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Semi-Decoupled Second-Order Consistency Correction for Smoothed Particle Hydrodynamics

Cristian V. Achim · Roberto E. Rozas · Pedro G. Toledo

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Abstract We present an approximate second-order consistent smoothed particle hydrodynamics method which uses the 1D solutions to approximate the 2D second order derivatives. The numerical tests of the analytic functions show that the method is exact for regular arrangements of interpolation points, while in the disordered areas the accuracy is lower than the exact solution of the second-order consistent modified smoothed particle hydrodynamics, but still better that the standard version or the so-called decoupled finite particle method. We applied the new model to the flow of a fluid around a circular solid obstacle and found that the use of a corrected semi-decoupled second-order consistent SPH gives better accuracy for lower resolutions allowing for a more efficient numerical model and also easier to extend to 3D.

Keywords MSPH · consistency · driven flow of solid-gas systems

1 Introduction

The modified smoothed particle hydrodynamics (MSPH) [1] and finite particle method (FPM) [2,3] are meshfree particle methods based on smoothed particle hydrodynamics (SPH) [4,5] with kernel corrections that improve the accuracy of the derivatives by imposing high order consistency. An n-order consistency C^n imposes that polynomials and their derivatives up to the nth order are exactly described for any distribution of the interpolation points. In these methods, the field variables and their derivatives are simultaneously obtained via inversion of corrective matrices which are computed at every time step for each SPH point. The FPM imposes C^1 and uses third order matrices in 2D and fourth order matrices in 3D, while the MSPH has C^2 with sixth order matrices in 2D and tenth order in 3D. While the standard SPH has been tuned and calibrated to work for modeling of incompressible fluids [6], some specific applications, such as driven flows of gas-solid mixtures, require higher order consistency [7].

C. V. Achim
Water Research Center for Agriculture and Mining (CRHIAM), Universidad de Concepción, Concepción, Chile
E-mail: cristian.v.achim@gmail.com

R. E. Rozas
Department of Physics, Universidad del Bío-Bío, Concepción, Chile
E-mail: rrozas@ubiobio.cl

P. G. Toledo
Department of Chemical Engineering, Universidad de Concepción, Concepción, Chile
E-mail: petoledo@udec.cl

However imposing a higher order consistency is in general accompanied by an increase in the accuracy [8], matrix inversions can be ill conditioned and the numerical error can be very large. This can happen in regions of space with extremely disordered configurations or for free-surfaces when the number of neighbors is too small. In order to avoid the matrix inversion problems Zhang et al [9] suggested a decoupled FPM which approximated the corrective matrix by neglecting its off-diagonal elements. This solution worked for the free-surface application, however for the driven flow of gas around solid obstacles Achim et al [10] showed that it was not enough and they presented a new way to construct decoupled corrections with C^1 using the 1D (FPM) solutions with results very close to the FPM [10] at a small computational cost. The approximations in our previous work [10] involves semi-decoupled FPM (SDFPM) and corrected semi-decoupled FPM (CSDFPM).

We expand the work done in Ref. [10] to impose second order consistency. The decoupling of the derivatives is computed using the normalized version of the kernel and its derivatives in which some of the non-diagonal elements in the correction matrices are exactly zero, while some non-diagonal cross terms are neglected. An extra advantage of our method is that in one dimension (1D) it is exact. Similar to Ref. [9], the effective quasi-diagonal matrices have no condition problems. We present two versions of SDMSPH and find that the corrected SDMPSH (CSDMSPH) gives very good results in practical applications and it can successfully replace the lower order methods, such as standard SPH and FPM for the modeling of pure fluid flows and solid-fluids flows. We present numerical tests of the second order derivatives for various selected analytic functions and finally, more important, we solve numerically the flow of a weakly compressible fluid around a solid obstacle using corrected gradients based on the new semi-decoupled MSPH (SDMSPH). The computational cost of the SDMSPH is the same as the FPM, but with higher accuracy.

2 Smoothed Particle Hydrodynamics and its Corrected Variants

In the SPH the relevant fields are interpolated from a set of points that move with the fluid [11]. In the continuum limit, for any field $f(\mathbf{r})$ the smoothed value is defined as [12,13]:

$$f(\mathbf{r}) \approx \int f(\mathbf{r}')W(|\mathbf{r}-\mathbf{r}'|,h)d\mathbf{r}', \tag{1}$$

where W is a kernel, a probability distribution function, and h is the smoothing length. The range of the kernel function can be infinite as in the case of a Gaussian function or it's limited to a few κ multiples of h, i.e. κh. All given values of the fields are approximated by the above formula in discrete form. The accuracy of the SPH is $O(h^2)$ [5,4,8].

2.1 Standard SPH

In the standard SPH [12,13], the integral in Eq. (1) can be expressed in a discrete form as follows

$$(f)_i^{SPH} \approx \sum_j f_j W(|\mathbf{r}_i - \mathbf{r}_j|,h)V_j, \tag{2}$$

where the index j goes over all particles in the range of the point where the evaluation is taking place, $\mathbf{r}_j$ denotes the position of the jth point, V_j is the associated volume, and f_j the value of the field at the respective point. Another advantage of the method is that in Eq. (2) the differential operators are applied to the kernel function, but not to f_j. The first order derivatives at $\mathbf{r}_i$ are

$$\begin{aligned}(\partial_x f)_i^{SPH} &= \sum_j f_j \partial_{x,i} W_{ij} V_j \\ (\partial_y f)_i^{SPH} &= \sum_j f_j \partial_{y,i} W_{ij} V_j,\end{aligned} \tag{3}$$

where $f_j = f(\mathbf{r}_j)$, $V_j = m_j/\rho_j$ and W_{ij} the kernel function at $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$.

Because $\sum_j \partial_{\alpha\beta,i} W_{ij} V_j$ (where α and β can be either component x or y) is zero only for the limit $h \to 0$ (when the kernel function W becomes the Dirac δ-function) [8], we use the following form for estimating the second order derivatives:

$$\begin{aligned} (\partial_{xx} f)_i^{SPH} &= \sum_j (f_j - f_i)\partial_{xx,i} W_{ij} \\ (\partial_{xy} f)_i^{SPH} &= \sum_j (f_j - f_i)\partial_{xy,i} W_{ij} \\ (\partial_{yy} f)_i^{SPH} &= \sum_j (f_j - f_i)\partial_{yy,i} W_{ij}, \end{aligned} \tag{4}$$

2.2 The Corrected Variants, FPM and MSPH

FPM and MSPH are derived similarly, the only difference between them is their order. It is sufficient to present the derivation of MSPH, because the FPM can be obtained from the same equations neglecting the higher order terms. The MSPH [1,3,2] is derived from the Taylor expansion of the field f up to the second order

$$f_j = (f)_i + (\partial_x f)_i x_{ji} + (\partial_y f)_i y_{ji} + \frac{1}{2}(\partial_{xx} f)_i x_{ji}^2 + (\partial_{xy} f)_i x_{ji} y_{ji} + \frac{1}{2}(\partial_{yy} f)_i y_{ji}^2 + \ldots \tag{5}$$

Next a linear system with six unknowns is obtained by multiplying the right and left terms by $W_{ij}V_j$, $\partial_{x,i}W_{ij}V_j$,$\partial_{y,i}W_{ij}V_j$, , $\partial_{xx,i}W_{ij}V_j$,$\partial_{xy,i}W_{ij}V_j$, or $\partial_{yy,i}W_{ij}V_j$n and performing the summation over j. Formally, the solution is:

$$\begin{bmatrix} (f)_i^{MSPH} \\ (\partial_x f)_i^{MSPH} \\ (\partial_{xx} f)_i^{MSPH} \\ (\partial_{xy} f)_i^{MSPH} \\ (\partial_{yy} f)_i^{MSPH} \end{bmatrix} = M^{-1} \sum_j \begin{bmatrix} f_j W_{ij} V_j \\ f_j \partial_{x,i} W_{ij} V_j \\ f_j \partial_{y,i} W_{ij} V_j \\ f_j \partial_{xx,i} W_{ij} V_j \\ f_j \partial_{xy,i} W_{ij} V_j \\ f_j \partial_{yy,i} W_{ij} V_j \end{bmatrix} \tag{6}$$

In practice, this is equivalent to replacing the kernel function and its derivatives with the effective corrected kernel

$$\begin{bmatrix} (W_{ij})^{MSPH} \\ (\partial_{x,i} W_{ij})^{MSPH} \\ (\partial_{y,i} W_{ij})^{MSPH} \\ (\partial_{xx,i} W_{ij})^{MSPH} \\ (\partial_{xy,i} W_{ij})^{MSPH} \\ (\partial_{yy,i} W_{ij})^{MSPH} \end{bmatrix} = M^{-1} \begin{bmatrix} W_{ij} \\ \partial_{x,i} W_{ij} \\ \partial_{y,i} W_{ij} \\ \partial_{xx,i} W_{ij} \\ \partial_{xy,i} W_{ij} \\ \partial_{yy,i} W_{ij} \end{bmatrix} \tag{7}$$

With the correction matrix

$$M = \sum_j \begin{bmatrix} 1 & x_{ji} & y_{ji} & \frac{1}{2}x_{ji}^2 & x_{ji}y_{ji} & \frac{1}{2}y_{ji}^2 \\ \partial_{x,i} & x_{ji}\partial_{x,i} & y_{ji}\partial_{x,i} & \frac{1}{2}x_{ji}^2\partial_{x,i} & x_{ji}y_{ji}\partial_{x,i} & \frac{1}{2}y_{ji}^2\partial_{x,i} \\ \partial_{y,i} & x_{ji}\partial_{y,i} & y_{ji}\partial_{y,i} & \frac{1}{2}x_{ji}^2\partial_{y,i} & x_{ji}y_{ji}\partial_{y,i} & \frac{1}{2}y_{ji}^2\partial_{y,i} \\ \partial_{xx,i} & x_{ji}\partial_{xx,i} & y_{ji}\partial_{x,i} & \frac{1}{2}x_{ji}^2\partial_{xx,i} & x_{ji}y_{ji}\partial_{xx,i} & \frac{1}{2}y_{ji}^2\partial_{xx,i} \\ \partial_{xy,i} & x_{ji}\partial_{xy,i} & y_{ji}\partial_{xy,i} & \frac{1}{2}x_{ji}^2\partial_{xy,i} & x_{ji}y_{ji}\partial_{xy,i} & \frac{1}{2}y_{ji}^2\partial_{xy,i} \\ \partial_{yy,i} & x_{ji}\partial_{yy,i} & y_{ji}\partial_{yy,i} & \frac{1}{2}x_{ji}^2\partial_{yy,i} & x_{ji}y_{ji}\partial_{yy,i} & \frac{1}{2}y_{ji}^2\partial_{yy,i} \end{bmatrix} W_{ij}V_j \tag{8}$$

The FPM is derived in the same way as the MSPH, but the second order terms in the Taylor expansion (5) and the second order derivatives are not included.

2.3 Decoupled MSPH

An ill conditioned matrix (which happens when there are not enough neighbors or the configuration of the interpolation is extremely disordered) can yield very poor results. In addition to this, Zhang & Liu noted in Ref. [9] that in most cases, the non-diagonal terms of the matrix M are small and a simplified correction can be used to achieve an accuracy similar to that of the FPM. In the new correction we make the approximations

$$M_{i,j} \simeq 0, i \neq j. \tag{9}$$

The corrected values are

$$\begin{aligned}
(f)_i^{DMSPH} &= \frac{\sum_j f_j W_{ij} V_j}{\sum_j W_{ij} V_j} \\
(\partial_x f)_i^{DMSPH} &= \frac{\sum_j f_j \partial_{x,i} W_{ij} V_j}{\sum_j x_{ji} \partial_{x,i} W_{ij} V_j} \\
(\partial_y f)_i^{DMSPH} &= \frac{\sum_j f_j \partial_{y,i} W_{ij} V_j}{\sum_j y_{ji} \partial_{y,i} W_{ij} V_j}
\end{aligned} \tag{10}$$

Same in the standard SPH, the following form for estimating the second order derivatives is used

$$\begin{aligned}
(\partial_{xx} f)_i^{DMSPH} &= \frac{\sum_j (f_j - f_i) \partial_{xx,i} W_{ij} V_j}{\frac{1}{2} \sum_j x_{ji}^2 \partial_{xx,i} W_{ij} V_j} \\
(\partial_{xy} f)_i^{DMSPH} &= \frac{\sum_j (f_j - f_i) \partial_{xy,i} W_{ij} V_j}{\sum_j x_{ji} y_{ji} \partial_{xy,i} W_{ij} V_j} \\
(\partial_{yy} f)_i^{DMSPH} &= \frac{\sum_j (f_j - f_i) \partial_{yy,i} W_{ij} V_j}{\frac{1}{2} \sum_j y_{ji}^2 \partial_{yy,i} W_{ij} V_j}
\end{aligned} \tag{11}$$

This method is very easy to implement in both 2D and 3D.

2.4 The Semi-Decoupled MSPH

The formulas presented in the previous section have the advantage of being simple, however the DFPM does not really have C^0, C^1 nor C^2 and effectively it becomes standard SPH with a normalization. As shown in Ref. [7] more than C^0 consistency is needed for the driven solid-gas systems. We start the derivation from the Taylor expansion (5), but multiplying each equation with one of the functions $\tilde{W}_{ij} V_j$, $\partial_{x,i} \tilde{W}_{ij} V_j$, $\partial_{y,i} \tilde{W}_{ij} V_j$, $\partial_{x,i} \left[\frac{\partial_{x,i} \tilde{W}_{ij}}{\sum_j x_{ji} \partial_{x,i} \tilde{W}_{ij} V_j} \right] V_j$, $\partial_{y,i} \left[\frac{\partial_{x,i} \tilde{W}_{ij}}{\sum_j x_{ji} \partial_{x,i} \tilde{W}_{ij} V_j} \right] V_j + \partial_{x,i} \left[\frac{\partial_{y,i} \tilde{W}_{ij}}{\sum_j y_{ji} \partial_{y,i} \tilde{W}_{ij} V_j} \right] V_j$, or $\partial_{y,i} \left[\frac{\partial_{y,i} \tilde{W}_{ij}}{\sum_j y_{ji} \partial_{y,i} \tilde{W}_{ij} V_j} \right] V_j$, with $\tilde{W}_{ij} = W_{ij}/S_i$ and $S_i = \sum_j W_{ij} V_j$. In the semi-decoupled MSPH (SDMSPH) the second order derivatives are

$$\begin{aligned}
(\partial_{xx} f)_i^{SDMSPH} &= \sum_j f_j \tilde{\tilde{\partial}}_{xx,i} \tilde{W}_{ij} V_j V_j \\
(\partial_{xy} f)_i^{SDMSPH} &= \sum_j f_j \tilde{\tilde{\partial}}_{xy,i} \tilde{W}_{ij} V_j V_j \\
(\partial_{yy} f)_i^{SDMSPH} &= \sum_j f_j \tilde{\tilde{\partial}}_{yy,i} \tilde{W}_{ij} V_j V_j.
\end{aligned} \tag{12}$$

These are practically the 1D MSPH solutions. The new derivative operators are defined as

$$\tilde{\tilde{\partial}}_{xx,i} = \frac{\partial_{x,i}\left[\frac{\partial_{x,i}}{\sum_j x_{ji}\partial_{x,i}\tilde{W}_{ij}V_j}\right]}{\frac{1}{2}\sum_j x_{ji}^2\partial_{x,i}\left[\frac{\partial_{x,i}\tilde{W}_{ij}}{\sum_j x_{ji}\partial_{x,i}\tilde{W}_{ij}V_j}\right]V_j} \tag{13}$$

$$\tilde{\tilde{\partial}}_{xy,i} = \frac{\partial_{y,i}\left[\frac{\partial_{x,i}}{\sum_j x_{ji}\partial_{x,i}\tilde{W}_{ij}V_j}\right] + \partial_{x,i}\left[\frac{\partial_{y,i}}{\sum_j y_{ji}\partial_{y,i}\tilde{W}_{ij}}\right]}{\sum_j x_{ji}y_{ji}\left\{\partial_{y,i}\left[\frac{\partial_{x,i}\tilde{W}_{ij}}{\sum_j x_{ji}\partial_{x,i}\tilde{W}_{ij}V_j}\right] + \partial_{x,i}\left[\frac{\partial_{y,i}\tilde{W}_{ij}}{\sum_j y_{ji}\partial_{y,i}\tilde{W}_{ij}V_j}\right]\right\}V_j}$$

$$\tilde{\tilde{\partial}}_{yy,i} = \frac{\partial_{y,i}\left[\frac{\partial_{y,i}}{\sum_j y_{ji}\partial_{y,i}\tilde{W}_{ij}V_j}\right]}{\frac{1}{2}\sum_j y_{ji}^2\partial_{y,i}\left[\frac{\partial_{y,i}\tilde{W}_{ij}}{\sum_j y_{ji}\partial_{y,i}\tilde{W}_{ij}V_j}\right]V_j}.$$

Now the SDMSPH first order derivatives are

$$\begin{aligned}(\partial_x f)_i^{SDMSPH} &= \sum_j f_j\tilde{\partial}_{x,i}V_j - (\partial_{xx}f)_i^{SDMSPH}\frac{1}{2}\sum_j x_{ji}^2\tilde{\partial}_{x,i}V_j \\ (\partial_y f)_i^{SDMSPH} &= \sum_j f_j\tilde{\partial}_{y,i}V_j - (\partial_{xx}f)_i^{SDMSPH}\frac{1}{2}\sum_j y_{ji}^2\tilde{\partial}_{y,i}V_j.\end{aligned} \tag{14}$$

The fist order derivative operators defined as:

$$\begin{aligned}\tilde{\partial}_{x,i} &= \frac{\partial_{x,i}\tilde{W}_{ij}}{\sum_j x_{ji}\partial_{x,i}\tilde{W}_{ij}V_j} \\ \tilde{\partial}_{y,i} &= \frac{\partial_{y,i}\tilde{W}_{ij}}{\sum_j y_{ji}\partial_{y,i}\tilde{W}_{ij}V_j}\end{aligned} \tag{15}$$

Finally the corrected values of the field are obtained using

$$\begin{aligned}(f)_i^{SDMSPH} &= \sum_j f_j\tilde{W}_{ij}V_j - (\partial_x f)_i^{SDMSPH}\sum_j x_{ji}\tilde{W}_{ij}V_j - (\partial_y f)_i^{SDMSPH}\sum_j y_{ji}\tilde{W}_{ij}V_j \\ &- \frac{1}{2}(\partial_{xx}f)_i^{SDMSPH}\sum_j x_{ji}^2\tilde{W}_{ij}V_j - (\partial_{xy}f)_i^{SDMSPH}\sum_j x_{ji}y_{ji}\tilde{W}_{ij}V_j \\ &- \frac{1}{2}(\partial_{yy}f)_i^{SDMSPH}\sum_j y_{ji}^2\tilde{W}_{ij}V_j.\end{aligned} \tag{16}$$

One of the advantage of the SDMPSH is that unlike the DFPM and DMSPH, the second derivative of a constant field is identically zero. In addition the corrected values of a field and its first order derivatives are coupled to the higher order derivatives. Finally the second order derivatives are almost exact when the non-diagonal terms, which we ignored in order to obtain Eq. (12), are small enough.

As shown in Ref. [10] the semi-decoupled forms can be further improved with few additional operations, but with significant improvement in results, by taking into account the non-diagonal terms

$$
\begin{aligned}
(\partial_{xx} f)_i^{CSDMSPH} &= (\partial_{xx} f)_i^{SDMSPH} - (\partial_y f)_i^{SDMSPH} \sum_j y_{ji} \tilde{\tilde{\partial}}_{xx,i} \tilde{W}_{ij} V_j \\
&- (\partial_{xy} f)_i^{SDMSPH} \sum_j x_{ji} y_{ji} \tilde{\tilde{\partial}}_{xx,i} \tilde{W}_{ij} V_j \\
&- \frac{1}{2} (\partial_{yy} f)_i^{SDMSPH} \sum_j y_{ji}^2 \tilde{\tilde{\partial}}_{xx,i} \tilde{W}_{ij} V_j \\
(\partial_{xy} f)_i^{CSDMSPH} &= (\partial_{xy} f)_i^{SDMSPH} - (\partial_x f)_i^{SDMSPH} \sum_j x_{ji} \tilde{\tilde{\partial}}_{xy,i} \tilde{W}_{ij} V_j \\
&- (\partial_y f)_i^{SDMSPH} \sum_j y_{ji} \tilde{\tilde{\partial}}_{xy,i} \tilde{W}_{ij} V_j - \frac{1}{2} (\partial_{xx} f)_i^{SDMSPH} \sum_j x_{ji}^2 \tilde{\tilde{\partial}}_{xy,i} \tilde{W}_{ij} V_j \\
&- \frac{1}{2} (\partial_{yy} f)_i^{SDMSPH} \sum_j y_{ji}^2 \tilde{\tilde{\partial}}_{xy,i} \tilde{W}_{ij} V_j \\
(\partial_{yy} f)_i^{CSDMSPH} &= (\partial_{yy} f)_i^{SDMSPH} - (\partial_x f)_i^{SDMSPH} \sum_j x_{ji} \tilde{\tilde{\partial}}_{yy,i} \tilde{W}_{ij} V_j \\
&- \frac{1}{2} (\partial_{xx} f)_i^{SDMSPH} \sum_j x_{ji}^2 \tilde{\tilde{\partial}}_{yy,i} \tilde{W}_{ij} V_j \\
&- (\partial_{xy} f)_i^{SDMSPH} \sum_j x_{ji} y_{ji} \tilde{\tilde{\partial}}_{xx,i} \tilde{W}_{ij} V_j .
\end{aligned} \tag{17}
$$

Similarly, for the first order derivatives we get

$$
\begin{aligned}
(\partial_x f)_i^{CSDMSPH} &= (\partial_x f)_i^{SDMSPH} - (\partial_y f)_i^{SDMSPH} \sum_j y_{ji} \tilde{\partial}_{x,i} \tilde{W}_{ij} V_j \\
&- (\partial_{xy} f)_i^{SDMSPH} \sum_j x_{ji} y_{ji} \tilde{\partial}_{x,i} \tilde{W}_{ij} V_j \\
&- \frac{1}{2} (\partial_{yy} f)_i^{SDMSPH} \sum_j y_{ji}^2 \tilde{\partial}_{x,i} \tilde{W}_{ij} V_j \\
(\partial_y f)_i^{CSDMSPH} &= (\partial_y f)_i^{SDMSPH} - (\partial_x f)_i^{SDMSPH} \sum_j x_{ji} \tilde{\partial}_{y,i} \tilde{W}_{ij} V_j \\
&- \frac{1}{2} (\partial_{xx} f)_i^{SDMSPH} \sum_j x_{ji}^2 \tilde{\partial}_{y,i} \tilde{W}_{ij} V_j \\
&- (\partial_{xy} f)_i^{SDMSPH} \sum_j x_{ji} y_{ji} \tilde{\partial}_{y,i} \tilde{W}_{ij} V_j .
\end{aligned} \tag{18}
$$

The corrected values of the field are obtained using Eq. (16), but with the CSDMSPH values for the first and second-order derivatives.

3 Error estimates for the gradients for the different methods

In general we are interested in a method that allows the accurate estimation of second order derivatives that appears for example as the temperature laplacian in the heat equation, and in the divergence of velocity and pressure in the Navier-Stokes equation [12,13]. The accuracy of our approximations is evaluated by computing the derivatives of several analytic functions. We also test the performance of the various methods for the flow around a cylinder. [7].

3.1 Errors estimates of analytic functions close to solid boundaries

We are mainly interested in the errors of the methods close to a circular boundary because in a driven flow the wall particles remain fixed, while the fluid particles move with a average velocity close to the inlet velocity. This can result in less ordered configurations which require kernel correction for the calculations of the gradients. For the evaluation of the analytic functions, we arranged the SPH particles in a triangular lattice (Fig. 1) with distance $\Delta x = (3/4)h$ between them. In this configuration a circular solid obstacle with diameter $D = 30h$ was placed in the middle of the simulation box, as in the previous work [7]. Inside the solid a layer of virtual particles of thickness $3h$ was created, which complete the kernel support for the SPH particles close to the boundary. The volumes are assigned so that a zero gradient condition is imposed normal to the surface and various degrees of disorder are created near the solid boundary. In Fig. 1 some of the SPH fluid have less neighbors or more than in the ordered areas. We are particularly interested in what appears in the early stages of the simulations of a flow around a fixed or moving obstacle. While, in general, additional reordering techniques can provide a better configuration of the SPH particles, we believe that testing the various kernel corrections on this configuration is sufficient for our applications. For the kernel function we use the quintic B-spline function [14]. A detailed analysis of the different kernels done in Ref. [15] indicated that this kernel has the best accuracy. The kernel $W(r;h)$ is defined by

$$W(r,h) = \alpha_D \begin{cases} \left(3-\frac{r}{h}\right)^5 - 6\left(2-\frac{r}{h}\right)^5 + 15\left(1-\frac{r}{h}\right)^5 & r \leq h \\ \left(3-\frac{r}{h}\right)^5 - 6\left(2-\frac{r}{h}\right)^5 & h \leq r < 2h \\ \left(3-\frac{r}{h}\right)^5 & 2h \leq r < 3h \end{cases} \tag{19}$$

with α_D a normalization constant.

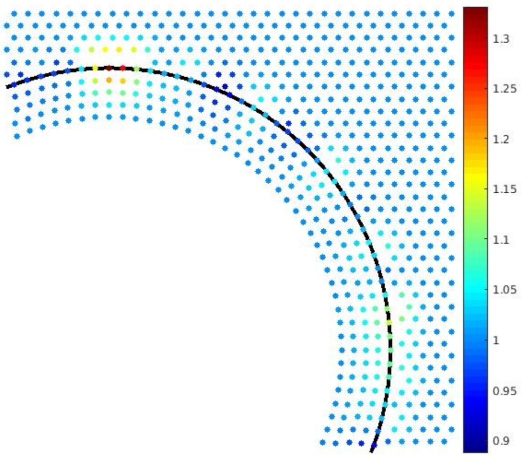

Fig. 1 The SPH interpolation points used to evaluate the gradients of the different functions. The color indicates the values of the volumes $V_j = m_j/\rho_j$. The black line gives the position of the solid boundary.

We analyzed three analytic functions, $1-(x/2-1/4)\cdot(y/2-1/4)$, $1-(x/2-1/4)^2\cdot(y/2-1/4)^2$, and $\exp[-(x/2-1/4)^2\cdot(y/2-1/4)^2]$.

The results are shown in Figures 2-7. Depending on the function, the SDMPSH and CSDMPSH can give up to one order of magnitude smaller errors. In addition we note that the SPH and

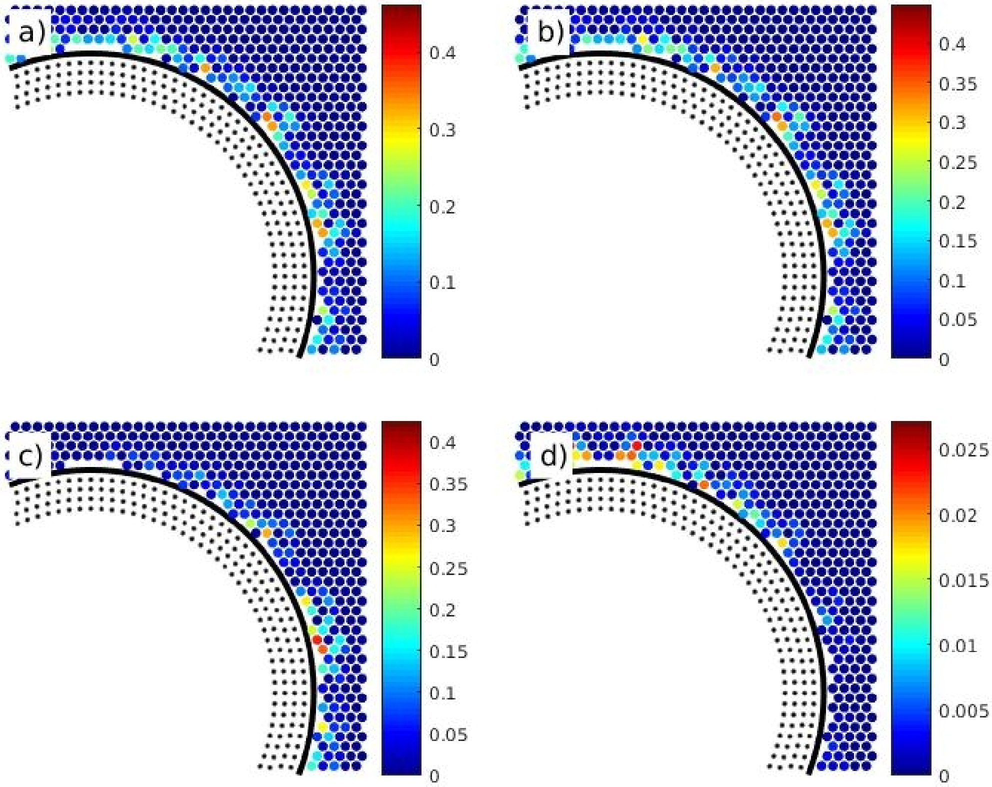

Fig. 2 Error evaluation when computing the second order x-derivative of the function $1-(x/2-1/4)\cdot(y/2-1/4)$ as given by: a) standard SPH, b) decoupled MSPH, c) semi-decoupled MSPH, and d) corrected semi-decoupled MSPH.

DMPSH are more sensitive to the particle configurations. The evaluated derivatives begin to present deviations as soon as the particle is closer than $3h$ to the solid surface.

3.2 Comparison of different methods for flow around a cylinder

Lastly, we tested the different approximations for a driven flow around a fixed circular obstacle using the same method as [7]. For each method, everything was kept the same as in FPM except for the gradients used in the equations of motion. While the SPH model of the Navier-Stokes equations can be written in a form that contains only first order derivatives, we expect that a higher order consistency will give a better accuracy to the lower derivatives as well. We present below the drag coefficients C_d for two regimes, $Re = 40$ and $Re = 100$. We present also the results for the FPM as shown in [7]. These cases are very useful to test the accuracy of computing the gradients. The results are summarized in Tables 1-4

When comparing the different methods we see that the CSDMSPH achieves convergence faster than the other methods. Plotting the drag coefficients as a function of the ratio $D/\Delta x$ with $\Delta x = 4h/3$ (Figures 8 and 9), we note that the CSDMPSH achieves convergence for $D/\Delta x = 25$. While the difference is small, the other methods require $D/\Delta x > 30$ to converge, for both the two Re numbers investigated.

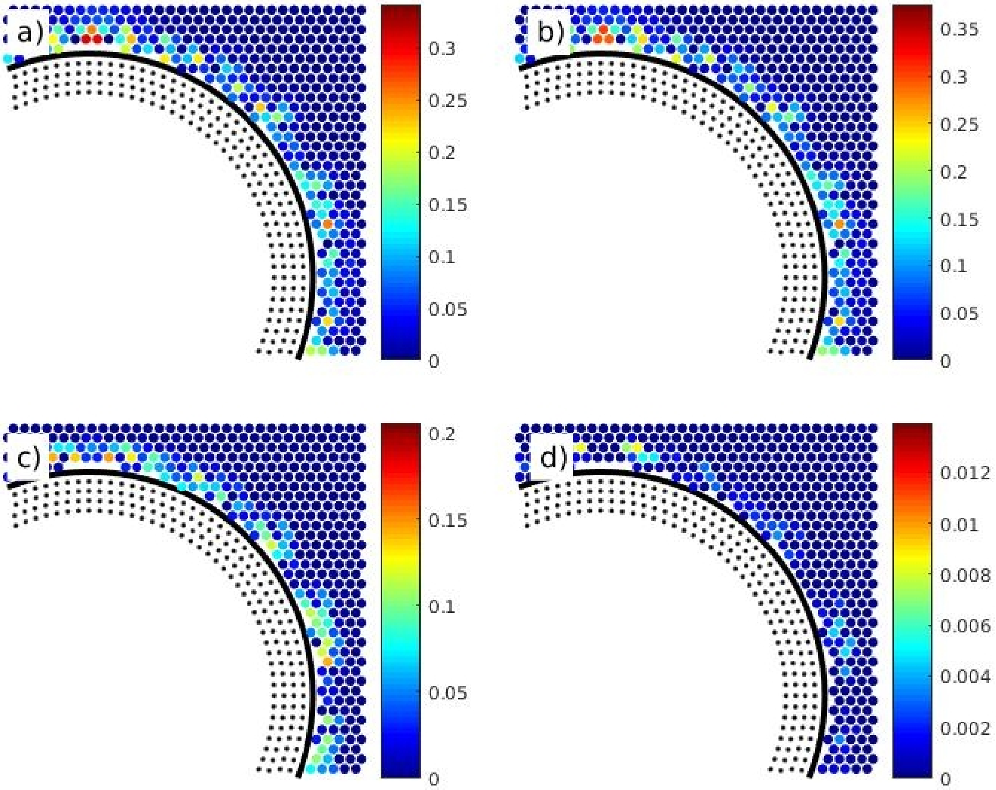

Fig. 3 Error evaluation when computing the second order xy-derivative of the function $1-(x/2-1/4)\cdot(y/2-1/4)$ as given by: a) standard SPH, b) decoupled MSPH, c) semi-decoupled MSPH, and d) corrected semi-decoupled MSPH.

Table 1 The drag coefficients for $Re = 40$ for the different methods as a function of the resolution keeping $h/\Delta x = 1.33$.

$D/\Delta x$	SDMSPH	CSDMSPH $h/\Delta x = 1$	CSDMSPH	FPM
10	1.6163 ± 0.0149	1.6161 ± 0.0220	1.6400 ± 0.0095	1.6024 ± 0.0075
15	1.6572 ± 0.0058	1.6583 ± 0.0153	1.6640 ± 0.0056	1.6440 ± 0.0054
20	1.6543 ± 0.0028	1.6638 ± 0.0114	1.6600 ± 0.0024	1.6531 ± 0.0029
25	1.6633 ± 0.0035	1.6650 ± 0.0088	1.6643 ± 0.0031	1.6588 ± 0.0035
30	1.6632 ± 0.0030	1.6600 ± 0.0083	1.6643 ± 0.0028	1.6584 ± 0.0031

Table 2 The drag coefficients for $Re = 100$ for the different methods as a function of the resolution keeping $h/\Delta x = 1.33$.

$D/\Delta x$	SDMSPH	CSDMSPH $h/\Delta x = 1$	CSDMSPH	FPM
10	1.1586 ± 0.0540	1.2080 ± 0.0350	1.1604 ± 0.0232	1.1151 ± 0.0160
15	1.3282 ± 0.0200	1.3685 ± 0.0210	1.3442 ± 0.0092	1.3252 ± 0.0093
20	1.3864 ± 0.0122	1.4031 ± 0.0134	1.3960 ± 0.0075	1.4013 ± 0.0080
25	1.4104 ± 0.0091	1.4141 ± 0.0121	1.4140 ± 0.0077	1.4158 ± 0.0076
30	1.4130 ± 0.0090	1.4166 ± 0.0115	1.4126 ± 0.0077	1.4129 ± 0.0080

4 Conclusions

Here we presented higher order corrections to impose C^2. The direct calculations of second order derivatives by CSDMSPH give smaller errors than other methods in the disordered regions. The

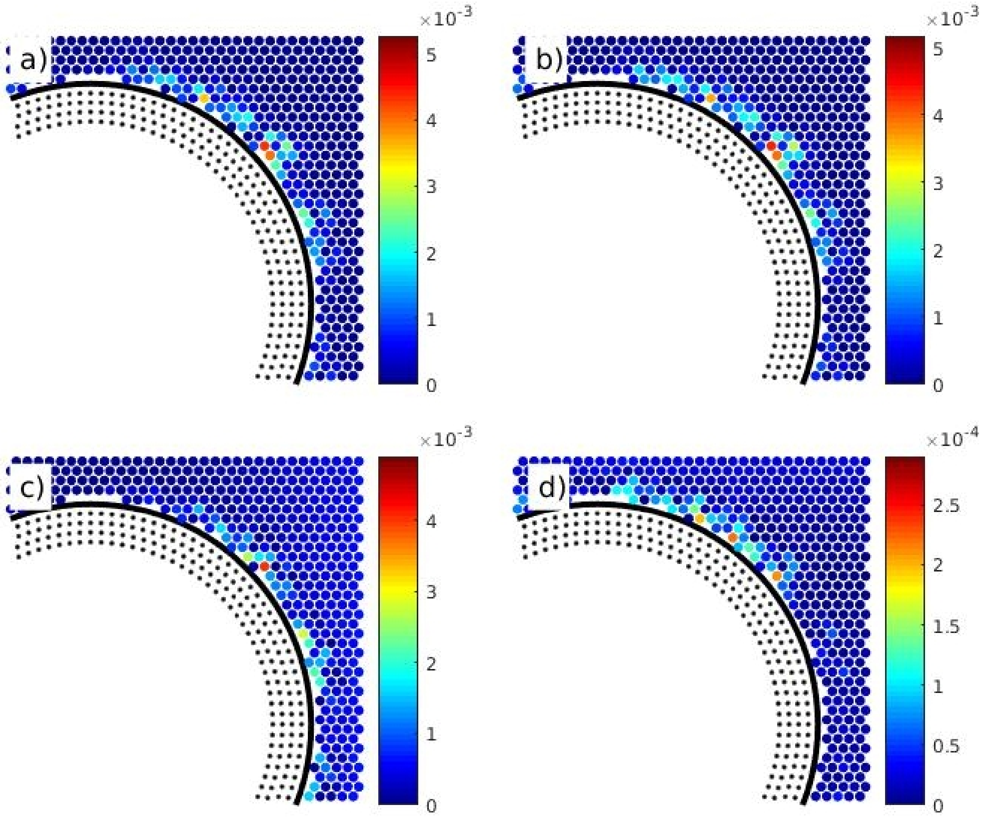

Fig. 4 Error evaluation when computing the second order x-derivative of the function $1-(x/2-1/4)^2 \cdot (y/2-1/4)^2$ as given by: a) standard SPH, b) decoupled MSPH, c) semi-decoupled MSPH, and d) corrected semi-decoupled MSPH.

Table 3 The lift coefficients for $Re = 100$ for the different methods as a function of the resolution keeping $h/\Delta x = 1.33$.

$D/\Delta x$	SDMSPH	CSDMSPH $h/\Delta x = 1$	CSDMSPH	FPM
10	±0.4380	±0.3547	±0.2452	±0.2072
15	±0.3636	±0.3877	±0.3357	±0.2962
20	±0.4075	±0.4219	±0.3617	±0.3738
25	±0.3747	±0.3830	±0.3694	±0.3700
30	±0.3807	±0.3982	±0.3644	±0.3655

Table 4 The Strouhal number for$Re = 100$ for the different methods as a function of the resolution keeping $h/\Delta x = 1.33$.

$D/\Delta x$	SDMSPH	CSDMSPH $h/\Delta x = 1$	CSDMSPH	FPM
10	0.1805	0.1751	0.1770	0.1744
15	0.1742	0.1742	0.1751	0.1758
20	0.1733	0.1751	0.1736	0.1738
25	0.1740	0.1752	0.1736	0.1735
30	0.1731	0.1747	0.1728	0.1736

corrections are useful in applications where the laplacian or bilaplacians are present in coupled equations. A surprising effect is the higher order accuracy results for the flow around a solid cylinder

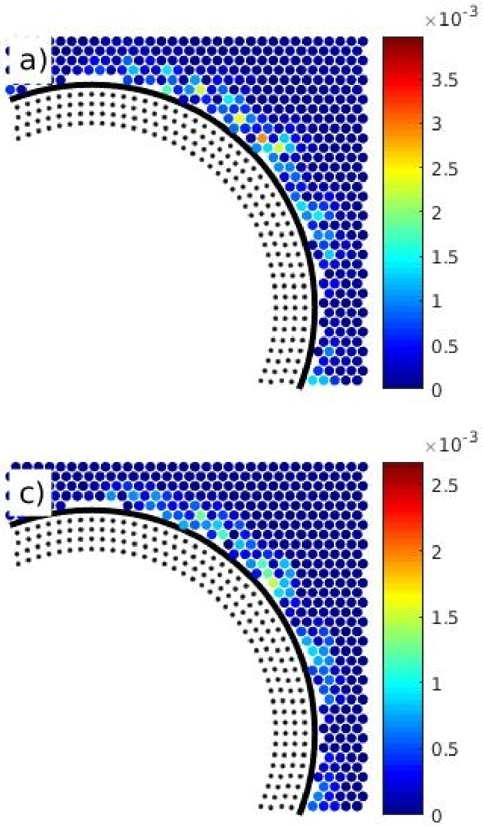

Fig. 5 Error evaluation when computing the second order xy-derivative of the function $1-(x/2-1/4)^2 \cdot (y/2-1/4)^2$ as given by: a) standard SPH, b) decoupled MSPH, c) semi-decoupled MSPH, and d) corrected semi-decoupled MSPH.

where the CSDMSPH proved to give higher accuracy than the FPM and it obtains convergence of the drag coefficients at lower values of $D/\Delta x$ for both cases studied, $Re = 40$ and $Re = 100$. In addition we tested the effect of changing the ratio $h/\Delta x$ while keeping $D/\Delta x$. Aside to having numerical stability, the drag coefficients had fairly good values. This is explain by the fact that the SPH configurations are fairly regularly due to the particle shifting technique [16, 17] which imposes the fluid particles to maintain distances close to the initial distance Δx. Unlike the case of standard δ-SPH [15] changing the ratio $h/\Delta x$ resulted in stable simulations with results very close to the FPM, but with higher standard errors in measuring the drag coefficients. This is important because decreasing the ratio $h/\Delta x$ results in decreasing the numbers of neighbors which can significantly accelerate the speed of simulation depending on the computer platform used.

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Conflict of interest

The authors declare that they have no conflict of interest.

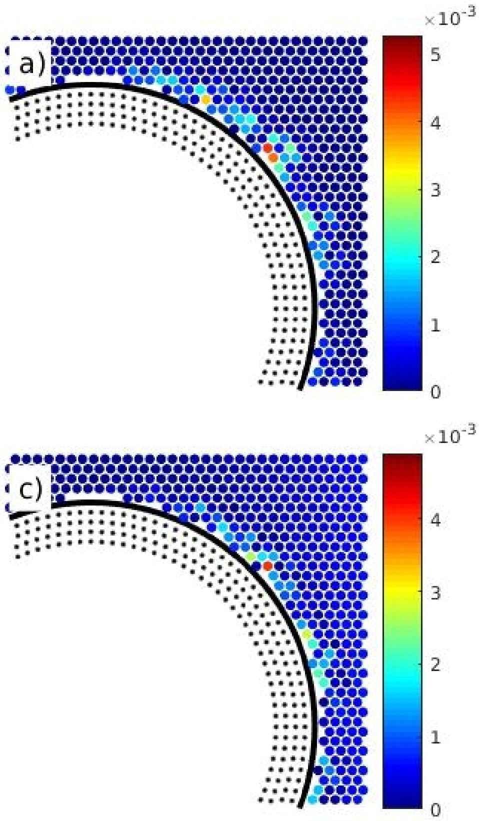

Fig. 6 Error evaluation when computing the second order x-derivative of the function $\exp[-(x/2-1/4)^2 \cdot (y/2-1/4)^2]$ as given by: a) standard SPH, b) decoupled MSPH, c) semi-decoupled MSPH, and d) corrected semi-decoupled MSPH.

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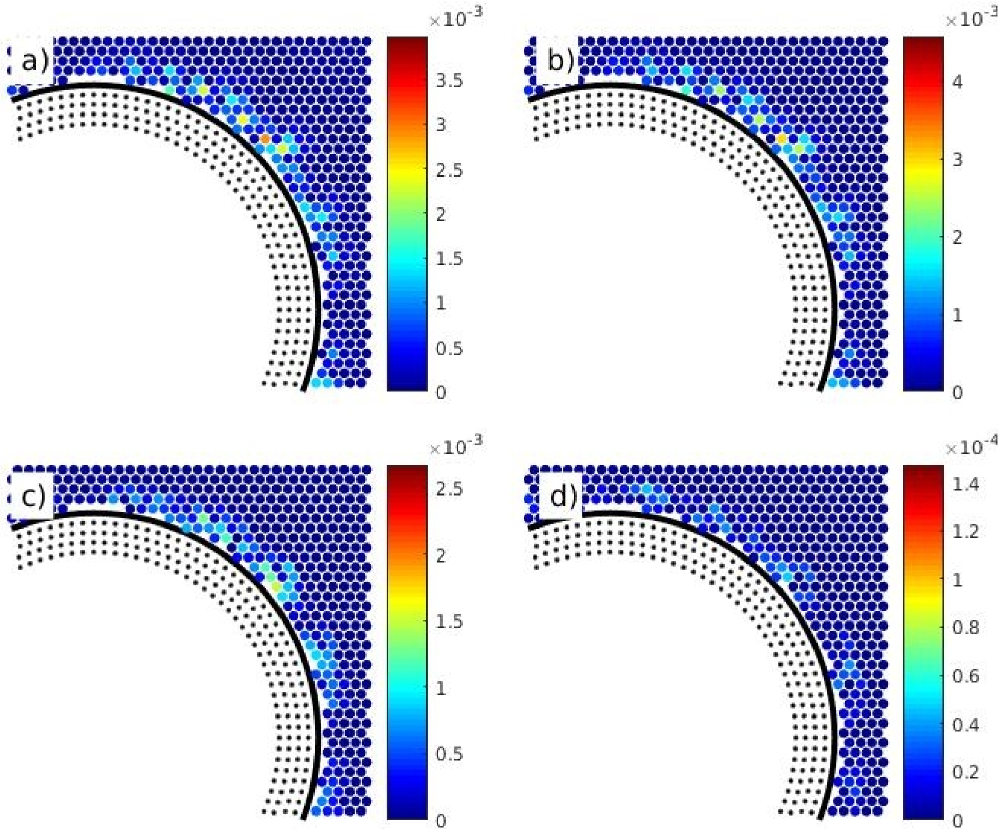

Fig. 7 Error evaluation when computing the second order xy-derivative of the function $\exp[-(x/2-1/4)^2 \cdot (y/2-1/4)^2]$ as given by: a) standard SPH, b) decoupled MSPH, c) semi-decoupled MSPH, and d) corrected semi-decoupled MSPH.

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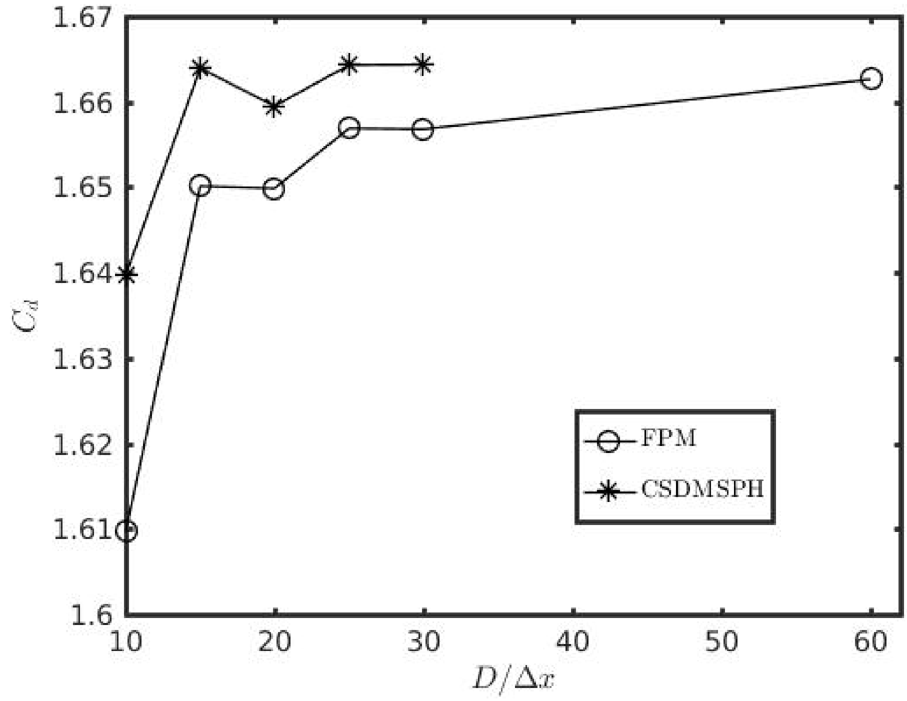

Fig. 8 The drag coefficient C_d as a function of the ratio between the particle diameter D and initial particle distance Δx for FPM ($-\circ$) and CSDMSPH ($-*$), $Re = 40$. The CSDMSPH reaches the saturated value of 1.6640 for $D/\Delta x = 15$. The ratio is significantly lower than in the case of FPM.

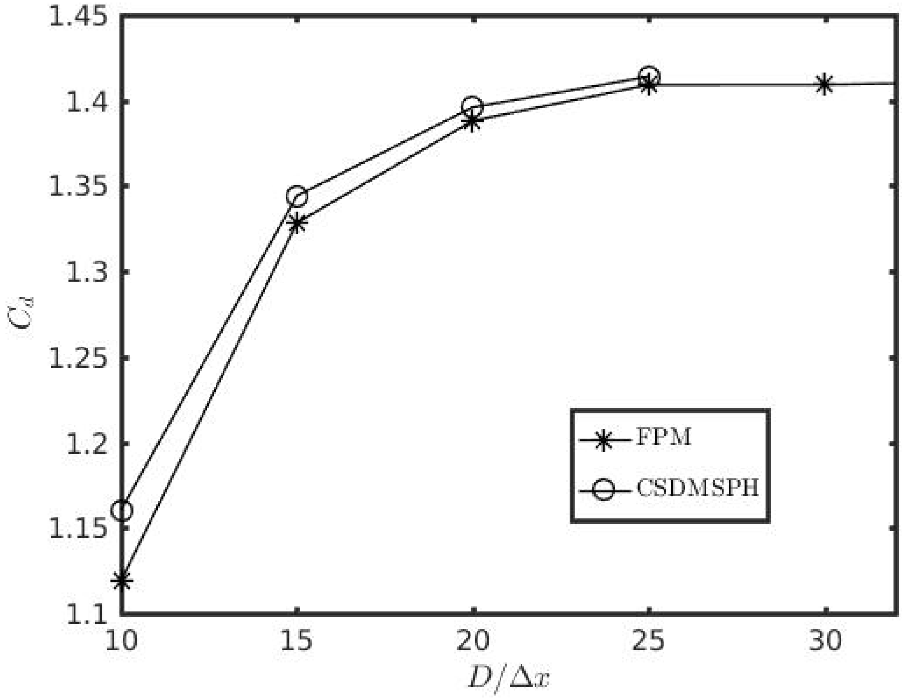

Fig. 9 The drag coefficient C_d as a function of the ratio between the particle diameter D and initial particle distance Δx for FPM ($-\circ$) and CSDMSPH ($-*$), $Re = 100$. The CSDMSPH reaches the saturated value of 1.414 for $D/\Delta x = 25$, very similar to the FPM, however for the lower resolution the values of the drag coefficient are closer to the saturated value than values obtained with FPM.