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STOCHASTIC SOLUTION OF GEOTECHNICAL PROBLEMS IN TRULY DISCRETE MEDIA

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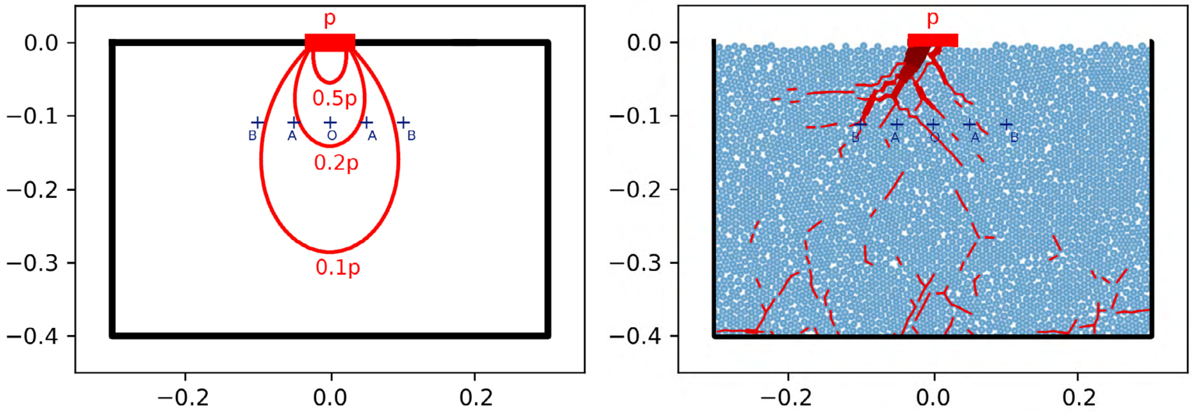
Abstract. This research deals with the solution of geotechnical problems on intermediate length scales, i.e. when the length scale of interest is larger than the size of the grains of the soil (or rockfill) but the medium cannot be considered as a continuous body. This is because on such scales, despite the large number of involved grains, the volumetric average stress fluctuates around the mean value and the fluctuation is due to the truly discrete nature of the soil. Then, the smooth stress field that would be predicted by continuum mechanics approaches is replaced by a stochastic system of interparticle forces forming force chains. The forces can be transformed into equivalent stresses by means of homogenization techniques, but the obtained fields are again non-smooth and stochastic. A classical statistical mechanics framework is followed to anticipate the probability distribution functions of equivalent (extensive) stresses according to the macroscopic constraints of the problem. In particular, we get stochastic models for two seminal problems in geotechnics: the at rest lateral earth pressure acting on a retaining wall and the vertical stress at a given point in the soil that is caused by a vertical surface load. The theory is validated through massive numerical simulation with the Discrete Element Method.
Mesoscale geotechnical analysis can find its main applications in the case of rockfill or other very coarse granular materials. However, it could be useful as well for laboratory, numerical and theoretical researches that are approached on small length scales. This theoretical framework contributes to fill the gap between micro and macro geotechnics and the resulting stochastic models may be useful for reliability analyses.

1 INTRODUCTION

There is a class of problems in soil mechanics that deals with the estimation of the stress field caused by the application of a certain load on the soil. The stress field is needed to verify whether these stresses can be withstood by the soil or to determine the deformations of the soil, which must usually remain limited.

In many seminal problems in soil mechanics the stress field was computed in the framework of the theory of elasticity [1]. For example, for the case of a vertical surface load,



(a) The solution in a continuous, homogeneous, isotropic and lineal elastic half-space obtained from continuum mechanics.

(b) A solution in a half-space made of discrete and elastic particles obtained from DEM simulation.

Figure 1: Flamant's problem (with uniform load of magnitude p on a strip of width $2a$).

Boussinesq [2], Flamant [3], Newark, etc. provided solutions by the end of the XIX$^{\text{th}}$ century. In all these cases the soil is supposed to be a half-space that is continuous, homogeneous, isotropic and linear elastic. Although this behavior may be a quite severe approximation, it can be sometimes useful as it gives reasonably accurate solutions. When this is not possible, rather complicated constitutive relationships are needed. Many constitutive models have been proposed to capture the inhomogeneous, anisotropic, non-linear or non-elastic behavior of soils [4]. These phenomenological laws are calibrated from laboratory experiments and, as common numerical methods are capable for using them, many geotechnical problems can be solved with noteworthy success. However it is not yet clear how geotechnical problems can be solved when the truly discrete nature of the soils cannot be ignored. Although such situation is not very common (since most of the time the length scale of interest is much larger than the typical size of soil particles and it behaves as a continuous body) there is no clear procedure to estimate the stresses (and their variability) in such circumstances. This could be the case of large particles (e.g. rockfill, rock blocks) or micromechanical approaches in which both length and grain scales come together and the particulated nature of soil has consequences. The most direct one is the existence of fluctuations of stresses (or forces [5]), with local values that may be much higher than the average.

Stress fluctuations are possible because the voids interrupt the continuity of the stress field from particle to particle. The support of the own weight or of any external load is provided by a system of interparticle forces, which form force chains [6, 7, 8, 9, 10, 11, 12, 13, 14]. The next two significant features invalidate continuum based approaches:

1. the stress field may sharply change from a particle to its immediate neighbors,
2. the problem is stochastic: if particles are packed in a different way, a new equilibrium is reached with a completely different system of interparticle forces.

This is the reason why the equivalent stress field of a particle can only be anticipated with some uncertainty. This work aims at establishing the statistical distribution of stress values expected at a given position, what will make it possible to get expected values with uncertainty intervals. This is done by following a statistical mechanics approach that is presented in section 2. Then this approach is applied to two seminal problems in geotechnics and verified trough numerical simulation (section 3). Results are shown in section 4 and discussed in section 5.

2 STATISTICAL MECHANICS

Statistical mechanics is the branch of physics that deals with systems made of a large number of constituents [15]. Although it was originally developed for thermal systems, several applications for granular media have been sought for [16, 17, 18, 19, 20, 21], starting from first Edward's model in 1989. Some of these approaches have been set up by considering the role played by the extensive stress (*i.e.* the product of the volumetric average of the stress within a region by the volume of that region) [19, 20, 21]. Following these ideas, a theoretical model for geotechnical applications has been set up [22]. This model is outlined in the following paragraphs:

1. There is vast number of ways of packing a granular system in static equilibrium objected to some body forces and boundary conditions. Each new random realization of an experiment will end up with one of these solutions. In the absence of any further information all the solutions are supposed to be equally likely.

2. Each packing can be partitioned into domains according to a Voronoi diagram. Each cell includes the space occupied by the particle and an associated part of void space.

3. The volumetric average of the stress field within a cell can be obtained from the interparticle forces that keep the corresponding particle in static equilibrium [23]:

$$\left\langle \sigma_{ij}^m \right\rangle = \frac{1}{V^m} \int \sigma_{ij}^m dV^m = \frac{1}{V^m} \sum_l x_i^{mn} F_j^{mn}, \tag{1}$$

   where V^m is the volume of the cell associated to particle m, F_j^{mn} is the j-component of the interaction force between particles m and n and x_i^{mn} is the i-component of the point of application of the force. The tensor product of forces by positions is the so-called extensive stress $\Sigma_{ij}^m = \sum_l x_i^{mn} F_j^{mn}$. This tensor is equal to the volumetric average of the stress field multiplied by the volume of the cell, $\Sigma_{ij}^m = \left\langle \sigma_{ij}^m \right\rangle V^m$. The extensive stress is expressed in energy units and is additive (the extensive stress of a composite body is equal to the sum of the extensive stress of its components).

4. The volumetric average of the stress field within a control volume V^c $(< V^m)$ located at $\mathbf{x}^c$, in the cell of particle m is supposed to be equal to the volumetric average of the stress of the cell. This assumption becomes true as V^c approximates V^m.

5. A statistical ensemble is an idealization consisting of a large number of virtual copies of the system randomly generated and driven according to the same procedure. Statistical samples can be generated by gathering the values of the extensive stress Σ_{ij} of a control volume V^c that is located at $\mathbf{x}^c$.

6. Normal extensive stresses (Σ_{xx},Σ_{yy},Σ_{zz}) are supposed to take any positive value provided that the average value over an ensemble is finite and corresponds to the solution of the equivalent boundary value problem (this is the value of the corresponding stress multiplied by the control volume: $\mu_{\Sigma ii} = \sigma_{ii} V^c$). Values obtained from different packings are uncorrelated and the three normal and shear components are uncorrelated from each other. Shear extensive stresses take any positive or negative value, provided that the distribution has a specified variance.

7. Under these constraints, the most probable statistical distribution of extensive normal components is an exponential distribution (similar to that of the Maxwell-Boltzmann statistics but with the extensive stress playing the role of energy):

$$f_{(\Sigma_{ii})} = \frac{1}{\mu_{\Sigma ii}} \mathrm{e}^{-\Sigma_{ii}/\mu_{\Sigma ii}}. \tag{2}$$

   For shear stresses, this model is incomplete. If either positive or negative values were possible and the variance was defined, then a normal distribution $\mathcal{N}_{\left(\mu_{\Sigma ij},\sigma_{\Sigma ij}\right)}$ would be expected, because this is the PDF of maximum entropy under such constraints. However a procedure to anticipate of such variance in a given problem is still missing.

3 METHODOLOGY

3.1 Estimation of the expected probability distribution function in two geotechnical problems

Two seminal problems in geotechnics have been analyzed in 2D, x-z plane, for the lack of simplicity: a half-space made of almost equal sized disks under its own weight and the same space supporting a vertical finite surface load.

Gravity: The gravity causes a stress field that at any point can be determined from the weight of the overlying material:

$$\sigma_{zz,\mathrm{g}} = \gamma z, \tag{3}$$

where γ is the unitary weight (in kN/m^3) and z is the depth. $\gamma = (1-n)\,\rho_\mathrm{s}\mathrm{g}$, ρ_s is the density of the material of the particles, g is the gravitational acceleration and n is the average porosity of the overlying packing.
The horizontal stress also increases with depth, but it does at a rate given the at-rest coefficient of lateral earth pressure:

$$\sigma_{xx,\mathrm{g}} = K_0 \sigma_{zz,\mathrm{g}}. \tag{4}$$

Just by the action of the gravity, vertical and horizontal stresses are aligned with principal stresses,

$$\sigma_{xz,\mathrm{g}} = 0.0. \tag{5}$$

Vertical surface load: The stress field caused by a vertical surface load is obtained from classical solutions (e.g. Boussinesq and Flamant problems, explained in [1]). In both cases the stresses depend not only on the depth but also on the horizontal distance to the applied load. In 2D, the stresses caused by a surface load p are given by:

$$\sigma_{zz,\mathrm{p}} = \frac{p}{\pi}\left[(\theta_1 - \theta_2) + \sin\theta_1\cos\theta_1 - \sin\theta_2\cos\theta_2\right], \tag{6}$$

$$\sigma_{xx,\mathrm{p}} = \frac{p}{\pi}\left[(\theta_1 - \theta_2) - \sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2\right], \tag{7}$$

and

$$\sigma_{xz,\mathrm{p}} = \frac{p}{\pi}\left[\cos{\theta_2}^2 + \cos{\theta_1}^2\right], \tag{8}$$

with $\theta_1 = \arctan{(x - X_1)/z}$ and $\theta_2 = \arctan{(x - X_2)/z}$ and X_1, X_2 the left and right limits of the surface load.

Gravity + Vertical surface load: As the material is supposed to be elastic, both solutions can be superposed in such a way that $\sigma_{ij} = \sigma_{ij,\mathrm{g}} + \sigma_{ij,\mathrm{p}}$. For any stress state, a shear indicator can be defined as the ratio of the maximum shear stress to the mean stress $s = (\sigma_1 - \sigma_3)\,/\,(\sigma_1 + \sigma_3)$.

Once the expected values of stresses are known, the PDFs of extensive stresses can be established. The mean values at any control volume are given by $\mu_{\Sigma ij} = \sigma_{ij}V^c$. For normal components, these values set the scale of the exponential distributions. Regarding extensive shear stresses, the PDF remains unknown.

Finally, when only the gravity acts, the at-rest coefficient of lateral pressure would follow the next ratio distribution:

$$f_{(K_0)} = \frac{\mu_{K0}}{(\mu_{K0} + K_0)^2}, \tag{9}$$

whose expected value would be $\mu_{K0} = \mu_{\Sigma xx}/\mu_{\Sigma zz}$.

3.2 Numerical validation

A series of numerical experiments were performed with the discrete element method [24], implemented in YADE-DEM [25][1]. A common frictional-Hookean DEM approach was followed. Two types of numerical experiments were performed: Case 1 (gravity) and Case 2 (gravity + surface load). The parameters used in the simulations are included in Table 1.

[1] `https://yade-dem.org/`.

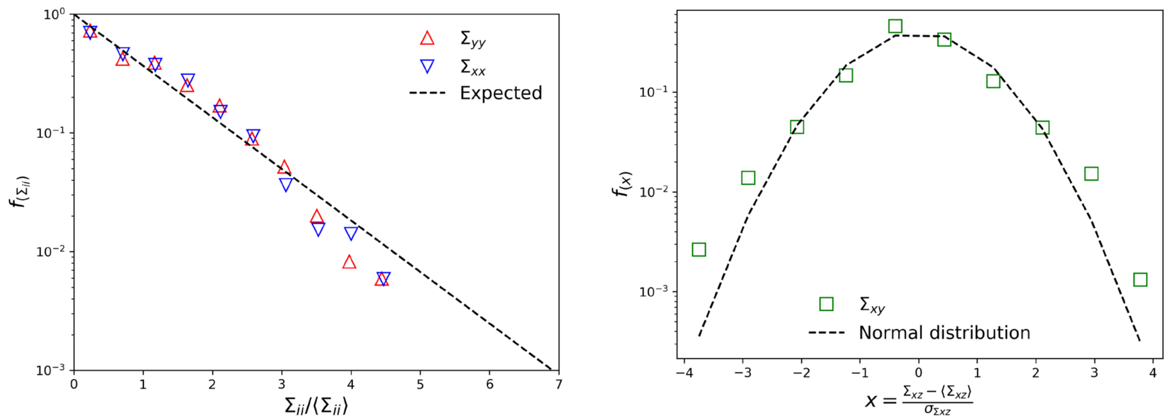
Table 1: Parameters used in the DEM numerical simulations to generate ensemble samples.

Parameter		Case1	Case2	Units
Number of particles	N	5000		-
Number of experiments	#	1812	5324	-
Simulation width	L	1.0	1.0	m
Mean diameter	D	0.01		m
Diameter dispersion	$\frac{\Delta D}{D}$	0.05		-
Young's modulus	E	$1.0 \cdot 10^7$		kPa
Material density	ρ_s	$2.6 \cdot 10^3$		kg.m^{-3}
Interparticle friction	Φ	$\pi/6$	0	rad
Loading width	$2a$	-	0.045	m
Surface load	p	-	44.4	kPa
Control point O	$(x^\mathrm{O}, z^\mathrm{O})$	$(0.00, 0.29)$	$(0.00, 0.10)$	m
Control point A	$(x^\mathrm{A}, z^\mathrm{A})$	-	$(0.08, 0.10)$	m
Control point B	$(x^\mathrm{B}, z^\mathrm{B})$	-	$(0.15, 0.10)$	m

Packings were generated by randomly pouring 5000 particles within a 1.0 m wide domain and waiting for an almost complete dissipation of the kinetic energy. The diameters of disks uniformly laid within the interval $D \pm \Delta D$. Gravity acted downwards with g = 9.81 m/s^2. Surface loads were applied by gently and vertically (downwards) moving a rigid body of length $2a$ and centered at $x = 0.0$. The simulation was stoped when the vertical reaction of the soil on the rigid element was equal to $2ap$.

A statistical sample of extensive stress values was measured at different control positions (see 1). In Case 1 the control point was located in the middle of the simulation box at a depth $z^c \pm \Delta z^c$ from the surface. In Case 2, three control points were considered: point O -right below the center of the surface load- and points A and B -located at the same depth than O but horizontally shifted (leftwards and rightwards) a certain distance-. These points were selected because the total stress induced there by the surface load $\sigma_{zz,\mathrm{p}}$ is noticeable, with respective ratios $\sigma_{zz,\mathrm{p}}/\sigma_{zz,\mathrm{g}}$ of 4.59, 2.15 and 0.65. The simulation box was large enough to ensure that the stress field caused by the surface loading p is below $0.05p$ at the boundaries. The control volume was $V^\mathrm{c} = 2.5 \cdot 10^{-5}$ m$^2 \simeq D^2/4$.

As the average height of the half-space H (and hence the porosity n) as well as the final position of the footing H_f slightly changed with the realization of the experiment, there are some uncertainties in the measurement. $H \pm \Delta H$ and $n \pm \Delta n$ were computed after performing a linear regression of the vertical stress with the depth (Eq. 3, with $z = H - h_i$). The final position of the footing and the actual surface load, with their variation intervals, are directly measured during the numerical experiments. An additional source of uncertainty is caused by the fact that the position of the control point may separate up to a distance $\simeq D/2$ from the center of the particle used to compute the extensive stress of the cell.



(a) Normal extensive stresses. (b) Shear extensive stress.

Figure 2: Expected and measured statistical distribution of extensive stresses in Case 1.

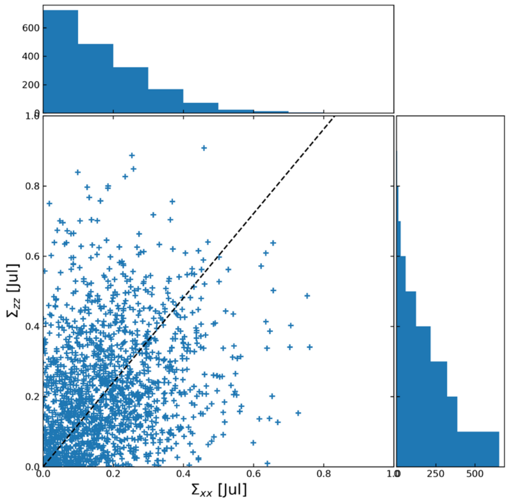
4 RESULTS

The obtained height of the half-space after pouring the particles under the action of gravity was $H = 0.49 \pm 0.01$ (average porosity $n = 0.22 \pm 0.01$) for interparticle friction angle $\phi = \pi/6$ and $H = 0.46 \pm 0.02$ m ($n = 0.15 \pm 0.03$) for frictionless particles. In Case 1, the measured at-rest coefficient of lateral earth pressure (sample mean) was $\langle K_0 \rangle = 0.83$. The expected vertical extensive stress was $\Sigma zz = (19.68 \pm 0.42) \cdot 10^{-2}$ Jul and the measured sample mean was $\langle \Sigma_{zz} \rangle = 19.42 \cdot 10^{-2}$ Jul, perfectly lying within the incertitude interval. In Fig 2 the statistical distribution of vertical and horizontal extensive stresses of the ensemble are compared to the expected exponential distributions.

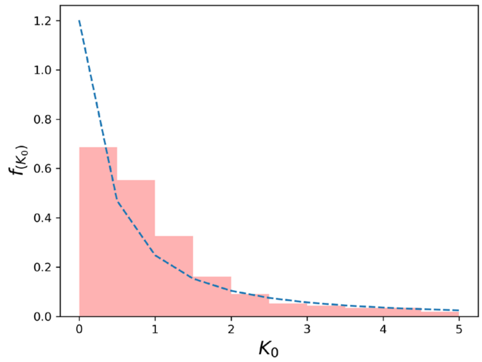
In Case 2, $\langle K_0 \rangle = 0.95$ after the gravity deposition. The action of the surface load increased shear ratios from $s_O = s_A = s_B = 0.023$ to $s_O = 0.693$, $s_A = 0.614$ and $s_B = 0.466$ and rotated the principal stressess 33.9° and 51.6° in points A and B, respectively, and did not rotate them in point O. The expected vertical extensive stress at points O, A and B were $\Sigma^O_{zz} = (33.17 \pm 1.00) \cdot 10^{-2}$ Jul, $\Sigma^A_{zz} = (18.73 \pm 1.40) \cdot 10^{-2}$ Jul and $\Sigma^B_{zz} = (9.81 \pm 1.01) \cdot 10^{-2}$ Jul. The sample mean at the control points were $\left\langle \Sigma^O_{zz} \right\rangle = 32.6 \cdot 10^{-2}$ Jul, $\left\langle \Sigma^O_{zz} \right\rangle = 17.8 \cdot 10^{-2}$ Jul and $\left\langle \Sigma^B_{zz} \right\rangle = 8.9 \cdot 10^{-2}$, lying within the interval in all the cases. In Fig. 4 the PDFs are plotted. In the three cases, the distributions seem to follow the exponential distribution predicted by the proposed model. The fitting with the exponential distribution is better in Case 2 than in Case 1, something that could be related to the higher shear ratios and the stress rotation.

5 DISCUSSION

The statistical distributions of extensive stress measured with DEM fit quite well those that were predicted under certain hypotheses: exponential distributions for extensive normal stresses and normal distribution for extensive shear stresses. The fitting is better



(a) Normal extensive stresses. The dotted line corresponds to the mean K_0 value.



(b) At-rest coefficient of earth trust. The doted line is the expected ratio distribution resulting from two independiently exponentially distributed normal stresses

Figure 3: Measured normal extensive stresses and K_0 in Case 1.

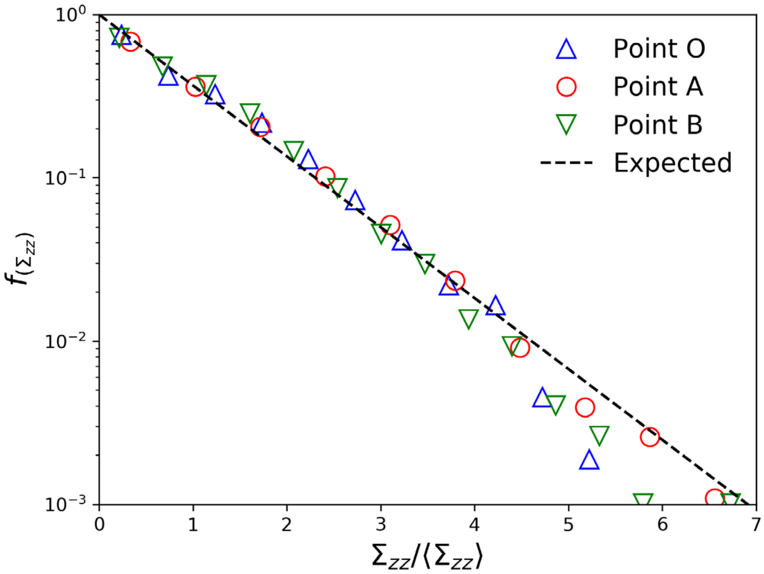


Figure 4: Expected and measured statistical distribution of vertical extensive stress in Case 2.

when a vertical surface load is acting after the deposition by gravity than in the the case that there is no such load. This could have to do with the fact that the stresses obtained after the gravity deposition were asymmetrically increased and modified during the application of the surface load. This higher level of shear and the stress rotation could have driven the distribution of forces and stresses towards the expected PDFs.
These result are interesting for geotechnical applications since they provide a way to solve geotechnical problems when particle and length scale of interest are close. For example, let be a rigid rectangular framework of width L covered with a layer of coarse granular fill (of depth H). Continuum based approaches would predict[2] that the total load on the top of the framework would correspond to the weight of the overlaying material γHL, being γ the unitary weight of the fill. However if the discrete nature of the filling is considered, the total load will fluctuate around the mean, especially when the number of particles interacting with the framework ($\mathcal{N} = L/D$) is small. The model here presented would anticipate that, with 100 particles interacting with the top of the framework, in 5% of cases the total pressure would be 20% higher than the mean value. Another example of interest could be the estimation of the total horizontal force acting on a block of an earth retaining wall that is supporting a rockfill.

6 CONCLUSIONS

- The solution of geotechnical problems in truly discrete media needs a stochastic model that provides interval estimations of the stress at a given point, rather than a point estimation.

- A simplistic model based on classical statistical mechanics has been set up to anticipate the probability distribution function of the extensive stresses (this is the average vertical stress field of a domain multiplied by its volume).

- This approach has been used to determine the PDFs of extensive stresses in two cases: an elastic half-space under its own weight and the same case with a vertical finite surface load. The mean value of the stresses is got from classical solutions.

- The model predicts that the PDF of normal components is an exponential distribution, while that of the shear extensive stresses could be normally distributed.

- Massive DEM simulation have been used to generate statistical samples of values of extensive stresses at several control points. The matching between expected and obtained PDFs is good, especially for practical purposes.

- Anticipating the PDF of extensive stresses can be very useful when the size of the discrete particles and the length scale of the problem come close. For example, this approach could provide the probability of finding stresses that double the value obtained from the corresponding continuum approach.

[2]Provided that the stiffness of the fill and framework are the same.)

- This research fills a gap between discrete and continuum geotechnical models and opens a way to treat other seminal problems in geotechnics.

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