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Finding Minimal and Non-Minimal Surfaces through the Natural Force Density Method

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Summary. This paper discusses the Natural Force Density Method, an extension of the well known Force Density Method for the shape finding of continuous membrane structures, which preserves the linearity of the original method, overcoming the need for regular meshes. The method is capable of providing viable membrane configurations, comprising the membrane shape and its associates stress field in a single iteration. Besides, if the NFDM is applied iteratively, it is capable of converging to a configuration under a uniform and isotropic plane stress field. This means that a minimal surface for a membrane can be achieved through a succession of viable configurations, in such a way that the process can be stopped at any iteration, and the result assumed as good. The NFDM can also be employed to the shape finding of non-minimal surfaces. In such cases, however, there is no guarantee that a prescribed, non-isotropic stress field can be achieved through iterations. The paper presents several examples of application of the NFDM to the shape finding of minimal and non-minimal membrane surfaces.

1 INTRODUCTION

Design and analysis of membrane structures constitute an integrate process, including procedures for shape finding, patterning and load analysis. The Finite Element Method is a versatile way to pose this overall process, directly providing, besides a viable shape, also a map of the stresses to which the structure is subjected. It is also adequate to determine the behavior of the structure under design loads, as well as to transfer data to the patterning routines. On the other hand, procedures based on the FEM or in other forms of structural analysis result in nonlinear analyses, and require specification of a convenient initial geometry, load steps and boundary conditions, which are not always known from start.

An alternative method for finding viable configurations, which avoids the problems related to nonlinear analysis, is given by the *force density method*, which was first proposed in the context of cable nets[1],[2],[3]. The method is routinely applied to shape finding of membrane surfaces, replacing the membrane by an equivalent cable network, which must be as regular as possible (otherwise is may become quite dubious which force densities should be prescribed to achieve a desired shape).

This paper discusses an extension of the force density method for the shape finding of continuous membrane structures, which preserves the linearity of the original method. The new NFDM was first suggested in 2006 by Pauletti[4], based on the natural approach introduced by Argyris for the Finite Element Method[5]. Afterwards, Pauletti and Pimenta[6] presented a more rigorous foundation for the method, recognizing that the imposition of natural force densities (NFD) is equivalent to the imposition of 2nd Piola-Kirchhoff (PK2) stresses to a reference mesh, a property first recognized by Bletzinger and Ramm[7] for the original force density method.

2 FORMULATION OF THE NATURAL FORCE DENSITY METHOD

Consider a three-node plane triangular finite element shown in Figure 1. Let ℓ_i^0, ℓ_i^r and ℓ_i, $i=1,2,3$, be the element side lengths at an undeformed, a reference and an equilibrium configurations, respectively. We define three "natural deformations" along the sides of the element, according to $\varepsilon_i = \left(\ell_i - \ell_i^0\right)\left(\ell_i^0\right)^{-1}$, and collect them in a *vector of natural deformations* $\boldsymbol{\varepsilon}_n = \begin{bmatrix}\varepsilon_1 & \varepsilon_2 & \varepsilon_3\end{bmatrix}^T$. There exists a linear relationship between $\boldsymbol{\varepsilon}_n$ and the linear Green strains $\boldsymbol{\varepsilon}_n = \mathbf{T}\boldsymbol{\varepsilon}$, from which we can define a *vector of natural stresses* $\boldsymbol{\sigma}_n = \mathbf{T}^{-T}\boldsymbol{\sigma}$, where $\boldsymbol{\sigma}$ is the vector of Cauchy stresses acting on the element. It can be shown that $\boldsymbol{\sigma}_n$ and $\boldsymbol{\varepsilon}_n$ are energetically conjugate.

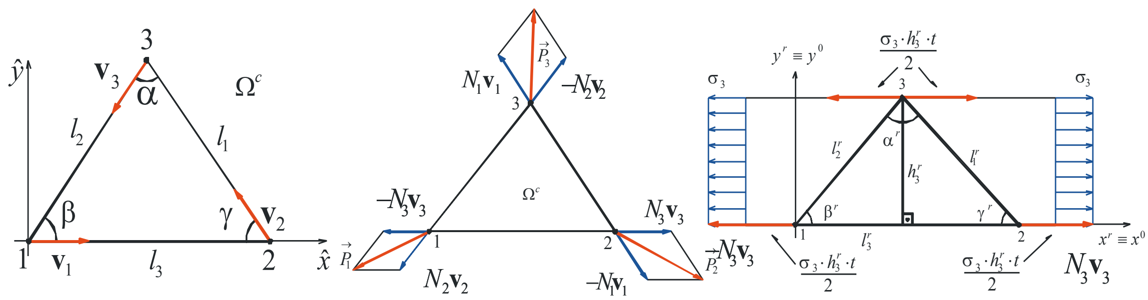


Figure 1. (a) Unit vectors $\mathbf{v}_i$, $i=1,2,3$, along the element edges; (b) internal nodal forces $\mathbf{p}_i$, decomposed into natural forces $N_i\,\mathbf{v}_i$; (c) determination of natural force N_3.

We also define three "natural forces" N_i acting along the sides of the element, according to $N_i = V\ell_i^{-1}\sigma_i$, where V is the volume of the element, and collect them into a *natural force vector* $\mathbf{p}_n = \begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}^T$. Furthermore, we define the *vector of the natural force densities* according to $\mathbf{n} = \begin{bmatrix}n_1 & n_2 & n_3\end{bmatrix}^T = \boldsymbol{\mathcal{L}}^{-1}\mathbf{p}_n = V\boldsymbol{\mathcal{L}}^{-2}\mathbf{T}^{-T}\boldsymbol{\sigma}$, where $\boldsymbol{\mathcal{L}} = \operatorname{diag}\left\{\ell_1 \quad \ell_2 \quad \ell_3\right\}$.

Thereafter, we show that, for a prescribed natural force densities vector $\mathbf{n}$, there is a linear relationship between the natural force vector $\mathbf{p}_n$ and element nodal coordinates $\mathbf{x} = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix}^T$, according to $\mathbf{P}_n = \mathbf{k}_n\mathbf{x}$, where $\mathbf{x}_i$, $i=1,2,3$, are the position vectors of the element nodes at the equilibrium configuration and $\mathbf{k}_n$ is a constant symmetric *element stiffness matrix,* given by

$$\mathbf{k}_n = \begin{bmatrix} (n_2+n_3)\mathbf{I} & -n_3\mathbf{I} & -n_2\mathbf{I} \\ -n_3\mathbf{I} & (n_1+n_3)\mathbf{I} & -n_1\mathbf{I} \\ -n_2\mathbf{I} & -n_1\mathbf{I} & (n_1+n_2)\mathbf{I} \end{bmatrix} \tag{1}$$

After assembling the load and stiffness contributions of all elements, we arrive to a linear problem at the structural level, which is completely independent of any reference configuration.

However, instead of prescribing directly some natural force densities vector $\mathbf{n}$ for each element, it may be more convenient calculate them from stresses $\boldsymbol{\sigma}_r$ defined in a reference configuration, according to $\mathbf{n}_r = V^r \mathcal{L}_r^2 \mathbf{T}_r^{-T} \boldsymbol{\sigma}_r$. It can be shown that $\boldsymbol{\sigma}_r$ corresponds to the 2nd P-K stresses associated to the final Cauchy stresses, calculated at the equilibrium configuration according to $\boldsymbol{\sigma} = \left(V^{-1}\mathcal{L}^2\mathbf{T}\right)\mathbf{n}_r$. A thorough derivation of the formulation outlined above is given in references [4] and [6].

If the NFDM is applied iteratively, always re-imposing a constant, uniform and isotropic 2nd P-K stress field, the method will converge to a configuration under a uniform, isotropic Cauchy stress field. This means that a minimal surface for a membrane can be achieved through a succession of viable configurations, in such a way that the process can be stopped at any iteration, and the result assumed as good. This is a clear advantage, if compared to Newton's Method, which may also converge to a minimal solution, but through a series of unfeasible, non-equilibrium configurations.

Moreover, the NFDM can also be applied to the shape finding of non-minimal membrane surfaces through the imposition of non-isotropic PK2 stress fields. In this case, however, even though a viable shape can still be obtained at every linear step, there is no guarantee that an arbitrary prescribed, non-isotropic Cauchy stress field can be achieved through iterations. Furthermore, since geometry varies during iterations, definition of principal stress directions becomes more complicate.

3 APPLICATIONS

3.1 Linear NFDM, isotropic initial stress field

As a first application of the linear NFDM, consider the transformation of the same square reference mesh into different surfaces, in a single NFDM step. The first row of Figure 2 shows the reference mesh transformed into different shapes, simply prescribing displacements to some selected nodes, along with a uniform isotropic PK2 stress fields on the membrane and uniform normal loads on the border cables. The resulting Cauchy stress fields at the equilibrium configurations are no longer uniform. This is fully coherent with the original FDM, which also has no control over the normal forces acting on cables, at the equilibrium configuration.

Since any minimal membrane surface is intrinsically associated to a uniform and isotropic

Cauchy stress field, clearly none of the equilibrium shapes shown in Figure 2 is minimal, although stress gradients are quite restricted to the membrane vertices, thus actually none of these shapes is too far from the corresponding minimal shapes.

Although the original mesh geometry is basically irrelevant, the *topological genus* of the surface has to be respected. Thus, in order the produce a conoidal surface, a hole must be cut into the original mesh, as shown in the Figure 2(d). Moreover, while the original FDM requires a two-directional layout of linear FDM elements, as regular as possible, as shown in Figure 3(a), the NFDM is capable to deal with irregular meshes, as the one shown in Figure 3(b). Figure 3(c) shows how an isotropic stress field $\hat{\boldsymbol{\sigma}}_0 = [1 \quad 1 \quad 0]^T$ defined onto a rectangle-triangle is converted into an equivalent natural force density $\mathbf{n}_0 = \frac{1}{2}[0 \quad 1 \quad 1]^T$.

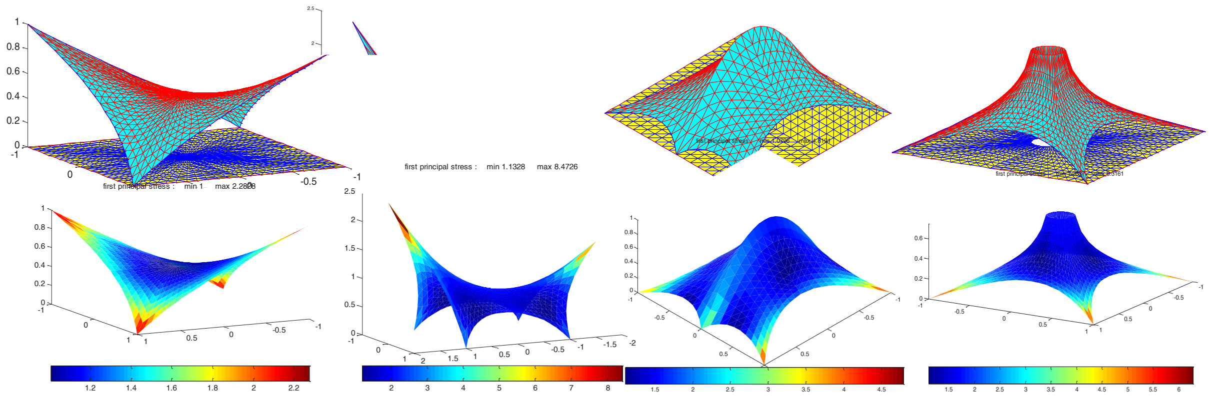


Figure 2: 1st row: viable configurations generated through the imposition of different sets of nodal displacements to the same plane squared reference mesh, with and without a hole; 2nd row: 1st principal Cauchy stress at the viable configurations.

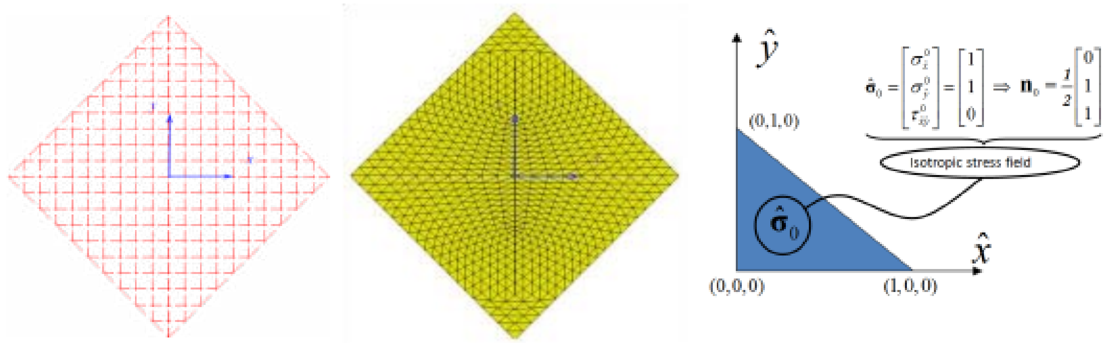


Figure 3: (a) a regular, two-directional layout of FDM line elements; (b) a generic mesh of NFDM surface elements; (c) isotropic stress field $\hat{\boldsymbol{\sigma}}_0$ defined onto a triangule and its equivalent NFD vector $\mathbf{n}_0$.

3.2 Linear NFDM, non-isotropic initial stress field

A broader class of shapes can be achieved prescribing non-isotropic PK2 stress fields to the reference configuration. In the case of the original FDM, a two-directional layout of line elements very conveniently provides two directions with respect to which different force densities can be prescribed (for instance $n_x \neq n_y$, in Figure 3(a)). On the other hand, in the case of the NFDM, it is necessary to define a convenient director plane Π, whose intersection with the surface Ω^e of a given element define the direction of one of the principal PK2 stresses acting onto the element.

Figure 4(a) shows how a horizontal plane may serve as director for a straight conoid, whilst a vertical plane adequately define a principal direction for every element of the *hypar* shown in Figure 4(b). Defining a unit vector $\vec{n} \perp \Pi$, and $\langle \hat{i}, \hat{j}, \hat{k} \rangle$ the unit vectors of the local coordinate system, with $\hat{k} \perp \Omega^e$, the principal stress directions are given by unit vectors $\hat{i}' = \hat{k} \times \vec{n} / \left\| \hat{k} \times \vec{n} \right\|$, and $\hat{j}' = \hat{k} \times \hat{i}'$, which are rotated with respect to the element local coordinate system by an angle $\theta = \arcsin\left(\left(\hat{i}' \times \hat{i} \right) \cdot \hat{k} \right)$.

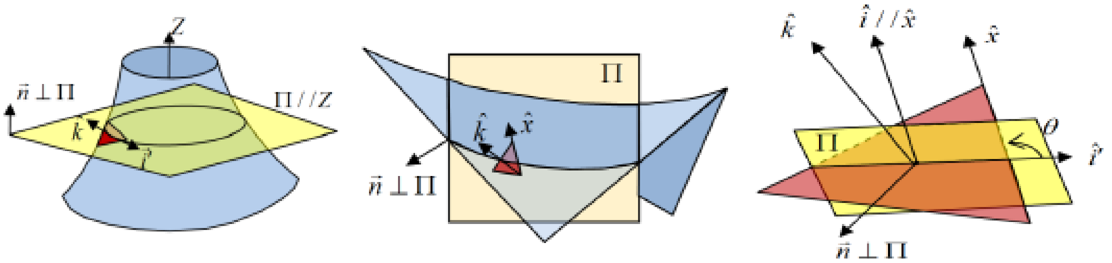
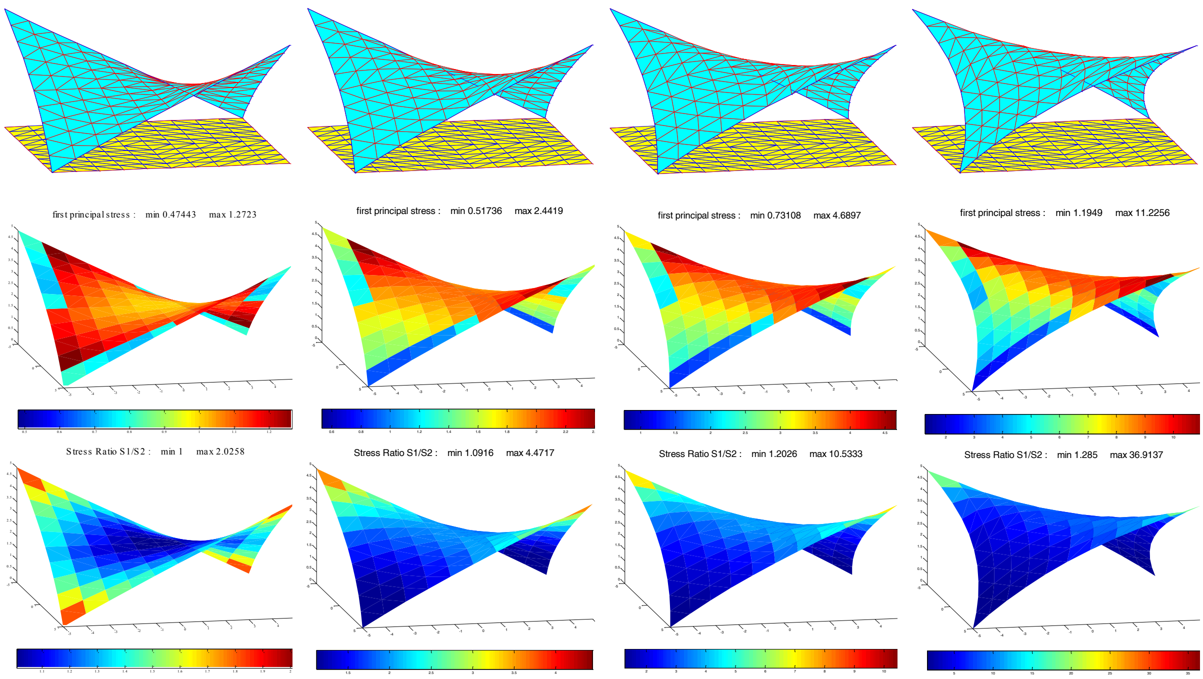


Figure 4: Definition of principal stress directions onto a NFD element requires conveneint director planes Π.



$\sigma_I^0 / \sigma_{II}^0 = 1$ $\sigma_I^0 / \sigma_{II}^0 = 2$ $\sigma_I^0 / \sigma_{II}^0 = 4$ $\sigma_I^0 / \sigma_{II}^0 = 10$

Figure 5: 11st row: non-minimal *hypar* surfaces, for different initial PK2 principal stress ratios; 2nd row: final 1st principal Cauchy stresses; 3rd row: final σ_I / σ_{II} stress ratios.

The first row of Figure 5 shows different *hypars* generated by the imposition of a non-isotropic PK2 initial stress field with uniform initial mean stress $\bar{\sigma}_0 = \frac{1}{2}\left(\sigma_I^0 + \sigma_{II}^0\right)$ an uniform initial stress ratio $\sigma_I^0 / \sigma_{II}^0$ onto an originally flat squared mesh, with director plane Π aligned with one of the square diagonals. All resulting geometries where obtained in a single iteration, thus the final 1st principal stresses and final σ_I / σ_{II} ratio vary over the surface, as can be seen in the second and third row of Figure 5, respectively.

3.3 Finding minimal surfaces with the Iterative NFDM

As a third example, inspired by a physical experiment illustrated by Isenberg[8], consider a helicoidal soap film, shown in Figure 6(a). The same previous square reference mesh is deformed such that sides S1 and S2 are transformed into small radial segments (see Figure 6(b/e)). Side S2 is displaced transversally to the reference plane. Side S3 is deformed into a helix. Side S4 is constrained to slip over the vertical axis. Figure 6(b) shows the initial square reference mesh and the resulting geometry, associated to a Cauchy stress field with quite high stress concentration close to borders S2, S3 and S4. Subsequent iterations do not alter the geometry significantly, but do smooth the stress field. After the 10th iteration, a practically uniform, isotropic Cauchy stress field is achieved, with the 1st principal Cauchy stress σ_I ranging from 1.005 to 1.063 (Figure 6(d)). Thus, the minimal surface associated with the prescribed boundary is in practice obtained.

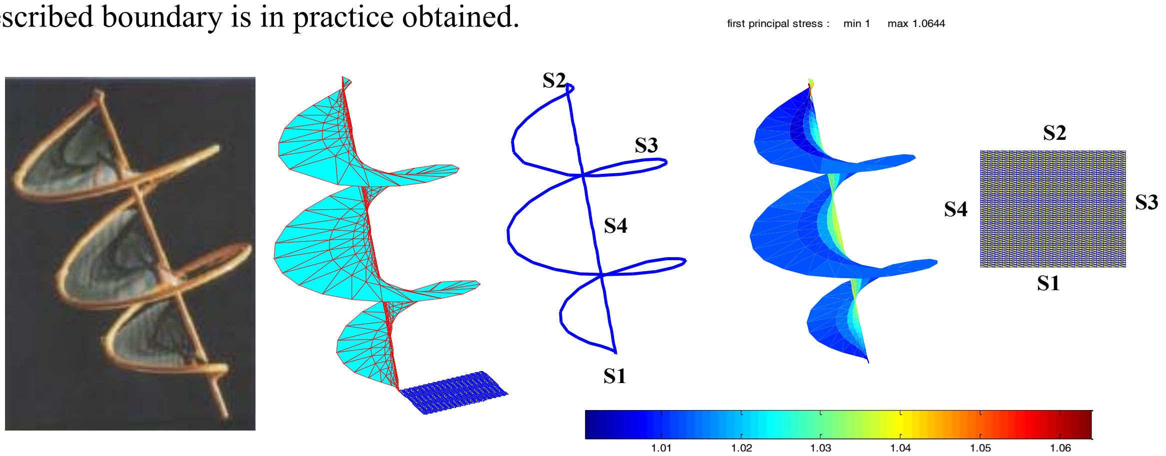


Figure 6: A helicoidal soap film

Next, we consider the generation of a minimal Costa's surface[9], starting from a non-minimal, non-smooth one (topologically, there is no distinction between them) and repeatedly imposing an isotropic PK2 stress field $\hat{\boldsymbol{\sigma}}_0 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T$. In the 1st row, Figure 7(a) shows a non-minimal Costa's surface connecting three fixed circular rings. Figs. 7(b/c) show the geometry obtained after the 1st and 6th iteration of the NFDM. It is seen that the 1st iteration of the NFDM already provides a fair approximation to the minimal surface. At the 2nd row, Figs. 7(d/e/f) show the σ_I fields resulting after the 1st iteration ($1.0288 \le \sigma_I \le 1.8086$), the 2nd iteration ($1.0015 \le \sigma_I \le 1.0594$) and the 6th iteration ($1.0001 \le \sigma_I \le 1.0124$). It is seen that after the 2nd iteration the σ_I field has already smoothed out any stress concentrations. It is also seen that geometry converges much faster than the imposed stress field, and, for practical purposes, the analysis could be stopped after a single iteration, or a couple of them, since there is no point in performing several iterations chasing a result (the imposed stress ratio) which is known *a priori*.

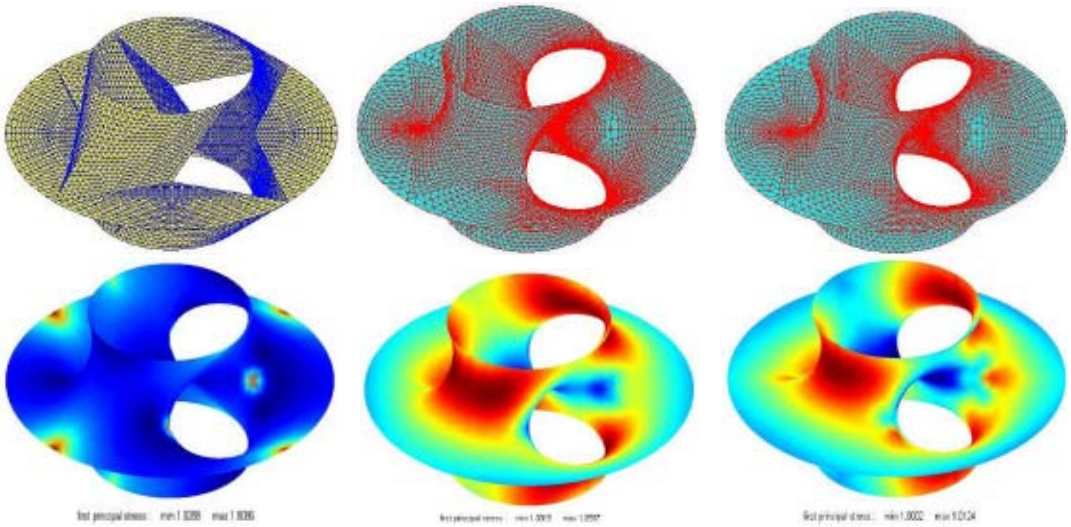


Figure 7: Numerical model of Costa's Surface

Figure 8(a) shows a physical realization of Costa's surface, exhibited at the atrium of the Civil Engineering building of the Polytechnic School of the University of São Paulo. Figure 8(b) shows the patterning used to produce the physical model.

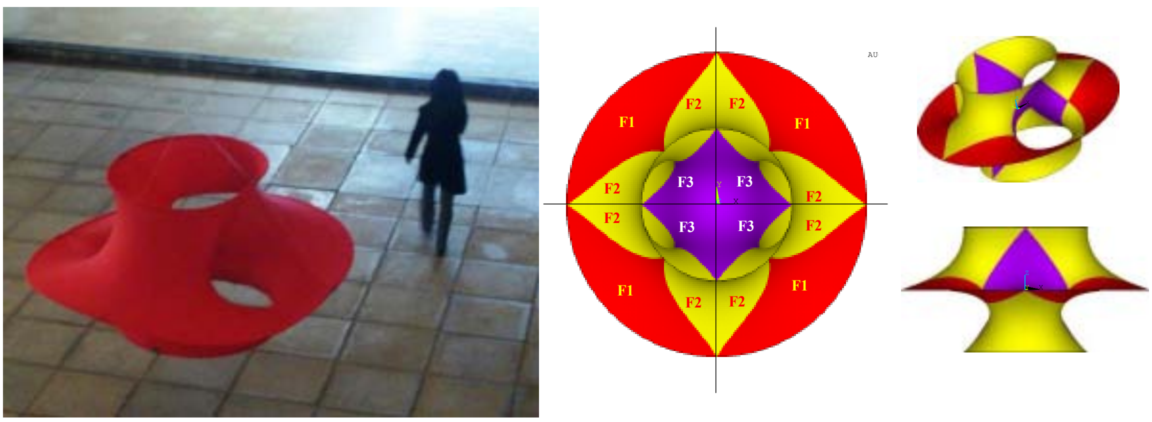
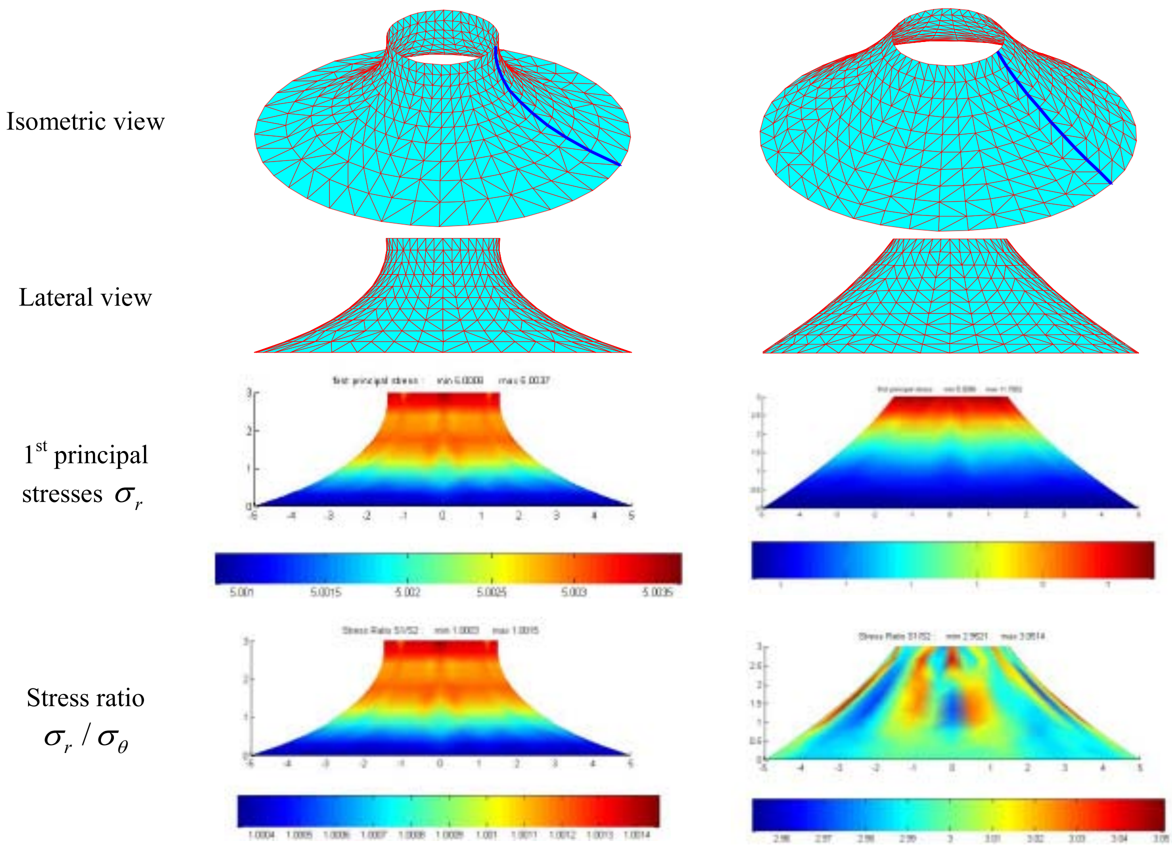


Figure 8: Physical model of Costa's Surface and corresponding fabric patterning

3.4 Finding non-minimal surfaces with the Iterative NFDM

As a final example, Figure 9 compares a minimal conoid (stress ratio $\sigma_r / \sigma_\theta = 1$ over the whole surface) to a non-minimal conoid ($\sigma_r / \sigma_\theta = 3$, arbitrary imposed over the whole surface). Both geometries were obtained after 10 NFDM iterations, required for convergence of the stress ratios. Results compare very well with analytical solutions, as shown in reference [10]. Once again, geometry converges much faster than stresses and, for practical purposes, the analysis could be stopped after a couple iterations.



minimal conoid ($\sigma_r / \sigma_\theta = 1$) *non-minimal conoid ($\sigma_r / \sigma_\theta = 3$)*

Figure 9. Comparison between minimal and non-minimal conoids

12 CONCLUSIONS

- The Natural Force Density Method is a convenient extension of the FDM for the shape finding of continuous membrane structures, which preserves the linearity of the original method. It is particularly adequate to deal with the general non-structured, irregular meshes provided by automatic mesh generators;
- The method provides quite convenient viable configurations, comprising both a viable geometry and the associated viable stress, field in a single iteration;
- Although the analyst has no absolute control over the final stress field, if a uniform isotropic

stress field is prescribed, the resulting geometry does not differ too much from a minimal surface;

- Besides, if a uniform isotropic stress field is repeatedly prescribed, the method quickly converges to the geometry of the minimal surface associated to the given boundary;
- Non-minimal shapes are also easily generated, through the imposition of non-isotropic stress fields at a reference configuration. This can be accomplished in a single linear step, without control of the final stress field, or again through iterations, repeatedly prescribing a given non-isotropic stress field;
- In this last case, however, even though a viable shape can still be obtained at every linear step, there is no guarantee that an arbitrary prescribed, non-isotropic Cauchy stress field can be achieved through iterations;
- It is worth to point out that, as the original FDM, the Natural Force Density method is an un-material method, simply providing a viable configuration, regardless of material properties. It is a method intended solely for shape finding.
- As far as load analysis is concerned, up to date the author does not know any good reason to supersede a proper nonlinear structural analysis by any sort of adapted force density method. But, of course, this statement is far from conclusive.

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