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ACCURATE SOLUTION OF THE BOUNDARY INTEGRAL EQUATION IN 2D LAGRANGIAN VORTEX METHOD FOR FLOW SIMULATION AROUND CURVILINEAR AIRFOILS

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Abstract. The problem of numerical solution of the boundary integral equation is considered for 2D case. Viscous vortex domains (VVD) method is used for flow simulation, so vorticity is generated on the whole surface line of the airfoil, and there are a lot of vortex elements close to the airfoil. The aim of the research is to provide high accuracy of numerical solution of the integral equation; at the same time the computational complexity of the numerical algorithm should be at rather low level. The third-order accuracy numerical scheme, based on piecewise-quadratic solution representation on the curvilinear panels is presented, approximate analytic expressions are obtained for the matrix coefficients.

These schemes work perfect in the case of potential flow simulation, when vorticity is absent and also when vortex elements are placed rather far from the airfoil surface line. A trivial way to the accuracy improvement for the closely located vortices, which consists in extremely fine surface line discretization, leads to unacceptably high numerical complexity of the algorithm. This problem is solved by developing semi-analytical correction procedure which makes it possible to achieve high accuracy at extremely coarse surface line discretization. For example, in the model problem of flow simulation around elliptical airfoil with 2:1 semiaxes ratio only 20 panels are required to achieve the error level less than 1 % for arbitrary position of the vortex element in the flow.

1 Introduction

Despite the fact that vortex methods are being developed for more than 50 years, there are a lot of problems to be solved. The most part of the researchers pay their attention to the problems, connected with vorticity evolution simulation in the flow, whereas vorticity generation on the streamlined surface is much less investigated area.

Even for 2D flows, which are much easier in comparison to three-dimensinal case due to orthogonality of vorticity and velocity vectors, the existing numerical schemes for flow simulation around airfoils sometimes are based on some semi-empirical hypotheses and not fully proven; their accuracy can be rather poor, that requires very detailed discretization of the airfoil surface line. However, even detailed and uniform discretization sometimes doesn't permit to achieve high accuracy. The source of such problems is connected with the properties of the mathematical model — the boundary integral equation (BIE). In well-known modifications of 2D vortex methods the singular BIE is usually considered with Hilbert-type kernel; the corresponding integrals are understood in Cauchy sense, and the numerical procedure of its calculation is non-trivial [1, 2]. Moreover, it is not easy to provide its correct calculation for non-uniform airfoil surface line discretization.

As it is mentioned in [2], it is impossible to develop higher-order numerical scheme without explicit taking into account the curvature of the airfoil surface line. In the present paper the other approach is developed which makes it possible to consider Fredholm-type BIE of the 2-nd kind with bounded (or absolutely integrated) kernel [3, 4, 5]. This allows arbitrary airfoil surface line discretization into panels, taking into account the curvilinearity of the airfoil, developing higher-order numerical schemes according to well-known Galerkin approach. Such schemes work perfect in the case of potential flow simulation, when vorticity in the flow domain is absent or it presents, but located rather far from the airfoil.

A successive attempt was made to derive approximate analytical expressions also for the integrals, arising in the right-hand side coefficients for closely placed vortex elements, at least for piecewise-constant and piecewise-linear numerical schemes [6, 7, 8], but such representation of the numerical solution doesn't permit one to approximate the exact solution with high accuracy in principally if there are vortex elements in the flow domain, placed at the distance smaller than the panel's length to the airfoil surface line. A trivial way to the accuracy improvement which consists in extremely fine surface line discretization, leads to unacceptably high numerical complexity of the numerical algorithm, especially for flow simulation around a system of movable airfoils. In order to solve this problem, semi-analytical approach can be used which makes it possible to achieve high accuracy even for extremely coarse surface line discretization. It consists in explicit addition of the terms, which correspond to the exact solution taking into account the influence of the vortex elements placed close to the panel, which, in turn, is approximately considered as the arc of an osculating circle.

2 The governing equations

Two-dimensional flow of the viscous incompressible media is described by the Navier — Stokes equations

$$\nabla \cdot \boldsymbol{V} = 0, \qquad \frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \nu \Delta \boldsymbol{V} - \frac{\nabla p}{\rho}, \tag{1}$$

where $\boldsymbol{V}$ is the flow velocity field; p is the pressure field; $\rho = \text{const}$ and ν are the density and kinematic viscosity coefficient, respectively.

For simplicity we consider the flow around immovable airfoil, however all the results can be applied to more general case of arbitrary movable and deformable airfoil or system of airfoils. The boundary condition on the airfoil surface line K is the no-slip condition:

$$\boldsymbol{V}(\boldsymbol{r},t)=\boldsymbol{0}, \quad \boldsymbol{r}\in K.$$

The unbounded flow domain is considered, and the perturbation decay conditions are satisfied on infinity:

$$\boldsymbol{V}(\boldsymbol{r})\to\boldsymbol{V}_\infty, \quad p(\boldsymbol{r})\to p_\infty, \quad |\boldsymbol{r}|\to\infty,$$

where $\boldsymbol{V}_\infty$ and p_∞ are the velocity and pressure in the incident flow.

The most efficient modification of 2D vortex methods is the Viscous Vortex Domains method (VVD), developed by prof. G.Ya. Dynnikova and described in [10, 11]. The vorticity is a primary computational variable, and the velocity field can be reconstructed in the flow domain by using the Biot — Savart law, which can be considered as a particular case of the Generalized Helmholtz Decomposition (GHD) [3]:

$$\boldsymbol{V}(\boldsymbol{r})=\boldsymbol{V}_\infty+\frac{1}{2\pi}\oint_K\frac{\boldsymbol{\gamma}(\boldsymbol{\xi})\times(\boldsymbol{r}-\boldsymbol{\xi})}{|\boldsymbol{r}-\boldsymbol{\xi}|^2}dl_\xi+\frac{1}{2\pi}\int_S\frac{\boldsymbol{\Omega}(\boldsymbol{\xi})\times(\boldsymbol{r}-\boldsymbol{\xi})}{|\boldsymbol{r}-\boldsymbol{\xi}|^2}dS_\xi, \tag{2}$$

where $\boldsymbol{\Omega}=\Omega\boldsymbol{k}$ is known vorticity distribution in the flow domain S; $\boldsymbol{\gamma}(\boldsymbol{\xi})=\gamma(\boldsymbol{\xi})\boldsymbol{k}$ is unknown intensity of the vortex sheet on the airfoil surface line K; $\boldsymbol{k}$ is unit vector orthogonal to the flow plane.

The GHD, being considered at the airfoil surface line and taking into account the no-slip boundary condition, makes it possible to write down the BIE with respect to unknown vortex sheet intensity $\boldsymbol{\gamma}(\boldsymbol{\xi})$, $\boldsymbol{\xi}\in K$. It is proven in [3], that in order to solve it, two approaches can be used:

- the equation can be projected onto outer normal direction, that leads to "traditional" numerical schemes of vortex methods with singular BIE of the 1-st kind; the disadvantages of such approach have been mentioned above;
- the equation can be projected onto tangent direction, that allows obtaining the 2-nd kind integral equation:

$$\oint_K\frac{(\boldsymbol{r}-\boldsymbol{\xi})\cdot\boldsymbol{n}(\boldsymbol{r})}{2\pi|\boldsymbol{r}-\boldsymbol{\xi}|^2}\gamma(\boldsymbol{\xi})dl_\xi-\frac{\gamma(\boldsymbol{r})}{2}=\underbrace{-\int_S\frac{(\boldsymbol{r}-\boldsymbol{\xi})\cdot\boldsymbol{n}(\boldsymbol{r})}{2\pi|\boldsymbol{r}-\boldsymbol{\xi}|^2}\Omega(\boldsymbol{\xi})dS_\xi-\boldsymbol{V}_\infty(\boldsymbol{r})\cdot\boldsymbol{\tau}(\boldsymbol{r})}_{f(\boldsymbol{r})},\ \boldsymbol{r}\in K. \tag{3}$$

Here $\boldsymbol{n}(\boldsymbol{r})$ and $\boldsymbol{\tau}(\boldsymbol{r})$ are unit outer normal vector and tangent vector, respectively.

The unique solution of the equation (3) can be selected with help of the additional condition [1]

$$\oint_K\gamma(\boldsymbol{r})dl_r=\Gamma, \tag{4}$$

where Γ is given value of the velocity circulation along the airfoil.

3 Galerkin approach to the boundary integral equation numerical solution

Due to the boundedness (or integrability in the traditional sense) of the kernel of the BIE (3), the most efficient way to its numerical solution is the use of Galerkin method [4, 5, 12]. Let us briefly describe its main ideas.

1. We consider, that the airfoil surface line is parameterized with the arc length, then the equation (3) takes the form

$$\int_0^L Q(s,\, \sigma)\gamma(\sigma)d\sigma - \frac{\gamma(s)}{2} = f(s), \quad s \in [0,\, L], \tag{5}$$

   where L is total length of the surface line.

2. The surface line is split into N parts, traditionally called “panels”, which endings correspond to arc length parameter values s_i, $i = 0, \ldots, N$, where $s_0 = 0$, $s_N = L$; the i-th panel corresponds to s in range $[s_{i-1},\, s_i]$.

3. The basis functions family $\{\phi_i^q(s)\}$, $i = 1, \ldots, N$, $q = 0, \ldots, m$ is introduced; we assume that the functions $\phi_i^q(s)$ can have non-zero values only at the i-th panel. The projection functions family $\{\psi_i^p(s)\}$ we choose coincide with the basis one.

4. The approximate solution has the following form:

$$\gamma(s) = \sum_{i=1}^{N}\sum_{q=0}^{m} \gamma_i^q \phi_i^q(s), \tag{6}$$

   where the coefficients γ_i^q are unknown and can be found from the orthogonality condition of the equation (3) residual to the projection functions:

$$\sum_{j=1}^{N}\sum_{q=0}^{m} \gamma_j^q \int_{s_{i-1}}^{s_i} \psi_i^p(s)ds \int_{s_{j-1}}^{s_j} Q(s,\, \sigma)\phi_j^q(\sigma)d\sigma - \frac{1}{2}\sum_{q=0}^{m} \gamma_i^q \int_{s_{i-1}}^{s_i} \psi_i^p(s)\phi_i^q(s)ds =$$
$$= \int_{s_{i-1}}^{s_i} \psi_i^p(s)f(s)ds, \quad i = 1, \ldots, N, \quad p = 0, \ldots, m. \tag{7}$$

The additional condition (4) now has the following form:

$$\sum_{i=1}^{N}\sum_{p=0}^{m} \gamma_i^p \int_{s_{i-1}}^{s_i} \phi_i^p(s)ds = \Gamma. \tag{8}$$

Thus, the initial BIE (3) and the unique solution condition (4) are discretized and represented as linear system (7)-(8). The main difficulty is the calculation of its coefficients.

This problem is considered in [4, 5, 7, 8] for rectilinear panels, where piecewise-constant and piecewise-linear basis functions have been used. The first and second order of accuracy numerical schemes are developed. However they are suitable only for close to uniform airfoil discretization, and it is impossible to raise the accuracy by introducing quadratic basis functions. In order to do it, we should take into account explicitly the curvature of the airfoil. It can't be done exactly; in [6, 8] the original technique is developed for calculation of the matrix coefficients for curvilinear panels, but only for piecewise-constant and piecewise-linear basis functions. Those approximate formulae are obtained as Taylor expansions with respect to the panel length L_i, and the only terms, proportional to L_i^3 are taken into account.

Now we consider 3 families of the basis functions:

- piecewise-constant and piecewise-linear, as in [5, 8]

$$\phi_i^0(s) = \begin{cases} 1, & s \in [s_{i-1},\, s_i], \\ 0, & s \notin [s_{i-1},\, s_i]; \end{cases} \qquad \phi_i^1(s) = \begin{cases} \dfrac{s(\boldsymbol{r}) - s(\boldsymbol{c}_i)}{L_i}, & s \in [s_{i-1},\, s_i], \\ 0, & s \notin [s_{i-1},\, s_i]; \end{cases}$$

- piecewise-quadratic

$$\phi_i^2(s) = \begin{cases} 4\left(\dfrac{s(\boldsymbol{r}) - s(\boldsymbol{c}_i)}{L_i}\right)^2 - \dfrac{1}{3}, & s \in [s_{i-1},\, s_i], \\ 0, & s \notin [s_{i-1},\, s_i]. \end{cases}$$

Here L_i is the length of the i-th panel, $\boldsymbol{c}_i$ is its center. Note, that the introduced in such a way basis functions are orthogonal.

The linear system (7)-(8) now has the following matrix form

$$\begin{pmatrix} A^{00} + D^{00} & A^{01} & A^{02} & I \\ A^{10} & A^{11} + D^{11} & A^{12} & O \\ A^{20} & A^{21} & A^{22} + D^{22} & O \\ L^0 & O & O & 0 \end{pmatrix} \begin{pmatrix} \gamma^0 \\ \gamma^1 \\ \gamma^2 \\ R \end{pmatrix} = \begin{pmatrix} b^0 \\ b^1 \\ b^2 \\ \Gamma \end{pmatrix},$$

where A^{pq} are matrix blocks of $N \times N$ size; D^{pp} are diagonal matrices; b^p is the right-hand side vector parts; $\gamma^p = (\gamma_1^p,\, \dots,\, \gamma_N^p)^T$ is vector of unknown coefficients, $p = 0,\, 1,\, 2$; I and O are the vectors/raws consist of units and zeros, respectively, L^0 is a raw consists of curvilinear panel lengthes; R is regularization variable [1].

The matrix and right-hand side coefficients are calculated as the following integrals:

$$A_{ij}^{pq} = \int_{K_i} \phi_i^p(s) ds \left(\int_{K_j} Q(s,\sigma) \phi_j^q(\sigma) d\sigma \right), \quad D_{ii}^{pp} = -\frac{1}{2} \int_{K_i} \phi_i^p(s) \phi_i^p(s) ds,$$

$$b_i^p = \int_{K_i} \phi_i^p(s) f(s) ds, \quad i,\, j = 1,\, \dots,\, N, \quad p,\, q = 0,\, 1,\, 2. \tag{9}$$

The diagonal coefficients D_{ii}^{pp} can be calculated exactly:

$$D_{ii}^{00} = -\frac{L_i}{2}, \quad D_{ii}^{11} = -\frac{L_i}{24}, \quad D_{ii}^{11} = -\frac{2L_i}{45}.$$

For the A_{ij}^{pq} coefficients approximate calculation Taylor expansions technique similar to [6] is used, but now term of order L_i^4 also should be taken into account.

We denote the "signed curvature" as $\varkappa(s) = (\boldsymbol{r}'(s) \times \boldsymbol{r}''(s)) \cdot \boldsymbol{k}$, where $\varkappa > 0$ for the convex parts of the airfoil surface line; the formulae, similar to Serret — Frenet ones can be easily obtained for the derivatives of the vectors $\boldsymbol{n}(s)$ and $\boldsymbol{\tau}(s)$.

For the diagonal components of the matrices A_{ii}^{pq}, $p, q = 0, 1, 2$ we obtain

$$A_{ii}^{00} \approx \frac{36L_i^2\varkappa_i + L_i^4\varkappa_i''}{144\pi}, \quad A_{ii}^{01} \approx \frac{\varkappa_i' L_i^3}{144\pi}, \quad A_{ii}^{02} \approx \frac{\varkappa_i'' L_i^4}{2160\pi},$$
$$A_{ii}^{10} \approx \frac{\varkappa_i' L_i^3}{72\pi}, \quad A_{ii}^{11} \approx \frac{\varkappa_i'' L_i^4}{3456\pi}, \quad A_{ii}^{12} \approx 0, \quad A_{ii}^{20} \approx \frac{\varkappa_i'' L_i^4}{720\pi}, \quad A_{ii}^{21} \approx 0, \quad A_{ii}^{22} \approx 0.$$

Here $\varkappa_i$ is the signed curvature at the center of the i-th panel, the prime mark denotes the derivative with respect to the arc length.

For non-diagonal coefficients, which calculation requires integration over different panels ($i \neq j$), we introduce auxiliary vector $\boldsymbol{d}_{ij} = \boldsymbol{c}_i - \boldsymbol{c}_j$, which connects centers of the corresponding panels, unit tangent vector $\boldsymbol{\tau}_i$ at the center of the i-th panel, and the angles α and β between the vectors $\boldsymbol{\tau}_i$, $\boldsymbol{\tau}_j$ and $\boldsymbol{d}_{ij}$, respectively (Fig. 1).

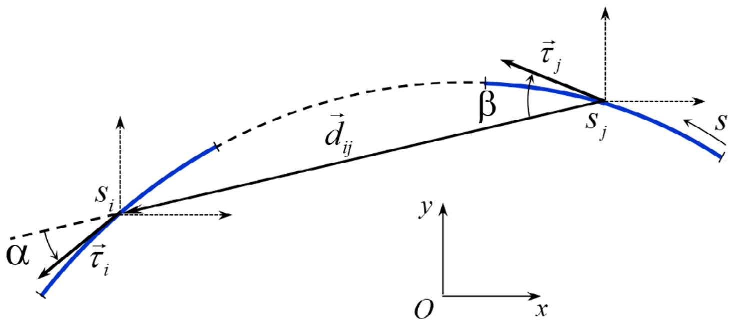

Figure 1: Two curvilinear panels, vector $\boldsymbol{d}_{ij}$ and the angles α and β

The resulting formulae have the following form:

$$A_{ij}^{00} \approx \frac{L_i L_j}{48\pi d^3}\Big[2\big(L_j^2 \sin(\alpha+2\beta) + 12d^2 \sin\alpha + L_i^2 \sin 3\alpha\big) +$$
$$+ d\Big(L_j^2 \varkappa_j \cos(\alpha+\beta) + L_i^2\big(d\varkappa_i' \cos\alpha - \varkappa_i(3\cos 2\alpha + d\varkappa_i \sin\alpha)\big)\Big)\Big],$$
$$A_{ij}^{01} \approx \frac{L_i L_j^2 \sin(\alpha+\beta)}{24\pi d^2}, \quad A_{ij}^{02} \approx \frac{L_i L_j^3\big(d\varkappa_j \cos(\alpha+\beta) + 2\sin(\alpha+2\beta)\big)}{180\pi d^3},$$
$$A_{ij}^{10} \approx \frac{L_i^2 L_j \cos\alpha\big(d\varkappa_i - 2\sin\alpha\big)}{24\pi d^2}, \quad A_{ij}^{11} \approx \frac{L_i^2 L_j^2\big(d\varkappa_i \cos(\alpha+\beta) - 2\sin(2\alpha+\beta)\big)}{288\pi d^3},$$
$$A_{ij}^{20} \approx \frac{L_i^3 L_j}{180\pi d^3}\Big[2\sin 3\alpha - d\big(\varkappa_i(3\cos 2\alpha + d\varkappa_i \sin\alpha) - d\varkappa_i' \cos\alpha\big)\Big],$$
$$A_{ij}^{12} \approx 0, \quad A_{ij}^{21} \approx 0, \quad A_{ij}^{22} \approx 0.$$

Here $\varkappa_i$ is the curvature at the center of the i-th panel, the prime mark denotes the derivative with respect to the arc length; d is the length of the vector $\boldsymbol{d}_{ij}$.

In case of smooth airfoil the following approximate formulae can be used for the matrix coefficients which calculation requires integration over the neighboring panels ($|i-j|=1$):

$$A_{ij}^{00} \approx \frac{L_i L_j}{288\pi}\Big[72\varkappa_{ij} \pm 12\big(L_j - 2L_i\big)\varkappa'_{ij} + \big(6L_i^2 - 3L_iL_j + 2L_j^2\big)\varkappa''_{ij}\Big],$$

$$A_{ij}^{01} \approx \frac{L_i L_j^2\big(4\varkappa'_{ij} \pm \big(L_j - L_i\big)\varkappa''_{ij}\big)}{576\pi}, \quad A_{ij}^{10} \approx \frac{L_i^2 L_j\big(8\varkappa'_{ij} \pm \big(L_j - 3L_i\big)\varkappa''_{ij}\big)}{576\pi},$$

$$A_{ij}^{02} \approx \frac{L_i L_j^3 \varkappa''_{ij}}{2160\pi}, \quad A_{ij}^{11} \approx \frac{L_i^2 L_j^2 \varkappa''_{ij}}{3456\pi}, \quad A_{ij}^{20} \approx \frac{L_i^3 L_j \varkappa''_{ij}}{720\pi}, \quad A_{ij}^{12} \approx A_{ij}^{21} \approx A_{ij}^{22} \approx 0.$$

Here $\varkappa_{ij}$ denotes the signed curvature of the airfoil at the common point of the neighboring panels; the prime, as earlier, means the derivative with respect to the arc length; sign "+" is used for $j=i+1$, and "−" for $j=i-1$.

Note, that in case of the airfoil with sharp edges [9], the formulae for rectilinear panels derived in [7, 8] are more suitable for the matrix coefficients corresponding to the panels which are adjacent to the angle points. Those formulae don't permit to take into account the curvilinearity of the panels, however, they are exact for rectilinear panels.

Let us consider firstly the potential flow when there is no vorticity in the flow domain and the right-hand side of the equation (5) has form $f(s) = -\boldsymbol{V}_\infty \cdot \boldsymbol{\tau}(s)$. In this case the right-hand side components of (3) can be calculated as following:

$$b_i^0 \approx -(\boldsymbol{V}_\infty \cdot \boldsymbol{\tau}_i)L_i + \frac{1}{24}\left((\boldsymbol{V}_\infty \cdot \boldsymbol{n}_i)\varkappa'_i + (\boldsymbol{V}_\infty \cdot \boldsymbol{\tau}_i)\varkappa_i^2\right)L_i^3,$$

$$b_i^1 \approx \frac{1}{12}(\boldsymbol{V}_\infty \cdot \boldsymbol{n}_i)\varkappa_i L_i^2, \qquad b_i^2 \approx \frac{1}{90}\left((\boldsymbol{V}_\infty \cdot \boldsymbol{n}_i)\varkappa'_i + (\boldsymbol{V}_\infty \cdot \boldsymbol{\tau}_i)\varkappa_i^2\right)L_i^3.$$

The described approach provides the 1^{st}, 2^{nd} and 3^{rd} order of accuracy for the piecewise-constant, linear and quadratic solution representation, respectively. In the Table 1 the number of panels is shown, which is required to achieve the accuracy 10^{-3} (for unit incident flow velocity, angle of incidence $\pi/6$).

Table 1: Number of panels required to achieve the accuracy 10^{-3} for elliptical airfoils

	rectilinear panels		curvilinear panels	
uniform discretization				
semiaxes ratio	piecewise-constant	piecewise-linear	piecewise-linear	piecewise-quadratic
2:1	9 600	244	136	52
5:1	13 400	760	360	196
10:1	24 000	2 000	900	640
non-uniform discretization				
2:1	7 800	132	104	36
5:1	8 700	152	144	60
10:1	9 000	186	184	80

Note, that it seems to be reasonable to use non-uniform discretization of the airfoil (length of the panels are inversely proportional to the square root of the curvature) since it permits to reduce number of panels significantly; usage of the piecewise-quadratic solution representation for curvilinear panels makes it possible to reduce additionally number of panels by 2.5–3 times.

4 Vortex elements influence accounting by numerical solution correction

In practice there are a lot of vortex elements in the flow, which simulate vorticity distribution. However, due to the linearity of the governing integral equation (5), the influences of separate vortices can be taken into account independently, so here we consider a model problem, when there is only one vortex element. In this case the only difference in numerical scheme is the form of the right-hand side term $f(s)$, for which the corresponding coefficients b_i^p of the linear system can be calculated either numerically (e.g., by using Gaussian quadrature formulae), of approximately analytically [6]. In the Fig. 2, the results are shown for the cases, when the vortex element is placed at the distance, which corresponds to 10 %, 25 %, 50 % and 100 % of the panel length (uniform discretization of the elliptical airfoil with 2:1 semiaxes ratio, split into 20 panels is considered).

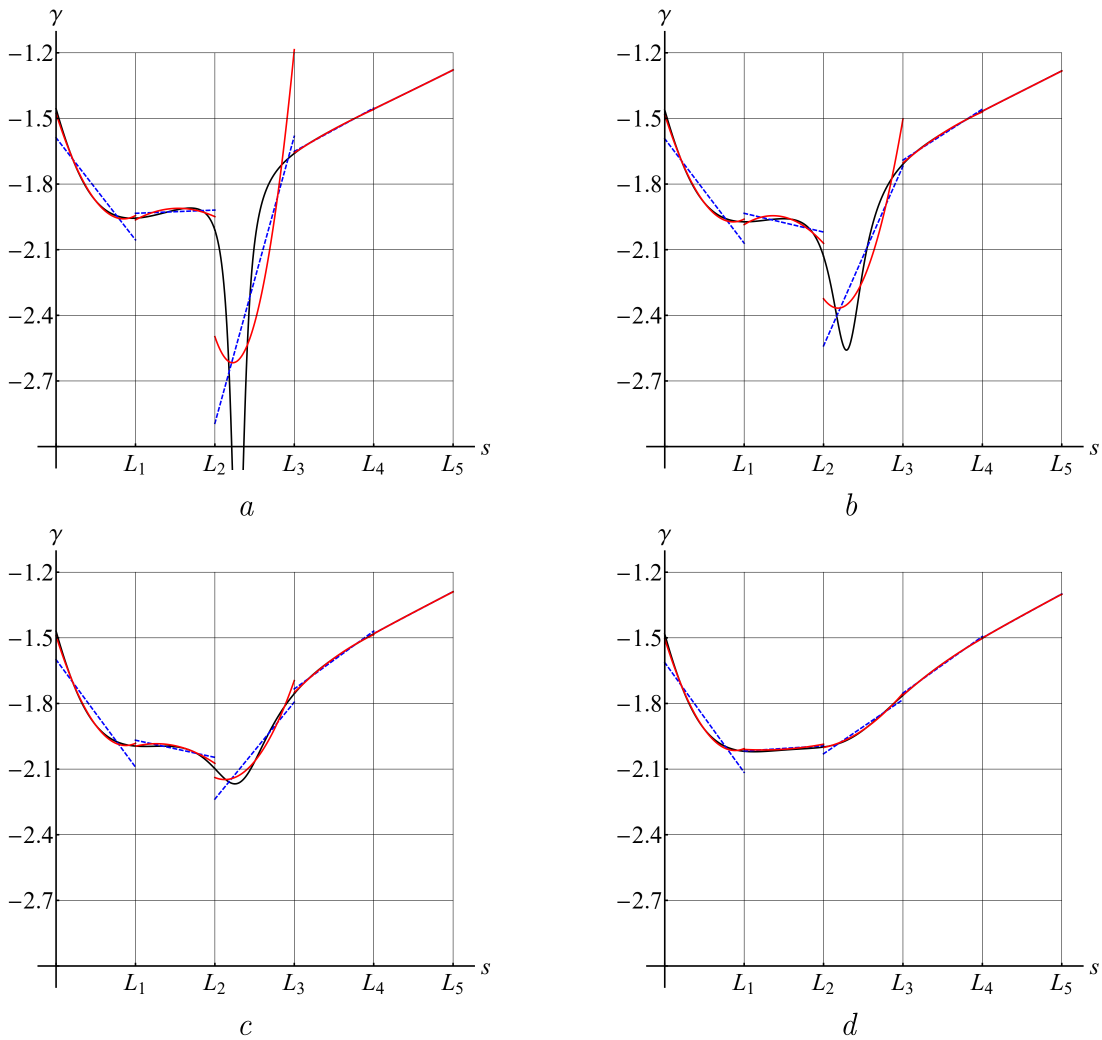

Figure 2: Exact solution (black solid line), piecewise-linear (blue dashed) and piecewise-quadratic (red solid) solutions for the vortex sheet intensity in presence of the vortex at the distance of 10 %, 25 %, 50 % and 100 % of panel size (*a*, *b*, *c*, *d*, respectively)

It is seen, that it is possible to obtain more or less correct numerical solution only for vortex elements, placed rather far from the airfoil surface line, i.e., at the distance which is not smaller than 50 % of the panel size. In practice, however, the typical distance from the vortex elements, which simulate the boundary layer, to the airfoil surface line has order of $10^{-6}\dots 10^{-5}$ (with respect to the chord). It means, that the required number of panels should have order of 10^5; for smaller number of panels it is impossible to reconstruct it correctly.

However, this issue can be overcome by implementing the correction procedure.

Note that for the vortex placed at the arbitrary point of the flow domain, the exact solution for the vortex sheet intensity is known for circular airfoil. It has the following form [4]

$$\tilde\gamma(s) = \Gamma_g\left(\frac{\boldsymbol{r}(s)-\boldsymbol{r}_g}{2\pi|\boldsymbol{r}(s)-\boldsymbol{r}_g|^2} - \frac{\boldsymbol{r}(s)-\boldsymbol{r}^m}{2\pi|\boldsymbol{r}(s)-\boldsymbol{r}^m|^2} + \frac{\boldsymbol{r}(s)-\boldsymbol{r}^c}{2\pi|\boldsymbol{r}(s)-\boldsymbol{r}^c|^2}\right)\cdot\boldsymbol{n}(s), \tag{10}$$

where $\boldsymbol{r}(s)$ is the point on the circle of radius R, $\boldsymbol{n}(s)$ in outer unit normal vector for the circle, $\boldsymbol{r}_g$ is position of the vortex, $\boldsymbol{r}^c$ is center of the circle, r^m is the position of the mirrored vortex,

$$\boldsymbol{r}_m = \boldsymbol{r}_c + \frac{R^2}{|\boldsymbol{r}_g-\boldsymbol{r}^c|^2}.$$

Now for the vortex, placed in neighborhood of the k-th panel we suppose that this panel can be approximately replaced with the osculating circle of radius $R_k = \varkappa_k^{-1}$, then we are able to take into account the influence of this vortex semi-analytically by explicit introducing to the numerical solution the term, similar to (10):

$$\gamma(s) = \sum_{i=1}^{N}\sum_{q=0}^{m}\gamma_i^q\varphi_i^q(s) + \sum_{k=k_b}^{k_f}\tilde\gamma_k(s)\varphi_k^0(s). \tag{11}$$

Here

$$\tilde\gamma_k(s) = \Gamma_g\left(\frac{\boldsymbol{r}(s)-\boldsymbol{r}_g}{2\pi|\boldsymbol{r}(s)-\boldsymbol{r}_g|^2} - \frac{\boldsymbol{r}(s)-\boldsymbol{r}_k^m}{2\pi|\boldsymbol{r}(s)-\boldsymbol{r}_k^m|^2} + \frac{\boldsymbol{r}(s)-\boldsymbol{r}_k^c}{2\pi|\boldsymbol{r}(s)-\boldsymbol{r}_k^c|^2}\right)\cdot\boldsymbol{n}_k(s)$$

is additional term, which determines the influence of the system of mirrored vortices with respect to the k-th panel, $\boldsymbol{r}_k^m$ is the position of the vortex, mirrored with respect to the k-th panel, $\boldsymbol{r}_k^c$ is the center of the osculating circle, $\boldsymbol{n}_k(s)$ unit outer normal vector for the osculating circle; $k_b\dots k_f$ is the range of panel numbers, for which the correction procedure is implemented. It can be easily shown, that the expression for $\tilde\gamma_k(s)$ can be simplified:

$$\tilde\gamma_k(s) = \Gamma_g\frac{\big(\boldsymbol{r}(s)-\boldsymbol{r}_g\big)\cdot\boldsymbol{n}_k(s)}{\pi|\boldsymbol{r}(s)-\boldsymbol{r}_g|^2}.$$

For such solution representation, the above described Galerkin approach remains applicable, but instead of the system (7) now we obtain the following linear system

$$\sum_{j=1}^{N}\sum_{q=0}^{m}\gamma_j^q\int_{s_{i-1}}^{s_i}\psi_i^p(s)ds\int_{s_{j-1}}^{s_j}Q(s,\,\sigma)\phi_j^q(\sigma)d\sigma-\frac{1}{2}\sum_{q=0}^{m}\gamma_i^q\int_{s_{i-1}}^{s_i}\psi_i^p(s)\phi_i^q(s)ds=$$
$$=-\sum_{\substack{k=k_b,\\k\neq i}}^{k_f}\int_{s_{i-1}}^{s_i}\psi_i^p(s)ds\int_{s_{k-1}}^{s_k}Q(s,\sigma)\tilde{\gamma}_k(\sigma)d\sigma+\sum_{\substack{k=k_b,\\k\neq i}}^{k_f}\int_{s_{i-1}}^{s_i}\psi_i^p(s)f_g(s)ds+$$
$$+\int_{s_{i-1}}^{s_i}\psi_i^p(s)f_v(s)ds,\quad i=1,\,\dots,\,N,\quad p=0,\,\dots,\,m,\quad (12)$$

where it is denoted

$$f_v(s)=\boldsymbol{V}_\infty\cdot\boldsymbol{\tau}(s),\quad f_g(s)=\frac{\Gamma_g\big(\boldsymbol{r}(s)-\boldsymbol{r}_g\big)\cdot\boldsymbol{n}(s)}{2\pi\big|\boldsymbol{r}(s)-\boldsymbol{r}_g\big|^2}.$$

The system (12) can be written down in the matrix form:

$$\begin{pmatrix}A^{00}+D^{00} & A^{01} & A^{02} & I\\ A^{10} & A^{11}+D^{11} & A^{12} & O\\ A^{20} & A^{21} & A^{22}+D^{22} & O\\ L^0 & O & O & 0\end{pmatrix}\begin{pmatrix}\gamma^0\\ \gamma^1\\ \gamma^2\\ R\end{pmatrix}=\begin{pmatrix}b_v^0+b_g^0+b_\gamma^0\\ b_v^1+b_g^1+b_\gamma^1\\ b_v^2+b_g^2+b_\gamma^2\\ \Gamma_w\end{pmatrix},$$

where the left-hand side remains the same as earlier (without correction), the coefficients b_v^p are connected with the incident flow influence, b_g^p — with influence of the vortices in the flow domain, which is taken account straightforwardly without correction; b_γ^p — additional terms, arising due to the correction procedure:

$$b_{v,i}^p=\int_{s_{i-1}}^{s_i}\psi_i^p(s)f_v(s)ds,\quad b_{g,i}^p=\sum_{\substack{k=k_b,\\k\neq i}}^{k_f}\int_{s_{i-1}}^{s_i}\psi_i^p(s)f_g(s)ds,$$
$$b_{\gamma,i}^p=-\sum_{\substack{k=k_b,\\k\neq i}}^{k_f}\int_{s_{i-1}}^{s_i}\psi_i^p(s)ds\int_{s_{k-1}}^{s_k}Q(s,\sigma)\tilde{\gamma}_k(\sigma)d\sigma,\quad i,\,j=1,\,\dots,\,N,\quad p,\,q=0,\,1,\,2.$$

For the last component of the right-hand side vector the following expression is obtained:

$$\Gamma_w=\Gamma-\sum_{k=k_b}^{k_f}\int_{s_{k-1}}^{s_k}\tilde{\gamma}_k ds.$$

For computation of all the integrals, arising in the right-hand side, Gaussian quadratures can be used, as well as approximate analytical expressions.

In the Fig. 3 the results of computations for the model problem, considered in the previous section, but with implemented correction procedure, are shown. The correction is performed not only for the panel, closest to the vortex, but also for the neighboring panels on both sides. Note, that the correction procedure preserves the accuracy order of the initial scheme. Moreover, it provides the more accurate results, the closer vortex element is located to the airfoil. It seems reasonable to use correction technique for the vortices placed at the distance of not more, than 75 % of the panel length.

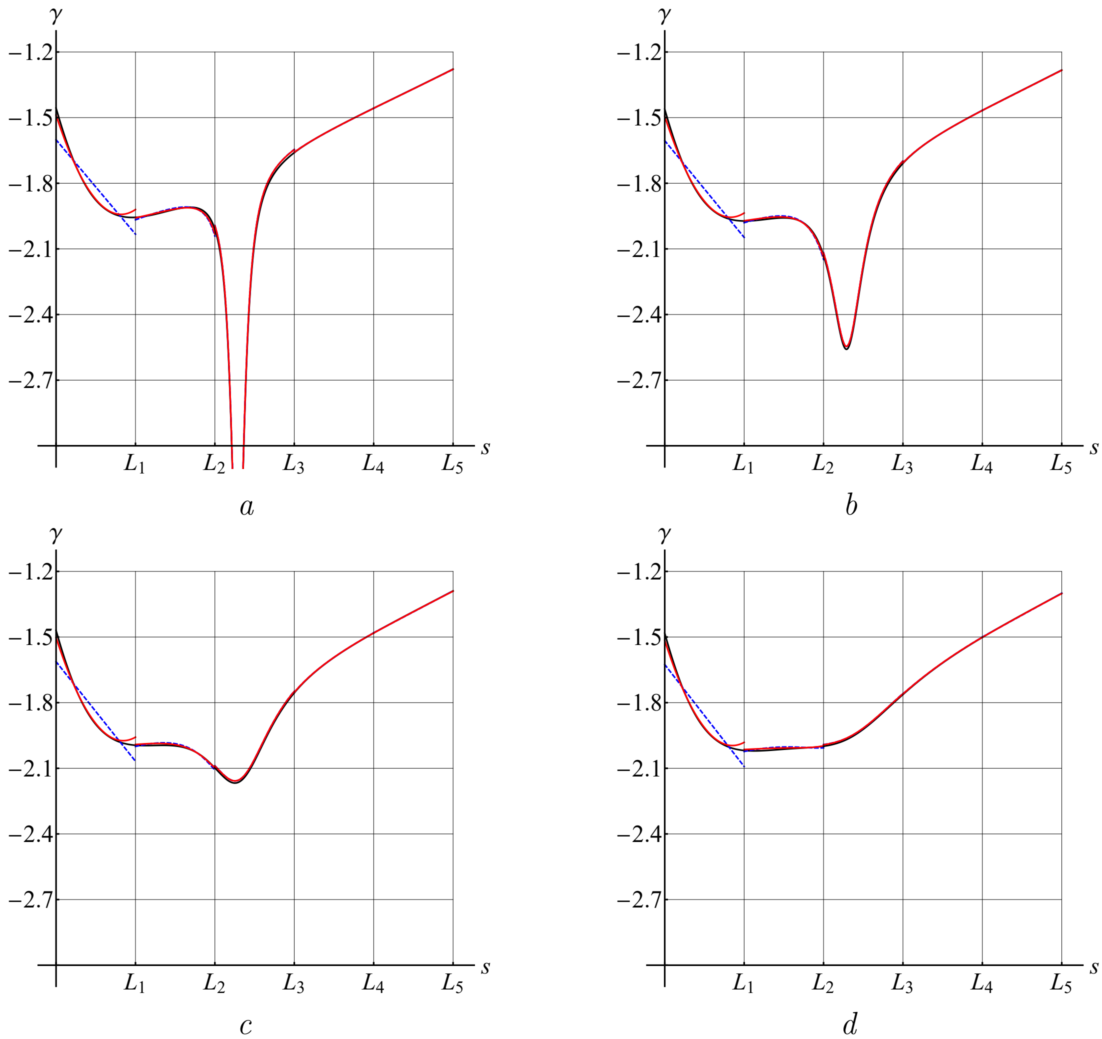

Figure 3: Exact solution (black solid line), piecewise-linear (blue dashed) and piecewise-quadratic (red solid) solutions for the vortex sheet intensity in presence of the vortex at the distance of 10 %, 25 %, 50 % and 100 % of panel size (a, b, c, d, respectively) after implementation of the correction procedure

5 CONCLUSIONS

In the present paper the numerical scheme of the third order of accuracy is developed by using the Galerkin approach, which takes into account the curvature of the surface line of the airfoil and piecewise-quadratic solution representation. This scheme makes it possible to deal with non-uniform discretization, moreover, considering panel lengths inverse proportional to the square root of the curvature permits to reduce number of panels significantly.

In order to take into account the influence of the vortex wake, simulated with separate vortices, the correction procedure is developed, which permits to consider arbitrary panel length with correct representation of the influence of closely placed vortices. Numerical experiments prove the properties of the developed scheme and the correction procedure.

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