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An alternate description of a $(q+1, 8)$-cage

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Abstract

Let $q \geq 2$ be a prime power. In this note we present an alternate description of the known $(q+1, 8)$-cages which has allowed us to construct small (k, g)–graphs for $k = q-1, q$ and $g = 7, 8$ in other papers on this same topic.

Keywords: Cages, girth, Moore graphs, perfect dominating sets.

Math. Subj. Class.: 05C35, 05C69, 05B25

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1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow the book by Bondy and Murty [14] for terminology and notation.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *girth* of G is the number $g = g(G)$ of edges in a shortest cycle. For every $v \in V$, $N_G(v)$ denotes the *neighbourhood* of v, i.e. the set of all vertices adjacent to v, and $N_G[v] = N_G(v) \cup \{v\}$ is the *closed neighbourhood* of v. The *degree* of a vertex $v \in V$ is the cardinality of $N_G(v)$. Let $S \subset V(G)$, then we denote by $N_G(S) = \cup_{s\in S} N_G(s) - S$ and by $N_G[S] = S \cup N_G(S)$.

A graph is called *regular* if all its vertices have the same degree. A (k, g)*-graph* is a k-regular graph with girth g. Erdős and Sachs [15] proved the existence of (k, g)-graphs for all values of k and g provided that $k \geq 2$. Since then most work carried out has focused on constructing a smallest (k, g)-graph (cf. e.g. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 16, 18, 20, 21, 22]). A (k, g)*-cage* is a k-regular graph with girth g having the smallest possible number of vertices. Cages have been intensely studied since they were introduced by Tutte [25] in 1947. More details about constructions of cages can be found in the recent survey by Exoo and Jajcay [17].

In this note we are interested in $(k, 8)$-cages. Counting the number of vertices in the distance partition with respect to an edge yields the following lower bound on the order of a $(k, 8)$-cage:

$$n_0(k, 8) = 2(1 + (k - 1) + (k - 1)^2 + (k - 1)^3). \tag{1.1}$$

A $(k, 8)$-cage with $n_0(k, 8)$ vertices is called a Moore $(k, 8)$*-graph* (cf. [14]). These graphs have been constructed as the incidence graphs of generalized quadrangles $Q(4, q)$ and $W(q)$ [12, 17, 24], which are known to exist for q a prime power and $k = q + 1$ and no example is known when $k - 1$ is not a prime power (cf. [11, 13, 19, 27]). Since they are incidence graphs, these cages are bipartite and have diameter 4. Recall also that if q is even, $Q(4, q)$ is isomorphic to the dual of $W(q)$ and viceversa. Hence, the corresponding $(q + 1, 8)$-cages are isomorphic.

In this note we present an alternate description of the known $(q+1, 8)$-cages with $q \geq 2$ a prime power as follows:

Definition 1.1. Let $\mathbb{F}_q$ be a finite field with $q \geq 2$ a prime power and ϱ be a symbol not belonging to $\mathbb{F}_q$. Let $\Gamma_q = \Gamma_q[W_0, W_1]$ denote a bipartite graph with vertex sets $W_i = \mathbb{F}_q^3 \cup \{(\varrho, b, c)_i, (\varrho, \varrho, c)_i : b, c \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_i\}$, $i = 0, 1$, and edge set defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{\Gamma_q}((a, b, c)_1) = \{(w,\ aw + b,\ a^2w + 2ab + c)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, a, c)_0\};$$
$$N_{\Gamma_q}((\varrho, b, c)_1) = \{(c, b, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, c)_0\};$$
$$N_{\Gamma_q}((\varrho, \varrho, c)_1) = \{(\varrho, c, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\};$$
$$N_{\Gamma_q}((\varrho, \varrho, \varrho)_1) = \{(\varrho, \varrho, w)_0 : w \in \mathbb{F}_q\} \cup \{(\varrho, \varrho, \varrho)_0\}.$$

Or equivalently,

For all $i,j,k \in \mathbb{F}_q$

$$N_{\Gamma_q}((i,j,k)_0) = \{(w,\ j-wi,\ w^2i-2wj+k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,j,i)_1\}$$
$$N_{\Gamma_q}((\varrho,j,k)_0) = \{(j,w,k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,j)_1\}$$
$$N_{\Gamma_q}((\varrho,\varrho,k)_0) = \{(\varrho,w,k)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\};$$
$$N_{\Gamma_q}((\varrho,\varrho,\varrho)_0) = \{(\varrho,\varrho,w)_1 : w \in \mathbb{F}_q\} \cup \{(\varrho,\varrho,\varrho)_1\}.$$

Note that ϱ is just a symbol not belonging to $\mathbb{F}_q$ and no arithmetical operation will be performed with it.

Theorem 1.2. *The graph Γ_q given in Definition 1.1 is a Moore $(q+1,8)$-graph for each prime power $q \geq 2$.*

The proof of the above theorem shows that the graph Γ_q described in Definition 1.1 is in fact a labelling for a $(q+1,8)$-cage, for each prime power $q \geq 2$. We need to settle this alternate description because it is used in [2, 3, 4] to construct small (k,g)-graphs for $k=q-1, q$ and $g=7,8$.

2 Proof of Theorem 1.2

2.1 Preliminaries: the graphs H_q and B_q

In order to prove Theorem 1.2 we will first define two q-regular bipartite graphs H_q and B_q (cf. Definitions 2.1 and 2.4). The graph H_q was also introduced by Lazebnik, Ustimenko and Woldar [20] with a different formulation.

Definition 2.1. Let $\mathbb{F}_q$ be a finite field with $q \geq 2$. Let $H_q = H_q[U_0,U_1]$ be a bipartite graph with vertex set $U_r = \mathbb{F}_q^3$, $r=0,1$, and edge set $E(H_q)$ defined as follows:

For all $a,b,c \in \mathbb{F}_q$

$$N_{H_q}((a,b,c)_1) = \{(w,\ aw+b,\ a^2w+c)_0 : w \in \mathbb{F}_q\}.$$

Note that throughout the proofs equalities and operations are intended in $\mathbb{F}_q$.

Lemma 2.2. *Let H_q be the graph from Definition 2.1. For any given $a \in \mathbb{F}_q$, the vertices in the set $\{(a,b,c)_1 : b,c \in \mathbb{F}_q\}$ are mutually at distance at least four. And, for any given $i \in \mathbb{F}_q$, the vertices in the set $\{(i,j,k)_0 : j,k \in \mathbb{F}_q\}$ are mutually at distance at least four.*

Proof. Suppose that there exists a path of length two between distinct vertices of the form $(a,b,c)_1\ (w,j,k)_0\ (a,b',c')_1$ in H_q. By Definition 2.1, $j = aw+b = aw+b'$ and $k = a^2w+c = a^2w+c'$. Combining the equations we get $b=b'$ and $c=c'$ which implies that $(a,b,c)_1 = (a,b',c')_1$ contradicting the assumption that the path has length two. Similarly suppose that there exists a path of length two $(i,j,k)_0\ (a,b,c)_1\ (i,j',k')_0$. Reasoning as before, we obtain $j = ai+b = j'$, and $k = a^2i+c = k'$ yielding $(i,j,k)_0 = (i,j',k')_0$ which is a contradiction. □

Proposition 2.3. *The graph H_q from Definition 2.1 is q-regular, bipartite, of girth 8 and order $2q^3$.*

Proof. For $q = 2$ it can be checked that H_2 consists of two disjoint cycles of length 8. Thus we assume that $q \geq 3$. Clearly H_q has order $2q^3$ and every vertex of U_1 has degree q. Let $(x, y, z)_0 \in U_0$. By definition of H_q,

$$N_{H_q}((x,y,z)_0) = \left\{(a,\ y - ax,\ z - a^2x)_1 : a \in \mathbb{F}_q\right\}. \tag{2.1}$$

Hence every vertex of U_0 has also degree q and H_q is q-regular. Next, let us prove that H_q has no cycles of length smaller than 8. Otherwise suppose that there exists in H_q a cycle

$$C_{2t+2} = (a_0, b_0, c_0)_1\ (x_0, y_0, z_0)_0\ (a_1, b_1, c_1)_1 \cdots (x_t, y_t, z_t)_0\ (a_0, b_0, c_0)_1$$

of length $2t + 2$ with $t \in \{1, 2\}$. By Lemma 2.2, $a_k \neq a_{k+1}$ and $x_k \neq x_{k+1}$ (subscripts being taken modulo $t + 1$). Then

$$\begin{aligned} y_k &= a_k x_k + b_k = a_{k+1} x_k + b_{k+1}, \quad k = 0, \ldots, t, \\ z_k &= a_k^2 x_k + c_k = a_{k+1}^2 x_k + c_{k+1}, \quad k = 0, \ldots, t, \end{aligned}$$

subscripts k being taken modulo $t + 1$. Summing all these equalities we get

$$\begin{aligned} \sum_{k=0}^{t-1} (a_k - a_{k+1}) x_k &= (a_0 - a_t) x_t, \quad t = 1, 2; \\ \sum_{k=0}^{t-1} (a_k^2 - a_{k+1}^2) x_k &= (a_0^2 - a_t^2) x_t, \quad t = 1, 2. \end{aligned} \tag{2.2}$$

If $t = 1$, then (2.2) leads to $(a_0 - a_1)(x_1 - x_0) = 0$. System (2.2) gives $x_0 = x_1 = x_2$ which is a contradiction to Lemma 2.2. This means that H_q has no squares so that we may assume that $t = 2$. The coefficient matrix of (2.2) has a Vandermonde determinant, i.e.

$$\begin{vmatrix} a_1 - a_0 & a_0 - a_2 \\ a_1^2 - a_0^2 & a_0^2 - a_2^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_0 & a_2 \\ a_1^2 & a_0^2 & a_2^2 \end{vmatrix} = \prod_{0 \leq k < j \leq 2} (a_j - a_k).$$

This determinant is different from zero because by Lemma 2.2, $a_{k+1} \neq a_k$ (the subscripts being taken modulo 3). Using Cramer's rule to solve it we obtain $x_1 = x_0 = x_2$ which is a contradiction to Lemma 2.2.

Hence, H_q has girth at least 8. Furthermore, when $q \geq 3$ the minimum number of vertices of a q-regular bipartite graph of girth greater than 8 must be greater than $2q^3$. Thus we conclude that the girth of H_q is exactly 8. □

Next, we will make use of the following induced subgraph B_q of Γ_q.

Definition 2.4. Let $B_q = B_q[V_0, V_1]$ be a bipartite graph with vertex set $V_i = \mathbb{F}_q^3$, $i = 0, 1$, and edge set $E(B_q)$ defined as follows:

For all $a, b, c \in \mathbb{F}_q$

$$N_{B_q}((a,b,c)_1) = \{(j,\ aj + b,\ a^2 j + 2ab + c)_0 : j \in \mathbb{F}_q\}.$$

Lemma 2.5. *The graph B_q is isomorphic to the graph H_q.*

Proof. Let H_q be the bipartite graph from Definition 2.1. Since the map $\sigma\colon B_q \to H_q$ defined by $\sigma((a,b,c)_1) = (a,b,2ab+c)_1$ and $\sigma((x,y,z)_0) = (x,y,z)_0$ is an isomorphism, the result holds. □

Hence, the graph B_q is also q-regular, bipartite, of girth 8 and order $2q^3$.

In what follows, we will obtain the graph Γ_q from the graph B_q by adding some new vertices and edges. We need a preliminary lemma.

Lemma 2.6. *Let B_q be the graph from Definition 2.4. Then the following hold:*

(i) The vertices in the set $\{(a,b,c)_1 : b,c \in \mathbb{F}_q\}$ are mutually at distance at least four for all $a \in \mathbb{F}_q$.

(ii) The vertices in the set $\{(i,j,k)_0 : j,k \in \mathbb{F}_q\}$ are mutually at distance at least four for all $i \in \mathbb{F}_q$.

(iii) The q vertices of the set $\{(x,y,j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least six for all $x,y \in \mathbb{F}_q$.

Proof. The proof of items (i) and (ii) is almost identical to that of Lemma 2.2.

(iii): By (ii), the vertices in $\{(x,y,j)_0 : j \in \mathbb{F}_q\}$ are mutually at distance at least four. Suppose by contradiction that B_q contains the following path of length four:

$$(x,y,j)_0\ (a,b,c)_1\ (x',y',j')_0\ (a',b',c')_1\ (x,y,j'')_0, \text{ for some } j'' \neq j.$$

Then $y = ax+b = a'x+b'$ and $y' = ax'+b = a'x'+b'$. It follows that $(a-a')(x-x') = 0$, which is a contradiction since, by the previous statements, $a \neq a'$ and $x \neq x'$. □

2.2 The conclusion

Figure 1 shows a spanning tree of Γ_q with the vertices labelled according to Definition 1.1. Note that the lower level of such a tree corresponds to the set of vertices of B_q.

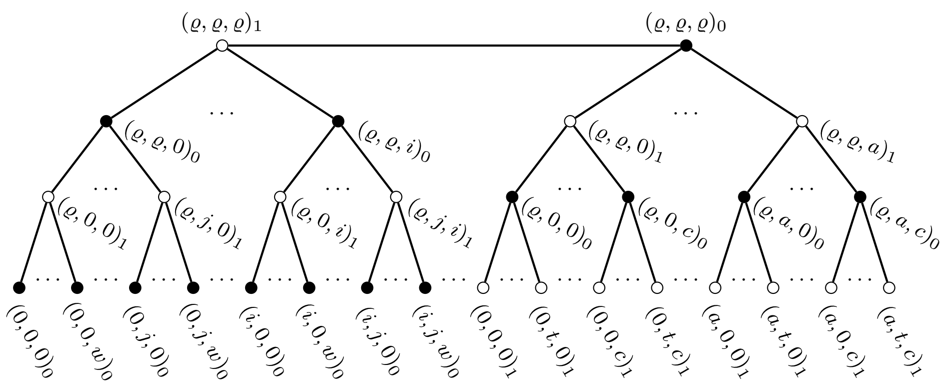

Figure 1: Spanning tree of Γ_q.

We are now ready to prove Theorem 1.2:

Proof of Theorem 1.2. Let $B'_q = B'_q[V_0, V'_1]$ be the bipartite graph obtained from $B_q = B_q[V_0, V_1]$ by adding q^2 new vertices to V_1 labeled $(\varrho, b, c)_1$, $b, c \in \mathbb{F}_q$ (i.e., $V'_1 = V_1 \cup \{(\varrho, b, c)_1 : b, c \in \mathbb{F}_q\}$), and new edges $N_{B'_q}((\varrho, b, c)_1) = \{(c, b, j)_0 : j \in \mathbb{F}_q\}$ (see Figure 1). Then B'_q has $|V'_1| + |V_0| = 2q^3 + q^2$ vertices, every vertex of V_0 has degree $q+1$, and every vertex of V'_1 has still degree q. Note that the girth of B'_q is 8 by Lemma 2.6(iii). The statements from Lemma 2.6 still partially hold in B'_q, as stated in the following claim.

Claim 1. *For any given $a \in \mathbb{F}_q \cup \{\varrho\}$, the vertices of the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four in B'_q.*

Proof. For $a = \varrho$, it is clear from Lemma 2.6(i), since the new vertices do not change the distance among the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$. For $a = \varrho$, the vertices in the set $\{(a, b, c)_1 : b, c \in \mathbb{F}_q\}$ are mutually at distance at least four since each vertex of the form $(i, j, k)_0$ has exactly one neighbour in this set, so the result follows from the bipartition of B'_q. □

Claim 2. *For all $a \in \mathbb{F}_q \cup \{\varrho\}$ and for all $c \in \mathbb{F}_q$, the q vertices of the set $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 6 in B'_q.*

Proof. By Claim 1, for all $a \in \mathbb{F}_q \cup \{\varrho\}$ the q vertices of $\{(a, t, c)_1 : t \in \mathbb{F}_q\}$ are mutually at distance at least 4 in B'_q. Suppose that there exists in B'_q the following path of length four:

$$(a, t, c)_1 \ (x, y, z)_0 \ (a', t', c')_1 \ (x', y', z')_0 \ (a, t'', c)_1, \text{ for some } t'' \neq t.$$

If $a = \varrho$, then $x = x' = c$, $y = t$, $y' = t''$ and $a' \neq \varrho$ by Claim 1. Then $y = a'x + t' = a'x' + t' = y'$ yielding that $t = t''$ which is a contradiction. Therefore $a \neq \varrho$. If $a' = \varrho$, then $x = x' = c'$ and $y = y' = t'$. Thus $y = ax + t = ax' + t'' = y'$ yielding that $t = t''$ which is a contradiction. Hence we may assume that $a' \neq \varrho$ and $a \neq a'$ by Claim 1. In this case we have:

$$\begin{aligned} y &= ax + t = a'x + t'; \\ y' &= ax' + t'' = a'x' + t'; \\ z &= a^2x + 2at + c = a'^2x + 2a't' + c'; \\ z' &= a^2x' + 2at'' + c = a'^2x' + 2a't' + c'. \end{aligned}$$

Thus,

$$(a - a')(x - x') = t'' - t; \tag{2.3}$$

$$(a^2 - a'^2)(x - x') = 2a(t'' - t). \tag{2.4}$$

If q is even, (2.4) leads to $x = x'$ and (2.3) leads to $t'' = t$ which is a contradiction with our assumption. Thus assume q is odd. If $a + a' = 0$, then (2.4) gives $2a(t'' - t) = 0$, so that $a = 0$ yielding that $a' = 0$ (because $a + a' = 0$) which is again a contradiction. If $a + a' \neq 0$, multiplying equation (2.3) by $a + a'$ and subtracting both equations we obtain $(2a - (a + a'))(t'' - t) = 0$. Then $a = a'$ because $t'' \neq t$, which is a contradiction to Claim 1. Therefore, Claim 2 holds. □

Let $B''_q = B''_q[V'_0, V'_1]$ be the graph obtained from $B'_q = B'_q[V_0, V'_1]$ by adding q^2+q new vertices to V_0 labeled $(\varrho, a, c)_0$, $a \in \mathbb{F}_q \cup \{\varrho\}$, $c \in \mathbb{F}_q$, and new edges $N_{B''_q}((\varrho, a, c)_0) = \{(a, t, c)_1 : t \in \mathbb{F}_q\}$ (see Figure 1). Then B''_q has $|V'_1| + |V'_0| = 2q^3 + 2q^2 + q$ vertices such that every vertex has degree $q+1$ except the new added vertices which have degree q. Moreover the girth of B''_q is 8 by Claim 2.

Claim 3. *For all* $a \in \mathbb{F}_q \cup \{\varrho\}$*, the* q *vertices of the set* $\{(\varrho, a, j)_0 : j \in \mathbb{F}_q\}$ *are mutually at distance at least* 6 *in* B''_q*.*

Proof. Clearly these q vertices are mutually at distance at least 4 in B''_q. Suppose that there exists in B''_q the following path of length four:

$$(\varrho, a, j)_0 \; (a, b, j)_1 \; (x, y, z)_0 \; (a, b', j')_1 \; (\varrho, a, j')_0, \text{ for some } j' \neq j.$$

If $a = \varrho$ then $x = j = j'$ which is a contradiction. Therefore $a \neq \varrho$. In this case $y = ax+b = ax+b'$ which implies that $b = b'$. Hence $z = a^2x+2ab+j = a^2x+2ab'+j'$ yielding that $j = j'$ which is again a contradiction. □

Let $B'''_q = B'''_q[V'_0, V''_1]$ be the graph obtained from B''_q by adding $q+1$ new vertices to V'_1 labeled $(\varrho, \varrho, a)_1$, $a \in \mathbb{F}_q \cup \{\varrho\}$, and new edges $N_{B'''_q}(\varrho, \varrho, a)_1 = \{(\varrho, a, c)_0 : c \in \mathbb{F}_q\}$, see Figure 1. Then B'''_q has $|V''_1| + |V'_0| = 2q^3 + 2q^2 + 2q + 1$ vertices such that every vertex has degree $q+1$ except the new added vertices which have degree q. Moreover the girth of B'''_q is 8 by Claim 3 and clearly these $q+1$ new vertices are mutually at distance 6. Finally, the graph Γ_q is obtained by adding to B'''_q another new vertex labeled $(\varrho, \varrho, \varrho)_0$ and edges $N_{\Gamma_q}((\varrho, \varrho, \varrho)_0) = \{(\varrho, \varrho, i)_1 : i \in \mathbb{F}_q \cup \{\varrho\}\}$. The graph Γ_q has $2(q^3+q^2+q+1)$ vertices, it is $(q+1)$-regular and has girth 8, so by the uniqueness of a $(q+1,8)$-cage (see e.g. [29]), Γ_q is indeed a $(q+1,8)$ Moore graph. □

Remark 2.7. Coordinatizations of classical generalized quadrangles $Q(4,q)$ and $W(q)$ in four dimensions are discussed in [23, 26, 28]. The alternate description of a Moore $(q+1,8)$-graph given in Theorem 1.2 in three dimensions is equivalent to this coordinatization.

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