PLASTIC TORSIONAL ANALYSIS OF STEEL MEMBERS

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Summary. A plastic torsional analysis of structural steel I-section members subject to torsion is presented in this paper. A method of plastic torsional analysis that is much simpler than elastic analysis is proposed, and it is validated against results obtained from advanced computational plastic models. The load factor at plastic collapse is obtained from the sum of the independent load factors for uniform-torsion plastic collapse and warping-torsion plastic collapse. The proposed plastic torsional analysis allows a method of plastic design to be used for torsion that is much simpler and more economical than first yield design. The use of plastic analysis and design will facilitate the design of steel torsion members and lead to more economical structures.

1 INTRODUCTION

Despite their importance, torsional actions are rarely considered in the design of steel structures, because of the difficulty in analysing them. Torsion in a thin-walled steel member is resisted by a combination of the resistance to uniform torsion developed by shear stresses, and the resistance to warping torsion developed by equal and opposite flange bending and shear actions. While an elastic theory for combining these two torsional actions is well developed, its solutions are sufficiently difficult to discourage its use in routine design.

Elastic analysis can be used for first-yield designs. However, this is likely to be extremely conservative, not only because of the significant difference between first yield in a cross-section and its full plasticity, but also because of the unaccounted for but significant reserve of strength. This situation contrasts strongly with that in the design braced steel beams, where the use of plastic analysis not only simplifies the analysis of redundant beams, but also allows due account to be taken of both the difference between first yield and full plasticity, and of the plastic redistribution as the collapse mechanism forms.

The results of advanced computational elastic-plastic analyses reported by Pi and Trahair[1, 2] and Pi et al.[3] have validated a simple method of predicting plastic torsional collapse. In this method, the full plastic collapse capacities of a member in torsion are evaluated separately for uniform and warping torsion, and then added together. This method allows a direct transfer of the methods of analysis for the plastic collapse of beams in bending to the plastic collapse of members under non-uniform torsion.

This paper explains and demonstrates the use of this method of analysing the plastic collapse of members in torsion. The method is simple to use, and comparisons with experimental results and computational non-linear elastic-plastic analysis[1-3] have demonstrated that it is conservative. The use of plastic analysis avoids the conservatism of first-yield analysis and design, because it accounts for the spread of plasticity across the critical sections and the redistribution of torsional actions. Design that is based on plastic analysis of torsion will lead to significant economies over first yield designs based on elastic analysis, and is much needed in design codes of practice for steel structures.

2 TORSIONAL BEHAVIOUR

2.1 Linear elastic behaviour

For the linear elastic torsional analysis of steel beams, the steel is assumed to be linear and the twist rotations are assumed to be small, so that the twists are proportional to the applied torques as shown in Fig. 1.

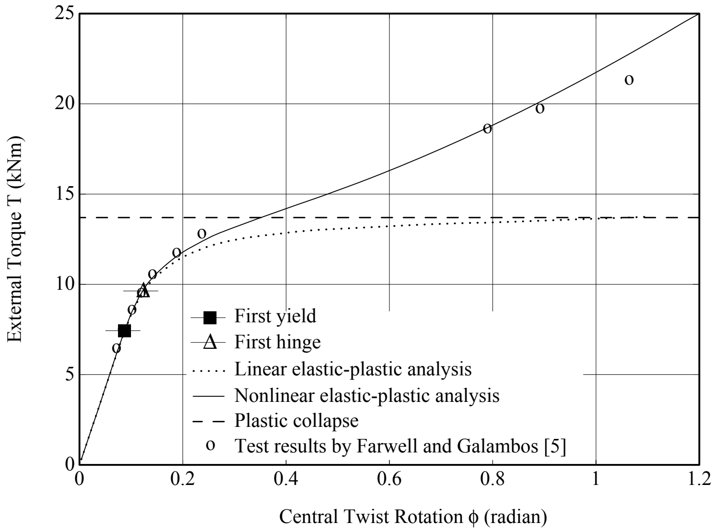

Figure 1: Torsional behaviour

The elastic methods of linear torsion analysis are well established[4]. The engineering method consists of two parts: cross-section analysis that relates the stresses to the stress resultants; and linear member analysis that relates the twist rotations and stress resultants to the applied torsional loading. The combination of these two parts allows the twist rotations and stresses to be predicted. The linear elastic method of torsion analysis is most logically used for serviceability design. Under service loading, most of the member remains elastic, and the linear elastic analysis closely predicts twist rotations. These and any related deflections can then be assessed by comparing them with what are considered to be limiting values. Linear elastic torsion analysis is used less logically for strength design, because yielding usually takes place well before the ultimate torsion capacity is reached. Nevertheless, the absence of accepted methods of failure prediction has forced designers to use the linear elastic method to predict the stresses caused by the strength design loads and to compare them with limiting values that are usually related to the yield stress. This method generally gives very conservative strength predictions.

When a member is subjected to a toque m_z distributed uniformly along its length, the differential equation of equilibrium for linear analysis can be written as[4]

$$GJ(\mathrm{d}^2\phi/\mathrm{d}z^2) - EI_w(\mathrm{d}^4\phi/\mathrm{d}z^4) = m_z \,, \quad (1)$$

where E and G are the Young's modulus and shear modulus of elasticity, J is the section torsional constant, I_w is the section warping constant, ϕ is the angle of twist rotation of the cross-section, and z is the coordinate along the member.

Eq. (1) can be solved for different boundary conditions, which may include the kinematic boundary conditions such as twist rotation prevented ($\phi = 0$), warping prevented ($\mathrm{d}\phi/\mathrm{d}z = 0$), warping free ($\mathrm{d}^2\phi/\mathrm{d}z^2 = 0$), and the static boundary condition given by

$$GJ(\mathrm{d}\phi/\mathrm{d}z) - EI_w(\mathrm{d}^3\phi/\mathrm{d}z^3) = M_z \quad (2)$$

where M_z is the toque acting at the beam ends.

The linear elastic behaviour of a member in torsion is terminated by the occurrence of first yield at a point in the member as indicated in Fig. 1. First yield in ductile materials under combined normal and shear stresses is usually modelled using the von Mises (circular) interaction equation. When applied to combinations at a point of warping normal stresses f_w due to warping torsion with shear stresses τ_u due to uniform torsion and warping shear stress τ_w due to warping torsion, this becomes

$$(f_w/f_y)^2 + [(\tau_u + \tau_w)/\tau_y]^2 = 1 \quad \text{with} \quad \tau_y = f_y/\sqrt{3} \quad (3)$$

where f_y is the normal yield stress and τ_y is the shear yield stress.

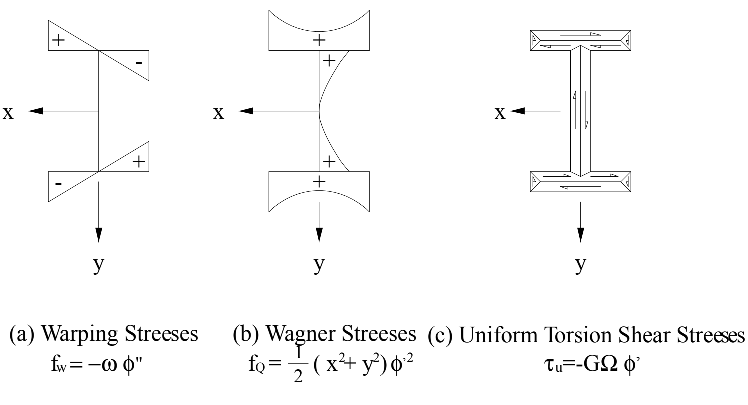

Figure 2: Elastic-torsion

The stresses f_w and τ_u vary in different ways around the cross-section (Fig. 2(a) and 2(c)) and along the member, and so the point at which first yield occurs may be quite difficult to determine. However, the different locations of the maximum values of f_w and τ_u often lead to a low value of one of these coinciding with the maximum value of the other. Consequently, a good approximation for first yield is often obtained by considering the separate conditions $f_w = f_y$ and $\tau_u = \tau_y$.

Numerical methods such as finite element methods can be developed based on Eq. (1) and material nonlinearity such as shown in Fig. 3 for the linear elastic-plastic torsional analysis (i.e. the geometrically linear and material nonlinear analysis). When strain hardening is neglected, the finite element result[2] of the linear elastic-plastic torsional analysis is shown in Fig.1 by the dotted line, which indicates that the torque approaches a limiting value (the broken line) as shown in Fig. 1, which corresponds to plastic collapse of the cross-section.

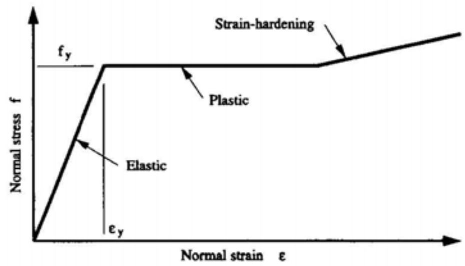

Figure 3: Stress-strain relationship for structural steel

2.2 Nonlinear Elastic-Plastic Behaviour and Large Twists and Failure

The normal stress/normal strain curve of structural steel is usually modelled as being elastic-plastic-strain hardening, as shown in Fig. 3. After first yield occurs, the torque-twist rotation relationship becomes nonlinear, as indicated in Fig. 1. The differential equations of equilibrium for nonlinear elastic-plastic torsion of I-section members are given by[2]

$$\mathrm{d}[AE(\mathrm{d}w/\mathrm{d}z) + EI_P(\mathrm{d}\phi/\mathrm{d}z)^2/2]/\mathrm{d}z = 0 \tag{4}$$

and

$$GJ(\mathrm{d}^2\phi/\mathrm{d}z^2) + \mathrm{d}\{[AE(\mathrm{d}w/\mathrm{d}z) + EI_P(\mathrm{d}\phi/\mathrm{d}z)^2/2)](\mathrm{d}\phi/\mathrm{d}z)\}/\mathrm{d}z - EI_w(\mathrm{d}^4\phi/\mathrm{d}z^4) = m_z \tag{5}$$

where A is the area of the cross-section and I_P is defined by $I_P = \int_A (x^2 + y^2)\mathrm{d}A$.

The corresponding static boundary conditions are given by

$$AE(\mathrm{d}w/\mathrm{d}z) + EI_P(\mathrm{d}\phi/\mathrm{d}z)^2/2 = 0\,, \tag{6}$$

and

$$GJ(\mathrm{d}\phi/\mathrm{d}z) + [AE(\mathrm{d}w/\mathrm{d}z) + EI_P(\mathrm{d}\phi/\mathrm{d}z)^2/2)](\mathrm{d}\phi/\mathrm{d}z) - EI_w(\mathrm{d}^3\phi/\mathrm{d}z^3) = M_z\,. \tag{7}$$

The finite element result based the differential equations of equilibrium for the nonlinear elastic-plastic analysis given by Eqs. (4) and (5) is shown in Fig. 1 by the solid line, which is much higher than the limiting value (the broken line). Hence, it is conservative to use the limiting value of the toque as the plastic collapse toque.

The assumption of linear twist rotation analysis ignores the secondary longitudinal Wagner stresses[5] f_Q associated with the relative extensions of the fibres of the cross-section away from the axis of twist (Fig. 3(b)). At small twist rotations these stresses are secondary and have little effect as shown in Fig. 1. However, at large twist rotations, they lead to significant longitudinal tensions in the regions of the flange tips, which increases the resistance of the member to torsion, as shown in Fig. 1. Final failure of the member is by tensile rupture at the flange tips[5] and at a torque that is significantly higher than the plastic collapse torque[2]. The plastic-collapse torque provides a quite conservative estimate of the strength of a member in torsion. Although this is not unlike the conservatism of bending plastic-collapse mechanisms of beams under moment gradient caused by strain-hardening effects, the degree of conservatism of torsion plastic-collapse mechanisms is significantly greater.

3 PLASTIC-COLLAPSE ANALYSIS OF TORSION

3.1 General

In the simple method described here of analysing the plastic collapse of a member in torsion, independent analyses are made for the plastic collapse in uniform torsion and in warping torsion to determine their collapse load factors λ_{up} and λ_{wp} and the actual plastic collapse load factor λ_p is approximated using

$$\lambda_p = \lambda_{up} + \lambda_{wp}\,. \tag{8}$$

This very simple approximation assumes no interaction at plastic collapse between uniform and warping torsion, and assumes that the separate plastic collapse capacities are additive. The errors due to these assumptions are on the unsafe side when compared with more accurate linear elastic-plastic analyses[2], but are very small and on the safe side when compared with accurate nonlinear elastic-plastic analyses[2], because (1) the warping torsion shear strains are small; (2) the yielding and plasticity interactions between normal and shear stresses, which are described by circular interaction equations given by Eq. (3), are small; (3) sections that are fully plastic due to warping torsion often occur at different locations along the member than those that are fully plastic due to uniform torsion. The unsafe errors caused by these assumptions are more than compensated for by the conservatism of ignoring the strengthening effects of strain hardening, and the strengthening effects of the Wagner stresses at large rotations.

3.2 Uniform-Torsion Plastic Collapse

When uniform torsion provides the only method of resisting applied torques, a collapse mechanism develops when a sufficient number of cross-sections of the member become fully plastic in uniform torsion, as shown for example, in Fig. 4.

These fully plastic sections are usually located at the supports where the reaction torques act. The sand-heap analogy[2, 4] can be used to analyse a thin-walled open section to approximate the uniform-torsion plastic torque

$$M_{up} \approx \tau_y \sum (bt^2/2) \tag{9}$$

in which b and t are the width and thickness of each rectangular element of the cross-section. For an I-section, the uniform torsion plastic torque is given by

$$M_{up} = \tau_y \left[b_f t_f^2 \left(1 - t_f / 3b_f\right) + b_w t_w^2 / 2 + t_w^3 / 6 \right] , \tag{10}$$

in which b_f is the flange width; t_f is the flange thickness; b_w is the web depth between flanges; and t_w is the web thickness.

An example of a uniform-torsion plastic-collapse mechanism is shown in Fig. 4. In this case, a general rigid-body twist rotation occurs when a uniform-torsion plastic hinge has formed at each end. In general, plastic hinges will develop progressively until the collapse mechanism forms. For the example shown in Fig. 4, the uniform-torsion plastic-collapse load factor is

$$\lambda_{up} = 2M_{up} / T . \tag{11}$$

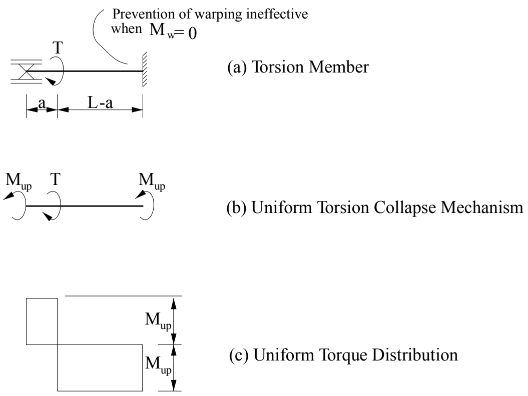

Figure 4: Uniform torsion plastic collapse

It is noted in Fig. 4 that preventing warping at the ends is ineffective in uniform torsion, because warping torsion is not accounted for. Other examples of uniform-torsion plastic-collapse mechanisms are shown in Fig. 5.

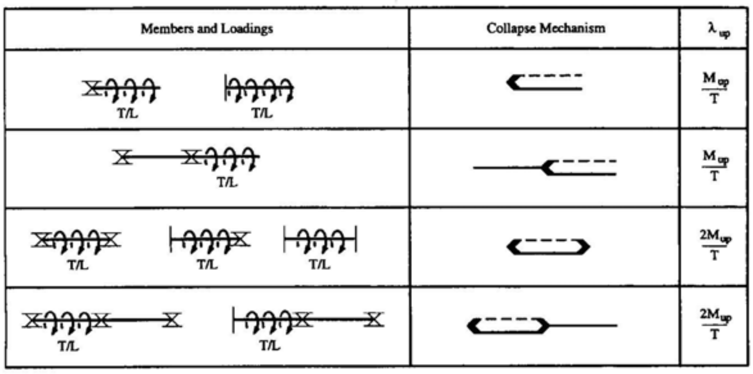

Figure 5: Uniform-torsion plastic-collapse mechanism

3.3 Warping Torsion Plastic Collapse

When warping torsion provides the only method of resisting applied torques, a collapse mechanism develops when there are a sufficient number of warping hinges (frictionless or plastic) to transform the member into a mechanism. In the case of an equal-flanged I-section member, these warping hinges transform each flange into a flexural collapse mechanism, as shown in Fig. 6. Warping hinges often occur at supports or at points of concentrated torque. When there are distributed torques, then the location of the warping hinges may lie between points of concentrated torque, and may not be conspicuous. In this case, the location may be guessed, and the upper- and lower-bound techniques of plastic analysis[4] are used to determine a sufficiently accurate collapse load factor.

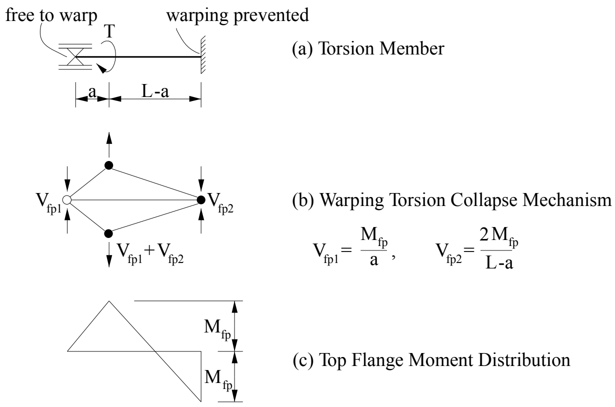

Figure 6: Warping torsion plastic collapse

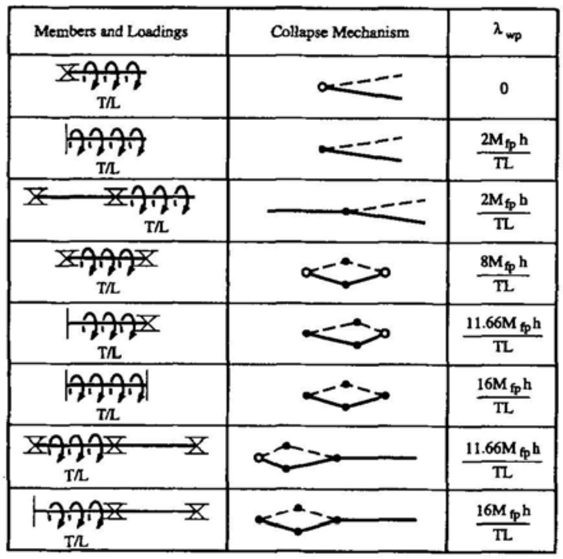

Figure 7: Warping torsion plastic collapse mechanism

The fully plastic bimoment B_p at which warping hinges form in an equal-flanged I-section is given as[4]

$$B_p = M_{fp} h \quad \text{with} \quad M_{fp} = f_y b_f^2 t_f / 4 \quad \text{and} \quad h = b_w + t_f \tag{12}$$

in which M_{fp} is the flange plastic moment and h is the distance between the flange centroids.

The warping-torsion plastic-collapse load factor λ_{wp} of an equal-flanged I-section torsion member can be found from the flexural plastic collapse loads for the flanges, which can be analysed by the methods of analysing plastic collapse in flexural structures[4]. An example of a warping-torsion plastic-collapse mechanism is shown in Fig. 6. In this case, there are frictionless hinges at the end that is free to warp, and general twist rotation occurs when each flange forms a collapse mechanism with plastic hinges at the other end and within the member length. In general, plastic hinges will develop progressively until the flange collapse mechanisms form. The warping plastic torque for the example in Fig. 6 is given by

$$M_{wp} = V_{fp} h \quad \text{with} \quad V_{fp} = \frac{\mathrm{d}M_f}{\mathrm{d}z} \tag{13}$$

with the gradient of the flange moment $\mathrm{d}M_f/\mathrm{d}z$ being given by M_{fp}/a and $2M_{fp}/(L\text{-}a)$ (Fig. 6(b)). Thus the warping torsion collapse load factor is given by

$$\lambda_{wp} = \frac{M_{wp}}{T} = \frac{M_{fp} h}{T}\left(\frac{1}{a} + \frac{2}{L-a}\right). \tag{14}$$

Other examples of warping-torsion plastic-collapse mechanisms are shown in Fig. 7.

4 A NUMERICAL EXAMPLE

A 4,500 mm long torsion member and its torsional loading are shown in Fig. 8. The I-section dimensions are b_f =420 mm; t_f =30 mm; b_w =400 mm; and t_w = 22 mm. The normal yield stress is f_y = 250 N/mm^2; and the uniformly distributed torque is 150 kNm/m.

From Eq. (3), the shear yield stress is obtained as $\tau_y = f_y/\sqrt{3} = 250/\sqrt{3} = 114.34$ N/mm^2.

The uniform torsion collapse toque can be obtained from Eq. (10), as

$$\begin{aligned} M_{up} &= \tau_y \left[b_f t_f^2 \left(1 - t_f / 3b_f\right) + b_w t_w^2 / 2 + t_w^3 / 6 \right] \\ &= 114.34 \times \{420 \times 30^2 \times [1 - 20/(3 \times 420)] + 400 \times 22^2 / 2 + 22^3 / 6\} = 53.81 \text{KNm} \end{aligned} \tag{15}$$

while the warping torsion collapse toque can be obtained from Eq. (12) as

$$M_{fp} = f_y b_f^2 t_f / 4 = 250 \times 420^2 \times 30 / 4 = 330.75 \text{KNm}. \tag{16}$$

The uniform torsion collapse factor is the obtained from Eq. (11) as

$$\lambda_{up} = 2M_{up} / T = 2 \times 53.81 \times 10^6 / (150 \times 10^3 \times 4500) = 0.1594. \tag{17}$$

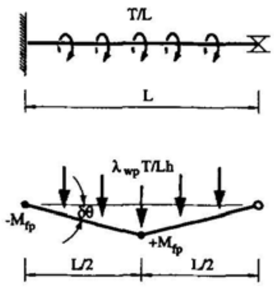

Figure 8: Worked example

The warping-torsion flange plastic-collapse mechanism is not obvious, but an upper-bound solution can be obtained by assuming that plastic flange hinges occur at mid-span, as shown in Fig. 8. In this case, a virtual work analysis of each flange mechanism leads to

$$\delta W = (\lambda_{wp} T / Lh)(L^2 \delta\theta / 4) \tag{18}$$

for the external work done by the distributed flange loads *T/Lh* , and

$$\delta U = 3M_{fp} \delta\theta \tag{19}$$

for the internal work absorbed at the plastic hinges, in which $\delta\theta$ is the virtual rotation of each half flange.[4] An upper bound is obtained from the inequality $\delta W \leq \delta U$, so that

$$\lambda_{wp} \leq 12 M_{wp} h / TL^2 . \tag{20}$$

A lower bound is found by determining that at this load factor, the maximum flange moment is $25M_{fp}/24$, so that

$$\lambda_{wp} \leq 12 M_{wp} h / TL^2 \times (24/25) = 11.52 M_{wp} h / TL^2 , \tag{21}$$

which is very close to the exact solution[4]

$$\lambda_{wp} = \frac{(6 + 4\sqrt{2}) M_{fp} h}{TL^2} \approx 11.66 M_{fp} h / TL^2 . \tag{22}$$

The warping torsion collapse factor can then be obtained as

$$\lambda_{wp} = \frac{11.66 M_{fp} h}{TL^2} = \frac{11.66 \times 330.75 \times 10^6 \times 430}{150 \times 10^3 \times 4500^2} = 0.546 . \tag{23}$$

Finally, the torsion collapse factor is obtained from Eq. (8) as

$$\lambda_p = \lambda_{up} + \lambda_{wp} = 0.1594 + 0.546 = 0.7054 . \tag{24}$$

5 CONCLUSIONS

- Hand methods for the linear elastic analysis of the non-uniform torsion of thin-walled steel members are difficult to carry out, but first-yield methods of design based on elastic analyses are unnecessarily conservative.

- This paper presents and demonstrates a very simple method of analyzing the plastic collapse of equal-flanged I-section members in torsion. The method can be used manually, and is no more difficult than the plastic-collapse analysis of beams in bending. Comparisons have demonstrated that the method produces predictions that are very close to those of more accurate linear elastic-plastic analyses, and substantially less than test results and predictions of nonlinear large-rotation elastic-plastic analyses.
- The use of plastic analysis avoids the conservatism of first yield analysis and design, because it accounts for the spread of plasticity across the critical sections and the redistribution of torque that occurs in redundant members after the first hinge forms. Designs based on plastic analyses of torsion will lead to significant economies over first yield designs based on elastic analysis.

ACKNOWLEDGMENTS

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