# Partition functions for states models on signed graphs

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In this thesis, we study various physical models based on graphs and find the partition function as an evaluation of a polynomial invariant of graphs. We will also see what polynomials preserve isomorphisms such as vertex switching or Reidemeister moves and characterize them.

The work is basically based in graph theory but also gives an interpretation to physical properties of the system modelled by the graph we are working with.

- Statistical physics
- Partition function
- Graphs
- Signed graphs
- Knot theory
- Tutte polynomial
- Potts model
- Vertex colourings

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# Contents

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# 0 Introduction

Many systems in statistical physics can be modelled using graphs, in which vertices represent sites at which atomic "spins" are located and an edge joins two sites for which interaction can occur between these spins. Theoretical properties of the model on this graph can be used to extract information about the physical system being modelled. For example, in the Ising model of ferromagnetism, in which two types of spin representing magnetic dipole moments are located on vertices of a square lattice, the temperature a phase transition (spontaneous magnetization) occurs corresponds to a zero of the partition function of the model.

In the Potts model, which generalizes the Ising model to any finite set of spins, not just the two possible north-south spins in magnetism, the information carried by the edges of the graph are weights that always have the same sign because the interactions they represent are always of the same type. (For the case of ferromagnetism, any pair of adjacent spins interact in the same way, no matter where they are located: if they are the opposite they interact with more energy than if they are aligned.) In order to model systems with two different types of (opposite) interaction, for example the exciting/inhibiting interaction between neurons, we may use signs on the edges in order to distinguish between the two types of interaction. This is where signed graphs come in.

In this thesis we shall study various physical models based on a graph (signed or unsigned) and whose partition function, which is one of the key parameters of the model, can be obtained as an evaluation of a polynomial invariant of (signed) graphs. By a graph invariant is meant a parameter of graphs preserved under isomorphism. For signed graphs, as well as isomorphism-invariance we consider two further types of invariance, one related to the matroid underlying a signed graph (analogous to the cycle matroid underlying a graph) and the other to the representation of knots and links by signed graphs.

In the case of unsigned graphs, we review some of the theory of the classical Potts model, for which the partition function is a specialization of the Tutte polynomial of the underlying graph. The Tutte polynomial is a polynomial graph invariant that satisfies an edge deletion-contraction recurrence (the Tutte polynomial of a given graph can be recovered from the Tutte polynomials of smaller graphs obtained by either deleting or contracting an edge), and contains the chromatic polynomial and many other combinatorial invariants of a graph as evaluations. (See Section 2)

In the case of signed graphs, we shall be interested in two Tutte-like polynomials that have been defined for them, each of which focuses on a different type of signed graph invariance (in addition to signed-graph-isomorphism invariance) while satisfying a deletion-contraction recurrence analogous to that of the Tutte polynomial of a graph. The first of these stems from a generalization of Kauffman's bracket polynomial of knots and links, for which invariance under Reidemeister moves (preserving knots up to ambient isotopy) is key, and the deletion-contraction recurrence corresponds to a skein relation on knots (see Section 5); the second stems from a generalization of the Tutte polynomial of the cycle matroid of a graph to the Tutte polynomial of a pair of matroids associated with a signed graph, and the deletion-contraction recurrence corresponds to matroid edge deletion-contraction (see Section 4).

The first Tutte polynomial (of Kauffman) leads us to consider invariants of signed graphs up to “Reidemeister moves” (translating the Reidemeister operations on links to operations on the signed graph representing the link); the second Tutte polynomial (of Goodall, Litjens, Regts and Vena) leads us to consider invariants of signed graphs up to “vertex switching” (signs on edges incident with a given vertex are all switched). In particular, we shall study the partition functions of statistical physics interaction models on signed graphs, defined similarly to the Potts model on graphs, and which are either invariant under vertex switching or invariant under Reidemeister moves. For the latter we review the work of Jones on this question; for the former we prove two new results. The first of these results is that the partition function of the “signed Potts model” is a specialization of the trivariate Tutte polynomial of the signed graph (Theorem 4.3), and the second gives a general family of vertex switching invariant interaction models that includes the signed Potts model with spins elements on a finite Abelian group: an exact formula is found for the number of independent parameters specifying this family (Theorem 4.4).

The thesis begins with graphs and a survey of properties of the Tutte polynomial, before reviewing the Potts model and its partition function. We then draw parallels between these graph invariants and models with those for signed graphs: we introduce the (seemingly new) “signed Potts model” and show its partition function is a specialization of the trivariate Tutte polynomial, and we produce a general family of weight systems, which include the signed Potts model, that are invariant under vertex switching, giving an exact enumeration of them. After this we review the analogous constructions for the Tutte polynomial of a signed graph (Kauffman) and interaction models invariant under Reidemeister moves (Jones). The thesis is structured in a way that we hope draws out the similarities and differences between the models on graphs and the models on signed graphs, and that make clear the difference between the trivariate Tutte polynomial and Kauffman Tutte polynomial of a signed graph.

# 1 Graphs

We will denote a graph by $\Gamma = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges. A colouring of the vertices of a graph $\Gamma = (V, E)$ is a map $\sigma : V \to [q]$, where $[q]$ is the set of colours $\{1, 2, \ldots, q\}$. We will call an edge $e = ab \in E$ *improper* if $\sigma(a) = \sigma(b)$ and *proper* otherwise; analogously, a colouring of a graph is proper if all of the edges are proper, and improper if there exists an improper edge.

For a colouring $\sigma$, we can define a *flow* as a function $\phi : E \to [q]$ such that, for every $e = ab \in E$, $\phi(e) = \sigma(b) - \sigma(a)$. It is usually defined for directed graphs, so there is no ambiguity (if the edge goes from $a$ towards $b$, $\phi(ab) = \sigma(b) - \sigma(a)$), however for undirected graphs we can define the flow both ways. In order to determine a unique flow for an undirected graph, we can name the vertices and make a list of preference and say that $a > b$ if we prefer $a$ over $b$, so then

$$\phi(ab) = \begin{cases} \sigma(a) - \sigma(b) & \text{if } a > b \\ \sigma(b) - \sigma(a) & \text{if } a < b \end{cases}$$

The notation $\Gamma \setminus e$ means that we are deleting the edge $e$ (i.e. $\Gamma \setminus e = (V, E \setminus e)$) and $\Gamma/e$ means that we are contracting the edge $e$, that is to merge both endpoints of $e$ and then delete $e$. For a subset of edges $F \subseteq E$, we denote as $\Gamma_F$ the subgraph spanned by F, that is $\Gamma_F = (V, F)$.

The number of connected components is denoted as $k(\Gamma)$; the *rank* of a graph is $r(\Gamma) = |V(\Gamma)| - k(\Gamma)$, which is the number of edges on $\Gamma$ if it were a forest, and the *nullity* of a graph is $n(\Gamma) = |E(\Gamma)| - r(\Gamma)$. For every of these parameters, we use the notation $k(F) = k(\Gamma_F)$, $r(F) = r(\Gamma_F)$, $n(F) = n(\Gamma_F)$ for $F \subseteq E$.

# 2 The Tutte polynomial of a graph

The Tutte polynomial of a graph $\Gamma$ is defined as the following subgraph expansion:

$$T_\Gamma(x,y) = \sum_{F\subseteq E} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} \tag{1}$$

We see that $r(E) - r(F) \geq 0$ for $F \subseteq E$ because the number of vertices is the same for both $\Gamma$, $\Gamma_F$ and this last one has trivially more connected components than the original graph. We can also conclude that $n(F) \geq 0$, since $r(F)$ is the number of edges on $\Gamma_F$ when it is a forest, and hence $|F| \geq r(F)$. The fact that, for every subgraph, the exponents on the term are positive means that $T_\Gamma(x,y) \in \mathbb{Z}[x,y]$.

**Proposition 2.1** *The Tutte polynomial satisfies the following deletion-contraction recurrence:*

$$T_\Gamma(x,y) = \begin{cases} T_{\Gamma\setminus e}(x,y) + T_{\Gamma/e}(x,y) & \text{if } e\in E \text{ an ordinary edge,} \\ xT_{\Gamma/e}(x,y) & \text{if } e\in E \text{ a bridge,} \\ yT_{\Gamma\setminus e}(x,y) & \text{if } e\in E \text{ a loop,} \\ 1 & \text{if } \Gamma \text{ has no edges,} \end{cases} \tag{2}$$

*Proof.* If we define $r', n'$ and $r'', n''$ as the ranks and nullities in the deleted and the contracted graph, respectively:

- If $E = \emptyset$, then $T_\Gamma = (x-1)^{r(\emptyset)-r(\emptyset)}(y-1)^{n(\emptyset)} = 1$, since $r(\emptyset) - r(\emptyset) = 0$ and $n(\emptyset) = |E| - r(\emptyset) = -|V| + k(\emptyset) = 0$ because every vertex is a connected component due to the fact that there are no edges.

- Take $e \in E$ a loop.

$$T_\Gamma(x,y) = \sum_{F\subseteq E} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} =$$

$$= \sum_{\substack{F=A\cup e\\ A\subseteq E\setminus e}} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} + \sum_{F\subseteq E\setminus e} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} =$$

$$= (y-1)\sum_{F\subseteq E\setminus e} (x-1)^{r'(E)-r'(F)}(y-1)^{n'(F)} + \sum_{F\subseteq E\setminus e} (x-1)^{r'(E)-r'(F)}(y-1)^{n'(F)}$$

$$= y\sum_{F\subseteq E\setminus e} (x-1)^{r'(E)-r'(F)}(y-1)^{n'(F)} = yT_{\Gamma\setminus e}(x,y)$$

The third equality comes from the fact that deleting a loop $e$ never changes the number of connected components nor the number of vertices (then $r = r'$ and then $n = n' + 1$).

- Take $e \in E$ a bridge.

$$T_\Gamma(x,y) = \sum_{F \subseteq E} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} =$$

$$= \sum_{\substack{F=A\cup e \\ A\subseteq E\setminus e}} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} + \sum_{F\subseteq E\setminus e} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} =$$

$$= \sum_{F\subseteq E\setminus e} (x-1)^{r''(E)-r''(F)}(y-1)^{n''(F)} + (x-1) \sum_{F\subseteq E\setminus e} (x-1)^{r''(E)-r''(F)}(y-1)^{n''(F)}$$

$$= x \sum_{F\subseteq E\setminus e} (x-1)^{r''(E)-r''(F)}(y-1)^{n''(F)} = xT_{\Gamma/e}(x,y)$$

The third equality comes from the fact that contracting a bridge $e$ never gives us another connected component and changes the number of vertices (then $r = r'' + 1$ and then $n = |E| - r = |E \setminus e| + 1 - (r'' + 1) = n''$).

- Take $e \in E$ an ordinary edge.

$$T_\Gamma(x,y) = \sum_{F \subseteq E} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} =$$

$$= \sum_{\substack{F=A\cup e \\ A\subseteq E\setminus e}} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} + \sum_{F\subseteq E\setminus e} (x-1)^{r(E)-r(F)}(y-1)^{n(F)} =$$

$$= \sum_{F\subseteq E\setminus e} (x-1)^{r''(E)-r''(F)}(y-1)^{n''(F)} + \sum_{F\subseteq E\setminus e} (x-1)^{r'(E)-r'(F)}(y-1)^{n'(F)} =$$

$$= T_{\Gamma\setminus e}(x,y) + T_{\Gamma/e}(x,y)$$

The third equality comes from the fact that deleting an ordinary edge $e$ does not change the number of vertices nor connected components ($r = r'$, and then $n = n' + 1$) and contracting it changes the number of vertices but not the number of connected components ($r = r'' + 1$ and then $n = n''$).

□

We are now going to check that (2) actually gives a second definition for the Tutte polynomial by proving that a graph has the same Tutte polynomial no matter the order in what we choose the edges for the recurrence.

**Lemma 2.1** *Let $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ be two graphs such that $|V_1 \cap V_2| \leq 1$, then*

$$T_{\Gamma_1 \cup \Gamma_2}(x, y) = T_{\Gamma_1}(x, y) T_{\Gamma_2}(x, y)$$

*Proof.*

$\underline{|V_1 \cap V_2| = 0}$

If we take the subgraph expansion definition of the Tutte polynomial, we get that:

$$T_{\Gamma_1 \sqcup \Gamma_2}(x, y) = \sum_{F \subseteq E_1 \sqcup E_2} (x-1)^{r(E_1 \sqcup E_2) - r(F)} (y-1)^{n(F)} = \sum_{\substack{F_1 \subseteq E_1 \\ F_2 \subseteq E_2}} (x-1)^{r(E_1 \sqcup E_2) - r(F_1 \sqcup F_2)} (y-1)^{n(F_1 \sqcup F_2)}$$

And since $r(F_1 \sqcup F_2) = |V_1 \sqcup V_2| - k(F_1 \sqcup F_2) = |V_1| + |V_2| - (k(F_1) + k(F_2)) = |V_1| - k(F_1) + |V_2| - k(F_2) = r(F_1) + r(F_2)$, and $n(F_1 \sqcup F_2) = |F_1 \sqcup F_2| - r(F_1 \sqcup F_2) = |F_1| - r(F_1) + |F_2| - r(F_2) = n(F_1) + n(F_2)$:

$$\sum_{\substack{F_1 \subseteq E_1 \\ F_2 \subseteq E_2}} (x-1)^{r(E_1 \sqcup E_2) - r(F_1 \sqcup F_2)} (y-1)^{n(F_1 \sqcup F_2)} = \sum_{\substack{F_1 \subseteq E_1 \\ F_2 \subseteq E_2}} (x-1)^{r(E_1) + r(E_2) - r(F_1) - r(F_2)} (y-1)^{n(F_1) + n(F_2)} =$$

$$= \left( \sum_{F_1 \subseteq E_1} (x-1)^{r(E_1) - r(F_1)} (y-1)^{n(F_1)} \right) \left( \sum_{F_2 \subseteq E_2} (x-1)^{r(E_2) - r(F_2)} (y-1)^{n(F_2)} \right) = T_{\Gamma_1}(x, y) T_{\Gamma_2}(x, y)$$

$\underline{|V_1 \cap V_2| = 1}$

If we take the subgraph expansion definition of the Tutte polynomial, we get that:

$$T_{\Gamma_1 \sqcup \Gamma_2}(x, y) = \sum_{F \subseteq E_1 \sqcup E_2} (x-1)^{r(E_1 \sqcup E_2) - r(F)} (y-1)^{n(F)} = \sum_{\substack{F_1 \subseteq E_1 \\ F_2 \subseteq E_2}} (x-1)^{r(E_1 \sqcup E_2) - r(F_1 \sqcup F_2)} (y-1)^{n(F_1 \sqcup F_2)}$$

And since $r(F_1 \sqcup F_2) = |V_1 \cup V_2| - k(F_1 \sqcup F_2) = |V_1| + |V_2| - 1 - (k(F_1) + k(F_2) - 1) = |V_1| - k(F_1) + |V_2| - k(F_2) = r(F_1) + r(F_2)$, and $n(F_1 \sqcup F_2) = |F_1 \sqcup F_2| - r(F_1 \sqcup F_2) = |F_1| - r(F_1) + |F_2| - r(F_2) = n(F_1) + n(F_2)$:

$$\sum_{\substack{F_1 \subseteq E_1 \\ F_2 \subseteq E_2}} (x-1)^{r(E_1 \sqcup E_2) - r(F_1 \sqcup F_2)} (y-1)^{n(F_1 \sqcup F_2)} = \sum_{\substack{F_1 \subseteq E_1 \\ F_2 \subseteq E_2}} (x-1)^{r(E_1) + r(E_2) - r(F_1) - r(F_2)} (y-1)^{n(F_1) + n(F_2)} =$$

$$= \left( \sum_{F_1 \subseteq E_1} (x-1)^{r(E_1) - r(F_1)} (y-1)^{n(F_1)} \right) \left( \sum_{F_2 \subseteq E_2} (x-1)^{r(E_2) - r(F_2)} (y-1)^{n(F_2)} \right) = T_{\Gamma_1}(x, y) T_{\Gamma_2}(x, y)$$

$\square$

Note that this result does not depend on the choice of the vertex in common. Hence we have the following result, which we shall need to prove that the deletion-contraction recurrence gives a well-defined polynomial (independent of how edges are chosen).

**Corollary 2.1** *Let $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$ be two disjoint graphs and consider the family $\Lambda$ of all the graphs that are the result of taking a vertex $v_1 \in V_1$ and another vertex $v_2 \in V_2$, and merge them into a new vertex $v$ that will have as neighbours those vertices that were neighbours of $v_1$ and $v_2$ (and the edges that had as an endpoint $v_1$ or $v_2$ will disappear with them). Then for every two graphs $\Gamma, \Gamma' \in \Lambda$:*

$$T_\Gamma(x, y) = T_{\Gamma'}(x, y)$$

**Proposition 2.2** *The polynomial defined by (2) is independent of the order in which the edges are chosen.*

*Proof.* Let us take $e, f \in E$, both non-ordinary edges (assume $e$ a loop and $f$ a bridge), then $xyT_{(\Gamma\backslash e)/f} = yT_{\Gamma\backslash e} = T_\Gamma = xT_{\Gamma/f} = xyT_{(\Gamma/f)\backslash e}$ which is the same because deleting a loop commutes with contracting or deleting any edge, and contracting a bridge also commutes with contracting another bridge.

Let's assume now that $e$ is a bridge and $f$ is an ordinary edge (if $e$ were a loop, the result will follow because of the commutativity between deleting a loop and deleting or contracting any other edge). By contracting $e$ we cannot make of $f$ a bridge in $\Gamma/e$ if it was an ordinary edge in $\Gamma$, because neither of the disjoint paths joining the endpoints of $f$ is disconnected by contracting $e$. If by contracting $e$, $f$ turns into a loop in $\Gamma/e$, then $e$ and $f$ had the same endpoints and then $e$ would not have been able to be a bridge.

Last, assume $e$ and $f$ are ordinary edges, if they do not have any endpoints in common the order is indifferent because deleting and contracting them will always commute and

$$T_\Gamma = T_{\Gamma\backslash e} + T_{\Gamma/e} = T_{(\Gamma\backslash e)\backslash f} + T_{(\Gamma\backslash e)/f} + T_{(\Gamma/e)\backslash f} + T_{(\Gamma/e)/f}$$

$$T_\Gamma = T_{\Gamma\backslash f} + T_{\Gamma/f} = T_{(\Gamma\backslash f)\backslash e} + T_{(\Gamma\backslash f)/e} + T_{(\Gamma/f)\backslash e} + T_{(\Gamma/f)/e}$$

If they share both endpoints, they are parallel edges and the deletion-contraction recurrence is symmetrical whether we start with one edge or the other, so the commutativity is trivial. And if they share an endpoint, it is impossible to turn one of them into a loop by contracting the other (and, obviously, neither a bridge) as it is impossible to turn one of them into a loop by deleting the other, so the only remaining case is that we delete one edge and the other one turns into a bridge. For the cases above, the graphs obtained by both paths (first delete-contract $e$ and then $f$ or vice versa) were the same, in the following case the graphs will not be the same in general, but their Tutte polynomials will:

$$T_\Gamma = T_{\Gamma\backslash e} + T_{\Gamma/e}$$

$$T_\Gamma = T_{\Gamma\setminus f} + T_{\Gamma/f}$$

Consider the two connected components $\Gamma_1 = (V_1, E_1), \Gamma_2 = (V_2, E_2)$ that we get when we delete $e$ and $f$, Corollary 2.1 assures us then that, if $\Gamma_e = (V_1 \sqcup V_2, E_1 \sqcup E_2 \sqcup e)$ and $\Gamma_f = (V_1 \sqcup V_2, E_1 \sqcup E_2 \sqcup f)$, $T_{\Gamma_e} = T_{\Gamma_f}$, so then by Lemma 2.1., $T_{\Gamma\setminus e} = T_{\Gamma\setminus f}$.

Let's now name the endpoints of $e = xy$ and $f = xz$ (so then, without loss of generality, $x \in V_1$ and $y, z \in V_2$), if we contract $e$, then $x = y$ and $f = xz = yz$, and $\Gamma_1$ and $\Gamma_2$ are joined by $V_1 \ni x = y \in V_2$. However, if we contract $f$, then $x = z$ and $e = xy = yz$, and $\Gamma_1$ and $\Gamma_2$ are joined by $V_1 \ni x = z \in V_2$. So if we consider the graph $\Gamma_2' = (V_2, E_2 \sqcup yz)$ and $\Gamma_1$, by Corollary 2.1. and Lemma 2.1., then $T_{\Gamma/e} = T_{\Gamma/f}$.

So the result is obtained and leaves (2) as a valid definition of the Tutte polynomial.

□

Thanks to Proposition 2.2, we have now shown that (1) and (2) are equivalent definitions. Then, by the definition (2) of $T_\Gamma(x, y)$, we can conclude that its coefficients are nonnegative. Moreover, some of the coefficients have simple interpretation. In fact, Tutte showed that each coefficient $t_{i,j}(\Gamma)$ of the Tutte polynomial of a connected graph $\Gamma$ represents the number of trees of $\Gamma$ of internal activity $i$ and external activity $j$ (see [1]). The following proposition is another example.

**Proposition 2.3** *For a graph $\Gamma$ with Tutte polynomial $T_\Gamma(x, y) = \sum t_{i,j}(G)x^i y^j$:*

1. *$t_{0,0}(\Gamma) = 0$ if $|E(\Gamma)| > 0$*
2. *if $\Gamma$ has no loops then $t_{1,0}(\Gamma) \neq 0$ if and only if $\Gamma$ is 2-connected*
3. *$x^k$ divides $T_\Gamma(x, y)$ if and only if $G$ has at least $k$ bridges, and $y^l$ divides $T_\Gamma(x, y)$ if and only if $\Gamma$ has at least $l$ loops*
4. *given $\Gamma$ has $k$ bridges and $l$ loops, if $i \geq r(\Gamma)$ or $j \geq n(\Gamma)$ then $t_{i,j}(\Gamma) = 0$ except when $i = r(\Gamma)$ and $j = l$, $i = k$ and $j = n(\Gamma)$, where we have $t_{r(\Gamma),l}(\Gamma) = 1 = t_{k,n(\Gamma)}$*

*Proof.*

1. If $|E(\Gamma)| = 1$, $T_\Gamma(x, y) = x$ and then $t_{0,0}(\Gamma) = 0$. So now we will use induction to prove the result. Let's suppose that every graph with less than $n$ edges satisfies that $t_{0,0}(\Gamma) = 0$.

   Take a graph $\Gamma$, $|E(\Gamma)| = n$ and an edge $e \in E$, by induction we know that $t_{0,0}(\Gamma \setminus e) = t_{0,0}(\Gamma/e) = 0$ because $E(\Gamma \setminus e) = E(\Gamma/e) < n$. Then, we have three options, due to the deletion-contraction recurrence:

   (a) If $e$ is a loop, $t_{0,0}(\Gamma) = t_{0,-1}(\Gamma \setminus e) = 0$ because there are no terms with negative powers of any variable.

   (b) If $e$ is a bridge, $t_{0,0}(\Gamma) = t_{-1,0}(\Gamma/e) = 0$, for the same reason as before.

(c) If $e$ is an ordinary edge (not a bridge nor a loop), $t_{0,0}(\Gamma) = t_{0,0}(\Gamma \setminus e) + t_{0,0}(\Gamma/e) = 0+0 = 0$.

2. See [1] for proof.

3. $x^k | T_\Gamma(x,y) \Leftrightarrow \Gamma$ has $k$ bridges

   ($\Rightarrow$) Take (2) for the bridges in $\Gamma$ and the result follows.

   ($\Leftarrow$) We will proceed by induction. A graph $\Gamma$ such that $T_\Gamma(x,y) = x^k$ consists in a forest with $k$ edges (all of them bridges). Let's suppose that for every polynomial with degree $< n$, with $n > k$, the statement holds. Consider a graph $\Gamma$ with Tutte polynomial satisfying $x^k | T_\Gamma(x,y)$, then by applying (2), if we use the recurrence for an ordinary edge, since all coefficients in the Tutte polynomial of a graph are positive, we will get to the fact that $x^k | T_{\Gamma \setminus e}(x,y), T_{\Gamma/e}(x,y)$. This is due to the fact that if $x^k$ does not divide one of the other polynomials, then there exists a term of the polynomial that has no $x^k$ as a factor (with nonnegative coefficient), and hence, $T_\Gamma(x,y)$ also has this factor because of the nonnegativity of the coefficients.

   If $\Gamma$ has no cycles, it is a tree and therefore every edge is a bridge, then the result follows. However, if $\Gamma$ has a cycle, by deleting-contracting we will eventually either a tree or a loop; in the first case, every edge is a bridge, and in the second case, we proceed as (2) indicates and $x^k | y T_{\Gamma' \setminus e}(x,y) \Rightarrow x^k | T_{\Gamma' \setminus e}(x,y)$, so $T_{\Gamma' \setminus e}(x,y)$ has $k$ bridges, but since $e$ is a loop in $\Gamma'$, by deleting it we do not change the number of bridges, and the result follows.

   $y^l | T_\Gamma(x,y) \Leftrightarrow \Gamma$ has $l$ loops

   Analogously.

4. See [1] for proof.

As well as its coefficients, the Tutte polynomial of a graph $\Gamma$ has evaluations that give us important enumerative data about $\Gamma$.

**Theorem 2.1** *Let $\Gamma$ be a connected graph. Then $T_\Gamma(1,1)$ is the number of spanning trees of $\Gamma$, $T_\Gamma(2,1)$ is the number of forests in $\Gamma$, $T_\Gamma(1,2)$ is the number of connected spanning subgraphs, and $T_\Gamma(2,2)$ is the number of spanning subgraphs of $\Gamma$.*

*Proof.* We consider which subgraphs of $\Gamma$ satisfy $r(E) - r(F) = 0$ and which ones satisfy $n(F) = 0$. Since $r(E) = |V| - k(E)$, $r(E) - r(F)$ vanishes whenever the subgraph spanned by the edges on $F$ has the same number of connected components as the original graph. And $n(F) = 0$ will be satisfied when $|F| = |V| - k(F)$, that is to say, when the subgraph of $\Gamma$ spanned by $F \subseteq E$ is a forest.

So then:

- $T_\Gamma(1,1)$

  In this case every term of the sum on (1) will be 0 except of those where both exponents are 0, in which case the terms are 1. Then, we will have a term 1 for every subgraph that has the same number of connected components as $\Gamma$ and is a forest, so then, $T_\Gamma(1,1)$ counts the number of spanning trees of $\Gamma$.

- $\underline{T_\Gamma(2,1)}$

  Now, $x-1=1$, so the terms that are 1 now are those associated to forests, thus $T_\Gamma(2,1)$ is the number of forests you can make with subsets of edges $F \subseteq E$.

- $\underline{T_\Gamma(1,2)}$

  In this case, $y-1=1$, so the terms that are 1 will be those with $r(E)-r(F)=0$, so $T_\Gamma(1,2)$ counts the number of connected spanning subgraphs.

- $\underline{T_\Gamma(2,2)}$

  Last, every term is 1 in (1), so we are counting all of the possible subsets of $E$, which is the same as count all of the possible spanning subgraphs of $\Gamma$ (that gives $2^{|E|}$).

By multiplicativity of the Tutte polynomial over disjoint unions (i.e. Lemma 2.1.), we have the following corollary:

**Corollary 2.2** *Let $\Gamma$ be a graph. Then $T_\Gamma(1,1)$ is the number of forests of $\Gamma$ in which every tree is a spanning tree of a coneected component in $\Gamma$; $T_\Gamma(2,1)$ is the number of forests in $\Gamma$; $T_\Gamma(1,2)$ is the number of graphs in which every connected component is a connected spanning subgraph of a connected component of $\Gamma$, and $T_\Gamma(2,2)$ is the number of spanning subgraphs of $\Gamma$.*

Another important property of the Tutte polynomial is the following theorem which is a variant of the recipe theorem in [4].

**Theorem 2.2** *"Recipe theorem" Let $\Lambda$ be a minor-closed class of graphs. There is a unique graph invariant $f : \Lambda \to \mathbb{Z}[x, y, \alpha, \beta, \gamma]$ such that for each graph $\Gamma = (V, E) \in \Lambda$*

$$f(\Gamma) = \begin{cases} \alpha f(\Gamma/e) + \beta f(\Gamma \setminus e) & e \textit{ ordinary edge of } \Gamma \\ x f(\Gamma/e) & e \textit{ a bridge in } \Gamma \\ y f(\Gamma \setminus e) & e \textit{ a loop in } \Gamma \\ \gamma^{|V|} & \Gamma \textit{ has no edges} \end{cases} \tag{3}$$

*This graph invariant can also be expressed in terms of the Tutte polynomial:*

$$f(\Gamma) = \gamma^{k(\Gamma)} \alpha^{r(\Gamma)} \beta^{n(\Gamma)} T_\Gamma\left(\frac{x}{\alpha}, \frac{y}{\beta}\right)$$

*Proof.* The uniqueness is given by induction on the number of edges, i.e. applying the recurrence and using as induction hypothesis that $f(\Gamma/e)$ and $f(\Gamma \setminus e)$ are unique.

For the expression in terms of the Tutte polynomial see Theorem 9.5 in [4].

$\square$

In other words, graph invariants that we can get by a deletion-contraction recurrence of the form (3) can be obtained as a specialization of the Tutte polynomial.

## 2.1 The chromatic polynomial

The number of proper colourings of a graph $\Gamma$ with $q$ colours is a polynomial in the number of colours $q$, called the chromatic polynomial. That it is a polynomial in $q$ can be seen by induction using the following deletion-contraction formula:

$$\mathcal{X}_\Gamma(q) = \begin{cases} \mathcal{X}_{\Gamma\setminus e}(q) - \mathcal{X}_{\Gamma/e}(q) & e \in E \\ q^{|V|} & \Gamma \text{ has no edges} \end{cases}$$

This deletion-contraction recurrence holds because of two facts. First, when a graph has no edges, every colouring is proper (there are no improper edges), then the number of proper colourings is the number of colourings which is the number of ways of colouring $|V|$ vertices with $q$ colours: $q^{|V|}$. And second, the number of proper colourings of $\Gamma$ is the number of proper colourings of $\Gamma \setminus e$ (where $e = ab \in E(G)$) with both $a$ and $b$ having different colours, which is the same as all the colourings of $\Gamma \setminus e$ without the ones that make $a$ and $b$ to have the same colour (and if they have the same colour they can become the same vertex so the number of colourings will not change); thus, $\mathcal{X}_\Gamma(q) = \mathcal{X}_{\Gamma\setminus e}(q) - \mathcal{X}_{\Gamma/e}(q)$.

**Proposition 2.4** *The Tutte polynomial gives an expression for the chromatic polynomial:*

$$\mathcal{X}_\Gamma(q) = (-1)^{r(\Gamma)} q^{k(\Gamma)} T_\Gamma(1-q, 0)$$

*Proof.* In the case of the chromatic polynomial, the parameters are $y = 0, x = q-1, \alpha = -1, \beta = 1, \gamma = q$ ($y = 0$ because if we have a loop there are no proper colourings, and $x = q-1$ comes from the fact that if $e$ is a bridge, $\mathcal{X}_\Gamma = \mathcal{X}_{\Gamma\setminus e} - \mathcal{X}_{\Gamma/e} = q\mathcal{X}_{\Gamma/e} - \mathcal{X}_{\Gamma/e} = (q-1)\mathcal{X}_{\Gamma/e}$, because for every proper colouring on $\Gamma/e$ we can build $q$ proper colourings in $\Gamma \setminus e$ changing the colour of the new vertex and keeping the relations between joined vertices, that is, keeping the flow), so then, using Theorem 2.2.:

$$\mathcal{X}_\Gamma(q) = q^{k(\Gamma)}(-1)^{r(\Gamma)} 1^{n(\Gamma)} T_\Gamma\left(\frac{q-1}{-1}, \frac{0}{1}\right) = (-1)^{r(\Gamma)} q^{k(\Gamma)} T_\Gamma(1-q, 0)$$

□

## 2.2 $d$-dimensional $q$-state Potts model

The Potts model is a model of interacting spins on an arbitrary graph, where two related particles with the same energy state (i.e. two vertices of a graph that are joined by an edge and have the same colour, which is an element of an Abelian group) will have some energy interaction, that will contribute to the Hamiltonian of the system modelled by the graph $\Gamma = (V, E)$. In practice, this arbitrary graph will be a part of a $d$-dimensional lattice, that is $\Gamma \subset \mathbb{Z}^d$, and the set of spins will be called $\Theta$, so the state space will be $\Theta^{|V|}$. For a state $\sigma : V \to \Theta$, the Hamiltonian has the following expression:

$$H(\sigma) = \sum_{ij \in E} J_{ij}(1 - \delta(\sigma_i, \sigma_j)),$$

where $\delta$ is the Kronecker delta function, and $J_{ij}$ is the energy of interaction between the particles $i$ and $j$, which, in this model, we assume is constant for every pair of particles, hence $J_{ij} = J, \forall i, j$, and then:

$$H(\sigma) = J(|E| - \sum_{ij \in E} \delta(\sigma_i, \sigma_j))$$

The partition function of this system of particles is defined as follows

$$Z(\Gamma) = \sum_{\sigma \in \Theta^{|V|}} e^{-\beta H(\sigma)}$$

where $\beta = \frac{1}{kT}$ is a temperature-depending parameter. For higher $\beta$ (lower temperatures) we can see that the partition function decreases, which implies a lower energy in the system; thus, for lower $\beta$ (higher temperatures) the energy in the system increases. This fact makes us understand that, for low enough temperatures, the particles will tend to have the same spin state in order to minimize the interaction between them, this process is called *phase transition* and we will talk about it later.

With the last definition of the Hamiltonian, we can rewrite the partition function as follows:

$$Z(\Gamma) = \sum_{\sigma \in \Theta^{|V|}} e^{-\beta J|E|} e^{\beta J \sum \delta(\sigma_i, \sigma_j)} = e^{-\beta J|E|} \sum_{\sigma \in \Theta^{|V|}} e^{\beta J(\#\text{improper edges})} = y^{-|E|} \sum_{\sigma \in \Theta^{|V|}} y^{\#\text{improper edges}}$$

where we have taken $y = e^{\beta J}$. So then, finding the partition function of the system reduces to compute the sum of some powers of $y$. If we let $P_\Gamma(y) = \sum_{\sigma \in \Theta^{|V|}} y^{\#\{ij \in E : \sigma_i = \sigma_j\}}$, we have the following proposition.

**Proposition 2.5** *Let $\Gamma$ be a finite graph, and $y = e^{\beta J}$ be a parameter. Then $P_\Gamma(y)$ satisfies the following deletion-contraction recurrence:*

$$P_\Gamma(y) = \sum_{\sigma \in \Theta^{|V|}} y^{\#\textit{improper edges}} = \begin{cases} P_{\Gamma \setminus e}(y) + (y-1)P_{\Gamma/e}(y) & e \in E \textit{ is an ordinary edge} \\ yP_{\Gamma \setminus e}(y) & e \in E \textit{ is a loop} \\ (y - 1 + q)P_{\Gamma/e} & e \in E \textit{ is a bridge} \\ q^{|V|} & \Gamma \textit{ has no edges} \end{cases}$$

*Proof.*

- <u>$e \in E$ is an ordinary edge</u>

$$P_\Gamma(y) = \sum_{\sigma \in \Theta^{|V|}} y^{\#\text{improper edges in } E} = \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)=\sigma(b)}} y^{\#\text{improper edges in } E} + \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)\neq\sigma(b)}} y^{\#\text{improper edges in } E} =$$

$$= y \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)=\sigma(b)}} y^{\#\text{improper edges in } E\setminus e} + \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)\neq\sigma(b)}} y^{\#\text{improper edges in } E\setminus e} =$$

$$= (y-1) \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)=\sigma(b)}} y^{\#\text{improper edges in } E\setminus e} + \sum_{\sigma \in \Theta^{|V|}} y^{\#\text{improper edges in } E\setminus e} = P_{\Gamma\setminus e}(y) + (y-1)P_{\Gamma/e}(y)$$

- $e \in E$ is a loop

$$P_\Gamma(y) = \sum_{\sigma \in \Theta^{|V|}} y^{\#\text{improper edges in } E} = \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)=\sigma(b)}} y^{\#\text{improper edges in } E} =$$

$$= y \sum_{\substack{\sigma \in \Theta^{|V|} \\ \sigma(a)=\sigma(b)}} y^{\#\text{improper edges in } E\setminus e} = yP_{\Gamma\setminus e}(y)$$

- $e \in E$ is a bridge

  We can follow the same steps as in where $e$ is an ordinary edge, but $P_{\Gamma\setminus e}(y) = qP_{\Gamma/e}$ due to the fact that for every colouring on $\Gamma/e$, we can build $q$ different colourings on $\Gamma \setminus e$ (the same way as in the chromatic polynomial subsection). And then

$$P_\Gamma(y) = P_{\Gamma\setminus e}(y) + (y-1)P(\Gamma/e) = (y-1+q)P(\Gamma/e)$$

- $\Gamma$ has no edges

  Since $\Gamma$ has no edges, #improper edges is 0 for every $\sigma \in \mathbb{Z}_q^{|V|}$, and then

$$P_\Gamma(y) = \sum_{\sigma \in \mathbb{Z}_q^{|V|}} y^0 = |\mathbb{Z}_q^{|V|}| = q^{|V|}$$

$\square$

Due to the deletion-contraction recurrence, the interesting property of this partition function is the fact that can be written as an evaluation of the Tutte polynomial:

$$\sum_{\sigma \in \Theta^{|V|}} y^{\#\text{improper edges}} = q^{k(\Gamma)}(y-1)^{r(\Gamma)} T_\Gamma\left(\frac{y-1+q}{y-1}, y\right)$$

This can be proved using again Theorem 2.2 by substituting the values $x = y-1+q, y = y, \alpha = y-1, \beta = 1, \gamma = q$ (getting then the deletion-contraction recurrence on Proposition 2.2.)

### 2.2.1 Phase transition on the square lattice $\mathbb{Z}^d$

One piece of most interest in the $d$-dimensional $q$-state Potts model is the phase transition, understood as the point where all particles align and present the same state due to some changes in the environment in which we locate the system (particularly, low temperatures). This phenomenon is related to the singularities in the logarithm of the partition function, so whenever the partition function vanishes, we have a phase transition. The main goal of this subsection will be to prove the existence of a phase transition in this Potts model.

In order to be able to do that, we are going to generalize the previous model. We want spins to interact different only depending on the fact that they are different or equal. For that purpose, we take the simplex $\mathbb{T}_q \subset \mathbb{R}^{q-1}$ containing $\vec{1} = (1, 0, \ldots, 0, 0)$ and having the origin as center of mass (that is, inscribed in $S^{q-2}$), and define a spin state space as $\mathbb{T}_q^{|V|}$. Then, we understand the interaction between two spins $\sigma_x, \sigma_y \in \mathbb{T}_q$ as the scalar product of them in $\mathbb{R}^{q-1}$:

$$\langle \sigma_x | \sigma_y \rangle = \begin{cases} 1 & \text{if } \sigma_x = \sigma_y \\ -\frac{1}{q-1} & \text{otherwise} \end{cases} \tag{4}$$

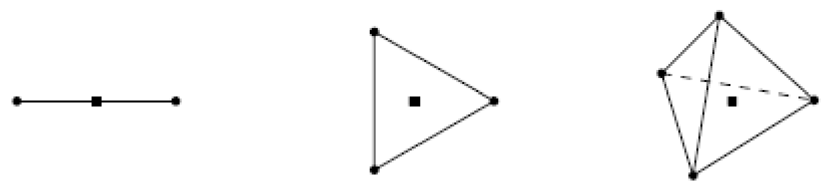


Figure 1: $\mathbb{T}_2, \mathbb{T}_3$ and $\mathbb{T}_4$ (the simplex in $\mathbb{R}, \mathbb{R}^2$ and $\mathbb{R}^3$, respectively), image from [6]

For instance, as we can see in figure (1), the possible states in $\mathbb{T}_2$ are $(1, 0)$ and $(-1, 0)$ and the possible interactions are $\langle (1,0)|(1,0)\rangle = \langle (-1,0)|(-1,0)\rangle = 1$ and $\langle (1,0)|(-1,0)\rangle = \langle (-1,0)|(1,0)\rangle = -1$, so we can see that the simplex satisfies (4). Another example would be $\mathbb{T}_3$, which has the possible states $(1, 0), \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$, so the possible interactions are:

$$\langle (1,0)|(1,0)\rangle = \langle \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) | \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\rangle = \langle \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) | \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\rangle = 1$$

$$\langle (1,0)| \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\rangle = \langle (1,0)| \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\rangle = \langle \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) | \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\rangle = -\frac{1}{2}$$

And, again, (4) is satisfied.

In this new model, the Hamiltonian is defined as:

$$H_\Gamma^{\text{free}}(\sigma) := -\sum_{xy \in E(\Gamma)} J \langle \sigma_x | \sigma_y \rangle$$

The Gibbs measure on $\Gamma$ at inverse temperature $\beta$ with free boundary conditions is defined as

$$\mu^{\text{free}}_{\Gamma,\beta,q}[f] := \frac{\displaystyle\int_{\mathbb{T}_q^{|V|}} f(\sigma) e^{-\beta H^{\text{free}}_\Gamma(\sigma)} d\sigma}{\displaystyle\int_{\mathbb{T}_q^{|V|}} e^{-\beta H^{\text{free}}_\Gamma(\sigma)} d\sigma} = \frac{\sum_{\mathbb{T}_q^{|V|}} f(\sigma) e^{-\beta H^{\text{free}}_\Gamma(\sigma)}}{\sum_{\mathbb{T}_q^{|V|}} e^{-\beta H^{\text{free}}_\Gamma(\sigma)}}$$

for every $f : \mathbb{T}_q^{|V|} \to \mathbb{R}$.

Let us introduce now some boundary conditions that will become important later:

$$H^1_\Gamma(\sigma) := H^{\text{free}}_\Gamma(\sigma) - \sum_{\substack{xy \in E(\mathbb{Z}^d) \\ x \in V(\Gamma) \\ y \notin V(\Gamma)}} J\langle \sigma_x | \vec{1} \rangle$$

This boundary condition makes all the vertices on the outside of the graph $\Gamma$ have spin state $\vec{1}$. Thus, we get a new measure:

$$\mu^{1}_{\Gamma,\beta,q}[f] := \frac{\displaystyle\int_{\mathbb{T}_q^{|V|}} f(\sigma) e^{-\beta H^{1}_\Gamma(\sigma)} d\sigma}{\displaystyle\int_{\mathbb{T}_q^{|V|}} e^{-\beta H^{1}_\Gamma(\sigma)} d\sigma} = \frac{\sum_{\mathbb{T}_q^{|V|}} f(\sigma) e^{-\beta H^{1}_\Gamma(\sigma)}}{\sum_{\mathbb{T}_q^{|V|}} e^{-\beta H^{1}_\Gamma(\sigma)}}$$

Fortuin-Kasteleyn percolation

We now introduce the Fortuin-Kasteleyn percolation in order to try to prove the existence of a phase transition and then couple it to the Potts model to see the existence of a phase transition there.

We take a subgraph $\Gamma \subset \mathbb{Z}^d$ of the $d$-dimensional lattice and a percolation configuration, which is a function $\omega : E(\Gamma) \to \{0, 1\}$ such that, if $\omega(e) = 1$ for an edge $e \in E(\Gamma)$, this edge is said to be open, else it is closed. We can also see this configuration as a subgraph of $\Gamma$ with $V(\omega) = V(\Gamma)$ and $E(\omega) = \{e \in E(\Gamma) : \omega(e) = 1\} \subseteq E(\Gamma)$.

We define a cluster as a maximal connected component of the graph $\omega$ (possibly an isolated vertex), that is to say, every two vertices $x, y$ which are connected are in the same cluster and it is denoted by $x \leftrightarrow y$. This notation can extend to subsets of vertices $A$ and $B$, $A \leftrightarrow B$, meaning that there exist a vertex of $A$ and another vertex of $B$ that are connected. In addition, we will consider one of this subsets to be $\infty$, so that $A \leftrightarrow \infty$ means that $A$ is in an infinite cluster.

Let $\Gamma$ be a finite subgraph of $\mathbb{Z}^d$, we define the boundary of $\Gamma$ as:

$$\partial\Gamma := \{x \in V(\Gamma) : \exists y \notin V(\Gamma), xy \in E(\mathbb{Z}^d)\}$$

Let $o(\omega)$ and $c(\omega)$ denote the open and the closed edges of the percolation configuration $\omega$, respectively. The boundary conditions $\xi$ are given by a partition $P_1 \sqcup P_2 \sqcup \ldots \sqcup P_k$ of the boundary of $\Gamma$, $\partial\Gamma$, and $k_\xi(\omega)$ is the number of clusters of the configuration with $\xi$ as boundary conditions. For every element of the partition $P_i$ we will say that all of the clusters that have a vertex in $P_i$ are the same cluster.

**Definition 2.1** *For every percolation configuration $\omega$, the probability measure $\phi^{\xi}_{\Gamma,p,q}$ of the Fortuin-Kasteleyn percolation on a graph $\Gamma$, with edge-weight $p \in [0,1]$, cluster weight $q > 0$ and boundary conditions $\xi$ is defined as*

$$\phi^{\xi}_{\Gamma,p,q}[\omega] = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_{\xi}(\omega)}}{Z^{\xi}_{\Gamma,p,q}}$$

*where $Z^{\xi}_{\Gamma,p,q}$ is a normalizing constant defined in order to get a sum over all the percolation configurations $\omega$ equal to 1.*

For a set $A$ of percolation configurations, we define its measure as

$$\phi^{\xi}_{\Gamma,p,q}[A] := \sum_{\omega \in A} \phi^{\xi}_{\Gamma,p,q}[\omega] = \sum_{\omega \in A} \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_{\xi}(\omega)}}{Z^{\xi}_{\Gamma,p,q}}.$$

There are two main examples of boundary conditions: "$\xi$ = free" denotes the one where every element of the partition contains just one vertex and therefore every different cluster touching the boundary is not joined to another cluster, and "$\xi$ = wired" denotes the one where there is only one element in the partition (containing all the vertices of the boundary) and therefore every different cluster touching the boundary is joined to everyone.

Now, we have arrived to the crucial point of this subsection: the coupling.

Let $q > 1$ be an integer and $\Gamma$ a finite graph. Given a percolation configuration $\omega : E(\Gamma) \to \{0,1\}$, we can obtain a spin configuration $\sigma : V(\Gamma) \to \mathbb{T}_q$ by assigning uniformly a spin state to every cluster obtained with $\omega$ and assign every vertex in the cluster the same spin state, except for the clusters touching the boundary $\partial\Gamma$, which will get automatically the spin state $\vec{1} \in \mathbb{T}_q$.

**Proposition 2.6** *Let $q \geq 2$ be an integer, $p \in (0,1)$ and $\Gamma \subset \mathbb{Z}^d$ a finite graph. If the percolation configuration $\omega$ is distributed according to $\phi^{wired}_{\Gamma,p,q}$, then the spin configuration $\sigma$ is distributed according to the q-state Potts measure $\mu^{1}_{\Gamma,\beta,q}$ where*

$$\beta = \beta(p,q) = -\frac{q-1}{q}\ln(1-p) \tag{5}$$

*Proof.* Let us consider the space of pairs $[(\omega,\sigma)]$ with $\omega : E(\Gamma) \to \{0,1\}$ and $\sigma : V(\Gamma) \to \mathbb{T}_q$ such that, for any edge $e = xy \in E(\Gamma)$:

$$\omega(e) = 1 \Leftrightarrow \sigma(x) = \sigma(y)$$

Consider now a measure $P$ on this space, where $\omega$ is a percolation configuration with wired boundary conditions and $\sigma$ is the corresponding spin configuration built as explained above. Then, for an element of this space $(\omega,\sigma)$, we have:

$$P(\sigma|\omega) = \frac{p^{o(\omega)}(1-p)^{c(\omega)}q^{k_{\text{wired}}(\omega)}}{Z^{wired}_{\Gamma,p,q}} \frac{1}{q^{k_{\text{wired}}(\omega)-1}} = \frac{q}{Z^{\text{wired}}_{\Gamma,p,q}} p^{o(\omega)}(1-p)^{c(\omega)}$$

because we assign uniformly one spin state to every cluster (except from the one touching the boundary which has spin state assigned $\vec{1}$), and then every spin state of the clusters has probability $q^{-k_{\mathrm{wired}}(\omega)-1}$.

Let us consider a new measure $\tilde{P}$ on the same space $[(\sigma,\omega)]$. Let $\tilde{\sigma}$ be a spin configuration distributed according to $\mu^1_{\Gamma,p,q}$, where $\beta = -\frac{q-1}{q}\ln(1-p)$. And we get the percolation configuration by opening an edge with probability $p$ (and closing it with probability $1-p$) if its endpoints have the same spin state and closing it if not. Then, for any $(\tilde{\omega},\tilde{\sigma})$:

$$\tilde{P}(\tilde{\omega}|\tilde{\sigma}) = \frac{e^{-\frac{q}{q-1}\beta r(\tilde{\sigma})}p^{o(\tilde{\omega})}(1-p)^{c(\tilde{\omega})-r(\tilde{\sigma})}}{Z} = \frac{1}{Z}p^{o(\tilde{\omega})}(1-p)^{c(\tilde{\omega})}$$

where $r(\tilde{\sigma})$ is the number of edges between vertices with different spin states and $Z$ is a normalizing constant. Since $P$ and $\tilde{P}$ are two measures on the same space and have the same distribution, they are the same measure. And the desired result is obtained, because the first marginal of the measure $P$ is $\phi^{\mathrm{wired}}_{\Gamma,p,q}$ and the second marginal is $\mu^1_{\Gamma,\beta,q}$.

□

Now that we have coupled the two models, we are going to prove the existence of the phase transition on the Fortuin-Kasteleyn percolation.

The space of percolation configurations $\{0,1\}^{|E(\Gamma)|}$ is naturally ordered (i.e. take two percolation configurations $w,w'\in\{0,1\}^{|E(\Gamma)|}$, then $w\geq w'$ means that $w(e)\geq w'(e),\forall e\in E(\Gamma)$). Now, we define an *increasing event* as a subset of the space of percolation configurations $A\subseteq\{0,1\}^{|E(\Gamma)|}$ where, for every $w\in A$, if $w'\in\{0,1\}^{|E(\Gamma)|}, w'\geq w$, then $w'\in A$. We are going to see now some properties of measures concerning increasing events.

Let $q\geq 1$, $\Gamma$ a finite graph, $p\leq p'$, $\xi\leq\psi$ (in the sense that every element of the partition $\xi$ is completely inside an element of the partition $\psi$) and $A,B$ be two increasing events, then we have the following set of inequalities:

$$\phi^{\xi}_{\Gamma,p,q}[A\cup B]\geq\phi^{\xi}_{\Gamma,p,q}[A]\phi^{\xi}_{\Gamma,p,q}[A] \text{ (FKG inequality, proof in [7] Theorem 4.14)}$$

$$\phi^{\xi}_{\Gamma,p',q}[A]\geq\phi^{\xi}_{\Gamma,p,q}[A] \text{ (monotonicity in p)}$$

$$\phi^{\psi}_{\Gamma,p,q}[A]\geq\phi^{\xi}_{\Gamma,p,q}[A] \text{ (comparison of boundary conditions, proof in [7] Corollary 4.19)}$$

$$\phi^{\mathrm{free}}_{\Gamma,p,q}[A]\leq\phi^{\xi}_{\Gamma,p,q}[A]\leq\phi^{\mathrm{wired}}_{\Gamma,p,q}[A] \tag{6}$$

Equalities on (6) will be achieved for every value of $p\in[0,1]$ except for a countable set because of the differentiability of the free energy defined, for $q\geq 1$ and $p\in[0,1]$, as :

$$f(p,q) = \lim_{n\to\infty} \frac{1}{|E_{\Lambda_n}} \log[Z^{\xi}_{\Lambda_n,p,q}]$$

where $(\Lambda_n)$ is a family of graphs such that $\lim_{n\to\infty} \Lambda_n = \mathbb{Z}^d$ (see [7] Theorem 4.30 for more details).

And one last proposition that will be useful for the final theorem:

**Proposition 2.7** *Let $p \in [0,1]$ and $q > 0$. Then the measures*

$$\phi^{free}_{p,q} = \lim_{\Gamma \nearrow \mathbb{Z}^d} \phi^{free}_{\Gamma,p,q}$$

$$\phi^{wired}_{p,q} = \lim_{\Gamma \nearrow \mathbb{Z}^d} \phi^{wired}_{\Gamma,p,q}$$

*are ergodic (i.e. any increasing event that is invariant under translation has probability either 0 or 1).*

*Proof.* See [7] Theorem 4.28.

**Theorem 2.3** *For $q, d \geq 1$, there exists a critical probability $p_c = p_c(q,d) \in [0,1]$ such that:*

- *For $p < p_c$, any infinite-volume measure has no infinite cluster almost surely.*
- *For $p > p_c$, any infinite-volume measure has an infinite cluster almost surely. Moreover, every vertex in the lattice will belong to this cluster almost surely.*

*Proof.* Let us define

$$p_c := \inf\{p \in [0,1] : \phi^{\text{free}}_{p,q}[0 \leftrightarrow \infty] > 0\} = \sup\{p \in [0,1] : \phi^{\text{free}}_{p,q}[0 \leftrightarrow \infty] = 0\}$$

where the second equality comes from the monotonicity in $p$ and the fact that $[0 \leftrightarrow \infty]$ is an increasing event.

- $\underline{p > p_c}$

  The $\phi^{\text{free}}_{p,q}$-probability is positive, and then, by the ergodicity of $\phi^{\text{free}}_{p,q}$ and the fact that $[0 \leftrightarrow \infty]$ is an increasing event, $\phi^{\text{free}}_{p,q}[0 \leftrightarrow \infty] = 1$. And (6) assures us that $\phi^{\xi}_{p,q}[0 \leftrightarrow \infty] = 1$ for any boundary condition $\xi$.

- $\underline{p < p_c}$

  Choose $p' \in (p, p_c)$ such that the infinite-volume measure is unique (Theorem 4.30 in [7]), which is possible due to the fact that there are only countably many values of $p$ for which the measure is not unique. Then, taking into account the definition of $p_c$ and that $p' < p_c$:

$$\phi^{\text{wired}}_{p',q}[0 \leftrightarrow \infty] = \phi^{\text{free}}_{p',q}[0 \leftrightarrow \infty] = 0$$

Hence, due to the monotonicity of $p$:

$$\phi_{p,q}^{\text{wired}}[0 \leftrightarrow \infty] \leq \phi_{p',q}^{\text{wired}}[0 \leftrightarrow \infty] = 0 \Rightarrow \phi_{p,q}^{\text{wired}}[0 \leftrightarrow \infty] = 0$$

This result and (6) assure us that $\phi_{p,q}^{\xi}[0 \leftrightarrow \infty] = 0$ for any boundary condition $\xi$.

$\square$

This last result translated to the Potts model with the coupling condition (5) means that there exists a $\beta_c$ (and, therefore, a temperature $T_c$) such that if $\beta > \beta_c$ (so $T < T_c$), then almost all of the vertices have the same spin state; and if $\beta < \beta_c$ (so $T > T_c$), then there is more than one significant spin state all over the vertices.

# 3 Signed graphs

A signed graph is a graph in which all edges have a sign. More formally, a signed graph $\Sigma = (\Gamma, \tau)$ is a graph $\Gamma = (V, E)$ with a function $\tau : E \to \{\pm 1\}$, called a signature. $E^+$ is the set of edges with image $+1$ by $\tau$, which are called positive edges; and $E^-$ is the set of edges with image -1: negative edges.

For a signed graph, a colouring is defined as a map $\sigma : V \to G$, where $G$ is an Abelian group. In a colouring, a positive edge $e = uv \in E$ is said to be improper if $\sigma(u) = \sigma(v)$, otherwise it is said to be proper; for negative edges we will say that they are improper whenever $\sigma(u) = -\sigma(v)$, otherwise they are proper. If all edges in a colouring of a signed graph are proper, the colouring is considered a proper colouring; if there exists an improper edge, however, we have an improper colouring.

The set of colours (or states) changes from that for unsigned graphs because, we only distinguished if the colour of two joined vertices is the same or different. However, for signed graphs we will also check if these colours are the opposite (particularly in negative edges), that is the reason why Zaslavsky [2] introduced the set $\{0, \pm 1, \pm 2, \dots, \pm n\}$, but in fact, an Abelian group also works. In this thesis we will use $\mathbb{Z}_{2n+1} = \mathbb{Z}/(2n+1)\mathbb{Z}$ instead, and, in some cases, extend it to any Abelian group.

## 3.1 Vertex switching

Two different orientations $\sigma_1, \sigma_2$ of the same unsigned graph $\Gamma$ are equivalent if they are identical except for a edge-cut set of $\Gamma$. Another definition of equivalency is that one can get from $\sigma_1$ to $\sigma_2$ by vertex switching. Vertex switching consists in taking a vertex $v$ of the signed graph and switching the sign of all the edges with an endpoint on $v$; as a remark, if we apply this vertex switching to a vertex with a loop, this loop will keep its sign. For a graph with coloured vertices, this vertex switching also implies switching the colour of the chosen vertex into its inverse in the group of colours, in order to preserve the improper or proper condition of the edges.

We want to find an object that is the same for graphs with two equivalent orientations because in some cases they will represent almost the same system; i.e. we want to find invariants under vertex switching.

## 3.2 Reidemeister moves

Let $X$ and $Y$ be Hausdorff spaces, and $f$ be a mapping $f : X \to Y$, we call $f$ an embedding if $f : X \to f(X)$ is a homeomorphism. Then, a knot is an embedding $f$ of $S^1$ into $S^3$. That is to say, a knot is a subset of $\mathbb{R}^3$ which is homeomorphic to a circle. A link with $k$ components is a subset of $\mathbb{R}^3$ that is homeomorphic to the disjoint union of $k$ circles.

We say that two knots $K, L$ are ambient isotopic if there exists a homotopy $h_t : S^3 \to S^3, 0 \leq t \leq 1$, such that $h_0 \equiv id$, $h_t$ is a homeomorphism for every $t$ and $h_1(K) = L$.

A knot is usually considered as its projection on the plane. We say that a projection is regular if it contains only finitely many multiple points (points with crossings) and all multiple points are double points, which will represent the crossings. Finally, a diagram is a projection in which every crossing point has an over/under crossing specification (i.e. which part of the knot goes over and which under).

Given a diagram of a link, we define the Tait colouring as a colouring of its faces into black and white, such that every face touching the exterior one is painted black and two touching faces have different colours. From a Tait colouring, we can define the Tait graph, the signed graph in which every vertex represents a black face and the crossings become positive or negative edges according to the rule in figure (2).

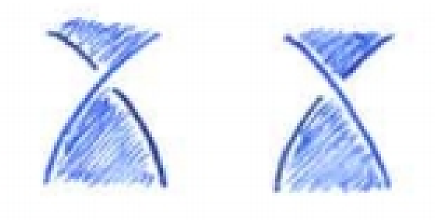


Figure 2: Left: positive edge. Right: negative edge

The Reidemeister moves are three local moves on a link diagram that, according to the Reidemeister theorem, can always relate two knot diagrams belonging to the same knot. They are usually defined over link diagrams, but we can also see them as transformations in the Tait graph. In the following figure we will see the comparison between the three Reidemeister moves in the knot diagrams and in the respective Tait graphs.

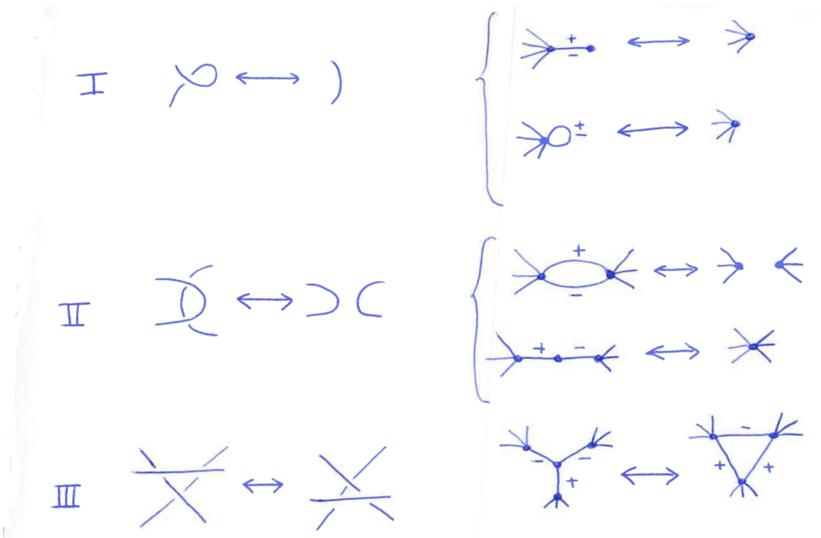



Figure 3: Left side: Knot diagram. Right side: Tait graph

The two different equivalences for types I and II are due to the two different possibilities in which we can find these phenomena. For instance, for the type I Reidemeister move, we do not get the same Tait graph if we colour the inside of the loop black (which would give the first case) or white (the second). This does not happen with the type III Reidemeister move, also called *star-triangle* or $\Delta - Y$, because it is symmetrical.

The desire to find invariants in knots brings us to find an object that is invariant between knot diagrams representing the same knot. That is, to find an invariant under the Reidemeister moves.

# 4 Vertex switching invariant

In this section we will focus on parameters of signed graphs that are invariant under vertex switching.

## 4.1 The trivariate Tutte polynomial

We represent cycles on a signed graph as a series of vertices and edges (alternating) $(v_1, e_1, v_2, \ldots, v_k, e_k, v_1)$. Then if we have a cycle such that $\prod_{i=1}^{k} \tau(e_i) = 1$, we say that this cycle is balanced (unbalanced otherwise). A signed graph $\Sigma$ is balanced if every cycle on $\Sigma$ is balanced (unbalanced otherwise). A vertex switching will always change the sign of an even number of edges in a cycle (in fact, if the vertex is in the cycle, the vertex switching will change the sign of two edges, and if it is outside the cycle, will change none), so then a vertex switching does not change the balanced or unbalanced condition of a cycle.

The number of connected components of $\Sigma = (\Gamma, \tau)$ is the number of edges of its underlying graph $\Gamma$, $k(\Sigma) = k(\Gamma)$; and we denote the number of balanced and unbalanced connected component as $k_b(\Sigma)$ and $k_u(\Sigma)$, respectively. Note that $k(\Sigma) = k_b(\Sigma) + k_u(\Sigma)$.

To a graph $\Gamma = (V, E)$ there corresponds a *matroid* $M(\Gamma)$ on ground set $E$ with rank function defined for $A \subseteq E$ by $r_M(A) = |V| - k(\Sigma \setminus A^c)$ (which is the definition we already had of the rank of a grpah). To a signed graph $\Sigma = (\Gamma = (V, E), \tau)$, however, is associated its *signed-graphic matroid* $F(\Sigma)$ on ground set $E$ with rank function defined for $A \subseteq E$ by $r_F(A) = |V| - k(\Sigma \setminus A^c)$

Deletion $(\Sigma \setminus e)$ and contraction $(\Sigma/e)$ of an edge in $\Sigma = (\Gamma, \tau)$ correspond to the deletion or contraction of the edge in $\Gamma$ and then take a signature $\tau' = \tau|_{E \setminus e}$. We are also going to define deletion and contraction of a subset of edges $A$: $\Sigma \setminus A$ denotes the deletion of this subset of edges, which is to delete all of them; and $\Sigma/A$ denotes the contraction of this subset of edges, that will only be meaningful when every edge in this subset is positive and represents the contraction of every edge in $A$.

With all this new notation, as we can find in [5], we can define a Tutte polynomial for signed graphs that corresponds in many ways to the Tutte polynomial of a graph.

**Definition 4.1** *Let $\Sigma = (\Gamma, \tau)$ be a signed graph with underlying graph $\Gamma = (V, E)$. Then the signed Tutte polynomial is defined by*

$$T_\Sigma(X, Y, Z) := \sum_{F \subseteq E} (X-1)^{k(\Sigma \setminus F^c)} (Y-1)^{|F|-|V|+k_b(\Sigma \setminus F^c)} (Z-1)^{k_u(\Sigma \setminus F^c)}$$

**Remark 4.1** *The trivariate Tutte polynomial is invariant under vertex switching because every balanced (unbalanced) cycle will be balanced (unbalanced) after a vertex switching.*

A property of this trivariate Tutte polynomial is that a certain specialization gives us the Tutte polynomial for unsigned graphs, hence we can see it as a generalisation of the Tutte polynomial.

Since the Tutte polynomial for unsigned graphs is multiplicative on one-point joints (Lemma 2.1.), one might presume that the trivariate Tutte polynomial will also satisfy this property. However, here is an example of this property not working.

Take a graph $\Sigma$ consisting in a vertex with two negative loops, trivially its trivariate Tutte polynomial is:

$$T_\Sigma(X,Y,Z) = (X-1)(Y-1)(Z-1) + 2(X-1)(Z-1) + (X-1) = (X-1)(YZ+Z-Y)$$

And now consider $\Sigma'$ two vertices, each one with a negative loop, then:

$$T_{\Sigma'}(X,Y,Z) = (X-1)^2(Z-1)^2 + 2(X-1)^2(Z-1) + (X-1)^2 = (X-1)^2 Z^2$$

Nevertheless, there is a recipe theorem for the trivariate Tutte polynomial. The following is a slight extension of the version given in [5]:

**Theorem 4.1** *"Recipe theorem". Let $R$ be an invariant of signed graphs invariant under vertex switching and multiplicative over disjoint unions. suppose that there are constants $\alpha, \beta, \gamma, x, y, z$, with $\gamma \neq 0$, such that, for a signed graph $\Sigma = (\Gamma = (V,E), \tau)$ and positive edge $e \in E$,*

$$R_\Sigma = \begin{cases} \alpha R_{\Sigma/e} + \beta R_{\Sigma\setminus e} & \text{if } e \text{ is ordinary in } \Gamma \text{ and in } \Sigma \\ \alpha R_{\Sigma/e} + \gamma R_{\Sigma\setminus e} & \text{if } e \text{ is ordinary in } \Gamma \text{ and } k_u(\Sigma\setminus e) < k_u(\Sigma) \\ \alpha R_{\Sigma/e} + \frac{\beta(x-\alpha)}{\gamma} R_{\Sigma\setminus e} & \text{if } e \text{ is a bridge in } \Gamma \text{ and a circuit path edge in } \Sigma \\ x R_{\Sigma/e} & \text{if } e \text{ is a bridge in } \Gamma \text{ that is not a circuit path edge in } \Sigma \\ y R_{\Sigma\setminus e} & \text{if } e \text{ is a loop in } \Gamma \text{ and in } \Sigma \end{cases}$$

*while if $\Sigma$ is a bouquet of $l \geq 1$ negative loops then*

$$R_\Sigma = \beta^{l-1}\gamma + (z-\gamma)\sum_{i=0}^{l-1} y^{l-1-i}\beta^i$$

*an $R_\Sigma = 1$ when $\Sigma$ is a single vertex with no edges.*

*Then,*

$$R_\Sigma = \alpha^{r_M(E)}\beta^{|E|-r_F(E)}\gamma^{r_F(E)-r_M(E)} T_\Sigma\left(\frac{x}{\alpha}, \frac{y}{\beta}, \frac{z}{\gamma}\right)$$

*a polynomial in $x, y, z, \alpha$ and $\beta$ with coefficients in $\mathbb{Z}[\gamma, \gamma^{-1}]$.*

And we can extend this recipe theorem to the following.

**Theorem 4.2** *"Extended Recipe Theorem" Let $U$ be an invariant of signed graphs invariant under vertex switching and multiplicative over disjoint unions. Suppose that there are constants $\alpha, \beta, \gamma, \delta, x, y$*

*and $z$, with $\gamma, \delta \neq 0$, such that, for a signed graph $\Sigma = (\Gamma, \sigma)$ with underlying graph $\Gamma = (V, E)$ and positive edge $e \in E$,*

$$U_\Sigma = \begin{cases} \alpha\, U_{\Sigma/e} + \beta U_{\Sigma\backslash e} & \textit{if } e \textit{ is ordinary in } \Gamma \textit{ and in } \Sigma, \\ \alpha\, U_{\Sigma/e} + \gamma U_{\Sigma\backslash e} & \textit{if } e \textit{ is ordinary in } \Gamma \textit{ and } k_u(\Sigma\backslash e) < k_u(\Sigma), \\ \alpha U_{\Sigma/e} + \frac{\beta(x-\alpha)}{\delta\gamma} U_{\Sigma\backslash e} & \textit{if } e \textit{ is a bridge in } \Gamma \textit{ and a circuit path edge in } \Sigma, \\ x U_{\Sigma/e} & \textit{if } e \textit{ is a bridge in } \Gamma \textit{ that is not a circuit path edge in } \Sigma, \\ y\, U_{\Sigma\backslash e} & \textit{if } e \textit{ is a loop in } \Gamma \textit{ and in } \Sigma, \end{cases}$$

*while if $\Sigma$ is a bouquet of $\ell \geq 1$ negative loops then*

$$U_\Sigma = \beta^{\ell-1}\gamma\delta + (z-\gamma)\delta \sum_{i=0}^{\ell-1} y^{\ell-1-i}\beta^i,$$

*and $U_\Sigma = \delta$ when $\Sigma$ is a single vertex with no edges.*

*Then,*

$$U_\Sigma = \alpha^{r_M(E)}\beta^{|E|-r_F(E)}\gamma^{r_F(E)-r_M(E)}\delta^{k(\Sigma)} T_\Sigma\left(\frac{x}{\alpha}, \frac{y}{\beta}, \frac{z}{\gamma}\right), \tag{7}$$

*a polynomial in $\alpha, \beta, \gamma, \delta, x, y$ and $z$ over $\mathbb{Z}$.*

*If $\alpha = 0$ or $\beta = 0$ then we use the subset expansion of the right-hand side of (7):*

$$U_\Sigma = \delta^{k(\Sigma)} \sum_{A \subseteq E} \alpha^{r_M(A)} \beta^{|E|-|A|+r_F(A)-r_F(E)} \gamma^{r_F(E)-r_F(A)-[r_M(E)-r_M(A)]} \cdot \\ \cdot (x-\alpha)^{r_M(E)-r_M(A)} (y-\beta)^{|A|-r_F(A)} (z-\gamma)^{r_F(A)-r_M(A)}. \tag{8}$$

*Proof.* By multiplying the equations in the recurrence for $U_\Sigma$ by $\delta^{-k(\Sigma)}$ we see that the invariant $R_\Sigma := \delta^{-k(\Sigma)} U_\Sigma$ satisfies

$$R_\Sigma = \begin{cases} \alpha\, R_{\Sigma/e} + \beta R_{\Sigma\backslash e} & \text{if } e \text{ is ordinary in } \Gamma \text{ and in } \Sigma, \\ \alpha\, R_{\Sigma/e} + \gamma R_{\Sigma\backslash e} & \text{if } e \text{ is ordinary in } \Gamma \text{ and } k_u(\Sigma\backslash e) < k_u(\Sigma), \\ \alpha R_{\Sigma/e} + \frac{\beta(x-\alpha)}{\gamma} R_{\Sigma\backslash e} & \text{if } e \text{ is a bridge in } \Gamma \text{ and a circuit path edge in } \Sigma, \\ x R_{\Sigma/e} & \text{if } e \text{ is a bridge in } \Gamma \text{ that is not a circuit path edge in } \Sigma, \\ y\, R_{\Sigma\backslash e} & \text{if } e \text{ is a loop in } \Gamma \text{ and in } \Sigma, \end{cases}$$

while if $\Sigma$ is a bouquet of $\ell \geq 1$ negative loops then

$$R_\Sigma = \beta^{\ell-1}\gamma + (z-\gamma) \sum_{i=0}^{\ell-1} y^{\ell-1-i}\beta^i,$$

and $R_\Sigma = 1$ when $\Sigma$ is a single vertex with no edges. By the Recipe Theorem with parameters $(x, y, z, \alpha, \beta, \gamma)$, the result follows.

## 4.2 Chromatic polynomial

As we have the definition of a proper colouring for signed graphs, we may raise the same question as in section 2 and try to count how many proper colourings for a certain graph $\Sigma = (\Gamma = (V, E), \tau)$ on a certain Abelian group $G$. Trivially, the chromatic polynomial is multiplicative on disjoint graphs, because the number of proper colourings of a graph is all the possible combinations of proper colourings of its connected components.

If we have a proper colouring and do a vertex switching, we will have a proper colouring of the new graph obtained; thus, the chromatic polynomial is invariant under vertex switching. Then we can define the following recurrence for a positive edge $e \in E^+$:

$$\mathcal{X}_\Sigma(G) = \begin{cases} \mathcal{X}_{\Sigma\setminus e}(G) - \mathcal{X}_{\Sigma/e}(G) & \text{if } e \text{ is an ordinary edge} \\ (|G|-1)\mathcal{X}_{\Sigma/e}(G) & \text{if } e \text{ is a bridge} \\ 0 & \text{if } e \text{ is a loop} \\ |G|^{|V|} & \text{if } \Sigma \text{ has no edges} \end{cases} \tag{9}$$

And another boundary condition we need is that one for a bouquet of negative loops, the number of colourings is $|G| - \frac{|G|}{|2G|}$, since only if we colour the vertex with a self-inverse of the group then the edges will be improper. We are defining $2G = \{2x : x \in G\}$, so then, by the First Isomorphism Theorem for groups, the homomorphism $x \longmapsto 2x$ from $G$ to $G$ has kernel the subgroup of self-inverse elements, and is isomorphic to $\frac{G}{2G}$.

**Theorem 4.3** *The chromatic polynomial can be expressed as a specialization of the trivariate Tutte polynomial:*

$$\mathcal{X}_\Sigma(G) = (-1)^{r(\Sigma)}|G|^{k(\Sigma)}T_\Sigma\left(1-|G|, 0, 1-\frac{1}{|2G|}\right)$$

*Proof.* Since the condition of $\Sigma$ having no edges is not that one in the recipe theorem (which is $\mathcal{X}_\Sigma = 1$), we are going to construct an auxiliary polynomial as $X_\Sigma(G) = |G|^{-k(\Sigma)}\mathcal{X}_\Sigma(G)$. Then, when $\Sigma$ has no edges, we obtain that $X_\Sigma(G) = |G|^{-k(\Sigma)}\mathcal{X}_\Sigma(G) = |G|^{-k(\Sigma)}|G|^{|V|}$, which is 1 because every vertex is a connected component. Using (9), we can find its deletion-contraction recurrence:

$$X_\Sigma(G) = \begin{cases} X_{\Sigma\setminus e}(G) - X_{\Sigma/e}(G) & \text{if } e \text{ is an ordinary edge} \\ (|G|-1)X_{\Sigma/e}(G) & \text{if } e \text{ is a bridge} \\ 0 & \text{if } e \text{ is a loop} \\ 1 & \text{if } \Sigma \text{ has no edges} \end{cases}$$

Using Theorem 4.1., we have $y = 0$, $\alpha = -1$, $\beta = 1$, $\gamma = 1$ and $x = |G| - 1$. A bouquet of $l$ negative loops gives us the following value

$$X_\Sigma(G) = |G|^{-k(\Sigma)}\mathcal{X}_\Sigma(G) = |G|^{-1}\left(|G| - \frac{|G|}{|2G|}\right) = 1 - \frac{1}{|2G|},$$

that should be equal to the value we get from the recipe theorem:

$$X_\Sigma(G) = \beta^{l-1}\gamma + (z-\gamma)\sum_{i=0}^{l-1} y^{l-1-i}\beta^i = 1 + (z-1) = z = 1 - \frac{1}{|2G|}.$$

So then, using the recipe Theorem 4.1.(4.1) with parameters $x = |G| - 1$, $y = 0$, $z = 1 - \frac{1}{|2G|}$, $\alpha = -1$, $\beta = 1$ and $\gamma = 1$, we get

$$|G|^{-k(\Sigma)}\mathcal{X}_\Sigma(G) = X_\Sigma(G) = (-1)^{r(\Sigma)} T_\Sigma\left(1 - |G|, 0, 1 - \frac{1}{|2G|}\right),$$

and the final result

$$\mathcal{X}_\Sigma(G) = (-1)^{r(\Sigma)}|G|^{k(\Sigma)} T_\Sigma\left(1 - |G|, 0, 1 - \frac{1}{|2G|}\right).$$

□

## 4.3 Signed Potts model

In the last section we defined the Potts model for an unsigned graph by summing over all possible states a certain parameter $y$ rise to the number of improper edges in the graph on every state. Now we define the signed Potts model the same way, but taking in count the definition of improper edge for signed graphs:

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = \sum_{\sigma:V\to G} y^{\#\{uv\in E^+:\sigma(u)=\sigma(v)\}+\#\{uv\in E^-:\sigma(u)=-\sigma(v)\}} \tag{10}$$

We can rewrite this polynomial as follows:

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = \sum_{\sigma:V\to G}\prod_{e=uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{e=uv\in E^-} w_-(\sigma(u),\sigma(v))$$

where $w_+, w_- : G\times G \to \mathbb{R}$ are the weight functions for positive and negative edges respectively and are defined as follows for $\alpha, \beta \in G$:

$$w_+(\alpha,\beta) = \begin{cases} y & \text{if } \alpha = \beta \\ 1 & \text{else} \end{cases}$$

$$w_-(\alpha,\beta) = \begin{cases} y & \text{if } \alpha = -\beta \\ 1 & \text{else} \end{cases}$$

The following step is to find, if it is possible, a deletion-contraction recurrence that this polynomial satisfies, such as the one we found for the unsigned Potts model. By analogy, one can conclude that the recurrence should be:

$$Z_\Sigma\left(|G|,\frac{|G|}{|2G|};y\right)=\begin{cases} Z_{\Sigma\setminus e}\left(|G|,\frac{|G|}{|2G|};y\right)+(y-1)Z_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right) & \text{if } e\in E \text{ not a loop}\\ yZ_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right) & \text{if } e\in E^+ \text{ loop}\\ \left(\frac{|G|}{|2G|}y^l+(|G|-\frac{|G|}{|2G|})\right) & \text{if } |V|=1 \text{ \& no positive loops}\end{cases} \tag{11}$$

In this context, $l$ is the number of (negative) loops with both ends in the only vertex in the given graph.

Moreover, the boundary condition is completed with the fact that this polynomial is multiplicative, i.e. given two disjoint graphs $\Sigma_1=(V_1,E_1,\tau_1)$, $\Sigma_2=(V_2,E_2,\tau_2)$ and a group $G$:

$$Z_{\Sigma_1\cup\Sigma_2}\left(|G|,\frac{|G|}{|2G|};y\right)=\sum_{\sigma:V_1\cup V_2\to G}\prod_{uv\in E_1^+\cup E_2^+}w_+(\sigma(u),\sigma(v))\prod_{uv\in E_1^-\cup E_2^-}w_-(\sigma(u),\sigma(v))\ =$$

$$=\sum_{\substack{\sigma_1:V_1\to G\\ \sigma_2:V_2\to G}}\prod_{uv\in E_1^+}w_+(\sigma(u),\sigma(v))\prod_{uv\in E_1^-}w_-(\sigma(u),\sigma(v))\prod_{uv\in E_2^+}w_+(\sigma(u),\sigma(v))\prod_{uvE_2^-}w_-(\sigma(u),\sigma(v))\ =$$

$$=\left(\sum_{\sigma_1:V_1\to G}\prod_{uv\in E_1^+}w_+(\sigma(u),\sigma(v))\prod_{uv\in E_1^-}w_-(\sigma(u),\sigma(v))\right)\left(\sum_{\sigma_2:V_2\Rightarrow G}\prod_{uv\in E_2^+}w_+(\sigma(u),\sigma(v))\prod_{uvE_2^-}w_-(\sigma(u),\sigma(v))\right)=$$

$$=Z_{\Sigma_1}\left(|G|,\frac{|G|}{|2G|};y\right)Z_{\Sigma_2}\left(|G|,\frac{|G|}{|2G|};y\right)$$

Let's check that the signed Potts model polynomial satisfies this deletion-contraction recurrence. However, before that, there are some things to take into account. Consider a signed graph $\Sigma=(V,E,\tau)$ and an edge $e\in E^+$.

On one hand, for the graph obtained by contracting $e=ab$ we get the following property (we call $a$ the vertex obtained by contracting $e$):

$$Z_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right)=\sum_{\sigma:V\setminus\{b\}\to G}\prod_{uv\in E^+\setminus e}w_+(\sigma(u),\sigma(v))\prod_{uv\in E^-}w_-(\sigma(u),\sigma(v))\ =$$

$$=\sum_{\substack{\sigma:V\to G\\ \sigma(a)=\sigma(b)}}\prod_{uv\in E^+\setminus e}w_+(\sigma(u),\sigma(v))\prod_{uv\in E^-}w_-(\sigma(u),\sigma(v))$$

This equality holds because the weight functions do not depend on the vertices we are focusing, only the colour we give them, so if we only split the vertex and keep the colour, both weight functions will give the same value for the original neighbours of $a$ in $\Sigma\setminus e$, no matter what vertices we attach to $a$ or $b$ after the splitting.

On the other hand, for the graph obtained by deleting $e=ab$ we get this other property:

$$Z_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; y\right) = \sum_{\sigma:V\to G}\prod_{uv\in E^+\setminus e} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

This happens because deleting the edge, we are cancelling the interaction between $a$ and $b$, thus we omit this term of the product.

For the negative case ($e \in E^-$) it is analogous.

Having seen this, we may now proceed checking the recurrence:

- <u>$e = ab \in E^+$ not a loop</u>

$$Z_{\Sigma}\left(|G|, \frac{|G|}{|2G|}; y\right) = \sum_{\sigma:V\to G}\prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= \sum_{\substack{\sigma:V\to G\\ \sigma(a)=\sigma(b)}}\prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) +$$

$$+ \sum_{\substack{\sigma:V\to G\\ \sigma(a)\neq\sigma(b)}}\prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= y\sum_{\substack{\sigma:V\to G\\ \sigma(a)=\sigma(b)}}\prod_{uv\in E^+\setminus e} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) +$$

$$+ \sum_{\substack{\sigma:V\to G\\ \sigma(a)\neq\sigma(b)}}\prod_{uv\in E^+\setminus e} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= yZ_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; y\right) + \sum_{\sigma:V\to G}\prod_{uv\in E^+\setminus e} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) -$$

$$- \sum_{\substack{\sigma:V\to G\\ \sigma(a)=\sigma(b)}}\prod_{uv\in E^+\setminus e} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= yZ_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; y\right) + Z_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; y\right) - Z_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; y\right) =$$

$$Z_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; y\right) + (y-1)Z_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; y\right)$$

- <u>$e = ab \in E^-$ not a loop</u>

$$Z_{\Sigma}\left(|G|, \frac{|G|}{|2G|}; y\right) = \sum_{\sigma:V\to G}\prod_{e=uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{e=uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= \sum_{\substack{\sigma:V\to G\\ \sigma(a)=-\sigma(b)}} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) +$$

$$+ \sum_{\substack{\sigma:V\to G\\ \sigma(a)\neq-\sigma(b)}} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= y \sum_{\substack{\sigma:V\to G\\ \sigma(a)=-\sigma(b)}} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-\setminus e} w_-(\sigma(u),\sigma(v)) +$$

$$+ \sum_{\substack{\sigma:V\to G\\ \sigma(a)\neq-\sigma(b)}} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-\setminus e} w_-(\sigma(u),\sigma(v)) =$$

$$= yZ_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right) + \sum_{\sigma:V\to G} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-\setminus e} w_-(\sigma(u),\sigma(v)) -$$

$$- \sum_{\substack{\sigma:V\to G\\ \sigma(a)=-\sigma(b)}} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-\setminus e} w_-(\sigma(u),\sigma(v)) =$$

$$= yZ_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right) + Z_{\Sigma\setminus e}\left(|G|,\frac{|G|}{|2G|};y\right) - Z_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right) =$$

$$Z_{\Sigma\setminus e}\left(|G|,\frac{|G|}{|2G|};y\right) + (y-1)Z_{\Sigma/e}\left(|G|,\frac{|G|}{|2G|};y\right)$$

- $e = ab \in E^+$ loop

$$Z_\Sigma\left(|G|,\frac{|G|}{|2G|};y\right) = \sum_{\sigma:V\to G} \prod_{uv\in E^+} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) =$$

$$= y \sum_{\sigma:V\to G} \prod_{uv\in E^+\setminus e} w_+(\sigma(u),\sigma(v)) \prod_{uv\in E^-} w_-(\sigma(u),\sigma(v)) = yZ_{\Sigma\setminus e}\left(|G|,\frac{|G|}{|2G|};y\right)$$

- $|V| = 1$ & no positive loops and $l$ negative loops

  Looking at the conditions for the values of the negative weight function, we can conclude that every negative loop will take value $y$ when the vertex is coloured with a colour $\alpha \in G$ such that $\alpha = -\alpha$, which is to say that there are exactly $\frac{|G|}{|2G|}$ colours that will make the negative loops contribute $y$; hence:

$$Z_\Sigma\left(|G|,\frac{|G|}{|2G|};y\right) = \sum_{\alpha\in G} \prod_{\text{loops in } E^-} w_-(\alpha,\alpha) = \frac{|G|}{|2G|}y^l + \left(|G| - \frac{|G|}{|2G|}\right)1^l = \frac{|G|}{|2G|}y^l + |G| - \frac{|G|}{|2G|}$$

As a remark, we can see that if the vertex we have is an isolated vertex, then $l = 0$, and the signed Potts model polynomial will give

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = \frac{|G|}{|2G|}y^0 + |G| - \frac{|G|}{|2G|} = \frac{|G|}{|2G|} + |G| - \frac{|G|}{|2G|} = |G|$$

as we should expect, since we could paint the vertex with any colour and there wouldn't be any improper edge.

**Theorem 4.4** *The partition function of the signed Potts model is a specialization of the trivariate Tutte polynomial:*

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = (y-1)^{r(\Sigma)}|G|^{k(\Sigma)}T_\Sigma\left(\frac{|G|+y-1}{y-1}, y, \frac{1}{|2G|}(y-1)+1\right)$$

*Proof.* As we did above with the chromatic polynomial, we are going to use an auxiliary polynomial $A_\Sigma = |G|^{-k(\Sigma)}Z_\Sigma$ because $Z_\Sigma$ does not satisfy the boundary conditions $Z_\Sigma = 1$ when $\Sigma$ has no edges. Since the partition function of the Potts model is vertex switching invariant, we can assume that the edges (that are not loops) are always positive, for a positive edge $e \in E^+$ (following (11)):

$$A_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = \begin{cases} A_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; y\right) + (y-1)A_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; y\right) & \text{if } e \text{ is an ordinary edge} \\ |G|A_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; y\right) + (y-1)A_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; y\right) & \text{if } e \text{ is a bridge} \\ yA_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}\right) & \text{if } e \text{ is a loop} \\ 1 & \text{if } \Sigma \text{ has no edges} \end{cases}$$

where the term $|G|$ comes due to the appearance of a new connected component (deleting a bridge). From this, we can say that $y = y$, $\alpha = y - 1$, $\beta = 1$, $\gamma = 1$ and $x = |G| + y - 1$,.

And we have the boundary condition that for a bouquet of $l$ negative loops, the partition function becomes:

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = \frac{|G|}{|2G|}y^l + \left(|G| - \frac{|G|}{|2G|}\right)$$

Comparing it to the value of the invariant defined by the recipe theorem we get:

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = \beta^{l-1}\gamma + (z-\gamma)\sum_{i=0}^{l-1} y^{l-1-i}\beta^i = 1 + (z-1)\sum_{i=0}^{l-1} y^{l-1-i} = \frac{|G|}{|2G|}y^l + \left(|G| - \frac{|G|}{|2G|}\right)$$

The value $z = \frac{1}{|2G|}(y-1)+1$, satisfies the last equality.

Last, if we use Theorem 4.1. with values $x = |G| + y - 1$, $y = y$, $z = \frac{1}{|2G|}(y-1)+1$, $\alpha = y-1$, $\beta = 1$ and $\gamma = 1$, we get the desired result:

$$Z_\Sigma\left(|G|, \frac{|G|}{|2G|}; y\right) = (y-1)^{r(\Sigma)}|G|^{k(\Sigma)}T_\Sigma\left(\frac{|G|+y-1}{y-1}, y, \frac{1}{|2G|}(y-1)+1\right)$$

□

The physical interpretation of these systems in this case is still a bit lacking, an that is the reason why there is not many literature about it yet. However, we could think about this model in a 2-dimensional lattice, where the vertical edges are positive and the horizontal ones are negative, due to some magnetic field acting on all the system or some angular dependence on the interaction energy. In such a system, a probability distribution similar to that of the Fortuin-Kasteleyn model can be defined by the same partition function of the model, except for a normalizing constant.

## 4.4 Invariance of the signed Potts model

The signed Potts model polynomial is an example of a vertex switching invariant since two signed graphs $\Sigma_1 = (\Gamma, \tau_1)$, $\Sigma_2 = (\Gamma, \tau_2)$, with equivalent orientations $\tau_1, \tau_2$ will have the same signed Potts model polynomial, since from the deletion-contraction recurrence we proceed the same way if we take a positive or a negative edge as long as it is not a loop, and loops are always the same sign in two signed graphs with equivalent orientations (because if you vertex switch the graph from a vertex with a loop, you will be changing twice the sign of the loop, hence leaving it untouched).

Now we are interested in vertex switching invariant polynomials with the following form:

$$f_\Sigma\left(G; w_+, w_-\right) = \sum_{\sigma: V \to G} \prod_{e=uv \in E^+} w_+(\sigma(u), \sigma(v)) \prod_{e=uv \in E^-} w_-(\sigma(u), \sigma(v)) =$$

where $w_+, w_- : G \times G \to \mathbb{R}$ are two weight functions defined for positive and negative edges, respectively, that satisfy the undirected graph condition $w_\pm(a, b) = w_\pm(b, a)$ and give different values depending on whether the edge is proper or improper, i.e. have the following form for $\alpha, \beta \in G$:

$$w_+(\alpha, \beta) = \begin{cases} y & \text{if } \alpha = \beta \\ z & \text{else} \end{cases} \tag{12}$$

$$w_-(\alpha, \beta) = \begin{cases} m & \text{if } \alpha = -\beta \\ n & \text{else} \end{cases} \tag{13}$$

**Proposition 4.1** *A polynomial $f_\Sigma$ with weight functions $w_-, w_+$, defined as in (12) and (13), is vertex swtiching invariant if, and only if,* $\forall a, b \in G$

$$w_+(a, b) = w_-(a, -b)$$

*Proof.* $\underline{\Rightarrow}$

Let's prove this right implication by contradiction.

Suppose that $\exists a, b \in G$ such that $w_+(a, b) \neq w_-(a, -b)$ (this is to say that $(y, z) \neq (m, n)$), and take a signed graph $\Sigma = (V, E, \tau)$. Due to the definition of the weight functions, $f_\Sigma$ satisfies the following deletion-contraction recurrence:

$$f_\Sigma\left(|G|, \frac{|G|}{|2G|}; w_+, w_-\right) = \begin{cases} zf_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; w_+, w_-\right) + (y-z)f_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; w_+, w_-\right) & \text{if } e \in E^+ \text{ not a loop} \\ nf_{\Sigma\setminus e}\left(|G|, \frac{|G|}{|2G|}; w_+, w_-\right) + (m-n)f_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; w_+, w_-\right) & \text{if } e \in E^- \text{ not a loop} \\ yf_{\Sigma/e}\left(|G|, \frac{|G|}{|2G|}; w_+, w_-\right) & \text{if } e \in E^+ \text{ loop} \\ \left(\frac{|G|}{|2G|}m^l + (|G| - \frac{|G|}{|2G|})n^l\right) & \text{if } |V| = 1 \text{ \& no pos.} \\ & \text{loops, } l \text{ neg. loops} \end{cases}$$

The following two cases cover all the possibilites that $\exists a, b \in G$ such that $w_+(a, b) \neq w_-(a, -b)$:

- if $y \neq m$, the degree of $f_\Sigma$ respect of $m$ is the number of negative edges
- if $z \neq n$, the degree of $f_\Sigma$ respect of $n$ is the number of negative edges

This implies that any vertex switching of any vertex in the signed graph $\Sigma$ will keep the same amount of negative and positive edges, since the polynomial must be the same and, hence, the degree of the polynomial respect any variable must be the same as well.

If there exists a vertex $v \in V$ which is an endpoint for a different number of positive than negative edges, by vertex switching it we would get a signed graph with a different polynomial because the number of positive and negative edges will change.

So we can now assume that all vertices in $V$ have the same amount of positive and negative edges. Let's take a vertex $v \in V$ and a neighbour of $v$, $u \in V$, which is joined by a different number of positive than negative edges; by vertex switching $v$ the number of positive and negative edges would not change; however, $u$ would be the endpoint for a different number of positive than negative edges and we would find ourselves in the previous case.

To sum up, the polynomial would only be invariant for those signed graphs in which, for every positive edge, exists a negative edge between the same endpoints, and viceversa; nevertheless, these graphs are the ones that stay the same by vertex switching (which is obvious since the same graph will always give the same polynomial). That concludes that $f_\Sigma$ wouldn't be vertex switching invariant. $\square$

$\Leftarrow$

Let us take $\Gamma = (V, E)$ an unsigned graph, $\Sigma = (\Gamma, \tau)$ and $\Sigma' = (\Gamma, \tau')$, such that $\tau'$ is the equivalent orientation of $\tau$ obtained by vertex switching $v \in V$. If we note $V_v$ the neighbours of $v$, then we have

$$\sum_{\sigma:V\to G} \prod_{ab\in E^+} w_+(\sigma(a), \sigma(b)) \prod_{ab\in E^-} w_-(\sigma(a), \sigma(b)) =$$

$$= \sum_{\sigma:V\to G} \prod_{ab\in E^+\setminus\{xv:x\in V_v\}} w_+(\sigma(a), \sigma(b)) \prod_{ab\in E^-\setminus\{xv:x\in V_v\}} w_-(\sigma(a), \sigma(b)) \prod_{ab\in E^+\cap\{xv:x\in V_v\}} w_+(\sigma(a), \sigma(b))$$

$$\prod_{ab\in E^-\cap\{xv:x\in V_v\}} w_-(\sigma(a), \sigma(b)) =$$

Now, doing the vertex switching on $v$ we switch the sign of the element of group which we previously coloured $v$ with, hence we obtain:

$$\sum_{\sigma:V\to G}\prod_{ab\in E^+\setminus\{xv:x\in V_v\}} w_+(\sigma(a),\sigma(b)) \prod_{ab\in E^-\setminus\{xv:x\in V_v\}} w_-(\sigma(a),\sigma(b)) \prod_{ab\in E^+\cap\{xv:x\in V_v\}} w_-(\sigma(a),-\sigma(b))$$

$$\prod_{ab\in E^-\cap\{xv:x\in V_v\}} w_+(\sigma(a),-\sigma(b)) =$$

$$= \sum_{\sigma:V\to G}\prod_{ab\in E^+\setminus\{xv:x\in V_v\}} w_+(\sigma(a),\sigma(b)) \prod_{ab\in E^-\setminus\{xv:x\in V_v\}} w_-(\sigma(a),\sigma(b)) \prod_{ab\in E^+\cap\{xv:x\in V_v\}} w_+(\sigma(a),\sigma(b))$$

$$\prod_{ab\in E^-\cap\{xv:x\in V_v\}} w_-(\sigma(a),\sigma(b)) =$$

$$= \sum_{\sigma:V\to G}\prod_{ab\in E^+} w_+(\sigma(a),\sigma(b)) \prod_{ab\in E^-} w_-(\sigma(a),\sigma(b))$$

Where the change on the sign of the weight functions only for the edges between $v$ and its neighbours is due to the change of the sign on the edges by vertex switching. □

## 4.5 Vertex switching invariance for general weights on any Abelian group

We just found out that an equivalence for vertex switching invariance of the polynomial $f_\Sigma$ for weight functions depending only on whether the edge is proper or improper, is that $\forall a,b\in G$, $w_\pm(a,b) = w_\pm(b,a)$ and $w_+(a,b) = w_-(a,-b)$. In this section we will look at more general weight functions (i.e. two colours interact independently), for example the general weight functions for $G=\mathbb{Z}_3$ are:

| $w_+$ | 0 | 1 | $-1$ |
|---|---|---|---|
| 0 | $a$ | $b$ | $c$ |
| 1 | $d$ | $e$ | $f$ |
| $-1$ | $g$ | $h$ | $i$ |

| $w_-$ | 0 | 1 | $-1$ |
|---|---|---|---|
| 0 | $j$ | $k$ | $l$ |
| 1 | $m$ | $n$ | $o$ |
| $-1$ | $p$ | $q$ | $r$ |

where each position in the table is the interaction between the two elements whose row/column intersects (e.g. $w_+(0,1) = b$, $w_-(-1,-1) = r$). Obviously, the polynomial we get with the previous weight functions has 18 variables.

Our next step is to find how many variables will have the polynomial for any Abelian group $G$ if we want it to be vertex switching invariant.

**Theorem 4.5** *The number of independent parameters in a vertex switching invariant partition function with an Abelian group $G$ as the group of spins is*

$$\frac{1}{4}\left(|G|^2 + \left(\frac{|G|}{|2G|}\right)^2\right) + \frac{|G|}{2}$$

*Proof.* Either of the two tables of interaction between the elements of a group $G$ will have $|G|^2$ parameters, the condition of symmetry will give us $\binom{|G|}{2}$ equations for every table and the condition of vertex switching will give us a total of $|G|^2$ equations. Following the previous example, we have the following equations:

$$\begin{array}{ll} b = d & c = g \\ & f = h \end{array} \qquad \begin{array}{ll} k = m & l = p \\ & o = q \end{array} \qquad \begin{array}{lll} a = j & b = l & c = k \\ d = m(*) & e = o & f = n \\ g = p & h = r & i = q \end{array}$$

However, there might be redundant (dependent) equations that can increase the number of variables. In the previous case, the equation marked with a (*) is redundant since with another equations we can get $d = b = l = p = g = c = k = m$, which already tell us that $d = m$. It is trivial to check that this is the only redundant equation. Hence:

$$\#\text{variables} = 2|G|^2 - 2\binom{|G|}{2} - |G|^2 + \#\text{redundant equations} = |G| + \#\text{redundant equations}$$

For our example case we would get that $\#\text{variables} = |G| + \#\text{redundant equations} = 3 + 1 = 4$, and we can check that there are actually 4 variables using the equations in order to substitute the parameters that are equal:

| $w_+$ | 0 | 1 | −1 |
|---|---|---|---|
| 0 | $a$ | $b$ | $b$ |
| 1 | $b$ | $e$ | $f$ |
| −1 | $b$ | $f$ | $e$ |

| $w_-$ | 0 | 1 | −1 |
|---|---|---|---|
| 0 | $a$ | $b$ | $b$ |
| 1 | $b$ | $f$ | $e$ |
| −1 | $b$ | $e$ | $f$ |

Then, the only issue left is how to compute the number of redundant equations for any Abelian group $G$. Let's assume that all the redundant equations are due to the vertex switching invariance (i.e. there are no redundant equations only due to the symmetry) which makes sense since every position has at most one equation and at the beginning every parameter is different.

Let's see a bigger example than before to follow these steps easier, for example $G = \mathbb{Z}_4$:

| $w_+$ | 0 | 1 | 2 | −1 |
|---|---|---|---|---|
| 0 | $a$ | $b$ | $c$ | $d$ |
| 1 | $b$ | $e$ | $f$ | $g$ |
| 2 | $c$ | $f$ | $h$ | $i$ |
| −1 | $d$ | $g$ | $i$ | $j$ |

| $w_-$ | 0 | 1 | 2 | −1 |
|---|---|---|---|---|
| 0 | $k$ | $l$ | $m$ | $n$ |
| 1 | $l$ | $o$ | $p$ | $q$ |
| 2 | $m$ | $p$ | $r$ | $s$ |
| −1 | $n$ | $q$ | $s$ | $t$ |

Since $w_+(x,0) = w_-(x,0)$, the first column of each chart is the same, so $(a,b,c,d) = (k,l,m,n)$. We will follow with $w_+(x,1) = w_-(x,-1)$, and so on. If we put the equations in a chart such that the first term of the equality stays in its position in the positive weight table, we get:

$$\begin{array}{llll} a=k & b=n & c=m(*) & d=l(*) \\ b=l & e=q & f=p & g=o \\ c=m & f=s & h=r & i=p(*) \\ d=n & g=t & i=s & j=q \end{array}$$

Now we classify these equations in a few classes:

- Those that involve $0 \in G$ (first row and column)

  The interaction between 0 and 0 has the following only relation $w_+(0,0) = w_-(0,0)$, which is counted once and, therefore, there are no redundant equations.

  For every non-zero self-inverse of $G$ we will get a redundant equation since $w_+(x,0) = w_-(x,0)$ and $w_+(0,x) = w_-(0,-x) = w_-(0,x)$, $\forall x \in \frac{G}{2G}$, which are the same equation due to the symmetry.

  And for every two non-zero non-self-inverse elements of $G$ that are inverse between each other, we will also get a redundant equation, because $w_+(x,0) = w_-(x,0)$, $w_+(-x,0) = w_-(-x,0)$, $w_+(0,x) = w_-(0,-x)$ and $w_+(0,-x) = w_-(0,x)$ which leads us to $w_+(0,x) = w_-(0,x) = w_+(0,-x) = w_-(0,-x)$, thus one of the previous equations is redundant.

  To sum up, the number of redundant equations in this section is:

$$\left(\frac{|G|}{|2G|} - 1\right) + \frac{1}{2}\left(|G| - \frac{|G|}{|2G|}\right)$$

  The number of equations used here is $2|G| - 1$.

- Those that involve the interaction between two different self-inverses of $G \setminus \{0\}$

  Let's take $x, y \in \frac{G}{2G} \setminus \{0\}$, then we have $w_+(x,y) = w_-(x,-y) = w_-(x,y)$ and $w_+(y,x) = w_-(y,-x) = w_-(y,x)$, which due to the symmetry are the same equation; so then, for every two non-zero self-inverses we get a redundant equation.

  Hence, the number of redundant equations in this case is:

$$\binom{\frac{|G|}{|2G|} - 1}{2}$$

  The number of equations use here is $2 \cdot \binom{\frac{|G|}{|2G|} - 1}{2}$.

- <u>Those that involve a non-zero element with the same</u>

  Let's take $x \in \frac{G}{2G} \setminus \{0\}$, then $w_+(x,x) = w_-(x,-x) = w_-(x,x)$ and since both values lay in the diagonal of the respective charts, there are no more equations involving these values, therefore there are no redundant equations.

  If $x \in G \setminus \frac{G}{2G}$, then $w_+(x,x) = w_-(x,-x)$ and if we look at the symmetric of $w_-(x,-x)$, $w_-(-x,x)$, we know it is equal to $w_+(-x,-x)$ which is another element of the diagonal, then $w_+(x,x) = w_-(x,-x) = w_-(-x,x) = w_+(-x,-x)$, which gives no redundant equations.

  The number of equations used here is $|G| - 1$.

- <u>Those that involve a non-self-inverse with its inverse</u>

  In this case we have, for $x \in G \setminus \frac{G}{2G}$, $w_+(x,-x) = w_-(x,x)$ and $w_+(-x,x) = w_-(-x,-x)$, so we get no redundant equations since $w_-(x,x)$ and $w_-(-x,-x)$ are unique due to the fact that they are at the diagonal of the negative chart.

  The number of equations used here is $|G| - \frac{|G|}{|2G|}$.

- <u>The rest of them</u>

  For the rest of elements, we get the following equations for $x, y \in G \setminus \frac{G}{2G}$:

$$w_+(x,y) = w_-(x,-y)$$
$$w_+(x,-y) = w_-(x,y)$$
$$w_+(-x,y) = w_-(-x,-y)$$
$$w_+(-x,-y) = w_-(-x,y)$$
$$w_+(y,x) = w_-(y,-x)$$
$$w_+(y,-x) = w_-(y,x)$$
$$w_+(-y,x) = w_-(-y,-x)$$
$$w_+(-y,-x) = w_-(-y,x)$$

  In the case in which one of the elements of the group (for instance, $y$) is a self-inverse we get what we can see in the example above: these equations relate 4 different parameters but we can pair the 8 equations since $y = -y$ (the first and the second ones are the same, and so on). Then we have 4 equations and 4 equal parameters (hence three independent equations), so we get a redundant equation for every 4 parameters.

  Else, we are in the case that none of both elements is self-inverse, then we have 8 equations involving 8 different parameters because of the following reasoning. Take these two series of equalities

$$w_+(x,y) = w_-(x,-y) = w_-(-y,x) = w_+(-y,-x) = w_+(-x,-y) =$$

$$= w_-(-x,y) = w_-(y,-x) = w_+(y,x) = w_+(x,y)$$

$$w_+(x,-y) = w_-(x,y) = w_-(y,x) = w_+(y,-x) = w_+(-x,y) =$$

$$= w_-(-x,-y) = w_-(-y,-x) = w_+(-y,x) = w_+(x,-y)$$

For each of them, we are using the previous equations and the symmetry. As we can see, we have two cycles, and each one of them compares 4 parameters. To sum up, we get a redundant equation every 4 parameters as well.

So, the number of equations used here is the total without the ones from other classes:

$$|G|^2 - (2|G|-1) - 2\binom{\frac{|G|}{|2G|}-1}{2} - (|G|-1) - \left(|G| - \frac{|G|}{|2G|}\right) = |G|^2 - \left(\frac{|G|}{|2G|}\right)^2 - 4|G| + 4\frac{|G|}{|2G|}$$

And for every 4 equations, we get a redundant one, then we have the following number of redundant equations:

$$\frac{|G|^2 - \left(\frac{|G|}{|2G|}\right)^2}{4} - \left(|G| - \frac{|G|}{|2G|}\right)$$

Finally, the total number of variables is:

$$\#\text{variables} = |G| + \left(\frac{|G|}{|2G|} - 1\right) + \frac{1}{2}\left(|G| - \frac{|G|}{|2G|}\right) + \binom{\frac{|G|}{|2G|}-1}{2} + \frac{|G|^2 - \left(\frac{|G|}{|2G|}\right)^2}{4} - \left(|G| - \frac{|G|}{|2G|}\right) =$$

$$= \boxed{\frac{1}{4}\left(|G|^2 + \left(\frac{|G|}{|2G|}\right)^2\right) + \frac{|G|}{2}}$$

$\square$

# 5 Reidemeister moves invariant

## 5.1 Kauffman bracket polynomial

The Kauffman bracket polynomial is obtained by the following skein relation recurrence

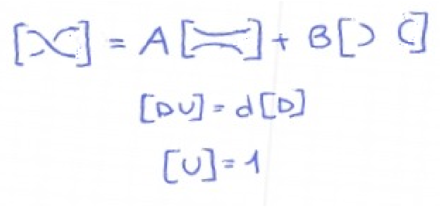


Figure 4: Skein relation for the Kauffman bracket polynomial

where $U$ denotes the knot with no crossings (i.e. a circle), and $A, B$ and $d$ are the parameters of the polynomial. This skein relation is the equivalent as the deletion-contraction of graphs (see figure 5) if we take the Tait graph of the link diagram.

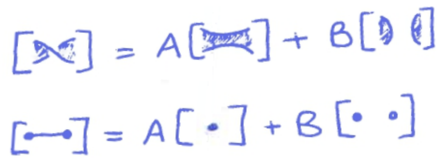
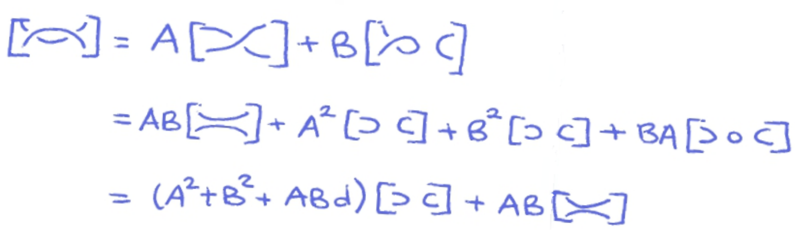


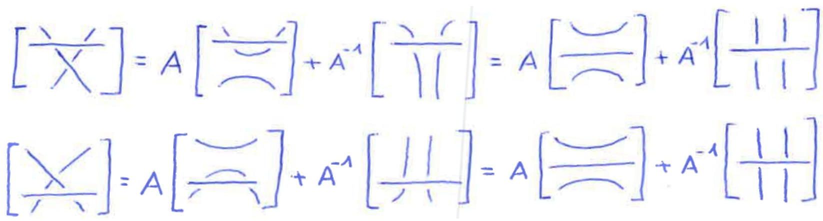
Figure 5: Relation between skein relation and deletion-contraction

We want this Kauffman bracket polynomial to be invariant under the Reidemeister moves. Let us begin with the conditions for type II Reidemeister move invariancy:



As we can see, in order to achieve this invariancy, we need two conditions: $AB = 1$ and $A^2 + B^2 + ABd = 0$. From the former, we obtain $B = A^{-1}$ and then, from the latter, $d = -\frac{A^2+B^2}{AB} = -A^2 - A^{-2}$.

We will follow by checking it also for the type III Reidemeister move:



Thus, the parameter conditions that made the bracket invariant under the type II Reidemeister move also makes it invariant under the type III. The problem appears when we check the invariancy under the type I Reidemeister move:

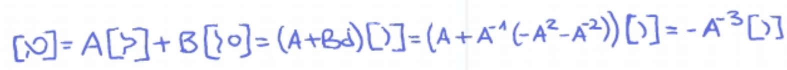


Figure 6: Kauffman bracket polynomial for a loop

where we have used the values $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

To get a result that actually satisfies invariance under Reidemeister moves we need to define the *writhe* of an oriented link:

**Definition 5.1** *The writhe $\omega(D)$ of an oriented link diagram $D$ is the sum of the signs at crossings, defined for every crossing as the figure below.*


Now we define the following polynomial:

$$f_D(A) = (-A^3)^{-\omega(D)}[D]$$

where $[D]$ refers to the Kauffman bracket of the disoriented link diagram. Hence, we have the following proposition:

**Proposition 5.1** *The polynomial $f$ is invariant under Reidemeister moves I, II & III.*

*Proof.* The Kauffman bracket polynomial is invariant under types II & III Reidemeister moves, and so is the writhe of a diagram so then the only step remaining is to check that, besides, $f$ is invariant under the type I Reidemeister move.

Following the skein relation of the Kauffman bracket polynomial for the type I Reidemeister move, we obtained figure (5.1). However, no matter the orientation of the link we take, this loop will always have negative writhe. Hence, after this skein relation, the writhe will be decreased by 1 and then, the $f$ polynomial will stay the same.

□

At last, as a contraposition to the vertex swtiching, we have found an invariant under another object: the Reidemeister moves.

## 5.2 Jones model

An example of a Reidemeister moves invariant model will be the following, extracted from Jones [11]:

Take a spin model $S = \{\Theta, w_\pm\}$ where $\Theta$ is a set of $n$ spins and $w_\pm : \Theta \times \Theta \to \mathbb{R}$ are the weight functions satisfying, for $a, b, c \in \Theta$:

$$w_\pm(a,b) = w_\pm(b,a) \tag{14}$$

$$w_\pm(a,a) = 1 \tag{15}$$

$$w_+(a,b)w_-(a,b) = 1 \tag{16}$$

$$\sum_{x\in\Theta} w_-(a,x)w_+(x,c) = n\delta(a,c) \tag{17}$$

where $\delta$ is the Kronecker delta,

$$\sum_{x\in\Theta} w_+(a,x)w_+(b,x)w_-(c,x) = \sqrt{n}w_+(a,b)w_-(b,c)w_-(c,a) \tag{18}$$

Given a signed graph $\Sigma = (V, E, \tau)$ define a state $\sigma$ to be any function $\sigma : V \to \Theta$ and the partition function $Z_\Sigma^S$ is defined by

$$Z_\Sigma^S = \left(\frac{1}{\sqrt{n}}\right)^{|V|-1} \sum_{\sigma:V\to\Theta} \prod_{ab\in E^+} w_+(a,b) \prod_{ab\in E^-} w_-(a,b) \tag{19}$$

Every equation, except from (14) that comes by the symmetry, can be associated to a type of Reidemeister move in the Tait graph (3). (15) is associated to the first and second type I move, (16) is associated to the first type II move, (17) is associated to the first type II and (18) is associated to the type III.

The main difference between this model and the vertex swtiching invariant model, different from the object that is invariant, is that, as well as we worked on a family of weights that assured us the equivalency between the vertex switching invariance and the conditions on the system, in this case we do not find this equivalency.

For instance, if we look at the condition (16) and the second type II Reidemeister move, denoting $uv_+, uv_-$ the positive and negative edges, respectively, we get the following:

$$Z_\Sigma^S = \sum_{\sigma:V\to\Theta} \prod_{ab\in E^+} w_+(\sigma(a),\sigma(b)) \prod_{ab\in E^-} w_-(\sigma(a),\sigma(b)) =$$

$$= \sum_{\sigma:V\to\Theta} w_+(\sigma(u),\sigma(v))w_-(\sigma(u),\sigma(v)) \prod_{ab\in E^+\setminus uv_+} w_+(\sigma(a),\sigma(b)) \prod_{ab\in E^-\setminus uv_-} w_-(\sigma(a),\sigma(b))$$

If condition (16) is satisfied, the partition function is invariant under the second type II Reidemeister move, however the reciprocal is false.

## 5.3 Kauffman-Tutte

If we go back to the skein relation of the bracket polynomial (figure 4), and we do not impose any condition on the parameters, the object we get is a polynomial in three variables, called the Kauffman-Tutte polynomial. Even though this polynomial comes from a knot invariant, a certain specialization gives us the Tutte polynomial.

So, we can think about the Kauffman-Tutte polynomial as a generalisation of the Tutte polynomial in the world of knot theory and Reidemeister moves, as the trivariate Tutte polynomial was a generalisation of the Tutte polynomial i the wotld of vertex switching invariants.

The Jones polynomial of a knot, which actually is the following specialization of the Kauffman bracket

$$V_L(t) = (-t)^{\frac{-3\omega(L)}{4}} [L]_{A=t^{-1/4}},$$

can be expressed in terms of the Tutte polynomial when applied to an alternating link. An alternating link is a link where all edges in the Tait graph have the same sign. This result has encouraged us historically to finding similar connections for links in general, and that is why the Tutte polynomial being a specialization of the Kauffman-Tutte polynomial has been studied.

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