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A MIXED ALGORITHM FOR INCREMENTAL ELASTOPLASTIC ANALYSIS

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Abstract. A new method for the incremental analysis of elastoplastic associated materials is presented. The method fully retains all the equations and variables of the problems at the same level and uses a sequential quadratic programming with equality constraints to solve in an efficient and robust fashion the elastoplastic step equations derived by means of a suitable mathematical programming formulation of the problem. The new proposal is compared with standard strain driven formulations which use a return mapping by closest point projection schemes. The numerical tests performed show a good performance and a great robustness of the proposed formulation also in the case of multi–surface elastoplasticity.

1 INTRODUCTION

The finite element incremental elastoplastic analysis is commonly performed by means of a strain driven (SD) step by step procedure in which each step implements a return mapping strategy. The latter is based on the formulation of a finite step holonomic equation obtained from the irreversible incremental elastoplastic laws by using an integration process which evaluates all the quantities at the end of the step starting from the known values at the beginning of the step and from a prescribed value of the displacement field (strain driven). Among the available integration processes the backward–Euler scheme is the most used.

In standard FEM implementations, the plastic flow rule and consistency conditions are solved exactly and, for an assigned value of the displacements, the return mapping process imposes these equations for each Gauss point of the element. The major advantage of this approach is that the inequality constraints arising from the constitutive laws are eliminated from the step equations using the closest point projection scheme which solves a small optimization problem on each Gauss point of the finite element, so defining the stress parameters in terms of the displacement ones. The finite step equations are so transformed into a nonlinear system of equations, without inequalities, easily solved by means of standard arc–length strategies. The global description of the algorithm is always performed in terms of displacement variables alone.

The use of descriptions based on displacement variables alone would not be the best choice and potentially more efficient and robust analysis algorithms could be obtained by directly solving the finite step equations, maintaining all the variables of the problems at the same level. Few attempts in this direction have been made and, among these we recall the use of nonsmooth Newton methods proposed by Cristiansen [1] or the use of interior point methods to solve an optimization problem exactly equivalent to the elastoplastic step of Krabbenhoft et al. [2].

In this work a new algorithm for the FEM elastoplastic analysis of structures is presented. The proposed algorithm uses the stress, displacement and plastic multiplier parameters introduced in the discretized form of the problem as primary variables. Adopting the mathematical programming point of view, which allows the use of a theoretically robust environment now endowed with several efficient solution algorithms, a sequential quadratic programming (SQP) formulation is proposed to solve the problem. All the equations describing the finite step are solved at the same level using an equality constraints sequential quadratic programming (EC-SQP) [3] which exploits the particular structure of the equations of the elastoplastic step in order to improve efficiency also for large dimensional problems. In particular an equality constrained sequential quadratic programming (EQ-SQP) is employed. The algorithm is subdivided in two phases: i) a suitable estimate of the active constraints at the current iteration is performed employing the closest point projection scheme; ii) the solution of a quadratic programming that retains only the active constraints is performed. In this way the solution of each QP problem is far easier than the general case and it also makes it possible to deal with very large dimension problems. In particular the solution of the QP subproblem can be performed after condensation of the locally defined quantities (stresses and plastic multipliers) so maintaining, at the global level of analysis, a pseudo compatible system that has the same structure used in standard elastoplastic analysis. The overall algorithm has then the same organization as standard SD-CPP ones and only a few modifications of the existing codes are required to implement the present proposal.

The finite elements used are of mixed type, see [4], but plastically enriched in order to work well also in the elastoplastic field. They are based on a three field interpolation and are so well suited for the application of the proposed algorithm. They also allow the new formulation to be tested in a more severe multi-surface case. The numerical results show how a great improvement in terms of robustness is achieved with respect to the standard SD-CPP algorithms. The proposed algorithms can painlessly undergo large steps sizes or singular yield conditions while the SD-CPP approach shows serious convergence difficulties or, also with respect to small step sizes, line search addiction is mandatory to obtain convergence.

2 THE DISCRETE EQUATIONS FOR THE ELASTOPLASTIC STEP

In the following, the elastoplastic step equations are derived by a backward Euler integration process and then are rewritten in terms of discrete algebraic expressions by introducing the finite element description. The chosen FEM format is based on the general three field interpolation presented in [4] but any other more usual finite element format could be considered by assigning the appropriate meaning to the discrete parameters used in the following.

2.1 The elastoplastic step equations for the continuum body

The nonlinear response of an elastoplastic body Ω subjected to bulk load $\boldsymbol{b}$ and tractions $\boldsymbol{t}$, increasing proportionally to a scalar multiplier λ, can be evaluated by using a standard step-by-step strategy based on the incremental computation of a sequence of discrete points along a time/loading process. In the following we will denote with a superscript $(n-1)$ the quantities relative to the current instant/load in which the solution is known and with a superscript (n) the unknown quantities at the new instant. The stress $\boldsymbol{\sigma}^{(n)}$ and the plastic multiplier $\gamma^{(n)}$ are evaluated by performing a time integration of the constitutive laws once the displacement field $\boldsymbol{u}^{(n)}$ at the end of the step is assigned. In this way the path–dependent elastoplastic behavior is transformed into a sequence of finite holonomic steps. In particular using a backward-Euler time integration scheme and omitting from now on the dependence of quantities on $\boldsymbol{x}$ for an easier reading, the n-th finite step equations can be written, using a standard vector notation, as follows:

$$
\begin{aligned}
&\text{Compatibility:}\\
&\begin{cases}
\boldsymbol{D}\Delta\boldsymbol{u} = \boldsymbol{\mathcal{C}}^{-1}\Delta\boldsymbol{\sigma} + \Delta\gamma \left.\dfrac{\partial\phi}{\partial\boldsymbol{\sigma}}\right|_n & \text{in } \Omega,\\
\boldsymbol{u}^{(n)} = \bar{\boldsymbol{u}} & \text{on } \partial\Omega_u;
\end{cases}\\
&\text{Admissibility and consistency:}\\
&\phi[\boldsymbol{\sigma}^{(n)}] \le 0 \ \ , \ \ \Delta\gamma \ge 0 \ \ , \ \ \Delta\gamma\phi[\boldsymbol{\sigma}^{(n)}] = 0.
\end{aligned}
\tag{1a}
$$

The symbol $\Delta(\cdot) = (\cdot)^n - (\cdot)^{n-1}$ will denote, from now on, the difference between quantities in (n) and $(n-1)$, $\boldsymbol{\mathcal{C}}^{-1}$ the elastic compliance operator, $\boldsymbol{D}$ the compatibility operator, ϕ the convex yield function and $\bar{\boldsymbol{u}}$ the prescribed displacement on $\in \partial\Omega_u$. In the previous equations and from now on the finite increment of plastic strain is evaluated using an associated flow rule.

The holonomic finite step is then completed with the equilibrium equations:

$$
\begin{cases}
\boldsymbol{D}^T\boldsymbol{\sigma}^{(n)} + \lambda^{(n)}\boldsymbol{b} = \boldsymbol{0} & \text{in } \Omega,\\
\boldsymbol{n}\boldsymbol{\sigma}^{(n)} = \lambda^{(n)}\boldsymbol{t} & \text{on } \partial\Omega_t
\end{cases}
\tag{1b}
$$

where $\boldsymbol{n}$ is the matrix collecting the normal to the loaded boundary $\partial\Omega_t$.

2.2 The weak form of the finite step equations

Following [5], the finite step equations (1a) can be rewritten in a weak form as

$$\int_\Omega \delta\boldsymbol{\sigma}^T \left(\boldsymbol{\mathcal{C}}^{-1}\Delta\boldsymbol{\sigma} - \boldsymbol{D}\Delta\boldsymbol{u} + \Delta\gamma \left.\frac{\partial\phi}{\partial\boldsymbol{\sigma}}\right|_n \right) = 0 \quad \forall\delta\boldsymbol{\sigma}, \tag{2a}$$

$$\int_\Omega \delta\gamma\phi[\boldsymbol{\sigma}^{(n)}] = 0 \quad \forall\delta\gamma \geq 0, \tag{2b}$$

where Eq. (2b) expresses, in a weak sense, the plastic admissibility condition for the material. In a similar fashion the weak statement of the equilibrium condition (1b) becomes

$$\int_\Omega (\boldsymbol{D}\delta\boldsymbol{u})^T \boldsymbol{\sigma}^{(n)} - \lambda^{(n)} \left(\int_\Omega \delta\boldsymbol{u}^T\boldsymbol{b} + \int_{\partial\Omega_t} \delta\boldsymbol{u}^T\boldsymbol{t} \right) = 0 \quad \forall\delta\boldsymbol{u}. \tag{2c}$$

Finally, the use of a path–following Riks algorithm to solve the step equations, requires the introduction of the arc–length parameters $\Delta\xi^{(n)}$. The following definition can be exploited

$$\delta\lambda \left(\int_\Omega \Delta\boldsymbol{u}^T\boldsymbol{b} + \int_{\partial\Omega_t} \Delta\boldsymbol{u}^T\boldsymbol{t} - \Delta\xi^{(n)} \right) = 0 \quad \forall\delta\lambda. \tag{2d}$$

2.3 The FE finite step equations

Following [4] where more details can be found, we adopt a finite element formulation based on the interpolation of three fields: displacement, stress and plastic multiplier. These interpolations can be expressed as:

$$\boldsymbol{u} := \boldsymbol{N}\boldsymbol{d}_e \qquad \boldsymbol{\sigma} := \boldsymbol{S}\boldsymbol{\beta}_e \qquad \gamma := \boldsymbol{G}\boldsymbol{\kappa}_e, \tag{3}$$

where $\boldsymbol{N}$, $\boldsymbol{S}$ and $\boldsymbol{G}$ are the matrices containing the interpolation functions and $\boldsymbol{d}_e$, $\boldsymbol{\beta}_e$ and $\boldsymbol{\kappa}_e$ are the vectors collecting the finite element parameters. The non–negativeness of the interpolation functions $\boldsymbol{G}$ allows the condition $\gamma \geq 0$ to be easily expressed by making $\boldsymbol{\kappa}_e \geq \boldsymbol{0}$, where, from now on, vector inequality will be considered in a componentwise fashion. Moreover an important aspect which will allow the nonlinear algorithm to be casted in the format described in the following regards the continuity order of the assumed interpolations, in particular the displacement field has to be capable of assuring the inter-element continuity while $\boldsymbol{\sigma}$ and γ can be defined locally inside the element.

From now on we omit reporting the superscript $()^{(n)}$ that defines the step.

2.3.1 Local equations

On the basis of the interpolations we obtain the discrete counterpart of the flow rule, plastic admissibility and consistency condition

$$\begin{cases} \boldsymbol{r}_\sigma \equiv \boldsymbol{H}_e\Delta\boldsymbol{\beta}_e - \boldsymbol{Q}_e\Delta\boldsymbol{d}_e + \boldsymbol{A}_e[\boldsymbol{\beta}_e]\Delta\boldsymbol{\kappa}_e = \boldsymbol{0} \\ \boldsymbol{r}_\mu \equiv \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^{(n)}] \leq 0, \quad \Delta\boldsymbol{\kappa}_e \geq 0, \quad \Delta\boldsymbol{\kappa}_e^T\boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^{(n)}] = 0, \end{cases} \tag{4}$$

where $\boldsymbol{A}_e[\boldsymbol{\beta}_e]$ represents the discrete form of the plastic flux direction and is defined by

$$\boldsymbol{A}_e[\boldsymbol{\beta}_e] := \int_{\Omega_e} \boldsymbol{S}^T \frac{\partial \phi}{\partial \boldsymbol{\sigma}}[\boldsymbol{\beta}_e]\boldsymbol{G}$$

while the discrete operators

$$\boldsymbol{Q}_e := \int_{\Omega_e} \boldsymbol{S}^T \boldsymbol{D}\boldsymbol{N} \qquad \boldsymbol{H}_e := \int_{\Omega_e} \boldsymbol{S}^T \boldsymbol{\mathcal{C}}^{-1}\boldsymbol{S} \tag{5}$$

are the usual compatibility/equilibrium and elastic flexibility matrices respectively and

$$\boldsymbol{\Phi}_e[\boldsymbol{\beta}_e] := \int_{\Omega_e} \boldsymbol{G}^T \phi[\boldsymbol{\beta}_e], \tag{6}$$

corresponds to the element representation of the yield function and it depends on the final value of the stress parameters. As Eqs. (4) are expressed in terms of quantities locally defined on the element, or at the Gauss points for standard finite element interpolations, they will be denoted from now on as *local equations*.

2.3.2 Global equations

The discrete form of the equilibrium equations and the arc length condition to be used in the numerical solution of the problem are:

$$\mathcal{A}_e \left\{\boldsymbol{Q}_e^T \boldsymbol{\beta}_e - \lambda \boldsymbol{p}_e\right\} = 0, \quad , \quad \mathcal{A}_e \left\{\Delta \boldsymbol{d}_e^T \boldsymbol{p}_e\right\} = \Delta\xi, \tag{7}$$

$\mathcal{A}_e$ being the standard assembling operator which takes into account the inter–element continuity conditions on the displacement field and

$$\boldsymbol{p}_e := \int_{\Omega_e} \boldsymbol{N}^T \boldsymbol{b} + \int_{\partial\Omega_e} \boldsymbol{N}^T \boldsymbol{t} \tag{8}$$

is the element load vector. For the sake of the following discussion eqs.(7) can be rewritten as

$$\begin{cases} \boldsymbol{r}_u \equiv \boldsymbol{Q}^T \boldsymbol{\beta} - \lambda \boldsymbol{p} = \boldsymbol{0} \\ r_\lambda \equiv \Delta \boldsymbol{d}^T \boldsymbol{p} - \Delta\xi = 0, \end{cases} \tag{9}$$

where $\boldsymbol{\beta}$, $\boldsymbol{d}$ and $\boldsymbol{p}$ denote the global vectors collecting all the stress parameters $\boldsymbol{\beta}_e$, the displacement parameters $\boldsymbol{d}_e$ and the applied loads $\boldsymbol{p}_e$.

2.4 The mathematical programming point of view

Noting that

$$\boldsymbol{A}_e[\boldsymbol{\beta}_e] := \left(\frac{\partial \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e]}{\partial \boldsymbol{\beta}_e}\right)^T$$

each finite step is characterized by the set of nonlinear equalities and inequalities defined by Eqs. (4) and (9) which represent the first order conditions of the following nonlinear convex optimization problem:

$$\begin{aligned} \text{maximize} \quad & \Delta\xi^{(n)}\lambda^{(n)} - \frac{1}{2}\sum_e (\Delta\boldsymbol{\beta}_e)^T \boldsymbol{H}_e \Delta\boldsymbol{\beta}_e, \\ \text{subject to} \quad & \boldsymbol{Q}^T\boldsymbol{\beta}^{(n)} = \lambda^{(n)}\boldsymbol{p} \\ & \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^{(n)}] \leq \boldsymbol{0} \qquad \forall \boldsymbol{\beta}_e. \end{aligned} \tag{10}$$

Assuming this point of view the actual solution strategy can be implemented on the basis of a nonlinear programming technique suitable for solving (10).

Note how more standard finite element formulations based on the numerical integration on the Gauss points could be easily framed inside the optimization problem defined by Eq. (10), by considering the quadratic terms of the objective function as the result of the sum of the contributions of the Gauss points of each element while the admissibility condition is imposed on each Gauss point.

3 A NEW SOLUTION SCHEME FOR ELASTOPLASTIC ANALYSIS

In the following section we will present an application of the SQP method to solve Eq. (10). The algorithm exploits the problem structure allowing its solution at the global level by means of a Newton (Riks) scheme which is characterized by minimal implementational differences with respect to standard SD-CPP formulations.

3.1 The linearized equations for the elastoplastic step and the sequential quadratic programming (SQP) formulation

The estimate of the unknowns relative to the new step, $\boldsymbol{z}^{(n)} = \{\lambda^{(n)}, \boldsymbol{\beta}^{(n)}, \boldsymbol{d}^{(n)}, \boldsymbol{\kappa}^{(n)},\}$, will be denoted by $\boldsymbol{z}^{j+1} = \boldsymbol{z}^j + \dot{\boldsymbol{z}}$ where, in order to make the notation simpler, the superscript relative to the step number has been dropped leaving only the indication for the current j–th iteration. The starting point for the new algorithm is the linearization of the finite step equation (10) which yields the local equations (4) again, i.e.

$$\begin{cases} -\boldsymbol{H}_{et}\dot{\boldsymbol{\beta}}^j + \boldsymbol{Q}_e\dot{\boldsymbol{d}} - \boldsymbol{A}_e^j\dot{\boldsymbol{\kappa}}_e = -\boldsymbol{r}_\sigma^j, \\ \boldsymbol{\Phi}_e^{j+1} \leq \boldsymbol{0}\ ,\ \boldsymbol{\kappa}_e^{j+1} \geq \boldsymbol{0}\ ,\ (\boldsymbol{\kappa}_e^{j+1})^T\boldsymbol{\Phi}_e^{j+1} = 0. \end{cases} \quad \forall e \tag{11a}$$

Where

$$\boldsymbol{H}_{et} \equiv \boldsymbol{H}_e + \sum_k \kappa_{ek}^j \left.\frac{\partial^2 \Phi_{ek}}{\partial \boldsymbol{\beta}_e^2}\right|_{\boldsymbol{\beta}_e = \boldsymbol{\beta}_e^j} \quad , \quad \boldsymbol{A}_j = \left.\frac{\partial \boldsymbol{\Phi}_e}{\partial \boldsymbol{\beta}_e}\right|_{\boldsymbol{\beta}_e = \boldsymbol{\beta}_e^j},$$

κ_{ek}^j and Φ_{ek} are the kth components of $\boldsymbol{\kappa}_e^j$ and $\boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^j]$ respectively, and

$$\boldsymbol{\Phi}_e^{j+1} \equiv \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^j] + (\boldsymbol{A}_e^j)^T(\boldsymbol{\beta}_e^{j+1} - \boldsymbol{\beta}_e^j) = \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^j] + (\boldsymbol{A}_e^j)^T\dot{\boldsymbol{\beta}}_e.$$

Moreover, the linearization of the global finite step equations (9) gives:

$$\begin{cases} \boldsymbol{Q}^T\dot{\boldsymbol{\beta}} - \dot{\lambda}\boldsymbol{p} = -\boldsymbol{r}_{uj}, \\ -\dot{\boldsymbol{d}}^T\boldsymbol{p} = -\boldsymbol{r}_{\lambda j}. \end{cases} \tag{11b}$$

Eq. (11) could also be obtained by applying a sequential quadratic programming (SQP) approach to (10) obtaining

$$\begin{aligned} &\text{maximize} && \Delta\xi\ \dot{\lambda} - \sum_e \left(\dot{\boldsymbol{\beta}}_e^T\boldsymbol{H}_e\Delta\boldsymbol{\beta}_e^j - \frac{1}{2}\dot{\boldsymbol{\beta}}_e^T\boldsymbol{H}_{et}\dot{\boldsymbol{\beta}}_e\right) \\ &\text{subject to} && \boldsymbol{Q}^T\dot{\boldsymbol{\beta}} - \dot{\lambda}\boldsymbol{p} + \boldsymbol{r}_u^j = \boldsymbol{0}, \\ & && (\boldsymbol{A}^j)^T\dot{\boldsymbol{\beta}} + \boldsymbol{\Phi}_e[\boldsymbol{\beta}^j] \leq \boldsymbol{0}, \end{aligned} \tag{12a}$$

whose solution gives the new estimate $\boldsymbol{z}^{j+1}$ in the form

$$\boldsymbol{z}^{j+1} = \{\lambda^j + \dot{\lambda}, \boldsymbol{\beta}^j + \dot{\boldsymbol{\beta}}, \boldsymbol{d}^j + \dot{\boldsymbol{d}}, \boldsymbol{\kappa}^{j+1}\}. \tag{12b}$$

However the solution of the QP sub-problems (12a) with a standard SQP algorithm requires however a great computational effort due to the coupling action exerted by the equilibrium constraints. A method to efficiently solve Eq. (11) or problem in (12a) will now be examined.

3.2 The EC-SQP formulation

First the SQP problem in (12) is solved by using an equality constraint sequential quadratic programming (EC-SQP) approach [3]. Each iteration of the EC-SQP approach consists of two phases: i) estimation of the active set of constraints; ii) solution of an equality constrained quadratic program that imposes the apparently active constraints and ignores the apparently inactive ones. The idea is to identify the active constraints for the actual estimate of the solution using information available at a point near to $\mathbf{z}^{j+1}$, a point which in the sequel will be denoted by $\bar{\mathbf{z}}^{j+1}$.

3.2.1 The detection of the active set of constraints

The estimation of the active constraints is performed by advocating the decomposition point of view, i. e. solving an optimization problem obtained by the original ones (11a) for a fixed, properly assumed, value of the displacements $\bar{\boldsymbol{d}}^{j+1} = \boldsymbol{d}^j$. The series of decoupled problems obtained in this way have the same form as a standard CPP scheme and it can be easily solved at the local element level in a way as efficient as, or also more, than the standard SD-CPP approach.

At the iteration $j+1$ then the active set of constraints is obtained by solving (11a) assuming $\bar{\boldsymbol{d}}_{j+1} \approx \boldsymbol{d}_j$ and so $\dot{\boldsymbol{d}} = \boldsymbol{0}$. The result is a problem that is now decoupled at the local level, i. e.

$$\begin{cases} -\boldsymbol{H}_{et}\dot{\bar{\boldsymbol{\beta}}}_e^j - \boldsymbol{A}_e^j\dot{\bar{\boldsymbol{\kappa}}}_e = -\boldsymbol{r}_\sigma^j, \\ \boldsymbol{\Phi}_e^{j+1} \leq \boldsymbol{0}\ ,\ \bar{\boldsymbol{\kappa}}_e^{j+1} \geq \boldsymbol{0}\ ,\ (\bar{\boldsymbol{\kappa}}_e^{j+1})^T\boldsymbol{\Phi}_e^{j+1} = 0, \end{cases} \quad \forall e \tag{13}$$

where the symbols with a bar denote the estimates of the new quantities. In particular Eqs.(13) are the first order conditions of the following QP problem:

$$\begin{cases} \min\limits_{(\dot{\boldsymbol{\beta}}_e)} : \dfrac{1}{2}(\dot{\bar{\boldsymbol{\beta}}}_e^j)^T\boldsymbol{H}_{et}\dot{\bar{\boldsymbol{\beta}}}_e^j + (\dot{\bar{\boldsymbol{\beta}}}_e^j)^T\mathbf{g}^j, \\ \text{subj.: } \boldsymbol{A}_j^T\dot{\bar{\boldsymbol{\beta}}}_e^j + \boldsymbol{\Phi}_e^j \leq \boldsymbol{0}, \end{cases} \quad \forall e \tag{14}$$

where $\boldsymbol{\Phi}_e^j = \boldsymbol{\Phi}_e[\boldsymbol{\beta}_e^j]$, $\mathbf{g}^j = \boldsymbol{H}_e(\boldsymbol{\beta}_e^j - \boldsymbol{\beta}_e^*)$. The solution of Eqs (14), which can be seen as the quadratic problems arising from an SQP approximation of the CPP projection scheme, gives the actual estimate of the stress and plastic multiplier parameters, $\bar{\boldsymbol{\beta}}_e^{j+1} = \boldsymbol{\beta}_e^j + \dot{\bar{\boldsymbol{\beta}}}_e$, $\bar{\boldsymbol{\kappa}}^{j+1}$. In particular the QP problem (14) is efficiently solved by using the Goldfarb-Idnani active set method, see [6] for further details.

3.2.2 The solution of the QP equality constraint scheme

After the detection of the set of active constraints, and assuming that this set is not void, we have to solve Eqs. (11) by means of the following system of equations in which only the residuals of the active constraints are considered:

$$\begin{bmatrix} \cdot & \boldsymbol{A}_e^{jT} & \cdot & \cdot \\ -\boldsymbol{A}_e^j & -\boldsymbol{H}_{et} & \boldsymbol{Q}_e & \cdot \\ \cdot & \boldsymbol{Q}_e^T & \cdot & -\boldsymbol{p}_e \\ \cdot & \cdot & -\boldsymbol{p}_e^T & \cdot \end{bmatrix} \begin{bmatrix} \dot{\boldsymbol{\kappa}}_e \\ \dot{\boldsymbol{\beta}}_e \\ \dot{\boldsymbol{d}}_e \\ \dot{\lambda} \end{bmatrix} = - \begin{bmatrix} \boldsymbol{r}_\mu^j \\ \boldsymbol{r}_\sigma^j \\ \boldsymbol{r}_u^j \\ r_\lambda^j \end{bmatrix} \quad , \quad \boldsymbol{z}^{j+1} = \boldsymbol{z}^j + \dot{\boldsymbol{z}}, \tag{15}$$

where the further condition $\boldsymbol{\kappa}_{j+1} \geq 0$ needs to be imposed.

System (15) is easily solved by static condensation of the local defined quantities. In particular, recalling that the QP scheme in (14) solves the first two equations of (11a) zeroing the displacements $\dot{\boldsymbol{d}}_e$, we obtain

$$\begin{cases} \dot{\boldsymbol{\beta}}_e = \boldsymbol{H}_{et}^{-1}\left(\boldsymbol{r}_\sigma^j + \boldsymbol{Q}_e\dot{\boldsymbol{d}}_e - \boldsymbol{A}_j\dot{\boldsymbol{\kappa}}_e\right) & = \dot{\bar{\boldsymbol{\beta}}}_e + \boldsymbol{H}_{et}^{-1}\boldsymbol{Q}_e\dot{\boldsymbol{d}}_e, \\ \dot{\boldsymbol{\kappa}}_e = \boldsymbol{W}\left(\boldsymbol{r}_\mu^j + \boldsymbol{A}_j^T\boldsymbol{H}_{et}^{-1}\boldsymbol{r}_\sigma^j + \boldsymbol{A}_j^T\boldsymbol{H}_{et}^{-1}\boldsymbol{Q}_e\dot{\boldsymbol{d}}_e\right) & = \dot{\bar{\boldsymbol{\kappa}}}_e + \boldsymbol{W}\boldsymbol{A}_j^T\boldsymbol{H}_{et}^{-1}\boldsymbol{Q}_e\dot{\boldsymbol{d}}_e, \end{cases} \tag{16}$$

where $\boldsymbol{W} = \left[\boldsymbol{A}_j^T\boldsymbol{H}_{et}^{-1}\boldsymbol{A}_j\right]^{-1}$.

At the global level then we have to assemble the condensed element contribution as

$$\mathcal{A}_e\left(\boldsymbol{Q}_e^T\boldsymbol{E}_t\boldsymbol{Q}_e\right)\dot{\boldsymbol{d}}-\dot{\lambda}\boldsymbol{p}=-\mathcal{A}_e\left(\tilde{\boldsymbol{r}}_u^j\right)\quad,\quad -\boldsymbol{p}^T\dot{\boldsymbol{d}}_e=-r_\lambda^j, \tag{17}$$

where

$$\tilde{\boldsymbol{r}}_u^j=\boldsymbol{r}_u^j+\boldsymbol{Q}_e^T(\boldsymbol{E}_t\boldsymbol{r}_\sigma^j-\boldsymbol{H}_{et}^{-1}\boldsymbol{A}_j\boldsymbol{W}\boldsymbol{r}_\mu^j)\quad\text{and}\quad \boldsymbol{E}_t=\boldsymbol{H}_{et}^{-1}-\boldsymbol{H}_{et}^{-1}\boldsymbol{A}_j\boldsymbol{W}\boldsymbol{A}_j^T\boldsymbol{H}_{et}^{-1}.$$

$\boldsymbol{E}_t$ has the same expression as the algorithmic tangent matrix evaluated by standard SD-CPP formulation.

System (17) is coincident with a standard SD-CPP iteration scheme except for the new definition of quantities $\tilde{\boldsymbol{r}}_u^j$. Note that $\boldsymbol{E}_t$ and $\boldsymbol{H}_{et}^{-1}\boldsymbol{A}_j\boldsymbol{W}$ are evaluated at each step of the QP problem, by the optimization algorithm used, so only the evaluation of $\tilde{\boldsymbol{r}}_u^j$ is required. In the case of an element with zero active constraints the solution is obtained from previous scheme by deleting the first row and column from system (15).

4 NUMERICAL RESULTS

A series of numerical tests, in plane stress/strain conditions has been performed in order to evaluate the performance of the proposed algorithm in the elastoplastic analysis of 2D problems under the action of various kinds of loads (traction tests characterized by stress concentration, in-plane bending actions) and for different materials (Von Mises and Drucker-Prager materials). The finite elements adopted are those proposed in [4] where the interpolation of the displacement, stress and plastic multiplier fields is adopted. In particular, among the elements proposed in the cited work, only the FC_4 element, with a piecewise-constant interpolation over 4 subareas into which the internal area of the element is divided, has been used. This choice allows to test the robustness of the proposed algorithm with respect to more severe and more nonlinear cases. Further details of the finite elements used can be found in [4].

The convergence to a new equilibrium point will be considered as achieved when the norm of residuals is less than a given tolerance, i.e. $\|\boldsymbol{r}_{\boldsymbol{u}}\|+\|\boldsymbol{r}_{\boldsymbol{\sigma}}\|+\|\boldsymbol{r}_\mu\|\leq$ toll, while the analysis is stopped when the displacement component of a specified point reaches a prescribed value. The number of points required by the Riks strategy to evaluate the equilibrium path will be denoted with steps while the iterations required for each step will be denoted with loops. The arc–length scheme adopted does not use any globalization technique, such as line search. In the case of convergence failure the algorithm simply restarts from the last point evaluated but with a reduced arc–length increment. A line search is performed in the return mapping process of the SD-CPP algorithm to allow the possibility of handling with large step sizes. Moreover, in order to test the robustness of the proposed EC-SQP algorithm, that is the possibility of convergence to an equilibrium point also starting very far from it, the analyses were repeated by increasing the value of the first step length $\Delta\xi^{(1)}$ selected in order to force the value of the observed component of the displacement to a prescribed amplitude. In this way the analysis works with larger step sizes, this last one being controlled through an extrapolation parameter calculated by the formula $\left(1 - \frac{1}{2}\frac{lps - lps_d}{lps + lps_d}\right)$, where lps is the number of loops required by the last step and lps_d the desired number of loops. However if the step is not closed within the maximum number of loops the analysis is restarted with a smaller initial step size. Then, for each test, a report is presented which shows the evaluated collapse load multiplier and the relative error, the number of steps, the total number of loops and the number of step failures (i. e. the steps not closed within lps_m). In this way a deep comparison of the SD-CPP and EC-SQP algorithm is performed. In the following only one test, representing well the behavior of the algorithm, will be presented.

4.1 Square plate with circular hole

This test is depicted in Fig. 1 and is known as square plate with circular hole. The analyses were performed in plane stress conditions on the basis of four different meshes of $(2^n \times 2^n)$ elements each denoted as mesh n and was stopped when the vertical component of node A reaches the value $5e-3$.

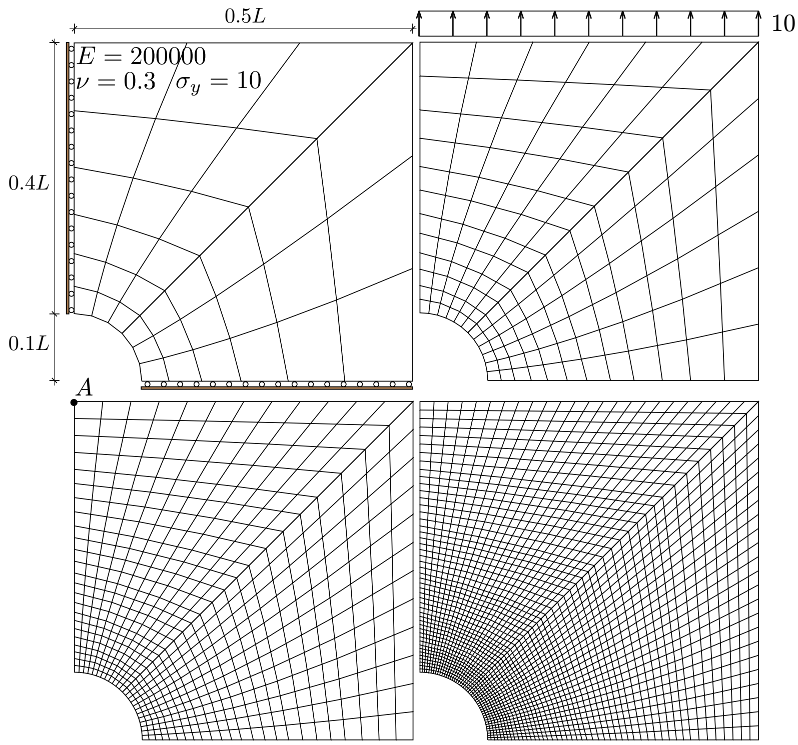

Figure 1: Plate with circular hole. Problem description and discretization meshes.

The results obtained on all the meshes using both the standard SD-CPP algorithm and the new EC-SQP algorithm are summarized in Tab. 4.1. The capability of the proposed algorithm to sustain even very large step sizes without affecting the computed collapse load, regardless of the mesh considered, is clear. In contrast the standard SD-CPP algorithm has no particular problem for the first values of the assigned initial increment of the monitored displacement component, i.e. $5e-5$ and $1e-4$. Afterward, for the bigger increments, the algorithm makes some false steps until the step size is not reduced to a value that can be sustained.

	Mesh 1 (dofs 98)					*Mesh 2 (dofs 338)*				
	λ_c	stps	lps	flrs	incr.	λ_c	stps	lps	flrs	incr.
SD-CPP	0.8100	20	58	0	5e-5	0.8030	23	77	0	5e-5
	0.8100	18	55	0	1e-4	0.8030	20	73	1	1e-4
	0.8100	19	55	4	1e-3	0.8030	21	72	4	1e-3
	0.8100	21	57	7	5e-3	0.8030	25	84	7	5e-3
	0.8100	22	65	8	1e-2	0.8030	25	86	8	1e-2
EC-SQP	0.8100	20	58	0	1e-5	0.8030	24	87	0	5e-5
	0.8100	18	58	0	1e-4	0.8030	21	83	0	1e-4
	0.8100	6	32	0	1e-3	0.8030	9	72	0	1e-3
	0.8100	2	37	0	5e-3	0.8030	2	54	0	5e-3
	0.8100	1	20	0	1e-2	0.8030	1	28	0	1e-2
	Mesh 3 (dofs 1250)					*Mesh 4 (dofs 4802)*				
	λ_c	stps	lps	flrs	incr.	λ_c	stps	lps	flrs	incr.
SD-CPP	0.8015	27	105	0	5e-5	0.8006	28	113	1	5e-5
	0.8015	22	91	0	1e-4	0.8006	24	109	1	1e-4
	0.8015	25	97	4	1e-3	0.8006	27	118	6	1e-3
	0.8015	28	106	7	5e-3	0.8006	32	136	8	5e-3
	0.8015	27	100	8	1e-2	0.8006	31	128	8	1e-2
EC-SQP	0.8015	27	111	0	5e-5	0.8006	34	157	0	5e-5
	0.8015	25	112	0	1e-4	0.8006	29	143	0	1e-4
	0.8015	9	91	0	1e-3	0.8006	10	103	0	1e-3
	0.8015	2	86	0	5e-3	0.8005	2	77	1	5e-3
	0.8015	1	37	0	1e-2	0.8005	1	43	1	1e-2

Table 1: Plate with circular hole. Analysis report, $v_{Amax} = 5e-3$, $toll = 1e-4$, $desired = 6$, $max = 50$.

5 CONCLUSIONS

In this paper a new method for the incremental elastoplastic analysis of structures has been presented. The method is based on a SQP approximation of the finite element representation of the holonomic step equations that retains as primary variables, and at each iteration, all the variables of the problems. In the solution process, based on the equality constrained approach, the set of active constraints is obtained by solving a simple quadratic programming problem which has the same structure and variables of a standard return mapping by closest point projection scheme, i.e. it is decoupled and it can be solved at a local level (finite element, Gauss point). The solution of the equality constraint problems is performed by means of a static condensation of the locally defined variables, that is stress and plastic multiplier parameters, for which the inter element continuity is not required so obtaining at the global level a nonlinear pseudo-compatible scheme of analysis that has the same structure as classic path following arc-length methods.

The numerical results are performed for plane stress/strain problems using both von Mises and Drucker-Pragher yield functions and adopting the finite element interpolation proposed in [4]. This finite element uses a three field interpolation and requires a multi-surface return mapping solution in the SD-CPP case, representing a good test for the robustness and efficiency of the incremental elastoplastic algorithm proposed here. A large number of numerical results performed for both single or multi–surface elastoplastic cases shows the great improvement in robustness and efficiency with respect to standard return mapping strain driven formulations.

The presentation and the application are limited to the perfect plasticity case but its extension to other more complex associated cases would be simple.

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