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EXPONENTIAL DECAY IN ONE-DIMENSIONAL TYPE II/III THERMOELASTICITY WITH TWO POROSITIES

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Abstract: In this paper we consider the theory of thermoelasticity with a double porosity structure in the context of the Green-Naghdi types II and III heat conduction models. For the type II, the problem is given by four hyperbolic equations and it is conservative (there is no energy dissipation). We introduce in the system a couple of dissipation mechanisms in order to obtain the exponential decay of the solutions. To be precise, we introduce a pair of the following damping mechanisms: viscoelasticity, viscoporosities and thermal dissipation. We prove that the system is exponentially stable in three different scenarios: viscoporosity in one structure jointly with thermal dissipation, viscoporosity in each structure, and viscoporosity in one structure jointly with viscoelasticity. However, if viscoelasticity and thermal dissipation are considered together, undamped solutions can be obtained.

Keywords: type II/III thermoelasticity with double voids, viscosity, viscoporosity, exponential decay.

1. Introduction

Porous materials are quite common in daily life. Think, for example, in something as simple and soft as a fairy cake or something more rigid as a piece of limestone travertine, like the ones shown in figure 1.

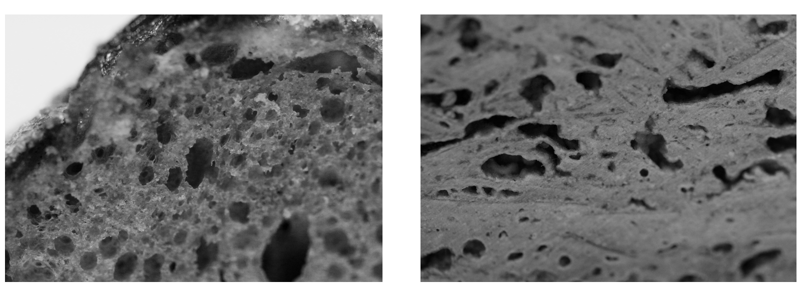

Figure 1. A fairy cake and a piece of travertine.

More sophisticated porous materials are being studied in biomedicine to repair injuries in bones. Zhang *et al.*[42] made a shape-memory polymer that molds itself to the shape of the bone defect

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without being brittle and supporting the growth of new bone tissue. The geometry of shape-memory polymers changes in response to heat. Therefore, it is very important to know the behavior ot these materials to the action of heat.

On the other side, porous materials can also have cracks or fissures in their skeletons. That means that two different voids structures should be considered to build a more accurate thermomechanical model which, later, can be mathematically analyzed. These two structures are known, respectively, as macro-porosity and micro-porosity.

Double porosity structures appear naturally in continuum mechanics. In fact, thermoelastic problems with double porosity have received an increasing attention in the last two decades. The first studies concerning this theory were related with the law of Darcy. In these first models, it is usual to work with displacement, pressures associated with the pores and the fissures.

The easiest extension of the classical theory of elasticity to this context was proposed by Nunziato and Cowin [35]. It describes the behavior of elastic solids with voids. It is supposed that the materials have a skeleton or matrix material that is elastic, and the interstices are voids of material. There is in the literature a huge quantity of publications analyzing this theory [3, 4, 6, 7, 8, 9, 20, 25, 26, 27, 28, 34, 36, 37, 40]. It has been applying to geological materials such as rocks or soils as well as to manufactured materials such as ceramics or pressed powders.

Iesan and Quintanilla [16] extended the above model to structures with double porosity. Obviously, both porosities have influence on the elastic deformations and on the heat conduction, and vice-versa. The interactions among the elasticity, both porosities and the temperature are described by means of the constitutive equations and, even in the static case, they are strongly coupled. This model has deserved a lot of research since it was proposed [1, 2, 15, 17, 18, 19, 41].

On the other hand, it is well known that the heat conduction theory based on the classical Fourier's law leads to a parabolic equation which, unfortunately, when it is solved allows the instantaneous propagation of the thermal waves. This behavior violates the *principle of causality* of physics. For this reason, many alternative heat conduction models have been developed. All of them try to overcome the aforementioned gap. As a matter of illustration, we recall that Green and Lindsay [11] and Lord and Shulman [24] proposed thermoelastic theories based on the Cattaneo-Maxwell heat conduction hyperbolic equation [5].

In the last decade of the twentieth century, Green and Naghdi [12, 13, 14] proposed three new thermoelastic theories based on a set of rational axioms. They named them as type I, type II and type III, respectively. The linear version of the first one agrees with the Fourier's model. However, for types II and III they introduced a new independent variable, the thermal displacement, that gives rise a new kind of equations. The Green and Naghdi models are being intensively studied [10, 21, 22, 29, 30, 33, 38, 39].

In this paper we analyze the thermoelastic problem that appears in solids with double porosity when the heat conduction is described by either the type II or the type III heat conduction theories. We restrict our attention to the one-dimensional case and we obtain several results which differ from the ones obtained for the classical thermoelasticity theory based on the Fourier law.

The thermomechanical reason for this behavior is the presence of the thermal displacement. This variable determines new and strong coupling between the macro and micro structures. From a strictly mathematical point of view, we study a system of four second-order in time equations and our aim is to propose a couple of dissipation mechanisms which could guarantee the exponential stability. The results that we obtain are somehow striking because with only two dissipation mechanisms we control four second-order in time equations. A similar behavior has been observed recently [20, 31, 32] for the types II/III thermoelasticity with voids: generically, the thermal dissipation is enough to bring the whole system to the exponential decay. This is also because the thermal displacement plays a role of *transmission band* between the macro and micro porosity structures.

We want to highlight a couple of facts. The first one is the novelty of the model, and the second one is that our approach is mainly theoretical. We believe that any theory needs a mathematical and physical analysis that allows to decide its applications to the real-world situations. We also want to remark the similarities, from a mathematical perspective, between the equations for elastic materials with double porosity and those for microstretch materials. That means that the equations that we study in this paper can also be viewed as the equations used to describe a mixture of microstretch materials if their macroscopic structures coincide.

The plan of the paper is the following. In Section 2 we set the problem and the hypotheses over the constitutive coefficients. In fact, we set four systems of equations with their respective initial and boundary conditions. For each one, we prove the existence and uniqueness of solutions. In the following sections we analyze, case by case, the behavior of the solutions with respect to the time.

2. Statement of the problem and well-posedness

We will study four different situations. In order to be rigorous but, at the same time, trying not to enlarge the paper very much, we will write in detail only the first situation. The other three are obtained in similar ways and are simply sketched, writing only the most important issues.

Case 1: viscoporosity and thermal dissipation

In the context of the one-dimensional type III thermoelasticity with two porous structures, and supposing that there exists porous dissipation in one of the porous components (it does not matter in which one), the system of equations is determined by the following evolution equations:

$$\begin{aligned} \rho\ddot{u} &= t_x, \\ J_1\ddot{\phi}_1 &= h_{1,x} + g_1, \\ J_2\ddot{\phi}_2 &= h_{2,x} + g_2, \\ \rho\dot{\eta} &= q_x. \end{aligned} \tag{2.1}$$

The constitutive equations are given by:

$$\begin{aligned} t &= \mu u_x + \gamma_1\phi_1 + \gamma_2\phi_2 - \beta\theta, \\ h_1 &= b_{11}\phi_{1,x} + b_{12}\phi_{2,x} + m_1\psi_x, \\ h_2 &= b_{12}\phi_{1,x} + b_{22}\phi_{2,x} + m_2\psi_x, \\ g_1 &= -\gamma_1 u_x + d_1\theta - \xi_{11}\phi_1 - \xi_{12}\phi_2 - \xi^*\dot{\phi}_1, \\ g_2 &= -\gamma_2 u_x + d_2\theta - \xi_{12}\phi_1 - \xi_{22}\phi_2, \\ \rho\eta &= \beta u_x + a\theta + d_1\phi_1 + d_2\phi_2, \\ q &= k\psi_x + m_1\phi_{1,x} + m_2\phi_{2,x} + k^*\theta_x. \end{aligned} \tag{2.2}$$

As usual ρ is the mass density, J_i $(i = 1, 2)$ are the products of the mass density by the equilibrated inertias, t is the stress, h_i are the equilibrated stresses, g_i are the equilibrated body forces, q is the heat flux, η is the entropy and the variables u, ϕ_i and ψ are the displacement, the volume fractions and the thermal displacement, respectively, and θ is the temperature.

After substitution of the constitutive equations into the evolution equations, we obtain the field equations for the one-dimensional problem:

$$\left\{\begin{aligned} \rho\ddot{u} &= \mu u_{xx} + \gamma_1\phi_{1,x} + \gamma_2\phi_{2,x} - \beta\dot{\psi}_x \\ J_1\ddot{\phi}_1 &= b_{11}\phi_{1,xx} + b_{12}\phi_{2,xx} + m_1\psi_{xx} - \xi_{11}\phi_1 - \xi_{12}\phi_2 + d_1\dot{\psi} - \gamma_1 u_x - \xi^*\dot{\phi}_1 \\ J_2\ddot{\phi}_2 &= b_{12}\phi_{1,xx} + b_{22}\phi_{2,xx} + m_2\psi_{xx} - \xi_{12}\phi_1 - \xi_{22}\phi_2 + d_2\dot{\psi} - \gamma_2 u_x \\ a\ddot{\psi} &= m_1\phi_{1,xx} + m_2\phi_{2,xx} + k\psi_{xx} - d_1\dot{\phi}_1 - d_2\dot{\phi}_2 - \beta\dot{u}_x + k^*\theta_{xx} \end{aligned}\right. \tag{2.3}$$

The parameters that appear in the system are related with the properties of the material and have to satisfy some thermomechanical restrictions. In particular, we assume that

$$J_i > 0\,(i=1,2),\ a>0,\ \rho>0,\ \xi^*>0,\ k^*>0, \tag{2.4}$$

and that the two following matrices are positive definite:

$$M_1 = \begin{pmatrix} b_{11} & b_{12} & m_1 \\ b_{12} & b_{22} & m_2 \\ m_1 & m_2 & k \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} \mu & \gamma_1 & \gamma_2 \\ \gamma_1 & \xi_{11} & \xi_{12} \\ \gamma_2 & \xi_{12} & \xi_{22} \end{pmatrix}. \tag{2.5}$$

Remark 2.1. Our assumptions are in agreement with the thermomechanical axioms and the empirical experiments. The assumptions concerning the mass density, the thermal capacity and the parameters J_i are obvious. The conditions on μ, ξ_{ij}, b_{ij}, k, m_i and γ_i can be understood with the help of the elastic stability. The conditions on the viscosity parameters k^* and ξ^* are the natural ones to guarantee the dissipation.

It is well known that the parameter β relates the displacement to the temperature. Furthermore, m_i relates the thermal displacement to the volume fractions. These parameters are responsible for the strong coupling between the variables.

To set the problem properly, we need to impose boundary and initial conditions. Thus, we assume that the solutions satisfy the following boundary conditions: for any $t>0$,

$$\begin{aligned} &u(0,t) = u(\pi,t) = 0, \\ &\phi_{1,x}(0,t) = \phi_{1,x}(\pi,t) = \phi_{2,x}(0,t) = \phi_{2,x}(\pi,t) = 0, \\ &\psi_x(0,t) = \psi_x(\pi,t) = 0. \end{aligned} \tag{2.6}$$

As for the initial conditions, we assume

$$\begin{aligned} &u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \phi_1(x,0) = \phi_{10}(x),\ \dot{\phi}_1(x,0) = \varphi_{10}(x), \\ &\phi_2(x,0) = \phi_{20}(x),\ \dot{\phi}_2(x,0) = \varphi_{20}(x),\ \psi(x,0) = \psi_0(x),\ \dot{\psi}(x,0) = \theta_0(x) \ \text{ for } \ x\in[0,\pi]. \end{aligned} \tag{2.7}$$

One of the aims of this paper is to determine the behavior of the solutions (with respect to the time) to the problem given by system (2.3), boundary conditions (2.6) and initial conditions (2.7). In fact, we want to prove that the solutions decay in an exponential way if appropriate damping mechanisms are considered in the system (the thermal dissipation and the porous dissipation on one porous structure are sufficient to bring the system to the exponential stability, as we will see in Section 3).

We note that there are solutions (uniform in the variable x) that do not decay. To avoid these cases, we will also assume that

$$\int_0^\pi \phi_{10}(x)\,dx = \int_0^\pi \varphi_{10}(x)\,dx = \int_0^\pi \phi_{20}(x)\,dx = \int_0^\pi \varphi_{20}(x)\,dx = \int_0^\pi \psi_0(x)dx = \int_0^\pi \theta_0(x)dx = 0.$$

We consider the Hilbert space

$$\mathcal{H} = \big\{(u,v,\phi_1,\varphi_1,\phi_2,\varphi_2,\psi,\theta) \in H_0^1 \times L^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2 \times H_*^1 \times L_*^2\big\}, \tag{2.8}$$

where

$$L_*^2 = \{f \in L^2, \int_0^\pi f(x)dx = 0\} \text{ and } H_*^1 = L_*^2 \cap H^1.$$

Taking into account that $\dot{u} = v$, $\dot{\phi}_i = \varphi_i$ and $\dot{\psi} = \theta$ and writing $D = \frac{d}{dx}$, we can restate system (2.3) in the following way:

$$\begin{cases} \dot{u} = v \\ \dot{v} = \dfrac{1}{\rho}\big(\mu D^2 u + \gamma_1 D\phi_1 + \gamma_2 D\phi_2 - \beta D\theta\big) \\ \dot{\phi}_1 = \varphi_1 \\ \dot{\varphi}_1 = \dfrac{1}{J_1}\big(b_{11}D^2\phi_1 + b_{12}D^2\phi_2 + m_1 D^2\psi - \xi_{11}\phi_1 - \xi_{12}\phi_2 + d_1\theta - \gamma_1 Du - \xi^*\varphi_1\big) \\ \dot{\phi}_2 = \varphi_2 \\ \dot{\varphi}_2 = \dfrac{1}{J_2}\big(b_{12}D^2\phi_1 + b_{22}D^2\phi_2 + m_2 D^2\psi - \xi_{12}\phi_1 - \xi_{22}\phi_2 + d_2\theta - \gamma_2 Du\big) \\ \dot{\psi} = \theta \\ \dot{\theta} = \dfrac{1}{a}\big(m_1 D^2\phi_1 + m_2 D^2\phi_2 + kD^2\psi - d_1\varphi_1 - d_2\varphi_2 - \beta Dv + k^* D^2\theta\big) \end{cases} \tag{2.9}$$

Moreover, if $U = (u,v,\phi_1,\varphi_1,\phi_2,\varphi_2,\psi,\theta)$, then our initial-boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}_1 U, \;\; U_0 = (u_0, v_0, \phi_{10}, \varphi_{10}, \phi_{20}, \varphi_{20}, \psi_0, \theta_0), \tag{2.10}$$

where $\mathcal{A}_1$ is the following 8×8-matrix

$$\mathcal{A}_1 = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & 0 & \frac{\gamma_1}{\rho}D & 0 & \frac{\gamma_2}{\rho}D & 0 & 0 & -\frac{\beta}{\rho}D \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1}{J_1}D & 0 & \frac{b_{11}D^2-\xi_{11}}{J_1} & -\frac{\xi^*}{J_1} & \frac{b_{12}D^2-\xi_{12}}{J_1} & 0 & \frac{m_1}{J_1}D^2 & \frac{d_1}{J_1} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ -\frac{\gamma_2}{J_2}D & 0 & \frac{b_{12}D^2-\xi_{12}}{J_2} & 0 & \frac{b_{22}D^2-\xi_{22}}{J_2} & 0 & \frac{m_2}{J_2}D^2 & \frac{d_2}{J_2} \\ 0 & 0 & 0 & 0 & 0 & I & & \\ 0 & -\frac{\beta}{a}D & \frac{m_1}{a}D^2 & -\frac{d_1}{a} & \frac{m_2}{a}D^2 & -\frac{d_2}{a} & \frac{k}{a}D^2 & \frac{k^*}{a}D^2 \end{pmatrix} \tag{2.11}$$

and I is the identity operator. We note that the domain of $\mathcal{A}_1$, that we will denote by $\mathcal{D}(\mathcal{A}_1)$, is dense in $\mathcal{H}$.

We now define an inner product in $\mathcal{H}$. If $U^*=(u^*,v^*,\phi_1^*,\varphi_1^*,\phi^*,\varphi_2^*,\psi_2^*,\theta^*)$, then

(2.12)

$$\langle U,U^*\rangle_{\mathcal{H}}=\frac{1}{2}\int_0^\pi\Big(\rho v\bar{v}^*+J_1\varphi_1\bar{\varphi}_1^*+J_2\varphi_2\bar{\varphi}_2^*+a\theta\bar{\theta}^*+\mu u_x\bar{u}_x^*+b_{11}\phi_{1x}\bar{\phi}_{1x}^*+b_{12}(\phi_{1x}\bar{\phi}_{2x}^*+\bar{\phi}_{1x}^*\phi_{2x})$$
$$+b_{22}\phi_{2x}\bar{\phi}_{2x}^*+\xi_{11}\phi_1\bar{\phi}_1^*+\xi_{12}(\phi_1\bar{\phi}_2^*+\bar{\phi}_1^*\phi_2)+\xi_{22}\phi_2\bar{\phi}_2^*+\gamma_1(\phi_1\bar{u}_x^*+\bar{\phi}_1^*u_x)$$
$$+\gamma_2(\phi_2\bar{u}_x^*+\bar{\phi}_2^*u_x)+k\psi_x\bar{\psi}_x^*+m_1(\phi_{1x}\bar{\psi}_x^*+\bar{\phi}_{1x}^*\psi_x)+m_2(\phi_{2x}\bar{\psi}_x^*+\bar{\phi}_{2x}^*\psi_x)\Big)dx.$$

As usual a superposed bar denotes the conjugate complex number. It is worth mentioning that this product is equivalent to the usual product in the Hilbert space $\mathcal{H}$.

Lemma 2.2. *For every $U\in\mathcal{D}(\mathcal{A}_1)$, $Re\langle\mathcal{A}_1U,U\rangle_{\mathcal{H}}\le 0$.*

Proof. Direct computation gives

$$\mathrm{Re}\langle\mathcal{A}_1U,U\rangle_{\mathcal{H}}=-\frac{1}{2}\int_0^\pi(k^*|\theta_x|^2+\xi^*|\varphi_1|^2)dx.$$

As we are assuming that k^* and ξ^* are positive, the lemma is proved. □

Lemma 2.3. 0 *belongs to the resolvent of $\mathcal{A}_1$ (in short, $0\in\rho(\mathcal{A}_1)$).*

Proof. We have to prove that for every $\mathcal{F}=(f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8)\in\mathcal{H}$ we can find $U\in\mathcal{H}$ such that $\mathcal{A}_1\mathcal{U}=\mathcal{F}$. That is, we have to prove that the following system

$$(2.13)\qquad\left.\begin{aligned}
v&=f_1\\
\frac{1}{\rho}\left[\mu D^2u+\gamma_1D\phi_1+\gamma_2D\phi_2-\beta D\theta\right]&=f_2\\
\varphi_1&=f_3\\
\frac{1}{J_1}\left[b_{11}D^2\phi_1+b_{12}D^2\phi_2+m_1D^2\psi-\xi_{11}\phi_1-\xi_{12}\phi_2+d_1\theta-\gamma_1Du-\xi^*\varphi_1\right]&=f_4\\
\varphi_2&=f_5\\
\frac{1}{J_2}\left[b_{12}D^2\phi_1+b_{22}D^2\phi_2+m_2D^2\psi-\xi_{12}\phi_1-\xi_{22}\phi_2+d_2\theta-\gamma_2Du\right]&=f_6\\
\theta&=f_7\\
\frac{1}{a}\left[kD^2\psi+m_1D^2\phi_1+m_2D^2\phi_2-d_1\varphi_1-d_2\varphi_2-\beta Dv+k^*D^2\theta\right]&=f_8
\end{aligned}\right\}$$

has a solution.

We will solve this system using Fourier series. We consider

$$f_i=\sum f_n^i\sin nx\ \text{ for }\ i=1,2\ \text{ and }\ f_i=\sum f_n^i\cos nx\ \text{ for }\ i=3,...,8.$$

We know that

$$(2.14)\qquad\sum n^2\left(f_n^i\right)^2<\infty,\ \text{for } i=1,3,5,7\ \text{ and }\ \sum\left(f_n^i\right)^2<\infty,\ \text{for } i=2,4,6,8.$$

We want to find

$$u=\sum u_n\sin nx,\ v=\sum v_n\sin nx,\ \phi_1=\sum\phi_n^1\cos nx,\ \varphi_1=\sum\varphi_n^1\cos nx,$$
$$\phi_2=\sum\phi_n^2\cos nx,\ \varphi_2=\sum\varphi_n^2\cos nx,\ \psi=\sum\psi_n\cos nx,\ \theta=\sum\theta_n\cos nx$$

in terms of the f_n^i.

From the first, third, fifth and seventh equations of system (2.13) it follows that $v_n = f_n^1$, $\varphi_n^1 = f_n^3$, $\varphi_n^2 = f_n^5$ and $\theta_n = f_n^7$. Hence, system (2.13) becomes

(2.15)
$$\left.\begin{aligned}\mu D^2 u + \gamma_1 D\phi_1 + \gamma_2 D\phi_2 &= \rho f_2 + \beta D f_7\\ b_{11}D^2\phi_1 + b_{12}D^2\phi_2 + m_1 D^2\psi - \xi_{11}\phi_1 - \xi_{12}\phi_2 - \gamma_1 Du &= J_1 f_4 - d_1 f_7 + \xi^* f_3\\ b_{12}D^2\phi_1 + b_{22}D^2\phi_2 + m_2 D^2\psi - \xi_{12}\phi_1 - \xi_{22}\phi_2 - \gamma_2 Du &= J_2 f_6 - d_2 f_7\\ m_1 D^2\phi_1 + m_2 D^2\phi_2 + kD^2\psi &= a f_8 + d_1 f_3 + d_2 f_5 + \beta D f_1 - k^* D^2 f_7\end{aligned}\right\}$$

Replacing each term by its Fourier series and simplifying, for each n the system reduces to
(2.16)
$$\left.\begin{aligned}u_n\mu n^2 + \phi_n^1\gamma_1 n + \phi_n^2\gamma_2 n &= -f_2\rho + \beta f_7 n\\ u_n\gamma_1 n + \phi_n^1\left(b_{11}n^2 + \xi_{11}\right) + \phi_n^2\left(b_{12}n^2 + \xi_{12}\right) + \psi_n m_1 n^2 &= d_1 f_7 - f_4 J_1 - f_3\xi^*\\ u_n\gamma_2 n + \phi_n^1\left(b_{12}n^2 + \xi_{12}\right) + \phi_n^2\left(b_{22}n^2 + \xi_{22}\right) + \psi_n m_2 n^2 &= d_2 f_7 - f_6 J_2\\ \phi_n^1 m_1 n^2 + \phi_n^2 m_2 n^2 + \psi_n k n^2 &= -a f_8 - d_1 f_3 - d_2 f_5 - f_7 k^* n^2 - \beta f_1 n\end{aligned}\right\}$$

Using Mathematica, we have found the following values:

$$u_n = \frac{k\gamma_2 A n^4 + k\gamma_2 p_3(n)}{k\gamma_2 n^2 q_4(n)}$$
$$\phi_n^1 = \frac{f_7 k^*\mu(b_{12}m_2 - b_{22}m_1)n^5 + f_1\beta\mu(b_{12}m_2 - b_{22}m_1)n^4 + r_3(n)}{nq_4(n)}$$
$$\phi_n^2 = \frac{f_7 k^*\mu(b_{12}m_1 - b_{11}m_2)n^5 + f_1\beta\mu(b_{12}m_1 - b_{11}m_2)n^4 + s_3(n)}{nq_4(n)}$$
$$\psi_n = \frac{f_7 k^*\mu(b_{11}b_{22} - b_{11}^2)n^6 + f_1\beta\mu(b_{11}b_{22} - b_{11}^2)n^5 + t_4(n)}{n^2 q_4(n)}$$

where
$$A = \beta\big((\gamma_1(b_{12}m_2 - b_{22}m_1) + \gamma_2(b_{12}m_1 - b_{11}m_2)\big)f_1 - \rho\det(M_1)f_2,$$
$p_3(n)$, $r_3(n)$, $s_3(n)$, $t_4(n)$ and $q_4(n)$ are polynomials of degree three and four (the subindex denotes the degree) whose coefficients involve the system coefficients and also f_i for $i = 1, ..., 8$.

It is important to analyze in detail polynomial $q_4(n)$, which appears in the denominator:
$$q_4(n) = \mu\det(M_1)n^4 + Bn^2 + k\det(M_2),$$
where
$$B = (\mu\xi_{11} - \gamma_1^2)(kb_{22} - m_2^2) + (\mu\xi_{22} - \gamma_2^2)(kb_{11} - m_1^2) - 2(\gamma_1\gamma_2 - \mu\xi_{12})(m_1m_2 - kb_{12}).$$

Notice that $q_4(n)$ has no real roots because its coefficients are all strictly positive. It is clear that $\mu\det(M_1) > 0$ and $k\det(M_2) > 0$ by assumptions (2.5). Coefficient B is also positive because (again by assumptions (2.5))
$$(m_1m_2 - kb_{12})^2 < (kb_{11} - m_1^2)(kb_{22} - m_2^2) \text{ and } (\gamma_1\gamma_2 - \mu\xi_{12})^2 < (\mu\xi_{11} - \gamma_1^2)(\mu\xi_{22} - \gamma_2^2).$$

We prove, for instance the first inequality:
$$m_1^2m_2^2 + k^2b_{12}^2 - 2m_1m_2kb_{12} < k^2b_{11}b_{22} - kb_{11}m_2^2 - kb_{22}m_1^2 + m_1^2m_2^2 \quad\Leftrightarrow$$
$$kb_{12}^2 - 2m_1m_2b_{12} < kb_{11}b_{22} - b_{11}m_2^2 - b_{22}m_1^2 \quad\Leftrightarrow\quad \det(M_1) > 0.$$

Hence,

$$\begin{aligned}
B &= (\mu\xi_{11}-\gamma_1^2)(kb_{22}-m_2^2)+(\mu\xi_{22}-\gamma_2^2)(kb_{11}-m_1^2)-2(\gamma_1\gamma_2-\mu\xi_{12})(m_1m_2-kb_{12})\\
&> (\mu\xi_{11}-\gamma_1^2)(kb_{22}-m_2^2)+(\mu\xi_{22}-\gamma_2^2)(kb_{11}-m_1^2)\\
&\quad -2\sqrt{(kb_{11}-m_1^2)(kb_{22}-m_2^2)(\mu\xi_{11}-\gamma_1^2)(\mu\xi_{22}-\gamma_2^2)}\\
&= \left(\sqrt{(\mu\xi_{11}-\gamma_1^2)(kb_{22}-m_2^2)}-\sqrt{(\mu\xi_{22}-\gamma_2^2)(kb_{11}-m_1^2)}\right)^2 \geq 0
\end{aligned}$$

That means that the denominators of u_n, ϕ_n^1, ϕ_n^2 and ψ_n do not vanish for any value of n. Moreover, from the values we have obtained and assumptions (2.14), it is not difficult to see that $\sum n^2(u_n)^2<\infty$, $\sum n^2(\phi_n^1)^2<\infty$, $\sum n^2(\phi_n^2)^2<\infty$ and $\sum n^2(\psi_n)^2<\infty$. Regularity conditions can also be checked.

It is worth noting that γ_2 can be 0. In this case, recalculating the solutions of system (2.16), we obtain

$$u_n=\frac{f_7\big(\beta\det(M_1)+\gamma_1k^*(b_{12}m_2-b_{22}m_1)\big)n^5+\overline{p}_4(n)}{n^2\overline{q}_4(n)},$$

where $\overline{p}_4(n)$ is a fourth degree polynomial in n and $\overline{q}_4(n)=q_4(n)_{|_{\gamma_2=0}}$.

As above, we also obtain that $\sum n^2(u_n)^2<\infty$ and the proof is complete. □

In view of these two lemmas and recalling the fact that the domain of the operator is dense, we can use the Lumer–Phillips corollary to the Hille–Yosida theorem to obtain the following result.

Theorem 2.4. *The operator given by matrix $\mathcal{A}_1$ generates a contraction C_0-semigroup $S(t)=\{e^{\mathcal{A}_1t}\}_{t\geq 0}$ in $\mathcal{H}$.*

The above theorem states that the problem defined by system (2.3), with boundary conditions (2.6) and initial conditions (2.7) is well posed in the sense of Hadamard.

Case 2: viscoporosities

We consider now a second system of equations. It corresponds to the type II thermoelasticity with two porous dissipations. For the sake of simplicity, we do not write the evolution neither the constitutive equations. Removing or adding what needs to be removed or added in the evolution and constitutive equations we obtain the following system of partial differential equations:

$$\begin{cases}
\rho\ddot{u}=\mu u_{xx}+\gamma_1\phi_{1,x}+\gamma_2\phi_{2,x}-\beta\dot{\psi}_x\\
J_1\ddot{\phi}_1=b_{11}\phi_{1,xx}+b_{12}\phi_{2,xx}+m_1\psi_{xx}-\xi_{11}\phi_1-\xi_{12}\phi_2+d_1\dot{\psi}-\gamma_1u_x-\xi_{11}^*\dot{\phi}_1-\xi_{12}^*\dot{\phi}_2\\
J_2\ddot{\phi}_2=b_{12}\phi_{1,xx}+b_{22}\phi_{2,xx}+m_2\psi_{xx}-\xi_{12}\phi_1-\xi_{22}\phi_2+d_2\dot{\psi}-\gamma_2u_x-\xi_{21}^*\dot{\phi}_1-\xi_{22}^*\dot{\phi}_2\\
a\ddot{\psi}=m_1\phi_{1,xx}+m_2\phi_{2,xx}+k\psi_{xx}-d_1\dot{\phi}_1-d_2\dot{\phi}_2-\beta\dot{u}_x
\end{cases}\tag{2.17}$$

As in the previous case, we assume that $J_i>0\,(i=1,2)$, $a>0$, $\rho>0$ and that matrices M_1 and M_2 are positive definite. Now we also impose that the matrix

$$\begin{pmatrix}\xi_{11}^* & \xi_{12}^*\\ \xi_{21}^* & \xi_{22}^*\end{pmatrix}\tag{2.18}$$

is positive definite. That is: $\xi_{11}^*>0$ and $4\xi_{11}^*\xi_{22}^*>(\xi_{12}^*+\xi_{21}^*)^2$.

We are interested in the asymptotic behavior of the solutions to the problem determined by (2.17) with the boundary and initial conditions proposed previously, (2.6) and (2.7), respectively (see Section 4). But we first prove that the problem is well posed.

We will follow the same guidelines that we have followed for the first system. In this case we can write

$$\frac{dU}{dt} = \mathcal{A}_2 U, \;\; U_0 = (u_0, v_0, \phi_{10}, \varphi_{10}, \phi_{20}, \varphi_{20}, \psi_0, \theta_0),$$

where $\mathcal{A}_2$ is the following 8×8-matrix

$$\mathcal{A}_2 = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & 0 & \frac{\gamma_1}{\rho}D & 0 & \frac{\gamma_2}{\rho}D & 0 & 0 & -\frac{\beta}{\rho}D \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1}{J_1}D & 0 & \frac{b_{11}D^2-\xi_{11}}{J_1} & -\frac{\xi_{11}^*}{J_1} & \frac{b_{12}D^2-\xi_{12}}{J_1} & -\frac{\xi_{12}^*}{J_1} & \frac{m_1}{J_1}D^2 & \frac{d_1}{J_1} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ -\frac{\gamma_2}{J_2}D & 0 & \frac{b_{12}D^2-\xi_{12}}{J_2} & -\frac{\xi_{21}^*}{J_2} & \frac{b_{22}D^2-\xi_{22}}{J_2} & -\frac{\xi_{22}^*}{J_2} & \frac{m_2}{J_2}D^2 & \frac{d_2}{J_2} \\ 0 & 0 & 0 & 0 & 0 & I & & \\ 0 & -\frac{\beta}{a}D & \frac{m_1}{a}D^2 & -\frac{d_1}{a} & \frac{m_2}{a}D^2 & -\frac{d_2}{a} & \frac{k}{a}D^2 & 0 \end{pmatrix}. \tag{2.19}$$

Lemma 2.5. *For every $U \in \mathcal{D}(\mathcal{A}_2)$, $Re\langle \mathcal{A}_2 U, U\rangle_{\mathcal{H}} \leq 0$.*

Proof. Using the inner product defined at (2.12), it not difficult to see that

$$\text{Re}\langle \mathcal{A}_2 U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^\pi (\xi_{11}^*|\varphi_1|^2 + \xi_2^*|\varphi_2|^2 + \frac{1}{2}(\xi_{12}^* + \xi_{21}^*)(\varphi_1\bar{\varphi}_2 + \varphi_2\bar{\varphi}_1))dx.$$

In view of the assumptions over the coefficients ξ_{ij}^*, it is clear that the operator is dissipative. □

Lemma 2.6. $0 \in \rho(\mathcal{A}_2)$.

Proof. The proof is similar to that of Lemma (2.3). □

As a consequence, we obtain again the existence of solutions.

Theorem 2.7. *The problem defined by system (2.17) with its corresponding boundary and initial conditions is well posed in the sense of Hadamard.*

Case 3: viscoporosity and viscoelasticity

In this third situation, we introduce again two dissipation mechanisms in the system. In this case, we assume to have viscoelasticity and porous dissipation in one of the porous structures for the type II thermoelasticity. From the corresponding evolution and constitutive equations we obtain the following system:

$$\begin{cases} \rho\ddot{u} = \mu u_{xx} + \gamma_1\phi_{1,x} + \gamma_2\phi_{2,x} - \beta\dot{\psi}_x + \mu^*\dot{u}_{xx} + \gamma^*\dot{\phi}_{1,x} \\ J_1\ddot{\phi}_1 = b_{11}\phi_{1,xx} + b_{12}\phi_{2,xx} + m_1\psi_{xx} - \xi_{11}\phi_1 - \xi_{12}\phi_2 + d_1\dot{\psi} - \gamma_1 u_x - \xi^*\dot{\phi}_1 - \gamma^{**}\dot{u}_x \\ J_2\ddot{\phi}_2 = b_{12}\phi_{1,xx} + b_{22}\phi_{2,xx} + m_2\psi_{xx} - \xi_{12}\phi_1 - \xi_{22}\phi_2 + d_2\dot{\psi} - \gamma_2 u_x \\ a\ddot{\psi} = m_1\phi_{1,xx} + m_2\phi_{2,xx} + k\psi_{xx} - d_1\dot{\phi}_1 - d_2\dot{\phi}_2 - \beta\dot{u}_x \end{cases} \tag{2.20}$$

In order to simplify the mathematical analysis, we change the boundary conditions. For any $t > 0$

$$
\begin{aligned}
&u_x(0,t) = u_x(\pi,t) = 0,\\
&\phi_1(0,t) = \phi_1(\pi,t) = \phi_2(0,t) = \phi_2(\pi,t) = 0,\\
&\psi(0,t) = \psi(\pi,t) = 0.
\end{aligned} \tag{2.21}
$$

As initial conditions we take again (2.7).

To study the time decay in this case it will be necessary to assume that

$$
\int_0^\pi u_0(x)\,dx = \int_0^\pi v_0(x)\,dx = 0, \tag{2.22}
$$

because, if not, solutions (uniform in the variable x) that do not decay can be found.

The conditions for the system coefficients are still (2.5), but we also assume that the matrix

$$
\begin{pmatrix} \mu^* & \gamma^* \\ \gamma^{**} & \xi^* \end{pmatrix} \tag{2.23}
$$

is positive definite. That is $\mu^* > 0$ and $4\mu^*\xi^* > (\gamma^* + \gamma^{**})^2$.

Once again the system can be written as

$$
\frac{dU}{dt} = \mathcal{A}_3 U, \quad U_0 = (u_0, v_0, \phi_{10}, \varphi_{10}, \phi_{20}, \varphi_{20}, \psi_0, \theta_0),
$$

using an appropriate Hilbert space. In this case, we take

$$
\mathcal{H} = H^1_* \times L^2_* \times H^1_0 \times L^2 \times H^1_0 \times L^2 \times H^1_0 \times L^2. \tag{2.24}
$$

$\mathcal{A}_3$ is the following matrix operator

$$
\mathcal{A}_3 = \begin{pmatrix}
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\mu}{\rho}D^2 & \frac{\mu^*}{\rho}D^2 & \frac{\gamma_1}{\rho}D & \frac{\gamma^*}{\rho}D & \frac{\gamma_2}{\rho}D & 0 & 0 & -\frac{\beta}{\rho}D \\
0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\
-\frac{\gamma_1}{J_1}D & -\frac{\gamma^{**}}{J_1}D & \frac{b_{11}D^2-\xi_{11}}{J_1} & -\frac{\xi^*}{J_1} & \frac{b_{12}D^2-\xi_{12}}{J_1} & 0 & \frac{m_1}{J_1}D^2 & \frac{d_1}{J_1} \\
0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
-\frac{\gamma_2}{J_2}D & 0 & \frac{b_{12}D^2-\xi_{12}}{J_2} & 0 & \frac{b_{22}D^2-\xi_{22}}{J_2} & 0 & \frac{m_2}{J_2}D^2 & \frac{d_2}{J_2} \\
0 & 0 & 0 & 0 & 0 & I & & \\
0 & -\frac{\beta}{a}D & \frac{m_1}{a}D^2 & -\frac{d_1}{a} & \frac{m_2}{a}D^2 & -\frac{d_2}{a} & \frac{k}{a}D^2 & 0
\end{pmatrix} \tag{2.25}
$$

Lemma 2.8. *For every $U \in \mathcal{D}(\mathcal{A}_3)$, $Re\langle \mathcal{A}_3 U, U\rangle_{\mathcal{H}} \leq 0$.*

Proof. In this case we have that

$$
\mathrm{Re}\langle \mathcal{A}_3 U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^\pi (\mu^*|u_x|^2 + \xi^*|\varphi_1|^2 + \frac{1}{2}(\gamma^* + \gamma^{**})(u_x\bar{\varphi}_1 + \varphi_1\bar{u}_x))dx.
$$

In view of the assumptions over the coefficients, the operator is dissipative. □

We can prove also that 0 is in the resolvent of $\mathcal{A}_3$. And, as a consequence, we obtain again an existence result.

Theorem 2.9. *The problem defined by system (2.20) with boundary conditions (2.21) and initial conditions (2.7) is well posed in the sense of Hadamard.*

CASE 4: VISCOELASTICITY AND THERMAL DISSIPATION

In the last situation that we study we introduce viscoelasticty in the system and we consider also thermal dissipation, that is, it agrees with the type III thermoelasticity. Using the corresponding evolution and constitutive equations, we obtain the following system:

$$
(2.26)\qquad \begin{cases} \rho\ddot{u} = \mu u_{xx} + \gamma_1\phi_{1,x} + \gamma_2\phi_{2,x} - \beta\dot{\psi}_x + \mu^*\dot{u}_{xx} \\ J_1\ddot{\phi}_1 = b_{11}\phi_{1,xx} + b_{12}\phi_{2,xx} + m_1\psi_{xx} - \xi_{11}\phi_1 - \xi_{12}\phi_2 + d_1\dot{\psi} - \gamma_1 u_x \\ J_2\ddot{\phi}_2 = b_{12}\phi_{1,xx} + b_{22}\phi_{2,xx} + m_2\psi_{xx} - \xi_{12}\phi_1 - \xi_{22}\phi_2 + d_2\dot{\psi} - \gamma_2 u_x \\ a\ddot{\psi} = m_1\phi_{1,xx} + m_2\phi_{2,xx} + k\psi_{xx} - d_1\dot{\phi}_1 - d_2\dot{\phi}_2 - \beta\dot{u}_x + k^*\theta_{xx} \end{cases}
$$

We assume that

$$
(2.27)\qquad J_i > 0\,(i=1,2),\ a>0,\ \rho>0,\ \mu^*>0,\ k^*>0.
$$

We take boundary conditions (2.6) and initial conditions (2.7).

Working in an appropriate Hilbert space, the system can be studied with the help of the matrix operator

$$
(2.28)\qquad \mathcal{A}_4 = \begin{pmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{\mu}{\rho}D^2 & \frac{\mu^*}{\rho}D^2 & \frac{\gamma_1}{\rho}D & 0 & \frac{\gamma_2}{\rho}D & 0 & 0 & -\frac{\beta}{\rho}D \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ -\frac{\gamma_1}{J_1}D & 0 & \frac{b_{11}D^2-\xi_{11}}{J_1} & 0 & \frac{b_{12}D^2-\xi_{12}}{J_1} & 0 & \frac{m_1}{J_1}D^2 & \frac{d_1}{J_1} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ -\frac{\gamma_2}{J_2}D & 0 & \frac{b_{12}D^2-\xi_{12}}{J_2} & 0 & \frac{b_{22}D^2-\xi_{22}}{J_2} & 0 & \frac{m_2}{J_2}D^2 & \frac{d_2}{J_2} \\ 0 & 0 & 0 & 0 & 0 & I & & \\ 0 & -\frac{\beta}{a}D & \frac{m_1}{a}D^2 & -\frac{d_1}{a} & \frac{m_2}{a}D^2 & -\frac{d_2}{a} & \frac{k}{a}D^2 & \frac{k^*}{a}D^2 \end{pmatrix}.
$$

Lemma 2.10. *For every $U \in \mathcal{D}(\mathcal{A}_4)$, $Re\langle \mathcal{A}_4 U, U\rangle_{\mathcal{H}} \le 0$.*

Proof. Using the inner product defined at (2.12) we obtain

$$
\mathrm{Re}\langle \mathcal{A}_4 U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^\pi (\mu^*|u_x|^2 + k^*|\theta_x|^2)dx,
$$

and, taking into account the assumptions over the coefficients, it is clear that the operator is dissipative. □

As in Lemma 2.3, it is also possible to prove that the operator $\mathcal{A}_4$ is exhaustive.

From these lemmas we obtain the existence result.

Theorem 2.11. *The problem defined by system (2.26) with boundary conditions (2.6) and initial conditions (2.7) is well posed in the sense of Hadamard.*

3. Case 1: exponential decay

In this section we will prove the exponential decay of the solutions for the first problem. That is, we consider two dissipation mechanisms: the viscoporosity and the thermal dissipation, or, in other words, the case corresponding to the type III thermoelasticity when there exists a dissipation mechanism in one of the porous structures.

Apart from the assumptions proposed over the constitutive coefficients, (2.4) and (2.5), in this section we also impose that the rank of the matrix

$$\begin{pmatrix} b_{11} & b_{12} \\ m_1 & m_2 \end{pmatrix} \tag{3.1}$$

is two and that β, m_2 and γ_2 are different from zero.

These new assumptions say that the coupling between the components of the problem is strong.

To prove the exponential decay of the solutions, we recall the characterization stated in the book of Liu and Zheng [23].

Theorem 3.1. *Let $S(t) = \{e^{\mathcal{A}t}\}_{t\geq 0}$ be a C_0-semigroup of contractions on a Hilbert space. Then $S(t)$ is exponentially stable if and only if the following two conditions are satisfied:*

(i) $i\mathbb{R} \subset \rho(\mathcal{A})$,
(ii) $\overline{\lim}_{|\lambda|\to\infty} \|(i\lambda\mathcal{I} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty$.

Lemma 3.2. *The operator $\mathcal{A}_1$ defined at (2.11) satisfies $i\mathbb{R} \subset \rho(\mathcal{A}_1)$.*

Proof. The proof follows the arguments given by Liu and Zheng ([23], page 25). It begins by supposing that the intersection of the imaginary axis and the spectrum is non-empty. Then, there exists a real number ϖ with $||\mathcal{A}^{-1}||^{-1} \leq |\varpi| < \infty$ such that the set $\{i\lambda, |\lambda| < |\varpi|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{||(i\lambda\mathcal{I} - \mathcal{A})^{-1}||, |\lambda| < |\varpi|\} = \infty$. Therefore, there exist a sequence of real numbers λ_n with $\lambda_n \to \varpi$, $|\lambda_n| < |\varpi|$ and a sequence of vectors $U_n = (u_n, v_n, \varphi_n, \phi_n, \psi_n, \theta_n)$ in the domain of the operator $\mathcal{A}$ and with unit norm such that

$$\|(i\lambda_n\mathcal{I} - \mathcal{A})U_n\| \to 0. \tag{3.2}$$

In our case, writing this condition term by term we get

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{3.3}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\mu D^2 u_n + \gamma_1 D\phi_{1n} + \gamma_2 D\phi_{2n} - \beta D\theta_n\right) \to 0 \text{ in } L^2, \tag{3.4}$$

$$i\lambda_n \phi_{1n} - \varphi_{1n} \to 0 \text{ in } H^1, \tag{3.5}$$

(3.6)
$$i\lambda_n\varphi_{1n} - \frac{1}{J_1}\left(-\gamma_1 Du_n + b_{11}D^2\phi_{1n} + b_{12}D^2\phi_{2n} - \xi_{11}\phi_{1n} - \xi_{12}\phi_{2n} + m_1 D^2\psi_n + d_1\theta_n - \xi^*\varphi_{1n}\right) \to 0 \text{ in } L^2,$$

$$i\lambda_n \phi_{2n} - \varphi_{2n} \to 0 \text{ in } H^1, \tag{3.7}$$

$$i\lambda_n\varphi_{2n} - \frac{1}{J_2}\left(-\gamma_2 Du_n + b_{21}D^2\phi_{1n} + b_{22}D^2\phi_{2n} - \xi_{21}\phi_{1n} - \xi_{22}\phi_{2n} + m_2 D^2\psi_n + d_2\theta_n\right) \to 0 \text{ in } L^2, \tag{3.8}$$

$$i\lambda_n \psi_n - \theta_n \to 0 \text{ in } H^1, \tag{3.9}$$

(3.10)
$$i\lambda_n\theta_n - \frac{1}{a}\left(-\beta Dv_n + m_1D^2\phi_{1n} + m_2D^2\phi_{2n} - d_1\varphi_{1n} - d_2\varphi_{2n} + kD^2\psi_n + k^*D^2\theta_n\right) \to 0 \text{ in } L^2.$$

In view of the dissipative term for the operator, we see that

$$D\theta_n, \varphi_{1n} \to 0 \text{ in } L^2. \tag{3.11}$$

Then $\lambda_n D\psi_n, \lambda_n\phi_{1n}$ also tends to zero in L^2.

We multiply now (3.6) by ϕ_{1n}. We find

$$b_{11}|D\phi_{1n}|^2 + b_{12}\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0.$$

In a similar way, if we multiply (3.10) by ϕ_{1n}, it turns into

$$m_1|D\phi_{1n}|^2 + m_2\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0.$$

As matrix (3.1) has rank two, we can conclude that $D\phi_{1n} \to 0$ in L^2 and that $\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0$.

We want to prove that $D\phi_{2n}$ also tends to zero. To this end, we now consider the product of (3.10) by ϕ_{2n}, and we find

$$m_2|D\phi_{2n}|^2 - i\lambda_n\beta\langle Du_n, \phi_{2n}\rangle - i\lambda_n d_2|\phi_{2n}|^2 \to 0.$$

Our claim will be proved if we show that $\langle Du_n, \phi_{2n}\rangle$ tends to a real number. We consider the product of (3.8) by ϕ_{2n}, and we see that

$$-J_2|\varphi_{2n}|^2 + b_{22}|D\phi_{2n}|^2 + \gamma_2\langle Du_n, \phi_{2n}\rangle + \xi_{22}|\phi_{2n}|^2 \to 0.$$

As we assume that γ_2 is different from zero, we see that $\langle Du_n, \phi_{2n}\rangle$ tends to a real number and then $D\phi_{2n}$ tends to zero. Now, from the previous convergence we also see that φ_{2n} tends to zero.

We now multiply (3.10) by $\lambda_n^{-1}Du_n$ and, taking into account that $\lambda_n^{-1}D^2u_n$ is bounded, we find that Du_n tends to zero. From here we also conclude that $v_n \to 0$.

We have thus obtained a contradiction and the lemma is proved.

□

Lemma 3.3. *The operator $\mathcal{A}_1$ satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A}_1)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. The proof also follows a contradiction argument. Suppose that the thesis is not true. We obtain the existence of a sequence of real numbers λ_n such that $|\lambda_n| \to \infty$ and a sequence of unit vectors in the domain of the operator in such a way that (3.2) holds. Therefore, conditions (3.3)–(3.10) still hold. Now we can use a similar argument to the one used in the proof of the previous lemma because the key point is that λ_n does not tend to zero. □

The two previous lemmas give rise to the following result.

Theorem 3.4. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}_1 t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)\| \leq M\|S(0)\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of Lemma 3.2, Lemma 3.3 and Theorem 3.1. □

It is worth noting that the behavior of the solutions for this model differs from the behavior in the one-dimensional classical thermoviscoelasticity with double voids. The exponential stability obtained in our case is a consequence of the strong coupling between the porosities and the temperature. These couplings are not present in the classical model and play a relevant role. This behavior is another striking effect of the type II/III thermoelasticity.

4. Case 2: Exponential decay

In this section we obtain the exponential decay of solutions for the second problem that we have considered: the system with two viscoporosities. To prove our result we use here the same arguments that we have used in the previous section. However, we need to impose two conditions: $\beta \neq 0$ and $|m_1| + |m_2| \neq 0$.

Lemma 4.1. *The operator $\mathcal{A}_2$ defined at (2.19) satisfies $i\mathbb{R} \subset \rho(\mathcal{A}_2)$.*

Lemma 4.2. *The operator $\mathcal{A}_2$ satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A}_2)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. We can prove both lemmas at the same time, and the proof is very similar to the one we have done above. We only write the main differences. Writing condition (3.2) term by term we now get:

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{4.1}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\mu D^2 u_n + \gamma_1 D\phi_{1n} + \gamma_2 D\phi_{2n} - \beta D\theta_n\right) \to 0 \text{ in } L^2, \tag{4.2}$$

$$i\lambda_n \phi_{1n} - \varphi_{1n} \to 0 \text{ in } H^1, \tag{4.3}$$

$$i\lambda_n\varphi_{1n} - \frac{1}{J_1}\left(-\gamma_1 Du_n + b_{11}D^2\phi_{1n} + b_{12}D^2\phi_{2n} - \xi_{11}\phi_{1n} - \xi_{12}\phi_{2n} + m_1 D^2\psi_n + d_1\theta_n - \xi_{11}^*\varphi_{1n} - \xi_{12}^*\varphi_{2n}\right) \to 0 \text{ in } L^2, \tag{4.4}$$

$$i\lambda_n \phi_{2n} - \varphi_{2n} \to 0 \text{ in } H^1, \tag{4.5}$$

$$i\lambda_n\varphi_{2n} - \frac{1}{J_2}\left(-\gamma_2 Du_n + b_{21}D^2\phi_{1n} + b_{22}D^2\phi_{2n} - \xi_{21}\phi_{1n} - \xi_{22}\phi_{2n} + m_2 D^2\psi_n + d_2\theta_n - \xi_{21}^*\varphi_{1n} - \xi_{22}^*\varphi_{2n}\right) \to 0 \text{ in } L^2, \tag{4.6}$$

$$i\lambda_n \psi_n - \theta_n \to 0 \text{ in } H^1, \tag{4.7}$$

$$i\lambda_n\theta_n - \frac{1}{a}\left(-\beta Dv_n + m_1 D^2\phi_{1n} + m_2 D^2\phi_{2n} - d_1\varphi_{1n} - d_2\varphi_{2n} + kD^2\psi_n\right) \to 0 \text{ in } L^2. \tag{4.8}$$

In view of the conditions on the parameters ξ_{ij}^* and the dissipative term for the operator, we see that

$$\varphi_{1n}, \varphi_{2n} \to 0 \text{ in } L^2. \tag{4.9}$$

Therefore, we also have that $\lambda_n\phi_{1n}$ and $\lambda_n\phi_{2n}$ tend to zero in L^2. Now, we want to prove that $D\phi_{in}$ $(i = 1, 2)$ also tend to zero. We multiply (4.8) first by ϕ_{1n} and after by ϕ_{2n}, and we obtain

$$m_1|D\phi_{1n}|^2 + m_2\langle D\phi_{2n}, D\phi_{1n}\rangle + k\langle D\psi_n, D\phi_{1n}\rangle \to 0,$$

and

$$m_1\langle D\phi_{1n}, D\phi_{2n}\rangle + m_2|D\phi_{2n}|^2 + k\langle D\psi_n, D\phi_{2n}\rangle \to 0,$$

respectively.

We now multiply (4.4) by ϕ_{1n} and (4.6) by ϕ_{2n}. We see that

$$b_{11}|D\phi_{1n}|^2 + b_{12}\langle D\phi_{2n}, D\phi_{1n}\rangle + m_1\langle D\psi_n, D\phi_{1n}\rangle \to 0,$$

and

$$b_{12}\langle D\phi_{1n}, D\phi_{2n}\rangle + b_{22}|D\phi_{2n}|^2 + m_2\langle D\psi_n, D\phi_{2n}\rangle \to 0.$$

We then obtain that

$$(b_{11} - \frac{m_1^2}{k})|D\phi_{1n}|^2 + (b_{12} - \frac{m_1m_2}{k})\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0 \tag{4.10}$$

and

$$(b_{12} - \frac{m_1m_2}{k})\langle D\phi_{1n}, D\phi_{2n}\rangle + (b_{22} - \frac{m_2^2}{k})|D\phi_{2n}|^2 \to 0. \tag{4.11}$$

We can obtain a couple convergencies more. We get

$$(b_{11} - \frac{m_1^2}{k})\langle D\phi_{1n}, D\phi_{2n}\rangle + (b_{12} - \frac{m_1m_2}{k})|D\phi_{2n}|^2 \to 0 \tag{4.12}$$

and

$$(b_{12} - \frac{m_1m_2}{k})|D\phi_{1n}|^2 + (b_{22} - \frac{m_2^2}{k})\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0. \tag{4.13}$$

Combining (4.10) and (4.13) and (4.11) and (4.12) and taking into account that the determinant of the matrix

$$\begin{pmatrix} b_{11} - \frac{m_1^2}{k} & b_{12} - \frac{m_1m_2}{k} \\ b_{12} - \frac{m_1m_2}{k} & b_{22} - \frac{m_2^2}{k} \end{pmatrix} \tag{4.14}$$

is different from zero we obtain that $D\phi_{1n}$ and $D\phi_{2n}$ tend to zero.

Our next step is to prove that ψ_n also tends to zero. We multiply (4.4) and (4.6) by ψ_n and (4.8) by $\lambda_n^{-1}\psi_n$ to arrive at

$$m_1|D\psi_n|^2 + i\lambda_n d_1|\psi_n|^2 - \gamma_1\langle Du_n, \psi_n\rangle \to 0,$$
$$m_2|D\psi_n|^2 + i\lambda_n d_2|\psi_n|^2 - \gamma_2\langle Du_n, \psi_n\rangle \to 0$$

and

$$-\lambda_n|\psi_n|^2 + \frac{k}{\lambda_n}|D\psi_n|^2 + i\beta\langle Du_n, \psi_n\rangle \to 0.$$

From the last one we see that $\langle Du_n, \psi_n\rangle$ tends to an imaginary number. Therefore, if we assume that $|m_1| + |m_2| \neq 0$, we conclude that $D\psi_n$ tends to zero. After multiplication of (4.8) by ψ_n we also see that $|\theta_n|^2$ goes to zero.

Now we want to prove that Du_n tends to zero. Paying attention to (4.2) we note that $\lambda_n^{-1}D^2u_n$ is bounded. We multiply (4.8) by $\lambda_n^{-1}Du_n$. As $\langle\lambda_n^{-1}D^2\psi_n, Du_n\rangle = -\langle D\psi_n, \lambda_n^{-1}D^2u_n\rangle$ and a similar equality holds for ϕ_{1n} and ϕ_{2n} we find that Du_n tends to zero. From (4.2) we also see that $v_n \to 0$.

□

Theorem 4.3. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}_2t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)\| \leq M\|S(0)\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of Lemma 4.1, Lemma 4.2 and Theorem 3.1. □

5. Case 3: Exponential decay

In this section we obtain the exponential decay of solutions for the third problem that we have considered: the system with viscoporosity and viscoelasticity. We use the same arguments again. We need the following conditions: $\gamma_2 \neq 0$, $\beta \neq 0$ and matrix (3.1) has rank two.

Lemma 5.1. *The operator $\mathcal{A}_3$ defined at (2.25) satisfies $i\mathbb{R} \subset \rho(\mathcal{A}_3)$.*

Proof. The proof is again similar to the one proposed for Lemma 3.2. We write condition (3.2) term by term for this new system:

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^1, \tag{5.1}$$

$$i\lambda_n v_n - \frac{1}{\rho}\left(\mu D^2 u_n + \gamma_1 D\phi_{1n} + \gamma_2 D\phi_{2n} - \beta D\theta_n + \mu D^2 v_n + \gamma^* D\phi_{1n}\right) \to 0 \text{ in } L^2, \tag{5.2}$$

$$i\lambda_n \phi_{1n} - \varphi_{1n} \to 0 \text{ in } H^1, \tag{5.3}$$

$$i\lambda_n\varphi_{1n} - \frac{1}{J_1}\left(-\gamma_1 Du_n + b_{11}D^2\phi_{1n} + b_{12}D^2\phi_{2n} - \xi_{11}\phi_{1n} - \xi_{12}\phi_{2n} + m_1 D^2\psi_n + d_1\theta_n - \xi^*\varphi_{1n} - \gamma^{**}Dv_n\right) \to 0 \text{ in } L^2, \tag{5.4}$$

$$i\lambda_n \phi_{2n} - \varphi_{2n} \to 0 \text{ in } H^1, \tag{5.5}$$

$$i\lambda_n\varphi_{2n} - \frac{1}{J_2}\left(-\gamma_2 Du_n + b_{21}D^2\phi_{1n} + b_{22}D^2\phi_{2n} - \xi_{21}\phi_{1n} - \xi_{22}\phi_{2n} + m_2 D^2\psi_n + d_2\theta_n\right) \to 0 \text{ in } L^2, \tag{5.6}$$

$$i\lambda_n \psi_n - \theta_n \to 0 \text{ in } H^1, \tag{5.7}$$

$$i\lambda_n\theta_n - \frac{1}{a}\left(-\beta Dv_n + m_1 D^2\phi_{1n} + m_2 D^2\phi_{2n} - d_1\varphi_{1n} - d_2\varphi_{2n} + kD^2\psi_n\right) \to 0 \text{ in } L^2. \tag{5.8}$$

In view of the conditions on the parameters and the dissipative term for the operator, we see that

$$Dv_n, \varphi_{1n} \to 0 \text{ in } L^2. \tag{5.9}$$

Therefore, we also have that $\lambda_n\phi_{1n}$ and $\lambda_n Du_n$ tend to zero in L^2.

If we integrate (5.2) and take into consideration the boundary conditions at $x = 0$ and the facts that u_n, v_n tend to zero in H^1 and ϕ_{1n}, φ_{1n} tend to zero in L^2, we see that

$$\gamma_2\phi_{2n} - \beta\theta_n \to 0 \text{ in } L^2.$$

Now, we want to prove that $D\psi_n$ tends to zero. We multiply (5.2) by $\lambda_n^{-1}D\psi_n$ and we obtain

$$\mu\langle D^2u_n, \lambda_n^{-1}D\psi_n\rangle + \gamma_1\langle D\phi_{1n}, \lambda_n^{-1}D\psi_n\rangle + \gamma_2\langle D\phi_{2n}, \lambda_n^{-1}D\psi_n\rangle$$
$$-\beta\langle D\theta_n, \lambda_n^{-1}D\psi_n\rangle + \mu^*\langle D^2v_n, \lambda_n^{-1}D\psi_n\rangle + \gamma^*\langle D\varphi_{1n}, \lambda_n^{-1}D\psi_n\rangle \to 0.$$

But, we have that $\langle D^2u_n, \lambda_n^{-1}D\psi_n\rangle = -\langle Du_n, \lambda_n^{-1}D^2\psi_n\rangle$, $\langle D^2v_n, \lambda_n^{-1}D\psi_n\rangle = -\langle Dv_n, \lambda_n^{-1}D^2\psi_n\rangle$, $\langle D\phi_{1n}, \lambda_n^{-1}D\psi_n\rangle = -\langle \phi_{1n}, \lambda_n^{-1}D^2\psi_n\rangle$ and $\langle D\varphi_{1n}, \lambda_n^{-1}D\psi_n\rangle = -\langle \varphi_{1n}, \lambda_n^{-1}D^2\psi_n\rangle$. On the other side, in view of (5.4), (5.6) and (5.8) we see that $\lambda_n^{-1}D^2\psi_n$ is bounded. We then obtain that

$$\gamma_2\lambda_n^{-1}\langle D\phi_{2n}, D\psi_n\rangle - i\beta|D\psi_n|^2 \to 0.$$

It is worth noting that $\langle D\psi_n, D\phi_{2n}\rangle$ tends to an imaginary number.

We now multiply (5.4) and (5.6) by ϕ_{2n}. We obtain that

$$b_{11}\langle D\phi_{1n}, D\phi_{2n}\rangle + b_{12}|D\phi_{2n}|^2 + \xi_{12}|\phi_{2n}|^2 + m_1\langle D\psi_n, D\phi_{2n}\rangle - d_1\langle \theta_n, \phi_{2n}\rangle \to 0$$

and

$$i\lambda_n J_2\langle \varphi_{2n}, \phi_{2n}\rangle + b_{12}\langle D\phi_{1n}, D\phi_{2n}\rangle + b_{22}|D\phi_{2n}|^2 + \xi_{22}|\phi_{2n}|^2 + m_2\langle D\psi_n, D\phi_{2n}\rangle - d_2\langle \theta_n, \phi_{2n}\rangle \to 0.$$

If we assume that $b_{11}m_2 - b_{12}m_1 \neq 0$ and recalling that $\gamma_2\phi_{2n} - \beta\theta_n \to 0$ in L^2, we see that $\langle D\psi_n, D\phi_{2n}\rangle$ tends to a real number. Therefore, $\langle D\psi_n, D\phi_{2n}\rangle$ must tend to zero. Then, we also obtain that $D\psi_n$ goes to zero. If we now multiply (5.8) by ψ_n we also see that θ_n tends to zero.

We now want to prove that $D\phi_{1n}$ converges to zero. We multiply (5.4) and (5.8) by ϕ_{1n}. We have

$$b_{11}|D\phi_{1n}|^2 + b_{12}\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0,$$

and

$$m_1|D\phi_{1n}|^2 + m_1\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0.$$

As we assume that $b_{11}m_2 - b_{12}m_1 \neq 0$ we conclude the convergence of $D\phi_{1n}$. In fact, we also see that $\langle D\phi_{2n}, D\phi_{1n}\rangle \to 0$.

The proof will be complete if we prove that φ_{2n} and $D\phi_{2n}$ also tend to zero. To this end, we multiply (5.8) by ϕ_{2n}. We obtain

$$m_2|D\phi_{2n}|^2 + i\lambda_n d_2|\phi_{2n}|^2 \to 0.$$

Taking the real part we obtain the convergence of $D\phi_{2n}$ to zero, and going back to (5.6) we also obtain that φ_{2n} converges to zero.

□

Lemma 5.2. *The operator $\mathcal{A}_3$ satisfies*

$$\overline{\lim_{|\lambda|\to\infty}} \|(i\lambda\mathcal{I} - \mathcal{A}_3)^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty.$$

Proof. Let us assume that the thesis of the lemma is not true. Then, there exist a sequence, λ_n, with unbounded absolute value and a sequence of vectors U_n in the domain of the operator $\mathcal{A}_3$ and with unit norm such that (5.1)–(5.8) hold. Using again the dissipative terms we see that φ_{1n} and Dv_n tend to zero. Hence $\lambda_n\phi_{1n}$ and $\lambda_n Du_n$ also tend to zero.

We multiply again (5.2) by $\lambda_n^{-1}D\psi_n$. Using the same argument as in the previous lemma we see that

$$\gamma_2\lambda_n^{-1}\langle D\phi_{2n}, D\psi_n\rangle - i\beta|D\psi_n|^2 \to 0.$$

As λ_n becomes unbounded but $D\phi_{2,n}$, $D\psi_n$ are bounded, we see that $D\psi_n$ tends to zero. From this point we can follow the same arguments used in the previous lemma to obtain a contradiction.

□

Theorem 5.3. *The C_0-semigroup $S(t) = \{e^{\mathcal{A}_3 t}\}_{t\geq 0}$ is exponentially stable. That is, there exist two positive constants M and α such that $\|S(t)\| \leq M\|S(0)\|e^{-\alpha t}$.*

Proof. The proof is a direct consequence of Lemma 5.1, Lemma 5.2 and Theorem 3.1. □

6. Case 4: Undamped solutions

The problem determined by system (2.26) with boundary conditions (2.6) and initial conditions (2.7) has undamped solutions. This fact suggests that the exponential decay cannot be expected in the general case.

To prove the existence of undamped solutions we prove that solutions of the form

$$(0, Ae^{\omega t}\cos nx, Be^{\omega t}\cos nx, 0)$$

for $n = 1, 2, 3...$ and certain $\omega \in \mathbb{R}$, $\omega \neq 0$, can be found.

If solutions of this form exist, then the following system of equations must be satisfied:

$$\left.\begin{array}{r} \gamma_1 A + \gamma_2 B = 0 \\ \left(b_{11}n^2 + J_1\omega^2 + \xi_{11}\right) A + \left(b_{12}n^2 + \xi_{12}\right) B = 0 \\ \left(b_{12}n^2 + \xi_{12}\right) A + \left(b_{22}n^2 + J_2\omega^2 + \xi_{22}\right) B = 0 \\ \left(d_1\omega + m_1 n^2\right) A + \left(d_2\omega + m_2 n^2\right) B = 0 \end{array}\right\} \tag{6.1}$$

This system has solutions. Take, for instance, $A = KB$ for a non zero real K and $\gamma_2 = -K\gamma_1$, $m_2 = -Km_1$ and $d_2 = -Kd_1$. Therefore, the system gives the following identities

$$\left.\begin{array}{r} \left(b_{11}n^2 + J_1\omega^2 + \xi_{11}\right) K + \left(b_{12}n^2 + \xi_{12}\right) = 0 \\ \left(b_{12}n^2 + \xi_{12}\right) K + \left(b_{22}n^2 + J_2\omega^2 + \xi_{22}\right) = 0 \end{array}\right\} \tag{6.2}$$

From the first identity, we obtain

$$\omega^2 = \frac{-K(b_{11}n^2 + \xi_{11}) - b_{12}n^2 - \xi_{12}}{J_1 K}.$$

And, from the second,

$$\omega^2 = \frac{-K(b_{12}n^2 + \xi_{12}) - b_{22}n^2 - \xi_{22}}{J_2}.$$

Therefore, if we select K in such a way that

$$\frac{K(b_{11}n^2 + \xi_{11}) + b_{12}n^2 + \xi_{12}}{J_1 K} = \frac{K(b_{12}n^2 + \xi_{12}) + b_{22}n^2 + \xi_{22}}{J_2},$$

then we will have solutions for the system.

Notice that the above expression is a second degree equation with respect to K. Its discriminant is

$$\Delta = \left(J_1(b_{22}n^2 + \xi_{22}) - J_2(b_{11}n^2 + \xi_{11})\right)^2 + 4J_1J_2(b_{12}n^2 + \xi_{12})^2 > 0.$$

In consequence, the existence of K is assured. In fact, we can take for K any of the two following values

$$K = \frac{J_2(b_{11}n^2 + \xi_{11}) - J_1(b_{22}n^2 + \xi_{22}) \pm \sqrt{\Delta}}{2J_1(b_{12}n^2 + \xi_{12})}.$$

Summarizing: if the constitutive coefficients satisfy

$$\frac{\gamma_2}{\gamma_1} = \frac{m_2}{m_1} = \frac{d_2}{d_1} = -\frac{J_2(b_{11}n^2 + \xi_{11}) - J_1(b_{22}n^2 + \xi_{22}) \pm \sqrt{\Delta}}{2J_1(b_{12}n^2 + \xi_{12})}$$

for $n = 1, 2, 3, \ldots$, therefore, undamped solutions can be found.

7. Conclusion

In this paper we have studied four different situations related to the system of equations that models the type II thermoviscoelasticity with two porous structures. For each one we have introduced in the model two dissipation mechanisms: viscoporosity and thermal dissipation, viscoporosity in both porous structures, viscoporosity and viscoelasticity, and viscoelasticity and thermal dissipation. When the thermal dissipation is considered, the model corresponds to the type III thermoviscoelasticity. We have proved that, under suitable hypotheses for the constitutive parameters, the solutions of the system of equations decay exponentially whenever one of the two mechanisms is the viscoporosity. This behavior differs from the one obtained for the classical theory.

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