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AN IMPROVED ACCURACY ANALYSIS OF ELASTOPLASTIC INTEGRATION ALGORITHMS

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Abstract. An improved accuracy analysis of elastoplastic integration algorithms is presented and proposed in this paper. The notion of the well–established isoerror maps is extended and polar plots are constructed for a wide range of stress points, algorithmically selected in the principal stress space. The selection of the stress points is independent of the yield surface and therefore a general procedure is obtained. The individual maps are then joined together to produce a complete view of the accuracy assessment of the stress update algorithm. The proposed procedure is validated in a characteristic multisurface yield criterion.

1 INTRODUCTION

Within the context of computational plasticity it is desirable that the stress update algorithms employed at the Gauss point level should be sufficiently accurate for strain increments as large as possible in order to ensure that the global finite element solution remains within reasonable bounds of accuracy for large load increments. Therefore, accuracy assessment of elastoplastic integration algorithms under finite steps becomes crucial.

A systematic approach to accuracy analyses of elastoplastic algorithms have been first developed by Krieg and Krieg [1] who constructed isoerror maps on a strain controlled homogeneous problem investigating the behaviour of integration algorithms for the Huber–von Mises perfectly plastic model. Although this technique should not be regarded as a replacement of a rigorous accuracy and stability analysis, it have been proved very effective and is generally accepted as a reliable tool for the accuracy assessment of integration algorithms [2].

Standard accuracy assessment employing isoerror maps follows a typical pattern. For a given yield surface, a range of possible stress points that reside on the surface and correspond to different stress states (eg. uniaxial loading, biaxial loading, pure shear stress etc.) must be identified. A sequence of yield surface and position dependant, normalized strain increments is then applied at each individual stress point and the error between the computed and the exact solution is obtained and plotted [3].

Clearly the accuracy assessment following the above procedure is a rather tedious work, which depends on the correct identification of all possible states, is limited only in the vicinity of these states and differs considerably in each yield surface.

In this paper an improved accuracy analysis of elastoplastic integration algorithms is presented and proposed. The isoerror maps are constructed for a wide range of stress points, algorithmically selected in the principal stress space. The selection of the stress points is independent of the yield surface and therefore a general procedure is obtained. The individual maps are then used together to produce a complete view of the accuracy assessment of the stress update algorithm. The proposed procedure is validated in the Mohr–Coulomb yield criterion.

2 METHODOLOGY

Let $P(\rho, \theta, z)$ denote a point in a cylindrical coordinate system and assume a tranformation that maps P to the Haigh–Westergaard stress space [4] as shown in **Fig. 1**, i.e.

$$\mathbf{T} : (\rho, \theta, z) \rightarrow (\sigma_1, \sigma_2, \sigma_3) \tag{1}$$

where σ_i are the principal stresses.

Now assume that the stress state corresponding to point P defines a *trial* stress state, i.e. $P \equiv \boldsymbol{\sigma}^{trial}$. Within the context of perfect plasticity, a scalar function $f : \mathbb{S} \rightarrow \Re$ is defined, which goes by the name *yield function* and constrains the admissible stresses to lie in the so called *elastic domain*, such as:

$$\mathbb{E}_{\boldsymbol{\sigma}} := \{\boldsymbol{\sigma} \in \mathbb{S} | f(\boldsymbol{\sigma}) \leq 0\} \tag{2}$$

If $\boldsymbol{\sigma}^{trial}$ violates the constraint defined by the yield function then a plastic correction is needed in order to bring back the trial state in the boundary of $\mathbb{E}_{\boldsymbol{\sigma}}$, namely $\partial\mathbb{E}_{\boldsymbol{\sigma}}$. From a numerical standpoint, a typical choice would be a fully–implicit integration algorithm, which will approximate the solution, yielding a stress state $\boldsymbol{\sigma}^{approx}$ on $\partial\mathbb{E}_{\boldsymbol{\sigma}}$.

However, it has been proved [5, 6] that a fully–implicit scheme tends to provide the exact solution $\boldsymbol{\sigma}^{exact}$ of the problem when $\boldsymbol{\sigma}^{trial}$ is divided into a sufficiently large number n of subincrements. Thus an error estimate can be defined as:

$$\epsilon(\%) = 100 \times \frac{\sqrt{(\boldsymbol{\sigma}^{exact} - \boldsymbol{\sigma}^{approx}) : (\boldsymbol{\sigma}^{exact} - \boldsymbol{\sigma}^{approx})}}{\sqrt{\boldsymbol{\sigma}^{exact} : \boldsymbol{\sigma}^{exact}}} \tag{3}$$

Therefore, one has to define a suitable search space $\mathcal{V}$ for (ρ, θ, z) and apply the error estimate for sufficiently large value of n, as described in algorithm **1**.

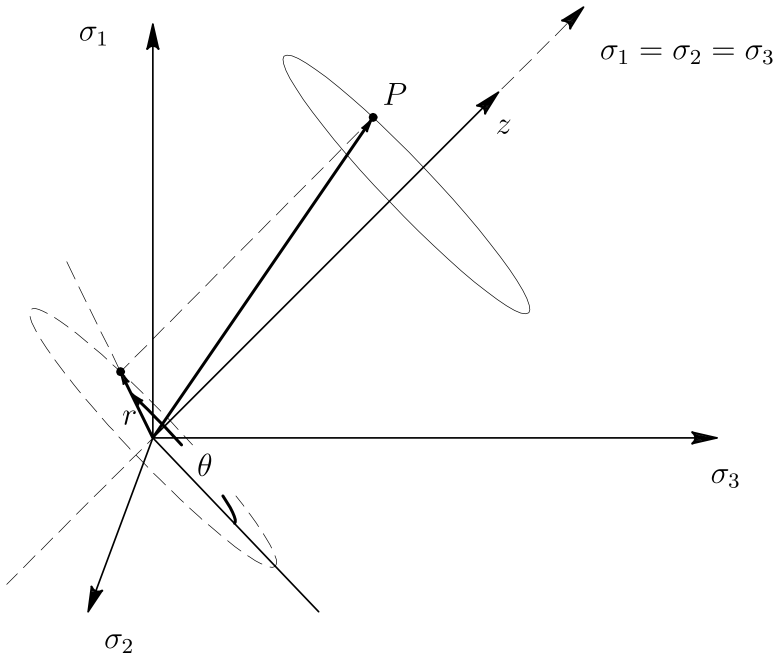


Figure 1: An arbitrary stress state P.

3 APPLICATION

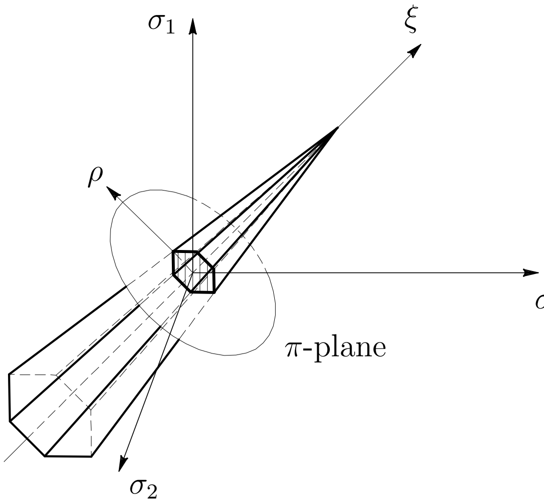
The above proposed methodology is applied in what follows to the Mohr–Coulomb yield criterion.

3.1 The Mohr–Coulomb yield criterion

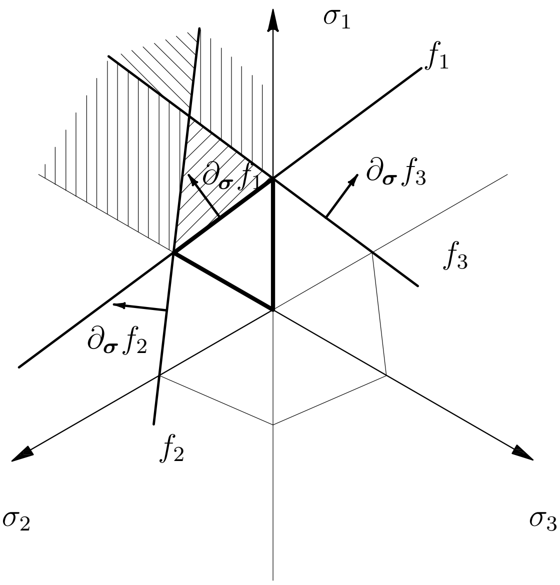
The Mohr–Coulomb yield criterion is frequently acknowledged as one of the first and most important criteria, widely used to describe the yield behavior of a wide range of materials. It is defined by six linear surfaces in the principal stress space (Fig. **2(a)**), assuming however and without any loss of generality that $\sigma_1 \geq \sigma_2 \geq \sigma_3$, only the following

Algorithm 1 Construction of improved isoerror maps.

for $z \in [z_0, z_1, \ldots]$ **do**
  for $\theta \in [-\pi/6, \pi/6]$ with step $d\theta$ **do**
    for $r \in [r_1, r_2]$ with step dr **do**
      transform (z, r, θ) to (σ_1, σ_2, σ_3)
      Find $\boldsymbol{\sigma}^{approx}$
      Find $\boldsymbol{\sigma}^{exact}$ using n subincrements
      Find error ϵ
    end for
  end for
end for



(a) Haigh–Westergaard space.



(b) Projection in the π–plane.

Figure 2: The Mohr–Coulomb yield criterion.

three surfaces (Fig. **2(b)**) can describe the elastic domain:

$$f_1(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_3) + (\sigma_1 + \sigma_3)\sin(\phi) - 2c\cos(\phi) \tag{4}$$
$$f_2(\sigma_1, \sigma_2, \sigma_3) = (\sigma_2 - \sigma_3) + (\sigma_2 + \sigma_3)\sin(\phi) - 2c\cos(\phi) \tag{5}$$
$$f_3(\sigma_1, \sigma_2, \sigma_3) = (\sigma_1 - \sigma_2) + (\sigma_1 + \sigma_2)\sin(\phi) - 2c\cos(\phi) \tag{6}$$

For the problem examined next an associative flow rule is assumed. The elastic properties are characterized by $E = 1000$, $\nu = 0.25$ while cohesion and internal friction angle are given as $c = 15$ and $\phi = 20°$.

3.2 The return mapping scheme

The return mapping scheme used here is thoroughly examined in [7, 8] and implemented in [9]. It is based on a spectral representation of stresses and strains and a return mapping scheme in principal stress directions. Because of the linearity of the yield surfaces in the principal stress space the return mapping reduces to a one step closest–point projection.

3.3 Accuracy assessment

For the accuracy assessment of the above implementation, it is chosen that $z = [0., -10., \ldots, -50.]$, $\rho = [0.01, 0.02, \ldots, 100.]$ and $\theta = [30°, 40°, \ldots, 90°]$. As described in the algorithm **1**, for a fixed z and θ both the $\boldsymbol{\sigma}^{approx}$ and $\boldsymbol{\sigma}^{exact}$ are recovered, the latter assuming a division into 1000 subincementations.

The results, along with the corresponding elastic domains for the given deviatoric plane, are plotted in **Fig.3.3**.

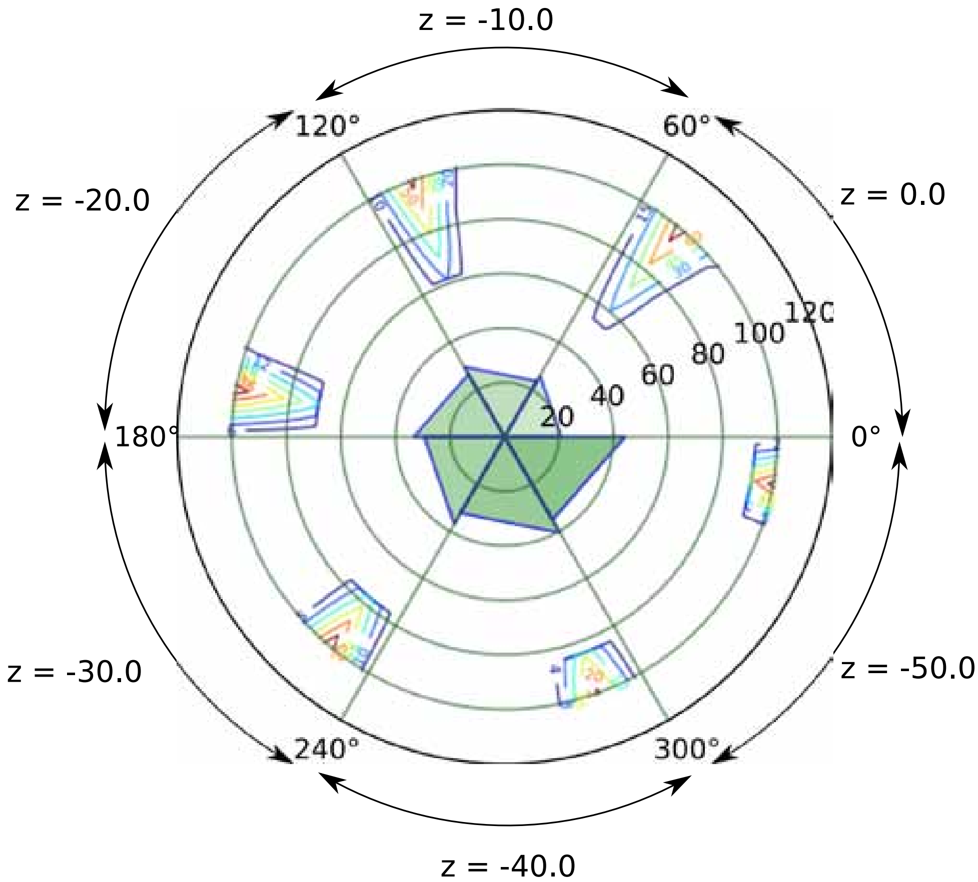


Figure 3: Isoerror maps for the Mohr–Coulomb yield criterion.

4 CONCLUSIONS

An improved accuracy analysis of elastoplastic integration algorithms is presented and proposed in this paper. The notion of the well–established isoerror maps is extended and polar plots are constructed for a wide range of stress states spreading over the principal stress space. The main advantages of this approach is that the selection of the stress states leads to a procedure that:

- is yield surface–agnostic,
- does not depend on the integration algorithm and
- is able to cover the entire range of possible stress states.

Contrary to typical isoerror maps, the individual equipotential diagrams produced are joined together and plotted in polar coordinates as to generate a complete picture of the accuracy assessment of the stress update algorithm. The proposed procedure is validated in a characteristic multisurface yield criterion, namely the Mohr–Coulomb yield criterion, composing an intuitive view of the selected integration scheme's accuracy.

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