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The eXtended Updated Reference Strategy for the form finding of tensile structures

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Summary. In this paper, the *eXtended Updated Reference Strategy* is presented. Starting from the established Updated Reference Strategy all related issues, which are involved for this methodology, are identified. It will be shown that the *eXtended Updated Reference Strategy* is able to solve the "correct" form finding problem in one non-linear iteration step. By applying the *eXtended Updated Reference Strategy* to well-known form finding problems the difference in convergence in comparison to establish methods like the force density method or the Updated Reference Strategy is discussed

1 INTRODUCTION

Tensile Structures are lightweight structures, which combine an optimal stress state of the material with an impressive language of shapes [1]. The shape of tensile structures is defined by the equilibrium of surface stress and cable edge forces in tension. Throughout the whole design process of tensile structures the variation of prestress constitutes the main shaping parameter. Due to this direct interaction of shape and prestress, the shape of a tensile structure cannot be set like for conventional structures (e.g. concrete bridges, wooden slaps, steel frames, etc.). The step of form finding is always necessary in order to find the final shape.

The first solutions for the task of form finding were made by using soap film and hanging models. From this approach some of the most challenging tensile structures were developed. Certainly, Frei Otto is one of the most important pioneers using physical models to solve the problem of form finding [2]. Nowadays the effort in research is mainly focused on the development of appropriate numerical methods for the form finding of tensile structures. This evolution is in large paths based on the huge impact of the introduction of Finite Element Methods (FEM) in engineering and the constantly growing computation capacities are the basis of this development. The starting point for the development of numerical methods for form finding of tensile structures is the work of Klaus Linkwitz with the well-known Force Density method (FD) [3].

In the following sections, starting from the correct continuum mechanical description of the form finding problem, the numerical issues in the solution process will be discussed. A method, which was introduced by [4] in order to solve the form finding of tensile structures, will be presented. Starting from this an extension will be derived which improves the convergence behaviour as well as the usability.

2 FORMFINDING OF TENSILE STRUCTURES

From a mathematical point of view the form finding of tensile structure is closely related to the well-known task of the determination of minimal surfaces. The connection between the pure geometrical and the mechanical model is the overall prestress in the surface, as minimal surfaces are characterized by an isotropic stress distribution. For centuries mathematicians have investigated research in the solution of minimal surfaces for different cases of boundary conditions [5]. Certainly, the experimental work of Joseph Plateau in the 19^{th} Century was one of the most important contributions to this research.

From a mechanical point of view the form finding of tensile structures is the task to find the shape of equilibrium w.r.t. a given surface stress state $\boldsymbol{\sigma}$ and natural (in terms of edge forces) or geometrical (e.g. clamped edges) boundary conditions. Additional loading, as e.g. internal pressure (cushions), has to be considered, too. Considering the non-linear kinematics of large deflections the equilibrium condition in the deformed, actual configuration is defined by the principle of virtual work. See Eq. (1):

$$\delta w = t\int_a \boldsymbol{\sigma} : \delta\mathbf{e}\, da - \int_a \mathbf{p}\,\delta\mathbf{u}\, da = 0 \qquad (1)$$

The total virtual work δw consists of the internal work given by the Cauchy stresses $\boldsymbol{\sigma}$, the virtual Euler-Almansi strains $\delta\mathbf{e}$, the thickness of the membrane t which is assumed to be thin and constant throughout the form finding process and the external work given by the external loading $\mathbf{p}$ and the virtual displacements $\delta\mathbf{u}$. Due the formulation of the equilibrium in the current configuration, the integration is carried out over the current domain da. In the following the external loading will be neglected. The discussion of the influence of the external load onto the governing equations is presented in [4] and holds for all of the derived equations in the later sections.
The equilibrium condition in Eq. (1) w.r.t the current configuration can also be transferred into the reference configuration by applying Nanson's relation which is given in Eq. (2).

$$a = \int_a da = \int_A \det \mathbf{F}\, dA \qquad (2)$$

In Eq. (2) det **F** represents the determinant of the deformation gradient **F** which connects the reference configuration to the current configuration. Inserting Eq. (2) in Eq. (1) leads to the equilibrium condition w.r.t. the reference configuration in Eq. (3).

$$\delta w = t\int_A \det \mathbf{F}\boldsymbol{\sigma} : \delta\mathbf{e}\, dA = 0 \qquad (3)$$

By applying the relation between the virtual Euler-Almansi strains $\delta\mathbf{e}$ and the Green-Langrange strains $\delta\mathbf{E}$ which is given in Eq. (4), the equilibrium can be written by values which are all defined w.r.t. the reference configuration.

$$\delta \mathbf{e} = \mathbf{F}^{-T} \delta \mathbf{E} \mathbf{F}^{-1} \tag{4}$$

Finally, after some rearrangements the equilibrium condition can be written as in Eq. (5).

$$\delta w = t \int_A \det \mathbf{F} \left(\boldsymbol{\sigma} \mathbf{F}^{-T} \right) : \delta \mathbf{F} dA = 0 \tag{5}$$

For the special case of minimal surfaces the prestress state can be expressed as a constant value s and the Identity Tensor **I** since it represents an isotropic stress distribution in the surface. See Eq. (6):

$$\boldsymbol{\sigma} = s\mathbf{I} \tag{6}$$

With Eq. (6) the equilibrium condition can be reformulated from Eq. (5) to the expression which is given in Eq. (7).

$$\delta w = st \int_A \det \mathbf{F} \mathbf{F}^{-T} : \delta \mathbf{F} dA = 0 \tag{7}$$

The derived equation up to this point is totally derived from continuum mechanics. In the following it will be shown that Eq. (7) describes a minimal surface in a mathematically correct way. From a mathematical point of view minimal surfaces are defined as surfaces of minimal area content between given boundaries. The minimum of area content can be derived by the vanishing variation δa of the area content a. See Eq. (8):

$$\delta a = \delta \int_a da = 0 \tag{8}$$

Again, using Nanson's relation (c.f. Eq. (2)) the variation of the area content can be formulated as given in Eq. (9).

$$\delta a = \delta \int_A \det \mathbf{F} dA = 0 \tag{9}$$

Herein, the variation of the determinant of the deformation gradient det **F** has to be formulated. This variation can be derived as shown in Eq. (10).

$$\delta (\det \mathbf{F}) = \det \mathbf{F} \mathbf{F}^{-T} : \delta \mathbf{F} \tag{10}$$

Inserting Eq. (10) in Eq. (9) the variation of the area content can be formulated as given in Eq. (11).

$$\delta a = \int_A \det \mathbf{F} \mathbf{F}^{-T} : \delta \mathbf{F} dA = 0 \tag{11}$$

Obviously, by applying an isotropic stress field to a tensile structure the equation which is derived from continuum mechanics is identical to the equation that is derived from a mathematical point of view. Both approaches are able to describe the task of finding a minimal surface.

In order to solve the problem stated in Eq. (1), standard numerical methods (e.g. FEM) can be used. In this context a discretization of the governing equation has to be done, in order to reduce the number of unknowns to a finite number. Furthermore, a geometrical nonlinear analysis is necessary due to the fact that the given problem includes large displacements. Trying to solve the given problem from Eq. (1) it turns out that the system matrix to evaluate the unknown discretization parameters is singular. The reason for this deficiency originates from the inverse character of the given problem where, stresses in the deformed configuration are given without considering material properties. This inverse character can be understood in comparison to standard structural analysis, where based on a defined reference configuration the deformation w.r.t. a certain load situation is computed. Therefor the stresses can be evaluated from displacements by applying the material law. In contrast to that, form finding already knows the stress and tries to determine the deformed geometry. Due to the prescribed stresses this can be done without defining any material property (see Fig. 1).

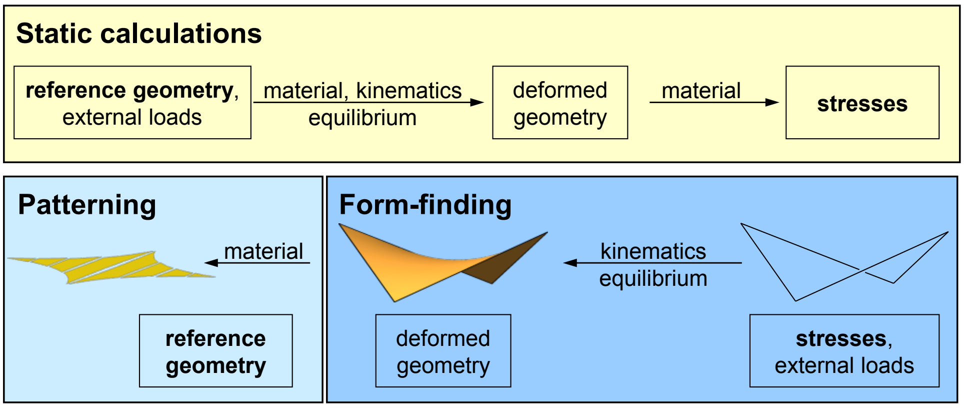

Figure 1: Inverse character of the form finding

On important feature is that surface stresses and strains are not related. As a consequence it turns out, that the position of the nodes on the surface cannot be evaluated uniquely, since it is possible to describe the same surface with differently shaped finite elements: The nodes can float freely on the surface. Hence, the fact, that the same surface can be described by an infinite number of discretizations leads to the singular system matrix (see Fig. 2).

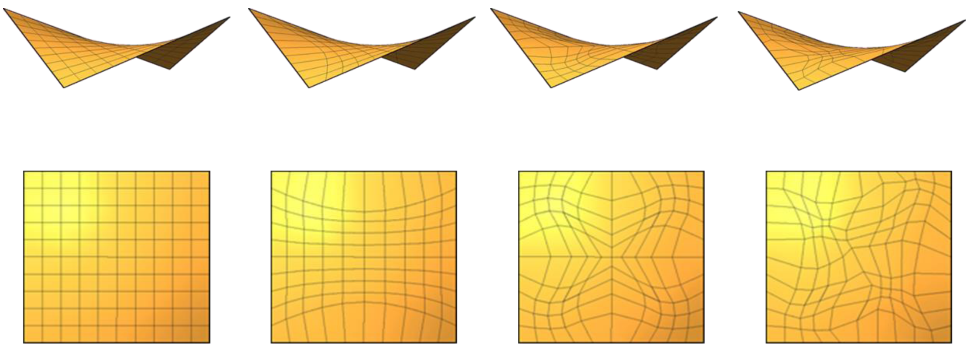

Figure 2: Floating meshes while describing the same surface

To eliminate this singularity various methods have been developed in the past, like e.g. the dynamic relaxation [6], [7] or the force density [3], [8]. All of them try to stabilize the singular system matrix by different kinds of approaches. In the following, a further, most general method is presented which is consistently derived from continuum mechanics.

3 UPDATED REFERENCE STRATEGY (URS)

The *updated reference strategy* (URS) uses general mathematical methods to stabilize the singular problem given in the previous section [4], [9]-[10]. The idea is to modify the original, singular problem by a related one which fades out as we approach the solution. Therefore Eq. (1) will be expanded by an additional term which describes an alternative formulation of the internal virtual work. See Eq. (12):

$$\delta w_{URS} = \underbrace{\lambda t \int_a \boldsymbol{\sigma} : \delta \mathbf{e} \, da}_{\text{Original}} + \underbrace{(1-\lambda) t \int_A \mathbf{S} : \delta \mathbf{E} \, dA}_{\text{Stabilization}} - \int_a \mathbf{p} \delta \mathbf{u} \, da = 0 \tag{12}$$

The first part of Eq. (12) is the original problem from Eq. (1). The second part represents the stabilization in terms of an added similar problem. The last part again represents the external virtual work. The stabilization term formulates the equilibrium condition w.r.t. the reference configuration where the true surface Cauchy stresses $\boldsymbol{\sigma}$ are replaced by $\mathbf{S}$, the 2nd Piola-Kirchhoff stresses. As they are artificially related to tangential deformation of the mesh a formulation using $\mathbf{S}$ does not suffer from singularity but the solution deviates from the intended one. On the other hand, if there is no deformation $\mathbf{S}$ and $\boldsymbol{\sigma}$ are identical. Obviously, the homotopy factor λ controls the solvability of the whole problem. For the choice $\lambda = 1$ only the original problem will be considered and for $\lambda = 0$ the pure stabilization term is solved. It is guaranteed that the system of equations is solvable as long as λ is small enough to stabilize the whole problem.

The biggest advantage of the URS is that the stabilization term becomes more and more alike the original problem as the reference configuration gets closer to the final shape. Hence, a further improvement of the method can be done, by using the solution of Eq. (12) with any arbitrary choice of λ as a new improved reference configuration for the next approximation. This means by solving Eq. (12) the reference configuration will be iteratively updated towards the optimal solution. The principal sequence of a form finding by applying the URS is shown in Figure 3.

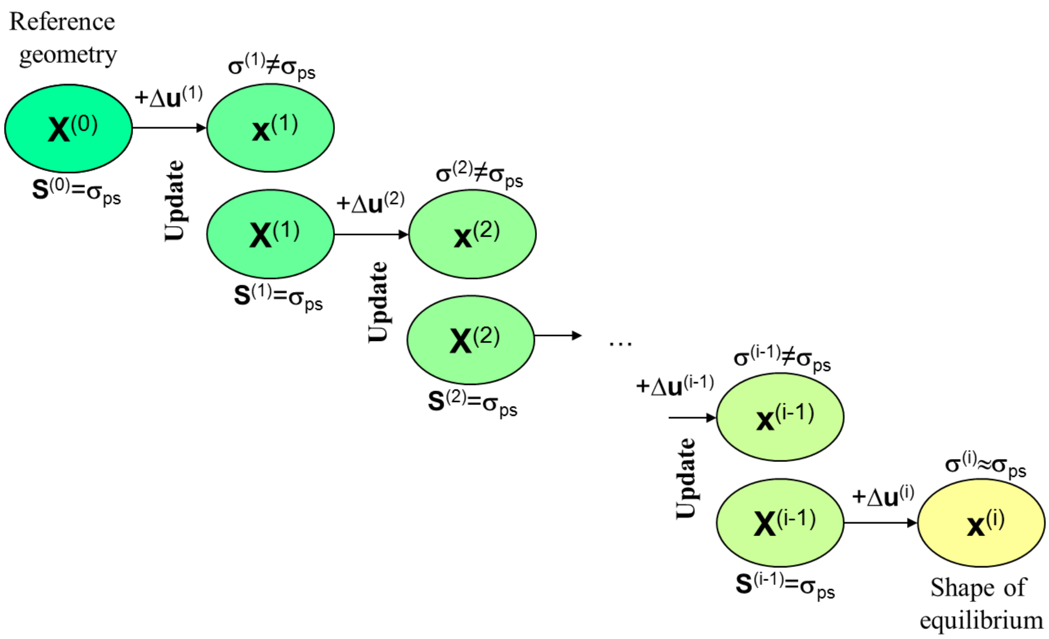

Figure 3: Principal sequence of the URS

Starting with an almost arbitrary reference configuration the 2nd Piola-Kirchhoff stresses **S** are assumed to be equal to the Cauchy stresses **σ**. Herein, the chosen reference configuration has to fulfill the boundary conditions. After solving the governing equation from Eq. (12) the resulting displacements are added to the reference configuration and set this state (= actual configuration) as the new reference configuration. The method will be repeated until the occurring displacements will converge to be small enough.
As the method is totally dependent on the choice of λ, the speed of convergence and the solvability is directly connected to this choice. In the following a new extension to the URS is presented which cancels out the drawback of choosing a value for λ in each form finding step by maximum possible convergence speed.

4 EXTENDED UPDATED REFRENCE STRATEGY (X-URS)

The idea of the *eXtended Updated Reference Strategy* is to modify the principle of virtual work in a way, that still the original problem is solved the singularity in the tangential direction are neglected. In the following, the terms which cause the singularity in the governing equation will be referred to as singular terms. To identify the singular terms the principle of virtual work which is given in Eq. (12) has to be stated in the linearized form. In Eq. (13) the residual form is shown. Herein the virtual work is linearized w.r.t. the virtual displacements δb_r which are points in the direction of the discretization parameters. For the sake of simplicity the external load is neglected in Eq. (13). The discussion of the influence of the external load onto the governing equations is presented in [4].

$$\frac{\partial w_{URS}}{\partial b_r}\delta b_r = R_r \delta b_r = \left(\underbrace{\lambda t \int_a \boldsymbol{\sigma} : \frac{\partial \mathbf{e}}{\partial b_r} da}_{R_{r,\sigma}} + \underbrace{(1-\lambda) t \int_A \mathbf{S} : \frac{\partial \mathbf{E}}{\partial b_r} dA}_{R_{r,S}} \right) \delta b_r = 0 \tag{13}$$

Again the residual forces for the respective parts of the original problem and the stabilization can be identified. $R_{r,\sigma}$ represents the residual force of the original problem in the direction of the discretization parameter b_r. The residual force of the stabilization term is given in $R_{r,S}$.
By investigating the forces w.r.t. to their influence on the singularity of the final stiffness matrix in the numerical solution process it can be identified that the singularity originates from the original term (which was already discussed in section 2). More precisely the singular term is related to the derivative of the residual force which points in the direction tangential to the surface. Figure 4 illustrates the different parts of the residual forces: $\mathbf{R}^n$ acts along the surface normal $\mathbf{n}$ and $\mathbf{R}^t$ along the tangential direction $\mathbf{t}$.

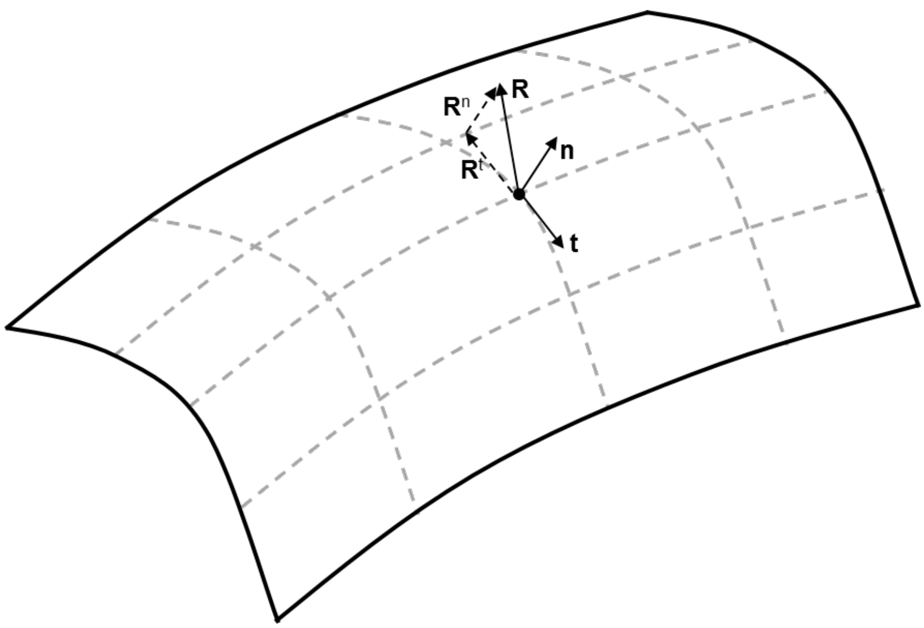

Figure 4: Separation of the residual force

To perform the separation of the forces in normal and tangential direction basic mathematic definitions can be used. Eq. (14) and Eq. (15) shows the residual force in normal and tangential direction.

$$\mathbf{R}^n = (\mathbf{n} \otimes \mathbf{n})\mathbf{R} \tag{14}$$

$$\mathbf{R}^t = \mathbf{R} - \mathbf{R}^n = (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{R} \tag{15}$$

The summation of the normal and tangential part of the residual force leads again to Eq. (13), but know the possibility to identify the singular terms is given. In Eq. (16) the original and the stabilization problem are shown by separating them to the normal and tangent direction.

$$\mathbf{R} = \lambda\underbrace{\left(\mathbf{R}_{\sigma}^{n} + \mathbf{R}_{\sigma}^{t}\right)}_{\mathbf{R}_{\sigma}} + (1-\lambda)\underbrace{\left(\mathbf{R}_{S}^{n} + \mathbf{R}_{S}^{t}\right)}_{\mathbf{R}_{S}} = 0 \qquad (16)$$

In Eq. (16) the singular term is fully related to the tangential residual force. The stabilization $\mathbf{R}^{s}$ ensures the solvability in the tangential direction. Due that it would be sufficient if the stabilization just affects the tangential residual force of the original problem. From Eq. (16) it can be seen that the stabilization also affects the residual force in normal direction. The splitted residual form know offers the opportunity to just take into account the terms of the residual force which are needed in order to solve the form finding problem. Obviously, the residual force of the original problem in the normal direction $\mathbf{R}_{\sigma}^{n}$ is needed to find the shape of equilibrium of the tensile structure. To stabilize the form finding problem the residual force of the stabilization term in tangential direction $\mathbf{R}_{S}^{t}$ is needed, too. The other two terms (R_{σ}^{t} and R_{S}^{n}) can be neglected as they are not of importance for the description of the form finding problem. Due to the absence of a singular term in the governing equation the homotopy blending is not necessary anymore and the given non-linear problem can be solved directly. Finally the governing equation of the *eXtended Updated Reference Strategy* can be given in Eq. (17).

$$\mathbf{R}_{X-URS} = \mathbf{R}_{\sigma}^{n} + \mathbf{R}_{S}^{t} = (\mathbf{n} \otimes \mathbf{n})\mathbf{R}_{\sigma} + (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{R}_{S} = 0 \qquad (17)$$

As the governing equation of the X-URS solves the original problem without any compromises in the normal direction of the surface, the solution after the first form finding step is identical to the analytical one. Additionally, the solution in normal direction might also be influenced by the deformation in the tangential direction (e.g. edge cables).

In Figure 5 the method and the convergence behavior is discussed for the Schwarz minimal surface. The surface is discretized with 4 finite elements, which results in 3 global degrees of freedom at the middle node. The reference configuration satisfies the boundary condition w.r.t. the straight edges. Obviously, the middle node just has to move in the vertical (here z-) direction. Due to that in the following the residual force R_z will be investigated.

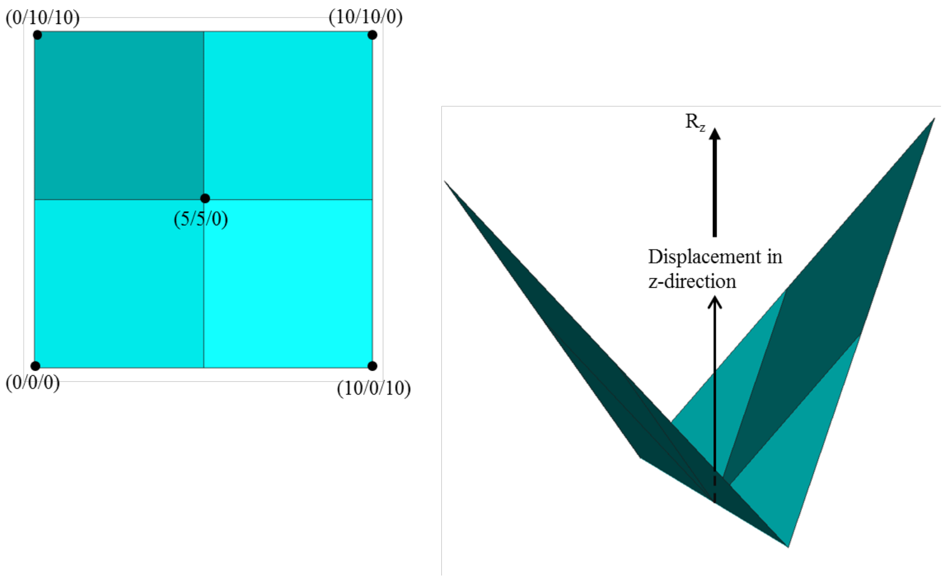

Figure 5: Schwarz minimal surface reference configuration; Top view (left); Iso view (right)

It is obvious that the final position of the middle node has to be at the half of the height of the surface (here 5). From a mechanical point of view, a "correct" residual equation should have a zero value of the residual forces for a displacement of 5 in the vertical direction. In the following the residual forces for the force density method (FD), for the URS and the X-URS are compared in the first form finding step. In Figure 6 the residual forces for the named methods are plotted. In case of the URS two different solutions are plotted for different choices of the homotopy factor λ.

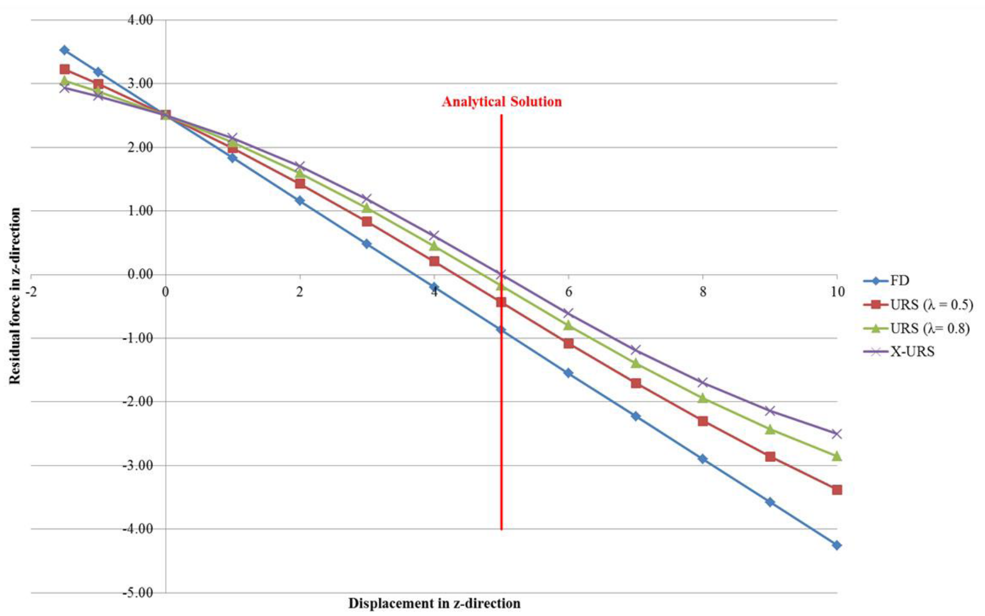

Figure 6: Residual forces for the Schwarz minimal surface for different form finding methods. Convergence is archived for displacement of 5 (red line).

From Figure 6 it can be seen that only the X-URS converges to the final solution within the first form finding step. The force density method shows the worst convergence within the first form finding step, while the URS convergences to the final solution with increasing homotopy factors λ. It has to be stated at this point that the non-linearity increases from force density to X-URS. When applying the force density method to a form finding problem in each form finding step just a linear system has to be solved. This advantage has the price of an increasing amount of form finding steps. In contrast the X-URS leads to a non-linear problem in each form finding step but with a decreased number (in the best case just one) of form finding steps. In the following, examples for the successful application of the X-URS are presented.

5 EXAMPLES

In this section two examples are shown. For both examples the comparison to the force density and URS are investigated.

5.1 Catenoid

The first example is the well-known Catenoid minimal surface. The observed displacement is at the half of the height (point M) of the height of the surface. The reference configuration was a cylinder. Due to that the analytical solution based on a catenary curve can be determined (see Fig. 7)

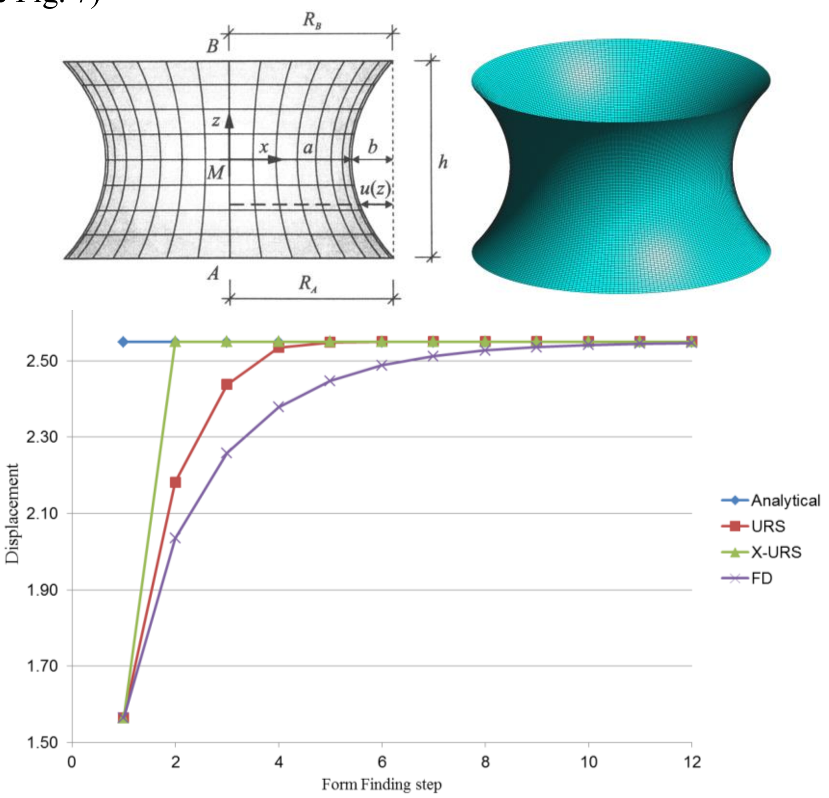

Figure 7: Catenoid (top left) [11]; FEM discretization (top right); Convergence graphs for FD, URS and X-URS for the displacement at the half of the height (distance b) (below)

5.2 Four Point Tent

The second example shows the well-known four point tent. Starting from an arbitrary reference configuration which satisfies the boundary conditions (in terms of high and low points) the final shape of the surface is determined. Again, the comparison to the force density method as well as to the URS is plotted (see Fig. 8).

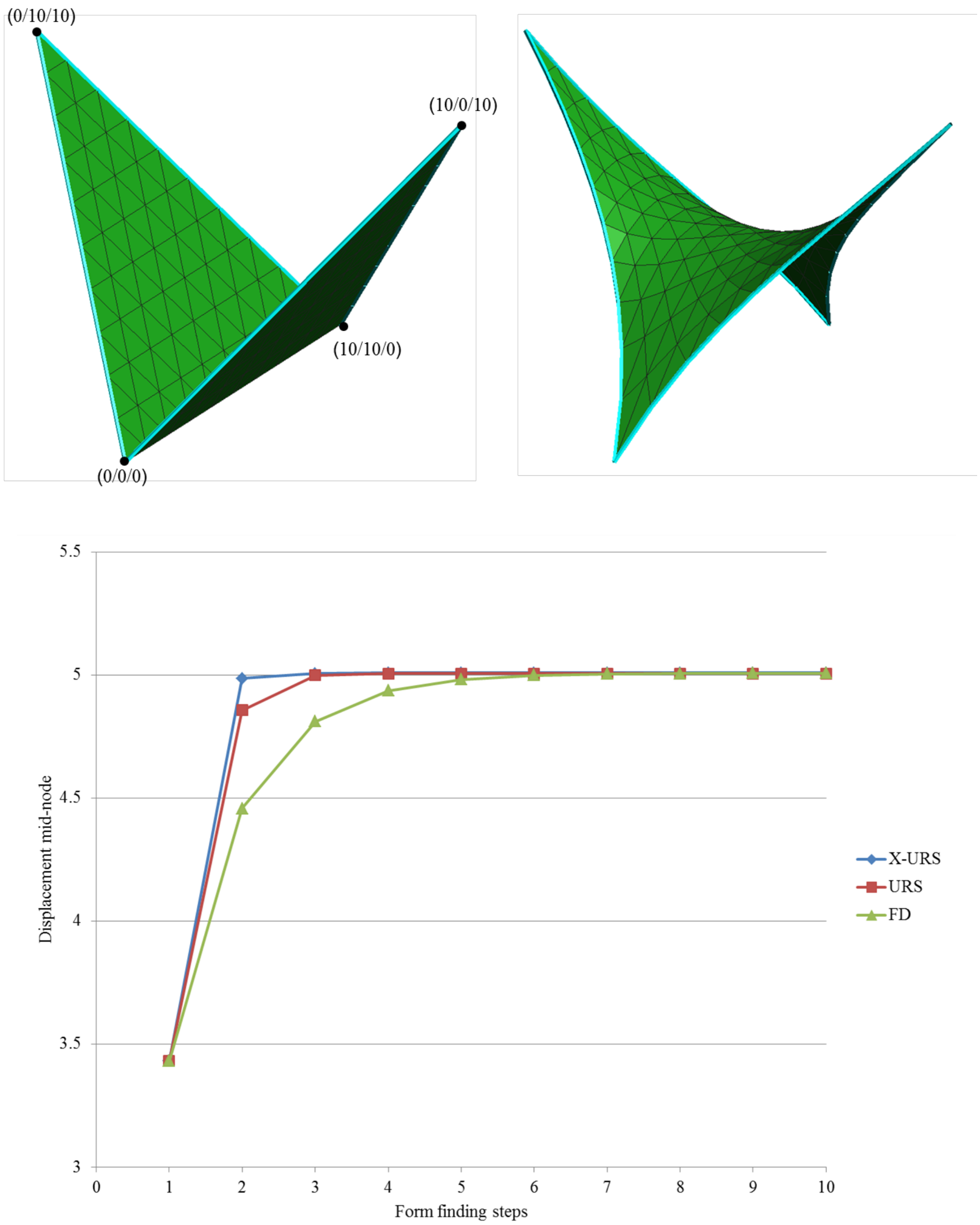

Figure 8: Initial reference configuration (top left); Converged state (top right); Convergence graphs for FD, URS and X-URS for the vertical displacement in the center of the surface (below)

6 CONCLUDING REMARKS

In the latter sections a new method for form finding of tensile structures has been presented. The *eXtended Updated Reference Strategy* describes the form finding problem in a non-linear residual equation. When solving this equation the final equilibrium shape is achieved. The additional introduction of pseudo time step iteration (form finding steps) is not necessary. The capacity of the method is illustrated on two well-known examples (Catenoid and Four Point Tent). For both of them, the convergence behavior of the respective displacements demonstrates the advantage of the method w.r.t. to nowadays established methods. The final shapes are achieved for both cases within the first form finding step. The improvement of the tangential stabilization w.r.t. the influence on the overall solution should be the topic of further discussions on the method.

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