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PLASTICITY DESCRIBED BY UNCERTAIN PARAMETERS - A VARIATIONAL INEQUALITY APPROACH -

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Abstract. In this paper we consider the mixed variational formulation of the quasi-static stochastic plasticity with combined isotropic and kinematic hardening. By applying standard results in convex analysis we show that criteria for the existence, uniqueness, and convergence can be easily derived. In addition, we demonstrate the mathematical similarity with the corresponding deterministic formulation which further may be extended to a stochastic variational inequality of the first kind. The aim of this work is to consider the numerical approximation of variational inequalities by a "white noise analysis". By introducing the random fields/processes used to model the displacements, stress and plastic strain and by approximating them by a combination of Karhunen-Loève and polynomial chaos expansion, we are able to establish stochastic Galerkin and collocation methods. In the first approach, this is followed by a stochastic closest point projection algorithm in order to numerically solve the problem, giving an intrusive method relying on the introduction of the polynomial chaos algebra. As it does not rely on sampling, the method is shown to be very robust and accurate. However, the same procedure may be applied in another way, i.e. by calculating the residuum via high-dimensional integration methods (the second approach) giving a non-intrusive Galerkin techniques based on random sampling—Monte Carlo and related techniques—or deterministic sampling such as collocation methods. The third approach we present is in pure stochastic collocation manner. By highlighting the dependence of the random solution on the uncertain parameters, we try to investigate the influence of individual uncertain characteristics on the structure response by testing several numerical problems in plain strain or plane stress conditions.

1 Introduction

The deterministic description of the inelastic behaviour [4, 10] is not applicable to heterogeneous materials due to the uncertainty of corresponding characteristics at the micro-structural level. Namely, the deterministic approach has one disadvantage: the description of the material parameters is given by the first order statistical moment called a mean value or mathematical expectation. However, such representation neglects the most important property of material characteristics — their random nature. Due to this reason, we consider a mathematical model which approximates material parameters as random fields and processes in order to closely capture the real nature of the random phenomena.

The history of the stochastic elastoplasticity begins with the work of Anders and Hori [1]. They declared elastic modulus as the source of the uncertainty and treated all following subsequent uncertainties with the help of a perturbation technique. Thereafter, Jeremić [6] introduced the Fokker-Plank equation approach based on the work of Kavvas [85], who obtained a generic Eulerian-Lagrangian form of the Fokker-Plank equation, exact to second-order, corresponding to any nonlinear ordinary differential equation with random forcing and random coefficient. In other words, Jeremic and his co-workers have obtained the deterministic substitute of the original stochastic partial differential equation. However, these methods are or mathematically very complicated to deal with or not enough accurate to be used for. Namely, the perturbation technique is limited only on the problems described by small variation of input properties. Its another disadvantage is known as a "closure-problem " or dependence of the lower-order moments on the higher-order moments. Similarly, the Fokker-Planck method predicates the mean behaviour exactly but it slightly over-predicates the standard deviation of the solution. The main reason for this are the Dirac delta initial conditions. The error may be minimised only by a better approximation of the Dirac initial condition on the expense of the computational cost.

In this paper we introduce the spectral stochastic finite element methods into the uncertainty quantification of stochastic elastoplastic material. The difficulty arising in this case comparing to other problems considered until now is the tensorial representation of some material characteristics such as constitutive tensor. Thus, we introduce the new method which is able to overcome this difficulty.

2 STRONG FORMULATION OF EQUILIBRIUM EQUATIONS

Consider a material body occupying a bounded domain $\mathcal{G} \in \mathbb{R}^d$ with a piecewise smooth Lipschitz continuous boundary $\partial\mathcal{G}$ on which are imposed boundary conditions in Dirichlet and Neumann form on $\Gamma_D \subseteq \partial\mathcal{G}$ and $\Gamma_N \subset \partial\mathcal{G}$ respectively, such that $\Gamma_D \cap \Gamma_N = \varnothing$ and $\partial\mathcal{G} = \bar{\Gamma}_N \cup \bar{\Gamma}_D$. The probability space is defined as a triplet $(\Omega, \mathcal{B}, \mathbb{P})$, with $\mathcal{B}$ being a σ-algebra of subsets of Ω and $\mathbb{P}$ a probability measure. The balance of momentum localized about any point x in domain $\mathcal{G}$ in time $t \in \mathcal{T} := [0, T]$ leads to an equilibrium equation and boundary conditions required to hold almost surely in ω, i.e. $\mathbb{P}$-almost everywhere:

$$\begin{aligned} \operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} &= \mathbf{0} \quad \text{on } \mathcal{G}, \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} &= \boldsymbol{g}, \quad \text{on } \Gamma_N, \\ \boldsymbol{u} &= \mathbf{0}, \quad \text{on } \Gamma_D \end{aligned} \tag{1}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ denote the displacement and velocity fields over $\mathcal{G}$, $\boldsymbol{f}$ the body force, $\boldsymbol{\sigma}$ stress tensor, $\boldsymbol{n}$ the exterior unit normal at $x \in \Gamma_N$, and $\boldsymbol{g}$ a prescribed surface tension. For the sake of simplicity we use homogeneous Dirichlet boundary conditions and under the assumptions of small deformation theory we introduce the strain $\boldsymbol{\varepsilon}(\boldsymbol{u}) = D\boldsymbol{u}$, with the linear bounded operator defined as a mapping $D : \boldsymbol{u}_1(x)\boldsymbol{u}_2(\omega) \to (\nabla_S \boldsymbol{u}_1(x))\boldsymbol{u}_2(\omega)$ [10, 5, 4].

3 VARIATIONAL FORMULATION

The strong formulation is not suitable for solving and thus one introduces the mixed formulation of elastoplastic problem, given by next theorem:

Theorem 3.1 *There are unique functions, $w \in H^1(\mathcal{T}, \mathcal{Z}^*)$ and $w^* \in H^1(\mathcal{T}, \mathcal{Z}^*)$ with $w(0) = 0$ and $w^*(0) = 0$, which solve the following problem a.e. $t \in \mathcal{T}$:*

$$\forall z \in \mathcal{Z} : \; a(w(t), z) + \langle\langle w^*(t), z \rangle\rangle = \langle\langle f(t), z \rangle\rangle \tag{2}$$

and

$$\forall z^* \in \mathcal{K} : \; \langle\langle \dot{w}(t), z^* - w^*(t) \rangle\rangle \leq 0. \tag{3}$$

Here $a(w(t), z)$ represents the bilinear form, $\boldsymbol{w}$ is the primal variable, $\boldsymbol{z}$ is the test function, $\boldsymbol{f}(t)$ the loading, $\boldsymbol{w}^*$ the dual variable and the duaility operator $\langle\langle \cdot, \cdot \rangle\rangle$ is defined as:

$$\langle\langle \boldsymbol{y}_1, \boldsymbol{y}_2 \rangle\rangle = \mathbb{E}\left(\int_{\mathcal{G}} \boldsymbol{y}_1 \cdot \boldsymbol{y}_2 \, \mathrm{d}y \right). \tag{4}$$

The first equation represents the equilibrium equation, while the second is the flow rule describing the rate of change of the plastic deformation. If the stress stays inside the domain $\mathcal{K}$ one has elastic response, otherwise the response is plastic.

4 STOCHASTIC CLOSEST POINT PROJECTION

Computationally the solution of the elastoplastic problem collapses to the (iterative) solution of a convex mathematical programming problem, which has goal to find the closest distance in the energy norm of a trial state to a convex set $\mathcal{K}$ of elastic domain, known as a closest point projection. In other words, one search for:

$$\boldsymbol{\Sigma}_n(\omega) = \underset{\boldsymbol{\Sigma}(\omega) \in \mathcal{K}}{\arg\min} \; \mathcal{I}(\omega), \tag{5}$$

where $\mathcal{I}$ is given as:

$$\mathcal{I} := \arg\min_{\mathbf{\Sigma}\in\mathcal{K}} \frac{1}{2}\langle\langle \mathbf{\Sigma}^{trial} - \mathbf{\Sigma}_n, \boldsymbol{A}^{-1} : (\mathbf{\Sigma}^{trial} - \mathbf{\Sigma}_n)\rangle\rangle \tag{6}$$

in the time step n described by an implicit Euler difference scheme. Here, $\mathbf{\Sigma}^{trial}$ describes the trial stress leading to the typical operator split of the closest point projection algorithm into two steps: elastic predictor and plastic corrector.

Predictor step The predictor step calculates the polynomial chaos expansion of displacement u_n^k (in iteration k) by solving the equilibrium equation Eq. (1) [9, 8]. The displacement is then used for the calculation of the strain increment $\triangle E_n^k$ and the trial stress $\Sigma_n^{k,tr}$ assuming step to be purely elastic. If the stress $\Sigma_n^{k,tr}$ lies outside of the admissible region $\mathcal{K}$ we proceed with the corrector step. Otherwise, $\Sigma_n^k = \Sigma_n^{k,tr}$ represents the solution and we may move to the next step.

Corrector step The purpose of the corrector step is to project the stress outside of admissible region back onto a point in $\mathcal{K}$. To do this, we define the corresponding Lagrangian to a minimisation problem Eq. (5):

$$\mathcal{L}(\omega) = \mathcal{I}(\omega) + \lambda(\omega)\varphi(\mathbf{\Sigma})(\omega), \tag{7}$$

where the function $\varphi(\mathbf{\Sigma})(\omega)$ represents the yield function describing the convex set $\mathcal{K} := \{\mathbf{\Sigma}(\omega) \in \mathcal{S} \mid \varphi(\mathbf{\Sigma}) \leq 0 \text{ a.s. in } \Omega\}$. Hence, the standard optimality conditions [7] become:

$$\mathbf{0} \in \partial_{\mathbf{\Sigma}}\mathcal{L} = \partial_{\mathbf{\Sigma}}\mathcal{I}(\omega) + \lambda\partial_{\mathbf{\Sigma}}\varphi(\omega) \quad \text{a.s..} \tag{8}$$

The problem of closest point projection becomes complicated since we deal with uncertain parameters, i.e. polynomial chaos variables (PCV), which require the introduction of the polynomial chaos algebra called PC algebra.

5 NUMERICAL RESULTS

Two test problems in plane strain conditions are considered: rectangular strip with hole Fig. (1) under extension and Cooke's membrane Fig. (2) excited by a shear force on the right edge. The finite element discretisation is done using eight-nodded quadrilateral elements. For random parameters are declared the shear and bulk modulus, yield stress and the isotropic hardening. Due to the positive definiteness of these properties, we model them as lognormal random fields, i.e. the piecewise exponential transformation of a Gaussian random field with prescribed covariance function and correlation lengths.

5.1 Plate with Hole

The geometry and the boundary conditions for this particular problem are given in Fig. (1). The extension force is of deterministic nature, and in the initial state doesn't depend on the parameter ω. However, in each iteration it gets mixed with the uncertainty

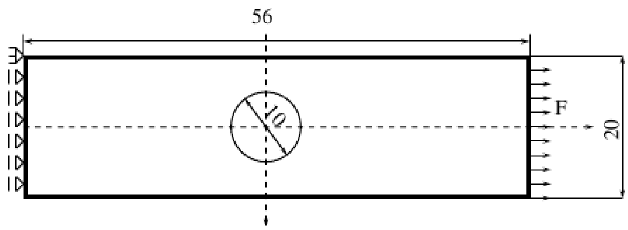

Figure 1: Geometry of the problem: plate with hole

of input parameters and hence becomes random. The randomness in input parameters depend on the choice of the values of the standard deviations as well as correlation lengths. The more large correlation length is, the less random field oscillates. Two representative examples of input random fields are given in Fig. (2), where the values of correlation lengths are chosen as moderate, 3 times less then the dimension of a plate.

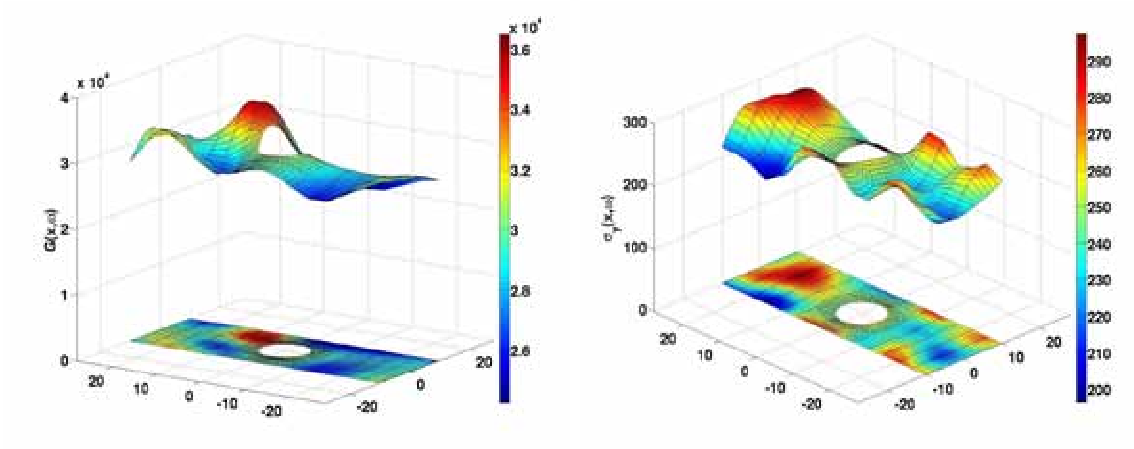

Figure 2: Two realisations of input random fields: shear modulus and yield stress

Solving the equilibrium equation, one obtains the displacement as a solution. In Fig. (3) we compared it with the initial configuration as well as with the deterministic value. Further more, we have calculated the shear stress, whose first two statistical moments are given in Fig. (4). Similarly, the Von Mises stress gives the statistical moments in Fig. (5).

The problem is solved in few different ways: by a pure sampling technique such as Monte Carlo or Latin Hypercube sampling [3], then by intrusive stochastic Galerkin method relying on white noise analysis and corresponding algebra [9] and non-intrusive variant of this method which uses the sparse grid collocation points [2]. The accuracy of these methods in the mean sense is almost the same, and hence we give the comparison of the variance convergence in Fig. (6). As one may notice, the normalised residual error is the smallest in a case of latin hypercube technique, while the intrusive method has the same convergence rate until certain error. In that point the method converges satisfying less strict criteria. The reason is the numerical error introduced by a polynomial algebra, as well as in the span space of the basis functions needed for the local Galerkin projections.

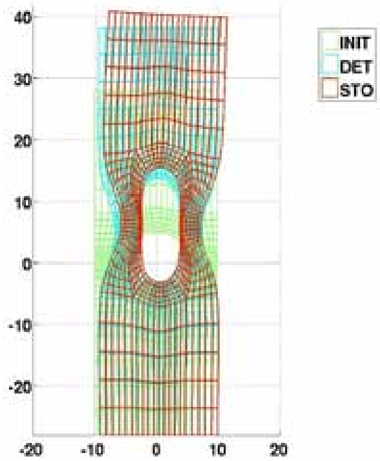

Figure 3: Comparison of the mean value of the total displacement in stochastic configuration with the deterministic and initial value

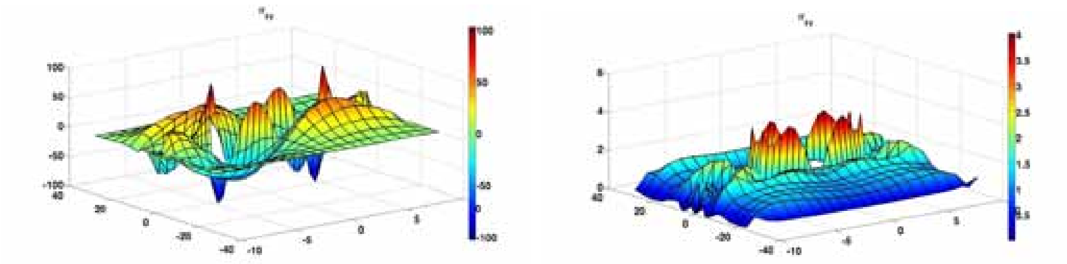

Figure 4: The shear stress σ_{xy}: mean vaalue and standard deviation

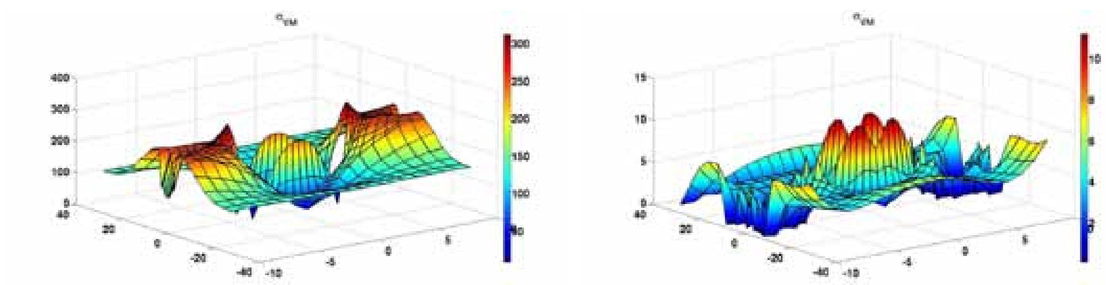

Figure 5: Von Mises stress: the mean value and standard deviation

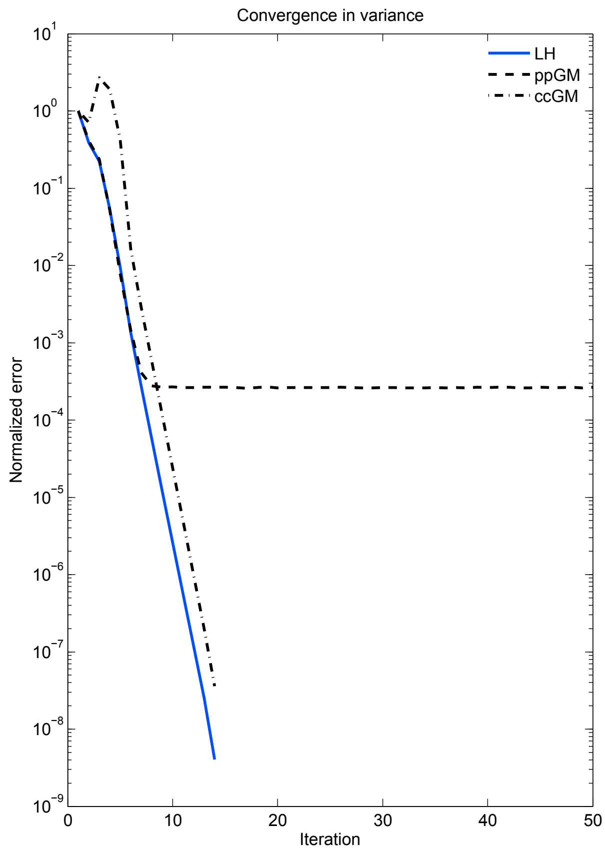

Figure 6: Comparison of the converegence of variance between latin hypercube sampling (LH), intrusive stochastic Galerkin method (ppGM) and non-intrusive stochastic Galerkin method (ccGM)

In practice very often one has to calculate the probability of stress taking the value less than some critical point. In Fig. (7) we show three probabilities schemes with respect to three different yield stress values. If the value is bigger, one has smaller probability to exceed the limit, which is expected.

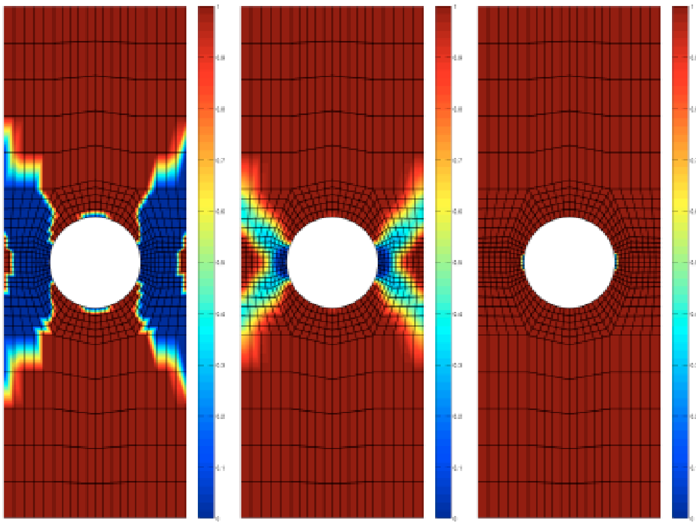

Figure 7: The probability excideence for different values of yield stress: $\sigma_y = 200$,$\sigma_y = 250$ and $\sigma_y = 300$

5.2 Cooke's membrane

The Cooke's membrane is subjected to load in y direction on the right boundary and constrained on the left as it is shown in Fig. (8). As in previous case, the random parameters are chosen in the same way, just with different mean values, and hence standard deviations.

The comparison of the mean displacements is given in Fig. (9), and as one may notice the difference is large enough not to be neglected.

The influence of the correlation lengths on the structure response is given in Fig. (10) and Fig. (11) for the xx component of deformation. In the case of large covariance lengths the random field fluctuates less and the response is more similar to deterministic one. However, in a case of small correlation lengths the field of compression grows into the larger area.

Besides the mean stastistics of the response structure, one may show the variance of the deformation ε_{xx} in Fig. (12).

6 CONCLUSION

The idea of random variables as functions in an infinite dimensional space approximated by elements of finite dimensional spaces has brought a new view to the field of stochastic

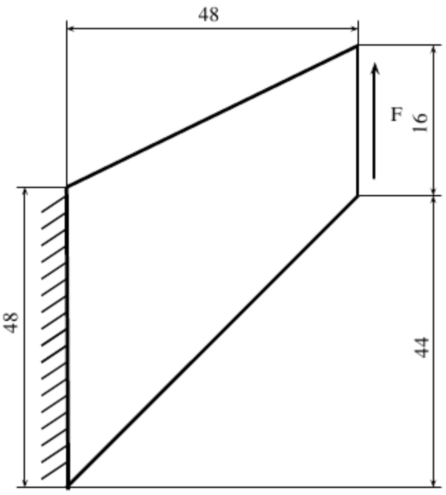

Figure 8: Geometry of the problem: Cooke's membrane

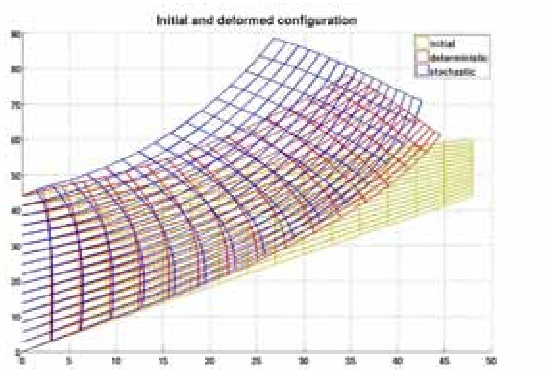

Figure 9: Comparison of the mean values of the displacement for initial, deterministic and stochastic configuration

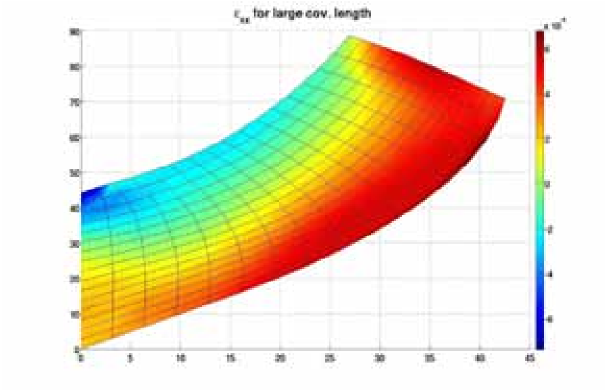

Figure 10: The mean value of deformation ε_{xx} for the large covariance length $l_c = 20$

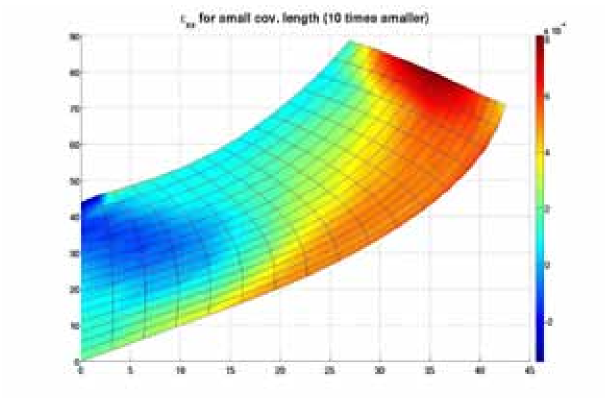

Figure 11: The mean value of deformation ε_{xx} for the small covariance length $l_c = 2$

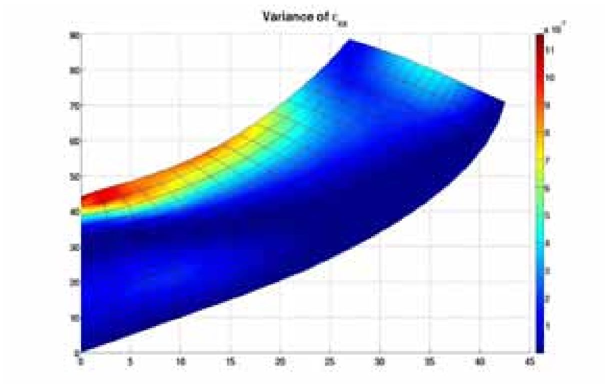

Figure 12:

elastoplasticity. In this paper, we have proposed an extension of stochastic finite element method and related numerical procedures to the resolution of inelastic stochastic problems in the context of Galerkin methods. In some way this strategy may be understood in a sense of model reduction technique due to the applied Karhunen Loève and polynomial chaos expansion. A Galerkin projection minimises the error of the truncated expansion such that the resulting set of coupled equations gives the expansion coefficients. If the smoothness conditions are met, the polynomial chaos expansion converges exponentially with the order of polynomials. In contrast to the Monte Carlo the Galerkin approach, when properly implemented, can achieve fast convergence and high accuracy and can be highly efficient in particular practical computations.

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