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MEMBRANE FORM FINDING BY MEANS OF FUNCTIONAL MINIMIZATION

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Abstract. This study deals with the methods of architectural design of structures made of textile membranes. We consider the problem of form finding of membranes strained by rigid skeletons and cables installed along the free edges of the membrane. First, we recall the methods used to solve the simple problem of finding form in the case of constant surface tension. Then we propose a method based on the minimization of the total potential energy. The problem is discretized using membrane triangular finite elements.

The potential considered is an energy density per unit area of uniform and constant surface tension. The minimization of this potential leads to the minimal surface solution. However the problem is singular with respect to the in-plane displacement. To handle this problem, the potential is enhanced by an elastic energy in order to regularize the numerical scheme and prevent the mesh degeneration. It is also enhanced by the elastic energy due to the cable tensions. The solution is obtained by minimizing the potential energy using the conjugate gradient method.

1 Introduction

The flexible structure like cables and membranes are characterized by form follower internal forces ; the stress vector remains axial to the cables and remains in-plane in case of membrane. The shape of such structures, when they are uniformly taut, is essentially defined by force equilibrium considerations. Conversely, the loads distribution in the membrane are strongly governed by the attained geometry. The structures without bending stiffness obey this principle.

The literature on the form optimisation may be classified into two main topics : structural optimization and form finding [8]. The first topics focuses on the search of the initial shape of the structure through a kinematic criterion or a resistance criterion, whereas the form finding focuses on the final form that can reached by a structure under a prescribed stress field. The first method is general and applies to any kind of structure as an inverse problem, whereas the second method is more than often used for structures made of stretched membranes and cables subjected to large deformations. It is necessary to clearly distinguish the objectives of these two approaches.

Membrane structures are characterized by a pure tensile in-plane stress state (i.e. without bending stress). The pure tension is governed by local equilibrium. When the stress state is in-plane, uniform and isotropic, the resulting geometry is defined by a minimal surface. This is the case, for example, of the soap film which exhibits a uniform surface tension and a minimum surface area.

Bletzinger [2] and Veenendaal [13] summarized methods of form finding developed in the last decades in three main families :

– Stiffness matrix methods that are based on using the standard elastic and geometric stiffness matrices [11, 6, 12].
– Geometric stiffness methods which are material independent, based on the force density method concept with some extensions [5, 1, 10].
– Dynamic relaxation methods which solve the problem to reach a steady-state solution, equivalent to the static equilibrium solution.

In this study, we will show that the force density method in the case of prescribed stress field can be formulated as an energy minimisation problem. Use will be made of the conjugate gradient method, which is a first order method, to minimise the total potential energy. It will be shown that this method is robust and efficient to solve the form finding problem.

2 Geometric model

In the force density approach, the membrane is represented as a geometric surface and not as a material one. The surface represents the midplane of the membrane and serves only to define the force field domain. The optimal form is defined by this surface when the local equilibrium of the force field is satisfied at each point of the whole surface. Seeking for the optimal form requires the definition of an initial surface $S \subset R^2$ which defines the surface state at time t_0. This surface evolves towards the optimal form $s \subset R^2$, at time $t\,(t > t_0)$, by a geometric transformation $\mathbf{\Phi}$.

We use the bijective mapping function $\mathbf{\Phi}$ to relate a point $\mathbf{X} \in S$ to a point $\mathbf{x} \in s$:

$$\mathrm{S} \ni \mathbf{X} \mapsto \mathbf{x} = \mathbf{\Phi}(\mathbf{X}, t) \in \mathrm{s}$$

The initial surface S is an approximation of the optimal solution s, in the sense that s is independent from the choice of S. Time t is any kinematic parameter. The material curvilinear coordinates (ξ^1, ξ^2) are introduced to describe the surface of the membrane, the third dimension is not represented geometrically but taken into account through the thickness denoted $\xi^3(h/2 \leq \xi^3 \leq h/2)$ and assumed to be uniform.

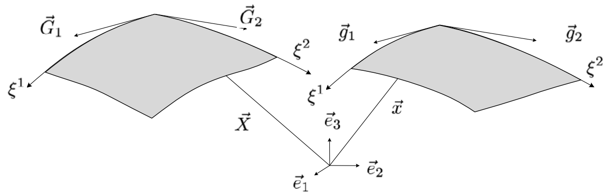


FIGURE 1: Transformation du repère matériel.

Representing the membrane by a surface requires both stress and strains fields constant along thickness ξ^3. Integration along the thickness is equivalent to multiplying the integrated quantity by h.

We denote $\mathbf{G}_\alpha = \partial\mathbf{X}/\partial\xi_\alpha$ the curvilinear base in the initial configuration, and $\mathbf{g}_\alpha = \partial\mathbf{x}/\partial\xi_\alpha$ the mapped base in the final configuration, where Greek indices take the values $\{1, 2\}$.

The metric tensor in the initial configuration is defined by $G_{\alpha\beta} = \mathbf{G}_\alpha \cdot \mathbf{G}_\beta$ and that in the actual configuration is defined by $g_{\alpha\beta} = \mathbf{g}_\alpha \cdot \mathbf{g}_\beta$. The deformation gradient tensor $\mathbf{F}$ writes

$$\mathbf{F}(\mathbf{X}, t) = \frac{\partial\mathbf{\Phi}(\mathbf{X}, t)}{\partial\mathbf{X}} \tag{1}$$

and the Green tensor is

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^T\mathbf{F} - \mathbf{I}\right) \tag{2}$$

We can also write the strain tensor as a function of the metric tensors in the current and reference configurations :

$$E_{\alpha\beta} = \frac{1}{2}\left(g_{\alpha\beta} - G_{\alpha\beta}\right) \tag{3}$$

3 Minimal surface method in form finding

A membrane uniformly and isotropically stretched on its rim takes the form which minimizes its surface. An example of such membranes is soap films. The surface tension of the film ensures a membrane retraction as much as possible until reaching the minimum area. The method of minimal surface amounts to investigate the shape of the membrane that achieves the minimum total surface s. The problem to solve is formulated as follows :

$$x = \arg\min_x s = \int_s d\mathbf{s} = \int_S J dS \tag{4}$$

where J is the determinant of tensor $\mathbf{F}$, s the surface in the current configuration and S the surface in the initial configuration. The surface stationarity condition is

$$\delta s = \int_S \delta J dS = \int_S J\mathbf{F}^{-T} : \delta \mathbf{F} dS = 0 \tag{5}$$

It should be noted here that the area s is not necessarily material and the transformation $\mathbf{F}$ is the mapping function that merely connects the two configurations occupied by the considered surface.

4 Potential energy method for prescribed stress field

In this section, we show that minimum surface finding – which is a purely geometrical method – can be formulated as a static equilibrium problem using the theorem of potential energy minimum. The energy considered results from a constant transversely isotropic stress field (e.g. a uniform surface tension on the membrane). One therefore seeks the form achieved by the membrane when it is stretched by a known plan stress field, represented by a Cauchy stress tensor σ prescribed on the whole membrane, of the form :

$$\sigma = \tau \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \tau \mathbf{I} \qquad \sigma_{i3} = 0, i = 1, 2, 3 \tag{6}$$

The stress field being prescribed on the current configuration, we seek the position field $\mathbf{x} = \mathbf{X} + \mathbf{U}$ that makes the membrane in equilibrium position under this load. The potential energy associated with σ is

$$\Pi^s = h \int_s \sigma : \mathbf{u}_{,x} ds \tag{7}$$

The equilibrium configuration makes the potential energy stationary for any displacement variation $\delta\mathbf{u}$. The corresponding deformation variation is

$$\delta\mathbf{u}_{,x} = \delta\mathbf{u}_{,X} \cdot \mathbf{X}_{,x} = \delta\mathbf{x}_{,X} \cdot \mathbf{X}_{,x} = \delta\mathbf{F} \cdot \mathbf{F}^{-T}$$

It follows that when the tension field is isotropic, $\sigma = \tau\mathbf{I}$, the stationarity condition of the potential energy can be written as

$$\delta\Pi^s = h\tau \int_s \mathbf{I} : \delta\mathbf{u}_{,x} ds = h\tau \int_S det\mathbf{F}\mathbf{F}^{-T} : \delta\mathbf{F} dS \tag{8}$$

which is exactly the minimal surface condition established in Equation (5).

Let us define the potential Π^s whose minimum defines the equilibrium surface configuration :

$$\Pi^s = h\tau s \tag{9}$$

The problem to solve is then formulated as follows :

$$x_{sol} = \arg\min_x \Pi^s = h\tau \arg\min_x \int_s d\mathbf{s} = h\tau \arg\min_x \int_S JdS \tag{10}$$

This result shows the equivalence, in the case of an isotropic stress field, between the minimum surface approach and the minimum of potential energy theorem. However, in the energy method, one can add other potentials of various loads like elastic potentials of deformable bodies (e.g. cables and flexible supporting structures).

5 Regularization of the form finding methods

The problem of form finding is to find the position $\mathbf{x} = \mathbf{X} + \mathbf{U}$, i.e. the actual configuration s, that makes Π^s minimum for any variation $\delta\mathbf{U}$.

This method has the particularity of being singular for degrees of freedom within the plane of the membrane. Indeed, for a given meshed surface, an arbitrary movement of nodes in the tangent plane of the membrane does not change the total area. From a numerical point of view, this can lead to an optimal solution with a very distorted mesh.

To avoid degeneration of the mesh (coincidence of two nodes for example), we should regularize the problem by limiting their in-plane movements. There are several methods to regularize the problem. One of them consists in projecting the displacements obtained at each iteration along the normal to the membrane.

In our case, we have supplemented the quantity to minimize Π^s with the elastic strain energy of the membrane Π^e, which plays the role of springs between nodes in the plane of the membrane. For this energy, the material is assumed hyperelastic and governed by the quadratic elastic potential of Saint-Venant Kirchhoff with a surface energy density Ψ^e :

$$\Pi^e = \int_V \Psi^e(\mathbf{E})dV \tag{11}$$

In plane stress condition, the out-of-plane stress components vanish :

$$\Sigma_{i3} = \frac{\partial\Psi(\mathbf{E})}{\partial E_{i3}} = 0 \tag{12}$$

Equation $(12)_c$, $\Sigma_{33} = 0$, establishes an implicit relationship between the components of the strain tensor.

$$\Sigma_{33} = \Sigma_{33}(E_{11}, E_{12}, E_{22}, E_{33}) = 0 \tag{13}$$

From this equation, the normal component E_{33} can be expressed in terms of the in-plane components of $\mathbf{E}$ as :

$$E_{33} = f(E_{11}, E_{12}, E_{22}) \tag{14}$$

It is then possible to reduce the volume energy density Ψ^e into a surface density. For this, we rewrite the potential Ψ as a function of E_{11}, E_{12} and E_{22} in the form :

$$\tilde{\Psi}^e\left(E_{11},\, E_{12},\, E_{22}\right) = \Psi\left(E_{11},\, E_{12},\, E_{22},\, E_{33}\right) \tag{15}$$

The plane stress condition has enabled one to eliminate E_{33} from the expression of the elastic potential which depends therefore only on the in-plane components of strain tensor **E**. The elastic potential energy of the membrane can be written as

$$\Pi^e = \int_V \Psi^e(\mathbf{E})dV = h\int_S \tilde{\Psi}^e(\mathbf{E})dS \tag{16}$$

The minimisation problem involves the quantity $\Pi = \Pi^s + \Pi^e$ and is the rewritten as

$$x_{sol} = \arg\min_x \left(\Pi^s + \Pi^e\right) \tag{17}$$

Adding the elastic energy Π^e to the energy of the surface tension Π^s preserves the structure of the mesh as long as the mesh is far from the optimal shape. However, when approaching the solution, this energy must be deactivated so that the optimal resulting shape is not altered by the added elastic energy.

6 Stain energy canceling

The addition of strain energy to the minimizing quantity introduces in-plane stiffness that disturbs the solution. The minimization leads, as can be seen in Figure 5 below, to the formation of wrinkles orthogonal to compressive stresses. It is therefore necessary to ensure that the quantity minimized in (17) leads to the minimum area of the membrane. To cancel the elastic energy at the end of the iterative process, we simply cancel the strain tensor. For this, we modify the strain tensor defining Π^e by updating the reference configuration. This idea was first proposed by Bletzinger [8] and proves to be efficient and robust. The process is repeated until convergence. For each minimization step n, the strain tensor formula (2) is modified by replacing the metric tensor **G** in the reference configuration by that in the configuration reached at the previous step $n-1$. At the end of the iterative process, the two configurations $n-1$ and n are close enough to each other, they become asymptotically the same and the strain tensor vanishes. We write

$$E_{kl}^{(n)} = \frac{1}{2}\left(g_{kl}^{(n)} - g_{kl}^{(n-1)}\right) \tag{18}$$

with $g_k^{(0)} = G_k$. At convergence, the strain energy Π^e, activated for the sole purpose of the problem regularization, will be automatically canceled.

7 Minimisation of the potential energy

The minimization formulation of the energy due to a uniform and isotropic stress field, allows extension of the formulation by adding any kind of potential energy to the quantity to be minimized. Thus, one can easily includes the energy Π^c due to deformable cables supporting the membrane edges, the energy due to the deformable elements bearing the structure, or the energy due to dead loads uniformly distributed over the membrane such as the snow. The problem then writes

$$x_{sol} = \arg\min_{x} \Pi^{tot} = \arg\min_{x} \left(\Pi^s + \Pi^e + \Pi^c + ...\right) \tag{19}$$

The total potential energy Π^{tot} is discretized using the finite element method and is written as a nonlinear function of the nodal unknown displacements $\{U\}$. Minimization is done either by the first order minimizing methods as the conjugate gradient or the second order which requires the linearization of the energy using the Taylor series expansion :

$$\delta\Pi^{tot} = \frac{\partial \Pi^{tot}}{\partial \{U\}}\{\delta U\} + \frac{1}{2}\{\delta U\}^T \frac{\partial^2 \Pi^{tot}}{\partial \{U\}^2}\{\delta U\} + O(\|\delta\{\delta U\}\|)^3 \tag{20}$$

$$= \left\{\nabla \Pi^{tot}\right\}^T \{\delta U\} + \frac{1}{2}\{\delta U\}^T [K] \{\delta U\} + O(\|\{\delta U\}\|)^3 \tag{21}$$

When using the second order methods of the Newton or quasi-Newton-type we simply require that the energy is stationary.

When using the first order methods, the solution is sought for with descent directions oriented in the opposite direction of the potential gradient $\{\nabla\Pi^{tot}\}$. This type of algorithm converges in all cases to a minimum whenever it exists. Their main disadvantage is that its convergence is linear.

The second order methods, like quasi-Newton ones, are preferable to the first order methods because of their quadratic convergence. However, these methods lose their advantage when the stiffness matrix is ill-conditioned as in the case of a significant loss of stiffness. In this situation, the first order methods take an advantage in that the algorithm works well, even if the critical points exist and making the stiffness matrix singular.

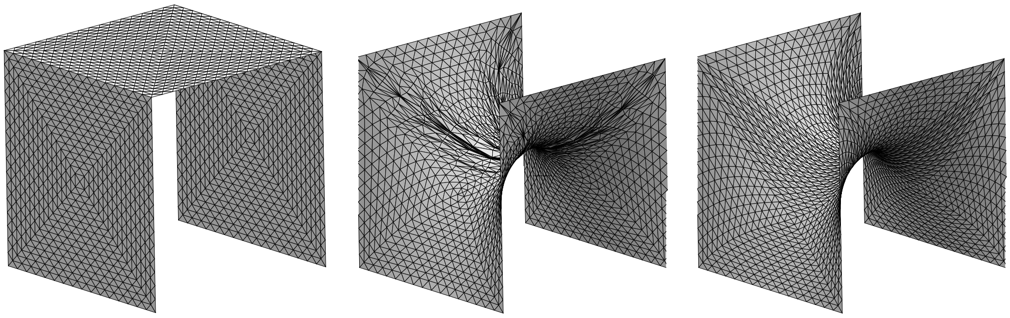
We used the conjugate gradient method to seek for the equilibrium position. This method uses only the gradient vector of the total potential energy and does not require specific processing of hypostatic kinematic modes. Numerical developments are implemented in the Brakke's Surface Evolver program [4].

8 Numerical examples

8.1 Scherk's problem

We consider here the classical test of the Scherk form finding problem. It is a minimal surface with boundaries described by the unit cube. The initial surface is made of three flat squares which is each meshed using 1024 linear triangular finite elements.

In order to accelerate the evanescence of the elastic potential, we introduce a weighting factor $\alpha = \lambda^n$ with $\lambda \in [0, 1]$ and n the computational step. The problem to solve is then written as



(a) Initial mesh (b) Mesh without regularization. (c) Elastic potential regularization.

FIGURE 2: Scherk's test meshes.

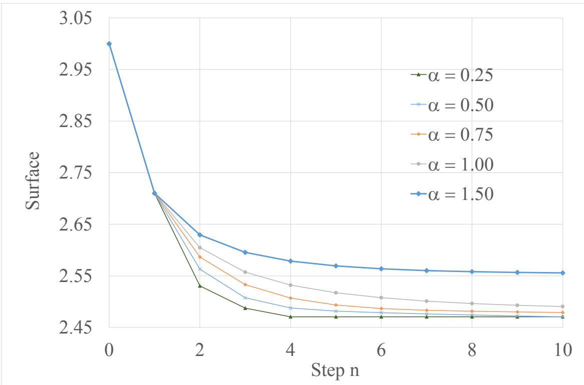


FIGURE 3: Convergence of central point z-position with respect to parameter α.

$$x_{sol} = \arg\min_x \Pi^{tot} = \arg\min_x \left(\Pi^s + \alpha\Pi^e + \Pi^c + ...\right) \tag{22}$$

For $\lambda = 1$, the problem is the same as in (19), for λ close to zero the coefficient α tends quickly to zero which reinforces the elastic energy cancellation. Figure 3 shows the effect of the coefficient α on the middle point position convergence.

8.2 Tent structure

The numerical example presented in what follows is a tent structure composed of an elastic membrane supported by cables, fixed anchors and rigid hoops. The structure has a wheelbase on the ground in rectangular form $2l \times 3l$ in the (x, y) plane, blocked on 8 anchors (A, B, ..., H), and surrounded by cables on the free edges. Two hoops are prescribed at $y = l$ and $y = 2l$, having the parabolic form $z = 2l\frac{x}{l}(\frac{x}{l} - 2)$ with $l = 0.5$ m.

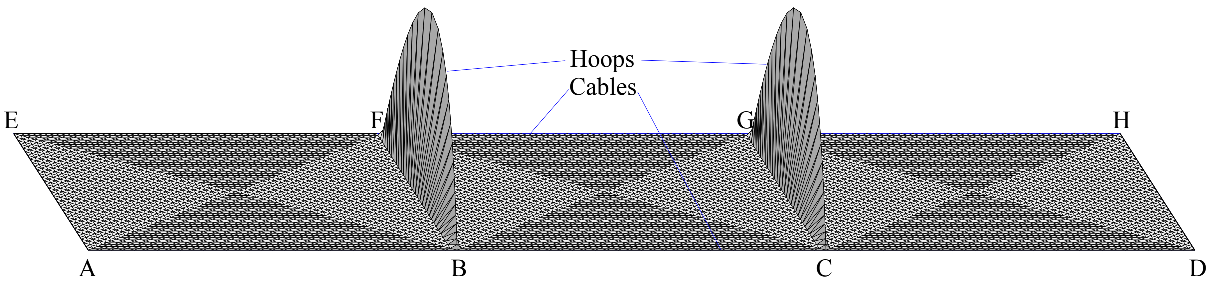


Figure 4: Initial mesh after geometric contraints.

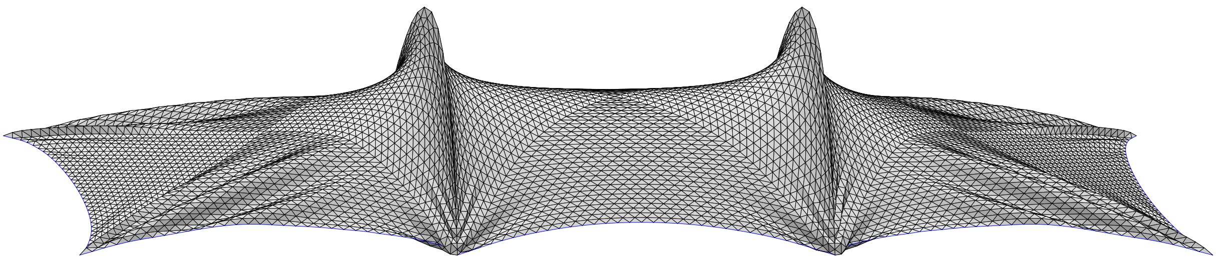


Figure 5: Deformed mesh without elastic energy canceling.

Figure 4 shows the mesh used and the displacement boundary conditions prescribed on the hoops.

The problem data are fixed as follows :

– For the membrane : surface tension $\tau = 1\ Pa$;
– For the elastic strain energy used for regularization : Young's modulus $5.10^5\ Pa$; Poisson's ratio 0.3 ; thickness 10^{-4} m.
– For cables : Young's modulus×section $ES = 10$ N.
– For the mesh : 1601 nodes, 3072 triangular finite elements.

Figure [5] shows the mesh obtained by minimizing functional Π^{tot} without canceling the elastic energy of the membrane Π^e. The deformed configuration, at this stage, has folds due to bifurcations arising from the compression of the membrane in certain directions.

By using the updated initial strategy, the elastic energy can be canceled iteratively. The membrane will gradually tend to a uniform and isotropic stress state. $\sigma = \tau\mathbf{I}$. Figure 6 shows the shape obtained after total cancellation of Π^e.

9 Conclusions

In this study we have transformed the stress field approach used in the form finding method to an energy minimisation problem. We have shown that the case of a uniform isotropic stress field is equivalent to the surface minimization. We have used the initial

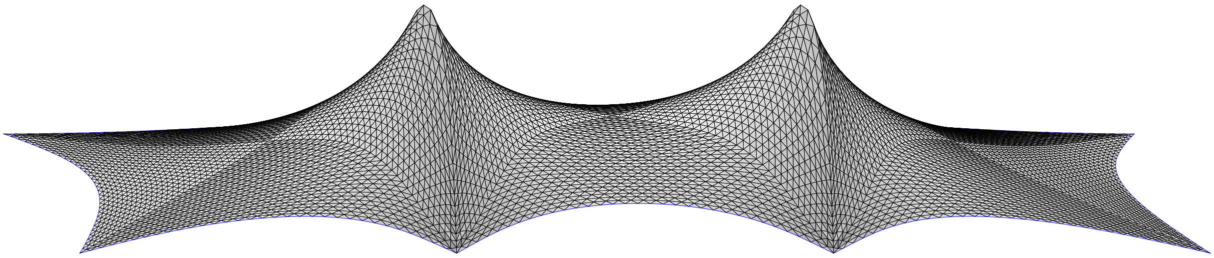


FIGURE 6: Deformed mesh with elastic energy canceled (Optimal shape).

configuration updating strategy to regularize the numerical scheme. This is done by modification of the metric tensor of the initial configuration.

The conjugate gradient method used to seek for the energy minimum proves to be very efficient to correctly handle hypostatic instabilities associated with membranes. Indeed, the mechanical model used is a pure membrane without bending stiffness. It is precisely this property of in-plane stress field that is formulated here as the criterion for form finding.

RÉFÉRENCES

[1] Bletzinger K.-U. and Ramm E., *A general finite element approach to the form finding of tensile structures by the updated reference strategy.* International Journal of Space Structures 14, 131Ű145.1999.

[2] Bletzinger K.-U., *Section 12.2 : Form finding and morphogenesis.* In : I. Munga, J.F. Abel (Eds.), Fifty Years of Progress for Shell and Spatial Structures, Multi-Science, pp. 459Ű482.

[3] Bonet J. M. J., *Form finding of membrane structures by the updated reference method with minimum mesh distorsion.* Int. Journal of solids and structures, vol. 38, p. 5469-5480, 2001.

[4] Brakke K.A. *The Surface Evolver.* Experimental Mathematics, 1, 141-165,1992.

[5] Haber R.B. and Abel, J.F., *Initial equilibrium solution methods for cable reinforced membranes*, Part I Ű Formulations. Computer Methods in Applied Mechanics and Engineering 30, 263Ű284, 1982.

[6] Haug E. and Powell G.H., *Analytical shape finding for cable nets.* In : Proceedings of the 1971 IASS Pacific Symposium Part II on Tension Structures and Space Frames, 1Ű5, Tokyo and Kyoto, Japan, pp. 83Ű92, 1972

[7] Kai-Uwe Bletzinger E. R., *Structural optimization and form finding of light weight structures*, Computers and structures, vol. 79, p. 2053-2062, 2001.

[8] Kai-Uwe Bletzinger . a., *Computational methods for form finding and optimizationof shells and membranes.*, Comput. Methods Appl. Mech. Engrg., vol. 194, p. 3438-3452, 2005.

[9] Maurin B. and Mautro R., *Structural optimization and form finding of light weight structures*, Computers and structures, vol. 20, nř 8, p. 712-719, 1998.

[10] Pauletti R.M.O. and Pimenta P.M., *The natural force density method for the shape finding of taut structures.* Computer Methods in Applied Mechanics and Engineering 197, 4419Ű4428. 2008

[11] Siev, A., Eidelman, J., *Stress analysis of prestressed suspended roofs*, Journal of the Structural Division,. Proceedings of the American Society of Civil Engineers,p. 103Ű121

[12] Tabarrok B. and Qin Z., *Nonlinear analysis of tension structures*, Computers and Structures 45, 973Ű984.

[13] Veenendaal D. and P. Block, *An overview and comparison of structural form finding methods for general networks*, International Journal of Solids and Structures, Volume 49, Issue 26, 15 December 2012, p. 3741-3753