+1} = \sigma_p^n + dt E \frac{\partial v^{n+1}}{\partial x}(x_p), x_p \in I_i. \tag{13}$$

In the case of GIMP the derivative $\frac{(v_{i+1}^{n+1} - v_i^{n+1})}{h}$ is replaced by a four point stencil.

$$\frac{\partial v^{n+1}}{\partial x}(x_p) = \sum_{j=i-1}^{i+2} \gamma_{j,i} v_j^{n+1}, x_p \in I_i. \ where \ \frac{-1}{h} \le \gamma_{j,i} \le \frac{1}{h}. \tag{14}$$

## 3 Stability of Time Integration Using the Spigler and Vianello Approach

Spigler and Vianello [12] consider ordinary and partial differential equations of the form

$$\dot{u} = f(t, u, u), 0 < t \le T, u(0) = u_0 \tag{15}$$

and apply the semi-implicit Euler method used by MPM to this as given by:

$$u^{n+1} = u^n + dt\ f(t_n, u^{n+1}, u^n). \tag{16}$$

It is assumed that the exact solution $\bar{u}$ to the PDE satisfies the perturbed equations given by

$$\bar{u}^{n+1} = \bar{u}^n + dt\ f(t_n, \bar{u}^{n+1}, \bar{u}^n) + \delta^{n+1}, \tag{17}$$

where $\delta^{n+1}$ is the local truncation error. Spigler and Vianello introduce a perturbed scheme given by

$$\bar{v}^{n+1} = \tilde{u}^n + dt\ f(t_{n+1}, \bar{v}^{n+1}, \tilde{u}^n) + \delta^{n+1}, \tag{18}$$

$$\tilde{u}^{n+1} = \bar{v}^{n+1} + \tilde{\sigma}^{n+1} \tag{19}$$

where $\tilde{\sigma}^{n+1}$ is a local error on the current timestep. Subtracting equation (17) from (18) and adding and subtracting a term then gives

$$\bar{v}^{n+1} - u^{n+1} = \bar{u}^n - u^n + dt\ f(t_{n+1}, \bar{v}^{n+1}, \tilde{u}^n) - dt\ f(t_{n+1}, \bar{v}^{n+1}, u^n) \tag{20}$$

$$+ dt\ f(t_{n+1}, \bar{v}^{n+1}, u^n) - dt\ f(t_{n+1}, u^{n+1}, u^n) + \delta^{n+1}. \tag{21}$$

Defining the error as

$$\varepsilon^n = \bar{v}^n - u^n. \tag{22}$$

taking the inner product of equation (21) with $\varepsilon^n$ ,using Cauchy-Schwartz on the right hand side of this equation, and taking norms and using a Lipshitz condition gives the error inequality [12]

$$||\varepsilon^{n+1}|| \le (1 + dtK_2)||\tilde{u}^n - u^n|| + dtK_1||\varepsilon^{n+1}|| + ||\delta^{n+1}||. \tag{23}$$

While the quantity $K_1$ is defined by [12] via a one-sided Lipshitz condition constant, here the stronger, but equivalent, condition [8] is used

$$||f(t_n, \bar{v}^{n+1}, u^n) - f(t_n, u^{n+1}, u^n)|| \le K_3||\bar{v}^{n+1} - u^{n+1}|| \tag{24}$$

that ensures that the one-sided condition also holds if $K_1$ is replaced by $K_3$. $K_2$ is defined by [12] as being a Lipshitz constant that satisfies the equation

$$||f(t_n, \bar{v}^{n+1}, u^n) - f(t_n, \bar{v}^{n+1}, \tilde{u}^n)|| \le K_2||\tilde{u}^n - u^n||. \tag{25}$$

Regardless of which approach is used we arrive at the equation (20) in [12]:

$$||\tilde{u}^{n+1} - u^{n+1}|| \le \frac{1 + dtK_2}{(1 - dtK_3)}||\tilde{u}^n - u^n|| + \frac{||\delta^{n+1}||}{(1 - dtK_3)} + ||\tilde{\sigma}^{n+1}||. \tag{26}$$

The stability condition stated by [12] is then given by

$$dt(K_2 + K_3) \le 1. \tag{27}$$

In showing how to apply such stability results to nonlinear problems Fekete and Farago [5, 6] reference extensive earlier work, that uses locally Lipshitz continuous functions, In this case it is necessary to find a constant $R$ such that a function, say, $f(x)$ satisfies a Lipshitz condition on an open ball of center $z$ and radius $L$ denoted by $B_R$ which may depend on the timestep where

$$B_L(z) := \{y \in \mathbb{R}^m : ||y - z|| \le L\} \tag{28}$$

and the Lipshitz condition is then given on this ball by

$$||f(x) - f(y)|| \leq K||x - y||, \forall x, y \in B_L. \tag{29}$$

In order to use the [12] theory, we now define vector quantities over the number of particles. Let the total number of particles be $npt$. Then vectors of particle velocities $\mathbf{v}_p^n$, and nodal velocities $\mathbf{v}_N^n$ are defined as:

$$\mathbf{v}_p^n = \left[v_1^n, ..., v_{npt}^n\right]^T, \tag{30}$$

$$\mathbf{v}_N^n = [v_1^n, ..., v_N^n]^T. \tag{31}$$

The vectors of particle positions $\mathbf{x}_p^n$, stresses $\sigma_p^n$ and deformation gradients $\mathbf{f}_p^n$ are given by

$$\mathbf{x}_p^n = \left[x_1^n, ..., x_{npt}^n\right]^T, \tag{32}$$

$$\sigma_p^n = \left[\sigma_1^n, ..., \sigma_{npt}^n\right]^T, \tag{33}$$

$$\mathbf{f}_p^n = \left[F_1^n, ..., F_{npt}^n\right]^T. \tag{34}$$

The MPM vectors that correspond to those used by [12] are now defined by:

$$u^n = \begin{bmatrix} \mathbf{v}_N^n \\ \mathbf{v}_p^n \\ \sigma_p^n \\ \mathbf{f}_p^n \\ \mathbf{x}_p^n \end{bmatrix}, \bar{v}^n = \begin{bmatrix} \bar{\mathbf{v}}_N^n \\ \bar{\mathbf{v}}_p^n \\ \bar{\sigma}_p^n \\ \bar{\mathbf{f}}_p^n \\ \bar{\mathbf{x}}_p^n \end{bmatrix} \; and \; \tilde{u}^n = \begin{bmatrix} \tilde{\mathbf{v}}_N^n \\ \tilde{\mathbf{v}}_p^n \\ \tilde{\sigma}_p^n \\ \tilde{\mathbf{f}}_p^n \\ \tilde{\mathbf{x}}_p^n \end{bmatrix}. \tag{35}$$

The vector norm used is given by the 2 norm given by

$$||\mathbf{y}_p^n||_2 = \sqrt{\sum_{i=1}^{N_{tot}} (y_i^n)^2}, \; where \; N_{tot} = N + 4Nm. \tag{36}$$

It is useful to have the elementary result

$$\left[\sum_{j=1}^{m} b_i\right]^2 \leq m \sum_{j=1}^{m} b_i^2, for \; b_i \geq 0. \tag{37}$$

## 4 MPM with Symplectic Euler A Integration (Stress Last)

The approach of [12] is now applied to the stress-last case as described by Bardenhagen [1] which uses the Euler-A symplectic scheme discussed by [9]. The vector form of the equations for the update of velocities, stresses and deformation gradients and then positions are given by the following equations. The vector form of equations (7, 8) and (7, 9) are:

$$\mathbf{v}_N^{n+1} = \mathbf{v}_N^n + dt\mathbf{H_N}(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n), \tag{38}$$

$$\mathbf{v}_p^{n+1} = \mathbf{v}_p^n + dt\mathbf{H_p}(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n). \tag{39}$$

The vector form of equations (13), (12) and (10) are written as:

$$\sigma_p^{n+1} = \sigma_p^n + dt\mathbf{S}(\mathbf{v}_N^{n+1}), \tag{40}$$

$$\mathbf{f}_p^{n+1} = \mathbf{f}_p^n + dt\mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1}), \tag{41}$$

$$\mathbf{x}_p^{n+1} = \mathbf{x}_p^n + dt\mathbf{v}_p^{n+1}. \tag{42}$$

Using this notation and that used to define the vectors (35) the MPM method may be written as

$$\begin{bmatrix} \bar{\mathbf{v}}_N^{n+1} \\ \bar{\mathbf{v}}_p^{n+1} \\ \bar{\sigma}_p^{n+1} \\ \bar{\mathbf{f}}_p^{n+1} \\ \bar{\mathbf{x}}_p^{n+1} \end{bmatrix} - \begin{bmatrix} \mathbf{v}_N^{n+1} \\ \mathbf{v}_p^{n+1} \\ \sigma_p^{n+1} \\ \mathbf{f}_p^{n+1} \\ \mathbf{x}_p^{n+1} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{v}}_N^{p+1} \\ \bar{\mathbf{v}}_p^{n} \\ \bar{\sigma}_p^{n} \\ \bar{\mathbf{f}}_p^{n} \\ \bar{\mathbf{x}}_p^{n} \end{bmatrix} - \begin{bmatrix} \mathbf{v}_N^{n+1} \\ \mathbf{v}_p^{n+1} \\ \sigma_p^{n} \\ \mathbf{f}_p^{n} \\ \mathbf{x}_p^{n} \end{bmatrix} + dt \begin{bmatrix} \mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_N(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n) \\ \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_p(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n) \\ \mathbf{0} \\ \mathbf{G}(\tilde{\mathbf{f}}_p^n, \mathbf{v}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1}) \\ \mathbf{0} \end{bmatrix}$$

$$+dt \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{S}(\bar{\mathbf{v}}_N^{n+1}) - \mathbf{S}(\mathbf{v}_N^{n+1}) \\ \mathbf{G}(\mathbf{f}_p^n, \bar{\mathbf{v}}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1}) \\ \bar{\mathbf{v}}_p^{n+1} - \mathbf{v}_p^{n+1} \end{bmatrix}. \tag{43}$$

## 4.1 Lipshitz constants

The results of [12] require the determination of the Lipshitz constants $K_2$ and $K_3$ where:

$$\left\| \begin{bmatrix} \mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_N(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n) \\ \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_p(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n) \\ \mathbf{0} \\ \mathbf{G}(\tilde{\mathbf{f}}_p^n, \mathbf{v}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1}) \\ \mathbf{0} \end{bmatrix} \right\| \le K_2 \left\| \begin{bmatrix} \bar{\mathbf{v}}_N^n - \mathbf{v}_N^n \\ \bar{\mathbf{v}}_p^n - \mathbf{v}_p^n \\ \bar{\sigma}_p^n - \sigma_p^n \\ \bar{\mathbf{f}}_p^n - \mathbf{f}_p^n \\ \bar{\mathbf{x}}_p^n - \mathbf{x}_p^n \end{bmatrix} \right\| \tag{44}$$

$$\left\| \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{S}(\bar{\mathbf{v}}_N^{n+1}) - \mathbf{S}(\mathbf{v}_N^{n+1}) \\ \mathbf{G}(\mathbf{f}_p^n, \bar{\mathbf{v}}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1}) \\ \bar{\mathbf{v}}_p^{n+1} - \mathbf{v}_p^{n+1} \end{bmatrix} \right\| \le K_3 \left\| \begin{bmatrix} \bar{\mathbf{v}}_N^{n+1} - \mathbf{v}_N^{n+1} \\ \bar{\mathbf{v}}_p^{n+1} - \mathbf{v}_p^{n+1} \\ \bar{\sigma}_p^{n+1} - \sigma_p^{n+1} \\ \bar{\mathbf{f}}_p^{n+1} - \mathbf{f}_p^{n+1} \\ \bar{\mathbf{x}}_p^{n+1} - \mathbf{x}_p^{n+1} \end{bmatrix} \right\|. \tag{45}$$

## 4.2 Bounding the Lipshitz Conditions $K_2$

At particle position $x_p \in I_i$, the local part of the equation for $L_2$ is

$$\left[\mathbf{G}(\mathbf{f}_p^n, \bar{\mathbf{v}}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1})\right]_p = F_p \frac{(\Delta v_i - \Delta v_{i-1})}{h} \tag{46}$$

where $\Delta v_i = \left[\bar{\mathbf{v}}_N^{n+1} - \mathbf{v}_N^{n+1}\right]_i$. Writing this as a vector equation and taking norms gives:

$$||\mathbf{G}(\mathbf{f}_p^n, \bar{\mathbf{v}}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1})|| = (\sum_{p=1}^{npt} (F_p \frac{(\Delta v_i - \Delta v_{i-1})}{h})^2)^{1/2},$$

$$||\mathbf{G}(\mathbf{f}_p^n, \bar{\mathbf{v}}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1})|| \leq \frac{F_{maxp}}{h} \sqrt{m} \sum_{i=1}^{N} (\Delta v_i - \Delta v_{i-1})^2)^{1/2},$$

$$\leq \frac{F_{maxp}}{h} 2\sqrt{m} ||\Delta \mathbf{v}||, \tag{47}$$

where $F_{maxp} = max_p |F_p|$ and where the factor of $\sqrt{m}$ appears as the same gradient is used at each of $m$ particles in an interval. Similarly at the same particle position

$$\left[\mathbf{S}(\bar{\mathbf{v}}_N^{n+1}) - \mathbf{S}(\mathbf{v}_N^{n+1})\right]_p = E \frac{(\Delta v_i - \Delta v_{i-1})}{h} \tag{48}$$

and so

$$||\mathbf{S}(\bar{\mathbf{v}}_N^{n+1}) - \mathbf{S}(\mathbf{v}_N^{n+1})|| \leq |E| \frac{2\sqrt{m}}{h} ||\Delta \mathbf{v}||. \tag{49}$$

The final equation of (45) is satisfies by a Lipshitz constant with value one. Combining these results, after noting that they apply to different parts of the right side of (45), gives

$$K_3 \leq max(1, \frac{2\sqrt{m}}{h}(|E| + F_{maxp})). \tag{50}$$

### 4.3 Defining the Lipshitz Conditions for the Function $G(...)$ in Equation (44)

From equation (46) at particle position $x_p \in I_i$

$$\left[\mathbf{G}(\bar{\mathbf{f}}_p^n, \mathbf{v}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1})\right]_p = (\bar{F}_p^n - F_p^n) \frac{(v_i^{n+1} - v_{i-1}^{n+1})}{h}. \tag{51}$$

Squaring both sides gives

$$|\left[\mathbf{G}(\bar{\mathbf{f}}_p^n, \mathbf{v}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1})\right]_p|^2 \leq |(\bar{F}_p^n - F_p^n)^2 (\frac{(v_i^{n+1} - v_{i-1}^{n+1})}{h})^2, p = 1, ...Nm, \tag{52}$$

where $i$ is defined by which $x_p \in I_i$. Summing over the number of particles $p$ and using a similar argument as in Section 4.2 gives

$$||\left[\mathbf{G}(\bar{\mathbf{f}}_p^n, \mathbf{v}_N^{n+1}) - \mathbf{G}(\mathbf{f}_p^n, \mathbf{v}_N^{n+1})\right]|| \leq K_2^* ||(\bar{\mathbf{f}}_p -, \mathbf{f}_p)|| \tag{53}$$

where

$$K_2^* = Max_i |\frac{(v_i^{n+1} - v_{i-1}^{n+1})}{h}|. \tag{54}$$

### 4.4 Defining the Lipshitz Conditions for the Function $H_N(...)$ in Equation (44)

Applying the triangle inequality to the first equation in the equations defined by (44) gives:

$$||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H_N}(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| \le ||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n) - \mathbf{H_N}(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| + \\ ||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| + ||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n)||. \tag{55}$$

This condition may be broken down into three parts

$$||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n) - \mathbf{H_N}(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| + \le K_{2,2}^N ||\tilde{\mathbf{x}}_p^n - \mathbf{x}_p^n||, \tag{56}$$

$$||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| \le K_{2,0}^N ||\tilde{\sigma}_p^n - \sigma_p^n||, \tag{57}$$

$$||\mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H_N}(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n)|| \le K_{2,1}^N ||\tilde{\mathbf{f}}_p^n - \mathbf{f}_p^n||. \tag{58}$$

For which by using the properties of vector norms if follows that

$$K_2 \le K_{2,0}^N + K_{2,1}^N + K_{2,2}^N. \tag{59}$$

The $i$th component of the left side of equation (57) may be written as

$$\left[\mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)\right]_i = \tilde{a}_i^{n+1} \tag{60}$$

where

$$\tilde{a}_i^{n+1} = \frac{1}{m}\left(\sum_{p \in I_i} D_{ip}^{n*} \delta\sigma_p^n |F_p^n| + \sum_{p \in I_{i-1}} D_{ip}^{n*} \delta\sigma_p^n |F_p^n|\right) \tag{61}$$

and

$$\delta\sigma_p^n = \tilde{\sigma}_p^n - \sigma_p^n. \tag{62}$$

Upon defining

$$DF^n = max_p |D_{ip}^{n*} F_p^n| \tag{63}$$

allows equation (60) to be written as

$$|\left[\mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)\right]_i| \le \frac{1}{m} DF^n \sum_{x_p \in I_i \cup I_{i-1}} |\delta\sigma_p^n|. \tag{64}$$

Squaring both sides, summing over $i$ nodes and using (37) gives

$$||\mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)||^2 \le (\frac{1}{m} DF^n)^2 2m \sum_p (\delta\sigma_p^n)^2 \tag{65}$$

which after taking the square root gives

$$||\mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_N(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| \le (\frac{1}{m} DF^n)\sqrt{2m}||\delta\sigma|| \tag{66}$$

and so

$$K^N_{2,0} \leq (\sqrt{\frac{2}{m}} DF^n. \tag{67}$$

For equation (58) the $p$th component of the vector $\delta \mathbf{f}$ is defined by

$$\delta f^n_p = \tilde{f}^n_p - f^n_p. \tag{68}$$

After defining

$$D\sigma^n = max_p |D^{n*}_{ip} \sigma^n_p|, \tag{69}$$

and a similar argument as above leads to

$$| \left[ \mathbf{H}_N(\tilde{\mathbf{x}}^n_p, \tilde{\sigma}^n_p, \tilde{\mathbf{f}}^n_p) - \mathbf{H}_N(\tilde{\mathbf{x}}^n_p, \tilde{\sigma}^n_p, \mathbf{f}^n_p) \right]_i | \leq \frac{1}{m} D\sigma^n \sum_{x_p \in I_i \cup I_{i-1}} |\delta F^n_p|. \tag{70}$$

A similar argument as in equations (65, 66,67) then gives

$$K^N_{2,1} \leq \sqrt{\frac{2}{m}} D\sigma^n. \tag{71}$$

In the case of equation (56) the original MPM method does not satisfy a Lipshitz condition. This is seen from the dependence of the mapping constants $D^{n*}_{ip}$ on the particles $\mathbf{x}^n_p$. Let

$$\delta D^{n*}_{ip} = D^{n*}_{ip}(\tilde{\mathbf{x}}^n_p) - D^{n*}_{ip}(\mathbf{x}^n_p), \tag{72}$$

then in the case of the original MPM method (see equation (5)),

$$D^{n*}_{ip}(x^n_p) = -1, x_p \in I_{i-1}$$
$$D^{n*}_{ip}(x^n_p) = 1, x_p \in I_i$$
$$D^{n*}_{ip}(x^n_p) = 0 x_p \notin I_{i-1} \; and \; x_p \notin I_i$$

and so the values of $\delta D^{n*}_{ip}$ are either 0 if the perturbed particle does not leave the interval of the unperturbed particle or $\pm 2/h$ or $\pm 1/h$ if the perturbed particle does, regardless of the gap between the particles. Given this jump discontinuity no Lipshitz constant is possible. In contrast for the GIMP method, see (29) and Figure 4b in [13], it follows that

$$|D^{n*}_{ip}(\tilde{x}^n_p) - D^{n*}_{ip}(x^n_p)| \leq \frac{2}{l} |\tilde{x}^n_p - x^n_p| \tag{73}$$

where $l$ is the nominal width associated with the particle. Let

$$\sigma F^n = max_p |F^n_p \sigma^n_p|, \tag{74}$$

then the change in acceleration in the left side of equation (56) as denoted by $\delta a^{n+1}_i$ is given by

$$\delta a^{n+1}_i = \frac{1}{m} \left( \sum_{p \in I_i} \delta D^{n*}_{ip} \sigma^n_p |F^n_p| + \sum_{p \in I_{i-1}} \delta D^{n*}_{ip} \sigma^n_p |F^n_p| \right). \tag{75}$$

and satisfies the inequality

$$|\delta a_i^{n+1}| \leq \frac{2}{lm}\sigma F^n \sum_{p \in I_i \cup I_{i-1}} |\delta x_p^n| \tag{76}$$

where

$$\delta x_p^n = \tilde{x}_p^n - x_p^n. \tag{77}$$

Similar arguments as in the previous section give the result

$$K_{2,2}^N \leq \frac{2}{l}\sqrt{\frac{2}{m}}\sigma F^n. \tag{78}$$

### 4.5 Defining the Lipshitz Conditions for the Function $H_p(...)$ in Equation (44)

Again this equation can be broken down into three parts

$$\begin{aligned} ||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_p(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| \leq ||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n) - \mathbf{H_p}(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n)||+ \\ ||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| + ||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n)|| \end{aligned} \tag{79}$$

and three Lipstitz constants used to bound the terms on the right side of this equation:

$$||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n) - \mathbf{H}_p(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| \leq K_{2,2}^p ||\tilde{\mathbf{x}}_p^n - \mathbf{x}_p^n|| \tag{80}$$

$$||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)|| \leq K_{2,0}^p ||\tilde{\sigma}_p^n - \sigma_p^n|| \tag{81}$$

$$||\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n)|| \leq K_{2,1}^p ||\tilde{\mathbf{f}}_p^n - \mathbf{f}_p^n|| \tag{82}$$

Equation (81) is considered first using the definition in (62). Let

$$\tilde{a}_i^{n+1} = \frac{1}{m}\left(\sum_{p \in I_i} D_{ip}^{n*}\delta\sigma_p^n |F_p^n| + \sum_{p \in I_{i-1}} D_{ip}^{n*}\delta\sigma_p^n |F_p^n|\right). \tag{83}$$

$$\tilde{a}_{i+1}^{n+1} = \frac{1}{m}\left(\sum_{p \in I_{i+1}} D_{i+1p}^{n*}\delta\sigma_p^n |F_p^n| + \sum_{p \in I_i} D_{i+1p}^{n*}\delta\sigma_p^n |F_p^n|\right). \tag{84}$$

and note that from equations (7,9) the $p$th component of equation (81 is

$$\left[\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)\right]_p = \left[\lambda_{ip}\tilde{a}_i^{n+1} + (1-\lambda_{ip})\tilde{a}_{i+1}^{n+1}\right]. \tag{85}$$

Using the same approach as in equations (64) to (66) gives the inequality

$$|\left[\mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n)\right]_p| \leq= \frac{1}{m}DF^n \sum_{x_p \in I_{i-1} \cup I_i \cup I_{i+1}} |\delta\sigma_p^n| \tag{86}$$

and, as above, summing over 3 intervals and 3m particles gives

$$K_{2,0}^p = \sqrt{\frac{3}{m}}DF^n. \tag{87}$$

For equation (82) a similar argument as above again leads to

$$| \left[ \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \tilde{\mathbf{f}}_p^n) - \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \tilde{\sigma}_p^n, \mathbf{f}_p^n) \right]_i | \leq \frac{1}{m} D\sigma^n \sum_{x_p \in I_{i-1} \cup I_i \cup I_{I+1}} |\delta F_p^n| \tag{88}$$

and then

$$K_{2,1}^p = \sqrt{\frac{3}{m}} D\sigma^n. \tag{89}$$

The final case gives

$$| \left[ \mathbf{H}_p(\tilde{\mathbf{x}}_p^n, \sigma_p^n, \mathbf{f}_p^n) - \mathbf{H}_p(\mathbf{x}_p^n, \sigma_p^n, \mathbf{f}_p^n) \right]_p | \leq \frac{1}{lm} \sigma F^n \sum_{p \in I_{i-1} \cup I_i \cup I_{i+1}} |\delta x_p^n| \tag{90}$$

resulting in

$$K_{2,2}^p = \sqrt{\frac{3}{ml}} \sigma F^n. \tag{91}$$

## 5 Summary

It is now possible to define the constants in the stability condition (27). The constant $K_3$ is defined by equation (54). Collecting together the different local Lipshitz conditions with respect to the vectors multiplied by those constants from equations (56,57,58,80,81,82) gives

$$K_2 \leq K_{2,0}^N + K_{2,0}^P + K_{2,1}^N + K_{2,1}^P + K_2^* + K_{2,2}^N + K_{2,2}^P) \tag{92}$$

Bringing together (67,71,78,87,89,91), then gives

$$K_2 \leq K_2^* + \alpha(DF_n + D\sigma_n + \frac{2}{l}\sigma F_n) \tag{93}$$

where $\alpha = \frac{\sqrt{2}+\sqrt{3}}{\sqrt{m}}$. The only part of this that is a conventional CFL type condition is the coefficient $K_2^*$, however even this term depends on the velocity gradients. This expression gives additional weight to the comments in [10] about how more than a conventional CFL condition is needed. As this approach uses a quite general ODE form and only general information about stencils. It is thus possible to extend the idea to GIMP,simply be redefining the stencil widths and associated coefficient values. An extension to multi-dimensions is also possible.

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## REFERENCES


[1] S. Bardenhagen, Energy conservation error in the material point method for solid mechanics, Journal of Computational Physics, 180, 2002, 383-403.

[2] S. Bardenhagen and E. Kober, The generalized interpolation material point method, Computer Modeling in Engineering and Science, 5 (2004), 477-495.

[3] J. Brackbill, The ringing instability in particle-in-cell calculations of low-speed flow, Journal of Computational Physics, 75, 1988, 469-492.

[4] G. A. Dilts, Moving-least-squares-particle hydrodynamics consistency and stability, International Journal for Numerical Methods in Engineering, 44, 1999, 1115-1155.

[5] I. Fekete and I. Farago, Stability concepts and their applications. Computers & Mathematics with Applications, 67,12, pp. 2158 - 2170, 2014.

[6] I. Farago, M.E. Mincsovics and I. Fekete, Notes on the basic notions in nonlinear numerical analysis, EJQTDE, Proc. 9th Coll. QTDE (2012), (6), pp. 122.

[7] C.E. Gritton and M. Berzins, Improving Accuracy In the MPM Methods by Using a Null Spaces Filter, Comp. Part. Mech. 2017, 4 131-142.

[8] W. Hundsdorfer and J.G Verwer, Numerical Solution of Time Dependent Advection-Diffusion-Reaction Equations. Springer Series in Computational Mathematics 33, 2003.

[9] B. Leimkuhler and S. Reich, Simulating Hamiltoniam dynamics. Canbridge Monographs on Applied and Computational Science. Cambridge University Press 2004.

[10] M. Gong, Improving the Material Point Method, Ph.D. thesis, The University of New Mexico (2015). http://hdl.handle.net/1928/30386

[11] M. Ortiz, A note on energy conservation and stability of nonlinear time-stepping algorithms. Computers and Structures 24, 1, 1986, 167-168.

[12] R. Spigler and M, Vianello, Convergence analysis of the semi-implicit Euler method for abstract evolution equations. Numer. Func. Anal. and Opt.16, 1995, 5-6, 785-803.

[13] M. Steffen, P.C. Wallstedt, J.E. Guilkey, R.M. Kirby and M. Berzins, Examination and Analysis of Implementation Choices within the Material Point Method (MPM), Computer Modeling in Engineering & Sciences, 2008, 31, 2, 107-127.

[14] P. C. Wallstedt and J. E. Guilkey, An evaluation of explicit time integration schemes for use with the generalized interpolation material point method. J. Comput. Phys. 2008, 227, 22, 9628-9642.