XI International Conference on Computational Plasticity. Fundamentals and Applications
COMPLAS XI
E. Oñate, D.R.J. Owen, D. Peric and B. Suárez (Eds)

A DETERMINISTIC FILTER FOR ESTIMATION OF PARAMETERS DESCRIBING INELASTIC HETEROGENEOUS MEDIA

HERMANN G. MATTHIES and BOJANA V. ROSIĆ

Institute of Scientific Computing
Technische Universität Braunschweig
38106 Braunschweig
e-mail: wire@tu-bs.de

Key words: Bayesian linear estimation, uncertainty updating, stochastic plasticity

Abstract. We present a new, fully deterministic method to compute the updates for parameter estimates of quasi-static plasticity with combined kinematic and isotropic hardening from noisy measurements. The materials describing the elastic (reversible) and/or inelastic (irreversible) behaviour have an uncertain structure which further influences the uncertainty in the parameters such as bulk and shear modulus, hardening characteristics, etc. Due to this we formulate the problem as one of stochastic plasticity and try to identify parameters with the help of measurement data. However, in this setup the inverse problem is regarded as ill-posed and one has to apply some of regularisation techniques in order to ensure the existence, uniqueness and stability of the solution. Providing the apriori information next to the measurement data, we regularize the problem in a Bayesian setting which further allow us to identify the unknown parameters in a pure deterministic, algebraic manner via minimum variance estimator. The new approach has shown to be effective and reliable in comparison to most methods which take the form of integrals over the posterior and compute them by sampling, e.g. Markov chain Monte Carlo (MCMC).

1 INTRODUCTION

The deterministic description of the properties of heterogeneous material and their parameter identification are not quite suitable having in mind the uncertainties arising at the micro-structural level. Thus, we may consider them as unknown and try to identify from the given experimental data in a Bayesian manner. By measuring some quantities of interest such as deformation, one cannot estimate the material characteristics (Young's modulus, shear and bulk modulus, etc) straightforwardly since the problem is ill-posed and suffers from the issues of the existence, uniqueness and stability of the solution. In order to regularize the problem, we may give additional information next to the measurement data in a form of distribution function of unknown characteristics — so called a priori information. This kind of regularization technique is known as Bayesian.

In recent studies [2, 8, 9, 11, 13, 21], the Bayesian estimates of the posterior density are taking the forms of integrals, computed via asymptotic, deterministic or sampling methods. The most often used technique represents a Markov chain Monte Carlo (MCMC) method [5, 9, 13], which takes the posterior distribution for the asymptotic one. This method has been improved by introducing the stochastic spectral finite element method [14] into the approximation of the prior distribution and corresponding observations [11, 13]. This group of methods is based on Bayes formula itself. Another group belongs to so-called 'linear Bayesian' [6] methods, which update the functionals of the random variables. The simplest known version represents the Kalman-type method [4, 1, 3, 22].

In order to avoid the sampling procedure required by previous methods, we describe the minimum variance estimator based on 'white noise' analysis. Starting with a probabilistic model for the uncertain parameters (the maximum entropy principle) we cast the identification problem in a direct algebraic estimation framework, which has shown to be effective and reliable. In this way not only that the sampling at any stage of the identification procedure is avoided but also the assumption of gaussianity.

The paper is organized as follows: in first section we briefly describe the forward problem [1], then we introduce the update procedure concentrating on one parameter, i.e. shear modulus and give the numerical results for two test problems in plain strain conditions.

2 FORWARD PROBLEM

Consider a material body occupying a bounded domain $\mathcal{G} \in \mathbb{R}^d$ with a piecewise smooth Lipschitz continuous boundary $\partial\mathcal{G}$ on which are imposed boundary conditions in Dirichlet and Neumann form on $\Gamma_D \subseteq \partial\mathcal{G}$ and $\Gamma_N \subset \partial\mathcal{G}$ respectively, such that $\Gamma_D \cap \Gamma_N = \varnothing$ and $\partial\mathcal{G} = \bar{\Gamma}_N \cup \bar{\Gamma}_D$ [20, 7, 10]. The probability space is defined as a triplet $(\Omega, \mathcal{B}, \mathbb{P})$, with $\mathcal{B}$ being a σ-algebra of subsets of Ω and $\mathbb{P}$ a probability measure. The balance of momentum localized about any point x in domain $\mathcal{G}$ in time $t \in \mathcal{T} := [0, T]$ leads to an equilibrium equation and boundary conditions required to hold almost surely in ω, i.e. $\mathbb{P}$-almost everywhere:

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} \quad \text{on } \mathcal{G}, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{g}, \quad \text{on } \Gamma_N, \\ \boldsymbol{u} &= \mathbf{0}, \quad \text{on } \Gamma_D \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ denote the displacement and velocity fields over $\mathcal{G}$, $\boldsymbol{f}$ the body force, $\boldsymbol{\sigma}$ stress tensor, $\boldsymbol{n}$ the exterior unit normal at $x \in \Gamma_N$, and $\boldsymbol{g}$ a prescribed surface tension. For the sake of simplicity we use homogeneous Dirichlet boundary conditions and under

[1] The paper with this subject is also part of the proceedings, search for authors

the assumptions of small deformation theory we introduce the strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) = D\boldsymbol{u}$, with the linear bounded operator defined as a mapping $D : \boldsymbol{u}_1(x)\boldsymbol{u}_2(\omega) \rightarrow (\nabla_S \boldsymbol{u}_1(x))\boldsymbol{u}_2(\omega)$.

The strong form of equilibrium equation given by Eq. (1) may be reformulated in a variational setting, which includes the uncertain parameters such as the elastic and hardening properties, yield stress, loading etc. These quantities are modeled as random fields/processes approximated by polynomial chaos and Karhunen Loève expansions [14], which further allow the use of the stochastic Galerkin projection procedures in low-rank and sparse format [16]. In other words, the problem collapses to the constrained-stochastic optimization one, where the closest distance in the energy norm of a trial state to the convex set of the elastic domain is found by a stochastic closest point projection algorithm[15, 19].

3 ESTIMATION OF SHEAR MODULUS VIA DIRECT GENERAL BAYESIAN APPROACH

In the scope of this paper we show the procedure of identifying one representative property q of elastoplastic material called shear modulus, often denoted by G. As this property is regarded as positive definite, we take the lognormal distribution with appropriate covariance function as a corresponding a priori information and solve the forward problem in Eq. (1) in order to obtain the solution (stress, displacement, etc.). The functional of the solution represents the 'forecast' measurement y further used in the update procedure. Besides this, one employs the information gathered by experimental (here simulated) measurements which are disturbed by some additional independent noise ϵ.

Let us define the random variable q as a measurable mapping [18]:

$$q : \Omega \rightarrow \mathcal{Q}, \tag{2}$$

where $\mathcal{Q}$ is a deterministic Hilbert space. If we denote the space of random variables with finite variance as $S := L_2(\Omega)$, then $\mathcal{Q}$-valued random variables belong to a space $L_2(\Omega, \mathcal{Q}) := \mathcal{Q} \otimes S$, obtained as a tensor product of corresponding deterministic and stochastic spaces.

According to previous definitions, we may define the linear measurement as a random variable/field:

$$z = \hat{y} + \epsilon, \tag{3}$$

where

$$\hat{y} = Y(q, u), \quad \hat{y} \in \mathcal{Y} \tag{4}$$

represents some functional of the solution linear in parameter q. The measurement error is here assumed to be Gaussian with prescribed covariance C_ϵ. Thus, the measurement is random and belongs to a subspace $\mathscr{Y}_0 \subseteq \mathscr{Y} := \mathcal{Y} \otimes S$.

In the same manner, the a priori information q_f belongs to a closed subspace $\mathscr{Q}_f \subset \mathscr{Q}$, allowing the prediction of the observation y via linear mapping $H : \mathcal{Q} \rightarrow \mathcal{Y}$, i.e.

$$y = Hq_f, \quad y \in \mathscr{Q}_0 = H^*(\mathscr{Y}_0). \tag{5}$$

Collecting these two informations, we are able to estimate the posterior $q_a \in \mathscr{Q}$ as the orthogonal or minimum variance projection of q onto the subspace $\mathscr{Q}_f + \mathscr{Q}_0$ [18, 17, 12]:

$$q_a(\omega) = q_f(\omega) + K(z(\omega) - y(\omega)), \quad K := C_{q_f y}\,(C_y + C_\epsilon)^{-1} \tag{6}$$

where q_f is the orthogonal projection onto $\mathcal{Q}_f$, K the "Kalman gain"operator and C appropriate covariances.

In order to numerically compute the previous estimate Eq. (6) one introduces a projection operator P, which projects the parameter set onto the subspace $\hat{\mathscr{Q}} := \mathcal{Q}_N \otimes S_J$. Here $\mathcal{Q}_N$ is a finite element discretization of a deterministic space $\mathcal{Q}$ and S_J the discretization of the stochastic space, obtained by taking the polynomial chaos expansion (PCE) of the solution space as the ansatz function. The projection of Eq. (6) then reads:

$$\hat{q}_a(\omega) = \hat{q}_f(\omega) + K(\hat{z}(\omega) - \hat{y}(\omega)), \tag{7}$$

giving the PCE of posterior $\hat{q}_a(\omega)$ as the final estimate, from which all other further properties, such as statistical moments, probability density functions, etc. are efficiently computed.

4 NUMERICAL RESULTS

Two test problems in plane strain conditions are considered: rectangular strip with hole Fig. (1) and Cooke's membrane Fig. (6). Due to lack of measurement data, we describe the virtual reality by constant value of shear modulus (deterministic truth), and simulate experiment by measuring the shear stress as the most suitable quantity (depends linearly on parameter). The measurements are preformed in all nodal points (including boundary conditions) obtained by finite element discretization of the domain of consideration with the help of eight-noded quadrilateral elements.

4.1 Plate with hole

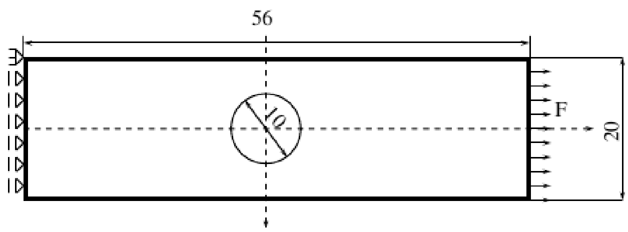

Figure 1: Geometry of the problem: plate with hole

The update procedure is performed as a sequential row of several measurements and updates. In each sequential step the loading is changed by its intensity and sign in prescribed manner. In this particular example the loading changes such that the force is first of extension type, and then compression. The identification starts by measuring the shear stress Fig. (2) in the sensors positions, assuming the a priori distribution Fig. (3) as a lognormal random field and then updating the parameter to q_a which is in the next update taken as a priori distribution. This new update is characterized by a new loading and hence new measurement set of data. This cycle repeats until the convergence is achieved.

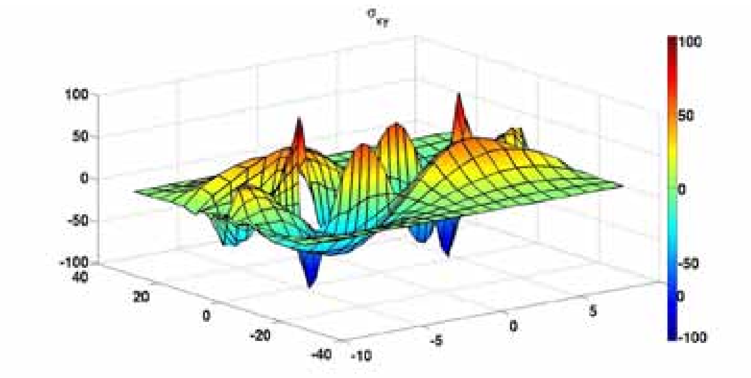

Figure 2: The shear stress as a measurement

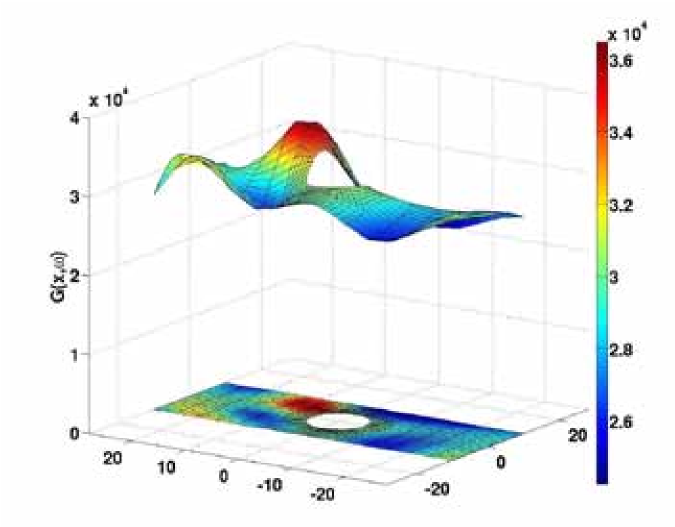

Figure 3: Apriori G - one realisation

The posterior distribution with each update changes in the direction of the truth, which is reflected in the convergence given by a RMSE (root mean squared error) between the

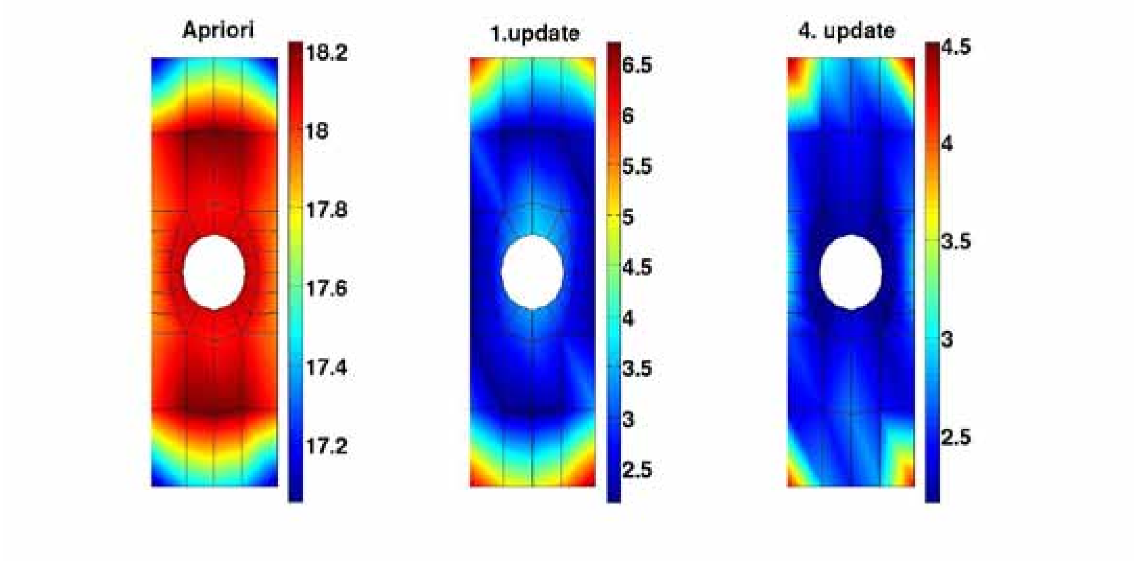

Figure 4: Relative RMSE of variance [%] after 4th update in 10% equally distributed measurement points

PCEs of the posterior $\hat{\boldsymbol{q}}_f^i$ and truth $\hat{\boldsymbol{q}}_t$ in each update i:

$$\varepsilon^i = \frac{\|\hat{\boldsymbol{q}}_f^i - \hat{\boldsymbol{q}}_t\|_{L_2(\Omega)}}{\|\hat{\boldsymbol{q}}_t\|_{L_2(\Omega)}}, \quad i = 1, .., I \tag{8}$$

As one may see in Fig. (4) the error decreases with each update, such that in 4th step one obtains the error circa 2% in almost all points besides the corners where it increases a bit. The possible explanation of such behavior is the existence of the boundary conditions imposed in these nodes.

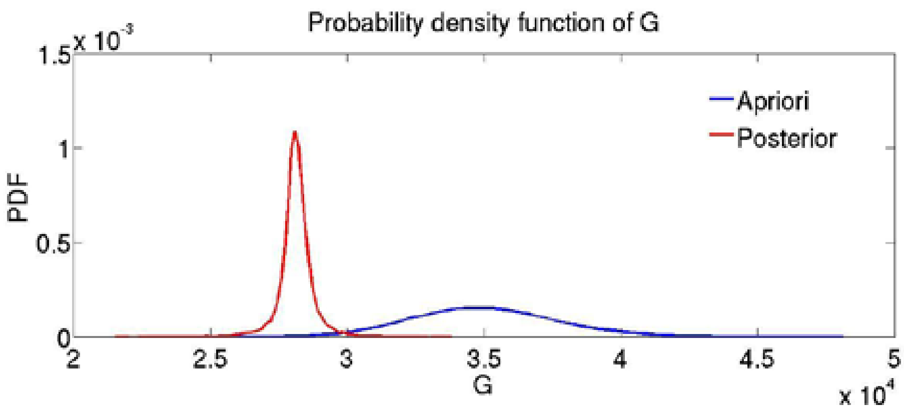

Figure 5: Comparison of apriori and posterior distribution

The comparison of probability density functions of prior and posterior after first update lead us to the same conclusion, since the posterior is much more narrowed and goes in the direction of the truth which is deterministic.

4.2 Cooke's membrane

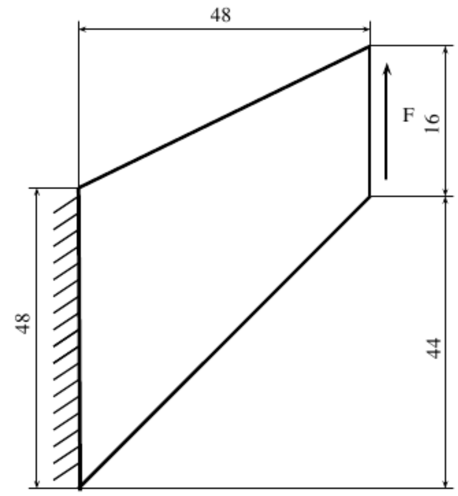

Figure 6: Geometry of the problem: Cooke's membrane

The Cooke's membrane is subjected to load in y direction on the right boundary and constrained on the left as it is shown in Fig. (6). The identification of shear modulus is done in the similar manner as in previous example, by sequential procedure. Thus, the measurement in each update changes since the loading alters according to some prescribed scheme. This change one may see in Fig. (7) from the first up to the third update.

After the third update in Fig. (8) one may notice that the initial lognormal field changes to a uniform value of shear modulus over spatial domain with the small variation on the boundary (see Fig. (9)). The same conclusion is made by calculating the root mean square error as before (see Fig. (10)), as well as comparison of probability density functions, see Fig. (11).

5 CONCLUSION

The mathematical formulation of the stochastic inverse elasto-plastic problem is recast via projection of the minimum variance linear Bayesian estimator onto the polynomial chaos basis. The update requires one solution of the stochastic forward problem via stochastic Galerkin method based on model reduction techniques together with the stochastic closest point projection method. The estimation is purely deterministic and doesn't require sampling at any stage, as well as assumption of linearity in the forward model and Gaussian statistics.

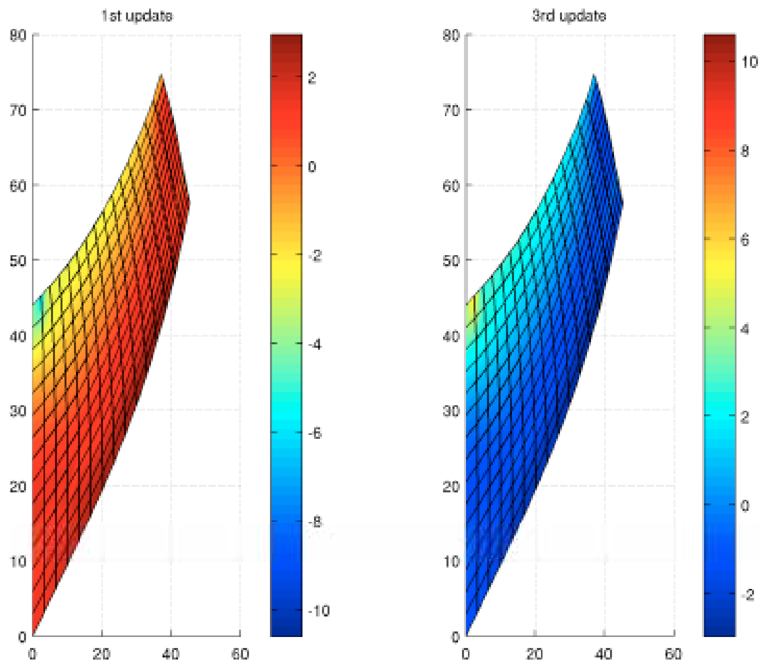

Figure 7: The measurment mean value in a case of plate with hole

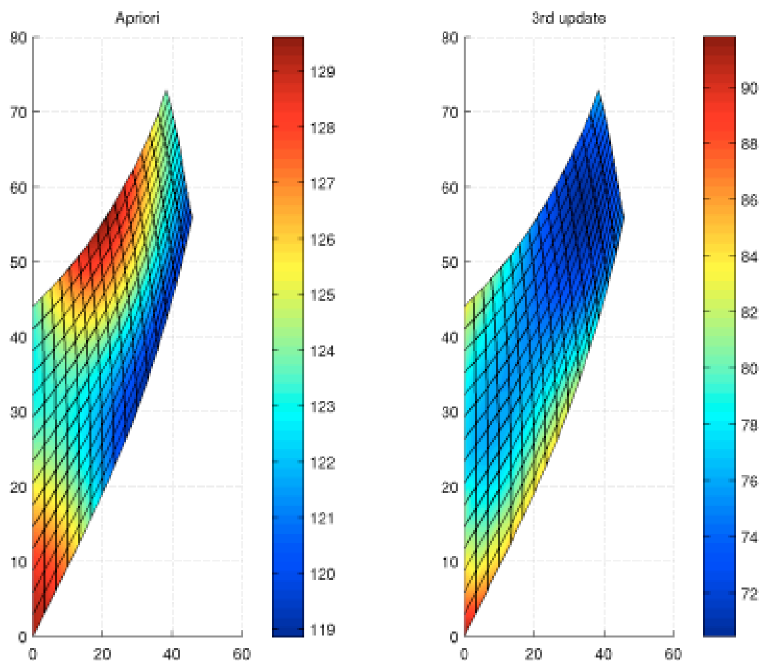

Figure 8: The realisations of apriori of shear modulus and its update

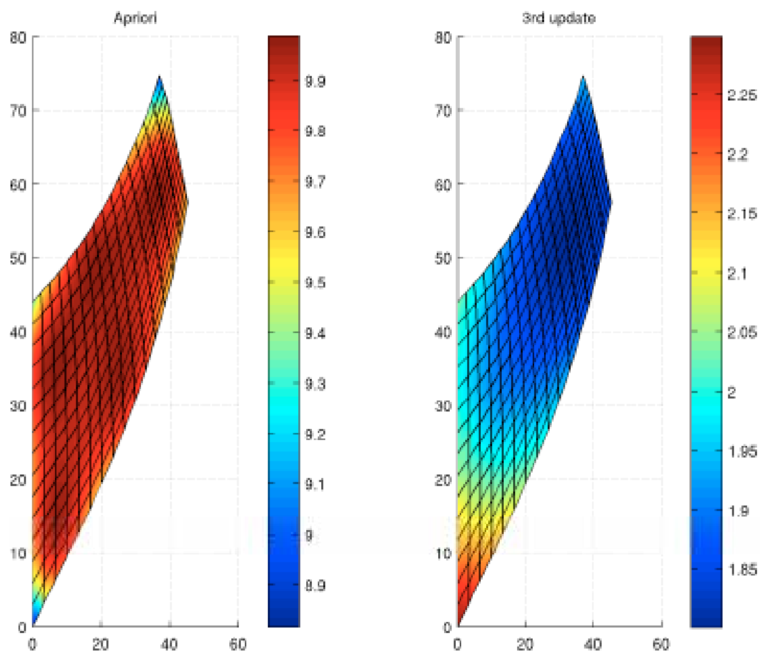

Figure 9: Change of variance of shear modulus from apriori to 3rd update

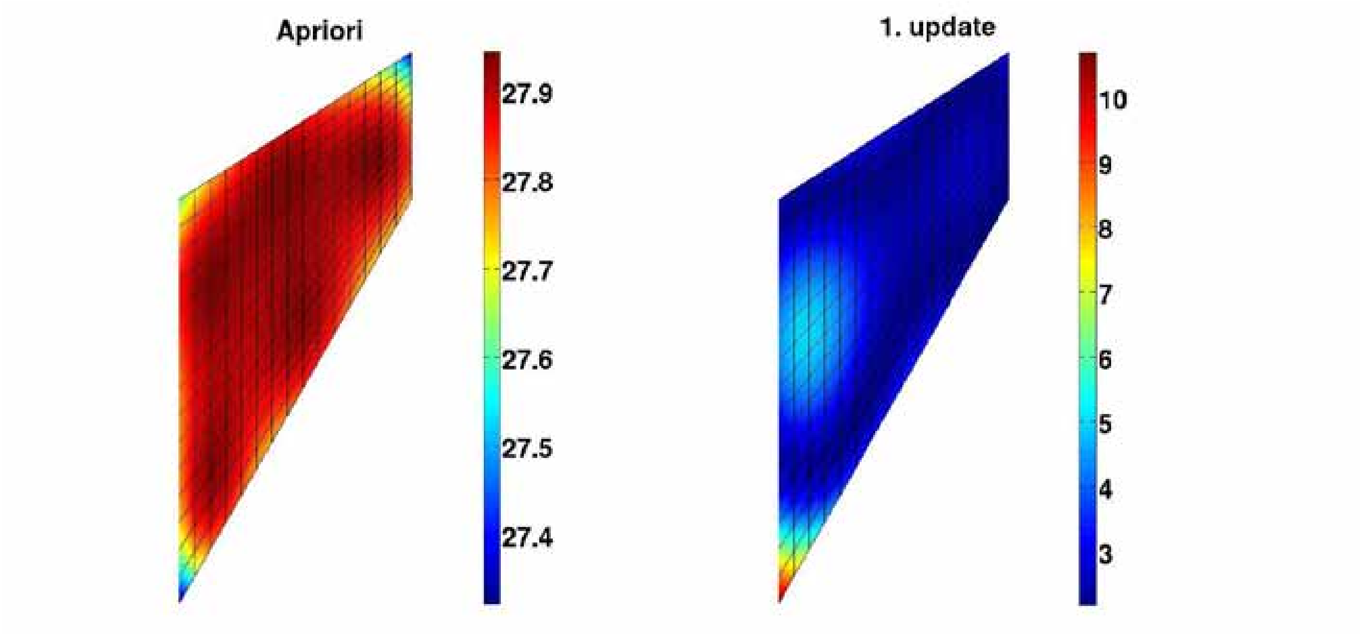

Figure 10: RMSE [%] of variance after first update for the case when mean is larger 30% than truth

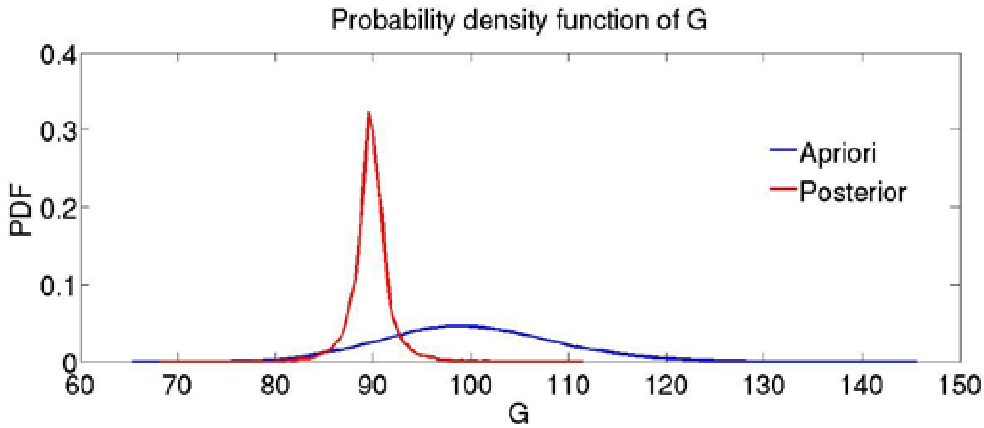

Figure 11: Comparison of apriori and posterior distribution

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