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ENERGY FRICTIONAL DISSIPATING ALGORITHM FOR RIGID AND ELLASTIC BODY'S CONTACT PROBLEMS

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Abstract. An Energy Frictional Dissipating Algorithm (EFDA) for time integration of Coulomb frictional impact–contact problems is presented. Using the Penalty Method, and in the context of a conserving framework, linear and angular momenta are conserved and energy is consistently dissipated.

Published formulations were stable, forcing the energy dissipation to be monotonic in order to prevent unstable energy growth. The shortcoming of many was that they were not able to reproduce the real kinematics and dissipation of physical processes, provided by analytical formulations and experiments. EFDA formulates a conserving framework based on a physical energy dissipation estimator. This framework uses an enhanced Penalty contact model based on a spring and a dashpot, enforcing physical frictional energy dissipation, controlling gap vibrations and modifying the velocities and contact forces during each time step. The result is that the dissipated energy, kinematics and contact forces are consistent with the expected physical behavior.

1 INTRODUCTION

The numerically accurate analysis of frictional dynamic contact problems has been a challenge for the last 30 years. Complex problems do not have analytical solution and due to their high nonlinearity, non–smooth unilateral restriction and the presence friction, they are hard to model. Therefore, numerical time–stepping schemes are developed to emulate the conservative properties of the corresponding continuous problem.

Previous authors have addressed frictionless contact problems, for instance [7] focused on iterative but no time–stepping formulations, [1] and [3] for implicit. These authors intended to create robust and stable algorithms for the enforcement of the contact constraints, while recent formulations have focused on frictional formulation and proposed unconditionally positive energy dissipation. Ref. [3] developed a positive energy dissipating algorithm, stable for friction with the Penalty Method and showed an artificial energy transfer between bodies–penalty springs in which the final energy was always lower than the initial for Stick and Slip cases. Therefore the behavior of the simulation was not consistent with the physical contact problem. Ref. [1] minimized that artificial energy transfer between body and penalty spring. For the non–sliding situation the energy after contact was equal to the initial and lower during contact while for the Slip case obtained a rigorous positive energy dissipation. The dissipation in both references was not based on a consistent conserving framework: the dissipation, although decreasing monotonically, was not in accordance to that of the continuous problem. Ref. [5] developed a conservative framework that enforced the impenetrability condition, eliminated the artificial energy transfer between body–penalty spring and took into account the frictional dissipation through an energy estimator. This formulation used a contact velocity that modified a predictor–corrector scheme, then the contact response agreed in velocities but not in forces and positions, not being accurate for persistent contact.

This article presents an Energy Frictional Dissipating Algorithm (EFDA) based on the frictionless algorithm of [2]. For Penalty contact problems, the new formulation conserves momenta, simulates the kinematics, contact forces and dissipates energy consistently, according to the physical problem in each time step. The algorithm key is a conservative framework based on updating contact forces and momenta for every contact. The framework takes into account dissipation by an energy estimator based on frictional Coulomb law, and is able to enforce energy conservation for the Stick contact and the right dissipation for the Slip contact.

2 DEFINITION OF THE PROBLEM AND GOVERNING EQUATIONS

2.1 Hamiltonian description of motion

The Hamiltonian Mechanics permit to obtain the equations of motion for multiple bodies that interact by contact. This subsection briefly describes the Hamiltonian equations for a continuous problem. Consider a manifold $\boldsymbol{Q}$ that describes the configuration of a mechanical system whose phase space is $\boldsymbol{P} = T^{\star}\boldsymbol{Q}$, the tangent space of $\boldsymbol{Q}$. This space is composed for each point of body i by positions $\boldsymbol{Q}^i(x, y, t)$ and linear momenta $\boldsymbol{P}^i(x, y, t)$ as function of time t. The Hamiltonian function $H\big[\boldsymbol{Q}^i(x, y, t), \boldsymbol{P}^i(x, y, t)\big]$ defines the total energy of the system and is assumed to be separable in kinetic $K(\boldsymbol{P}^i(x, y, t))$ and potential $V(\boldsymbol{Q}^i(x, y, t))$ energies, Eqs. 1.

$$
\begin{aligned}
H\big[\boldsymbol{Q}^i(x,y,t),\boldsymbol{P}^i(x,y,t)\big] &= \sum_{i=1}^{n_{bd}} \big[K\big(\boldsymbol{P}^i(x,y,t)\big) + V\big(\boldsymbol{Q}^i(x,y,t)\big)\big] \\
K\big(\boldsymbol{P}^i(x,y,t)\big) &= \frac{1}{2}\int_{\Omega^i} \frac{\boldsymbol{P}^i(x,y,t)^2}{\rho}\,\mathrm{d}\Omega
\end{aligned} \tag{1}
$$

where n_{bd} is the total number of bodies, Ω^i the domain of body i and ρ the density. The kinetic energy is a real function $K : \boldsymbol{P} \rightarrow R$ and the potential $V : \boldsymbol{Q} \rightarrow R$ is an arbitrary function. The motion is governed by the Hamiltonian canonical equations, Eqs. 2.

$$
\begin{aligned}
\dot{\boldsymbol{Q}}^i(x,y,t) &= \frac{\partial H(x,y,t)}{\partial \boldsymbol{P}^i} &= \int_{\Omega^i} \frac{\boldsymbol{P}^i(x,y,t)}{\rho}\,\mathrm{d}\Omega \\
\dot{\boldsymbol{P}}^i(x,y,t) &= -\frac{\partial H(x,y,t)}{\partial \boldsymbol{Q}^i} &= -\nabla V\big(\boldsymbol{Q}^i(x,y,t)\big)
\end{aligned} \tag{2}
$$

The continuum variables from Eqs. 2 may be discretized, giving Eqs. 3.

$$
\boldsymbol{Q}^i(x,y,t) = \sum_{A=1}^{n_{nod}} \boldsymbol{N}^A(x,y)\ \boldsymbol{q}^A(t)\ ; \qquad \boldsymbol{P}^i(x,y,t) = \sum_{A=1}^{n_{nod}} \boldsymbol{N}^A(x,y)\ \boldsymbol{p}^A(t) \tag{3}
$$

For the rigid bodies used in the current paper, this discretization is based on first order shape functions $N^A(x,y)$; for the elastic case they may be extended to higher order. For these first order functions, the discretization is based on only one node, $n_{nod} = 1$, usually defined at the center of gravity x^i, y^i of each particle. The nodal displacements and linear momenta of all bodies i to k are grouped in the vectors $\boldsymbol{q}(t)$, $\boldsymbol{p}(t)$. For each body i, the discretization Eqs. 3 applied to Eqs. 2 produce the system of equations:

$$
\dot{\boldsymbol{q}}^i = \boldsymbol{M}_i^{-1}\ \boldsymbol{p}^i\ ; \qquad \dot{\boldsymbol{p}}^i = \boldsymbol{f}_c^i + \boldsymbol{f}_{ext}^i \tag{4}
$$

where $\boldsymbol{M}_i$ is a diagonal mass matrix, with entries: $\boldsymbol{M}_i = \int_{\Omega^i} \rho\ [N^A(x,y)]^t N^A(x,y)\ \mathrm{d}\Omega$. Although contact forces $\boldsymbol{f}_c^i$ are applied in the contact points, the discretization considers an equivalent force applied to the nodes. The same thing can be said for the external forces $\boldsymbol{f}_{ext}^i$.

3 NEW ALGORITHM FORMULATION AND ENERGY–MOMENTUM CONSERVATION

The aim of this section is the discretization in time of Eqs. 4. The new equations will enforce the impenetrability condition and discretely inherit the conservation properties through the conserving framework of section 4. The main three characteristic of this algorithm are: i) energy conservation for normal contact, ii) consistent dissipation for tangential Slip and iii) conservation for tangential Stick.

3.1 Time–discrete formulation

The frictional development of EFDA is based on the Simo–Tarnow's algorithm from [6], an energy–momentum conserving time integration scheme. This scheme is a discrete approximation of a Hamiltonian system (Eqs. 5) in time configuration $n+1/2$. Considering the interval $[t_n, t_{n+1}]$, the first set of equations of this algorithm relates displacements $\boldsymbol{q}_n^i$, $\boldsymbol{q}_{n+1}^i$ and linear momenta $\boldsymbol{p}_n^i$, $\boldsymbol{p}_{n+1}^i$; the second, is the discrete approximation of second Newton's law at $n+1/2$:

$$\begin{aligned} \dot{\boldsymbol{q}}^i &= \boldsymbol{M}_i^{-1}\,\boldsymbol{p}^i & \rightarrow \quad \frac{\boldsymbol{q}_{n+1}^i-\boldsymbol{q}_n^i}{\Delta t} &= \boldsymbol{M}_i^{-1}\boldsymbol{p}^i_{n+\frac{1}{2}} \\ \dot{\boldsymbol{p}}^i &= \boldsymbol{f}^i_{cN}+\boldsymbol{f}^i_{cT} & \rightarrow \quad \frac{\boldsymbol{p}_{n+1}^i-\boldsymbol{p}_n^i}{\Delta t} &= \boldsymbol{f}^i_{cN\;n+\frac{1}{2}}+\boldsymbol{f}^i_{cT\;n+\frac{1}{2}} \end{aligned} \tag{5}$$

where $\Delta t = t_{n+1}-t_n$, $\boldsymbol{q}_n^i \approx \boldsymbol{q}^i(t_n)$, $\boldsymbol{p}_n^i \approx \boldsymbol{p}^i(t_n)$, $\boldsymbol{q}_{n+1}^i \approx \boldsymbol{q}^i(t_{n+1})$, $\boldsymbol{p}_{n+1}^i \approx \boldsymbol{p}^i(t_{n+1})$ and $\boldsymbol{p}_{n+1/2}^i = (\boldsymbol{p}_{n+1}^i+\boldsymbol{p}_n^i)/2$. The terms $\boldsymbol{f}^i_{cN\;n+1/2}$ and $\boldsymbol{f}^i_{cT\;n+1/2}$ are the discrete approximations of the resulting normal and tangential contact force vectors.

In order to obtain a conserving and a right kinematic response for contact between two bodies i,k, in EFDA additional linear momenta $\boldsymbol{p}^{ik}_{cN\;n+1/2}$, $\boldsymbol{p}^{ik}_{cT\;n+1/2}$ and contact forces $\boldsymbol{f}'^{ik}_{cN\;n+1/2}$, $\boldsymbol{f}'^{ik}_{cT\;n+1/2}$ (updating variables) are added to Eqs. 5, giving Eqs. 6. For these four variables, in the following the subscript $n+1/2$ will be omitted for simplicity. The role of these new variables is to enforce bodies' energy conservation for normal contact and conservation–consistent dissipation for tangential, respectively:

$$\begin{aligned} \frac{\boldsymbol{q}_{n+1}^i-\boldsymbol{q}_n^i}{\Delta t} &= \boldsymbol{M}_i^{-1}\left[\boldsymbol{p}^i_{n+\frac{1}{2}}+\sum_{\substack{k=1\\k\neq i}}^{n_{bd}}\left(\boldsymbol{p}^{ik}_{cN}+\boldsymbol{p}^{ik}_{cT}\right)\right] &= \boldsymbol{M}_i^{-1}\left(\boldsymbol{p}^i_{n+\frac{1}{2}}+\boldsymbol{p}^i_{cN}+\boldsymbol{p}^i_{cT}\right) \\ \frac{\boldsymbol{p}_{n+1}^i-\boldsymbol{p}_n^i}{\Delta t} &= \sum_{\substack{k=1\\k\neq i}}^{n_{bd}}\left(\boldsymbol{f}^{ik}_{cN}+\boldsymbol{f}'^{ik}_{cN}+\boldsymbol{f}^{ik}_{cT}+\boldsymbol{f}'^{ik}_{cT}\right) &= \boldsymbol{f}^i_{cN}+\boldsymbol{f}'^i_{cN}+\boldsymbol{f}^i_{cT}+\boldsymbol{f}'^i_{cT} \end{aligned} \tag{6}$$

The expressions for the updating variables are now formulated in local contact coordinates and transformed to global by the unit normal and tangential vectors $\boldsymbol{N}^{ik}_{n+1/2}$, $\boldsymbol{T}^{ik}_{n+1/2}$, both at the contact point. To obtain from Eqs. 6 a conservative solution for Stick and dissipative for Slip, the updating variables must fulfill the discrete conserving equations defined in section 4. These variables are defined for both contact directions as:

$$
\begin{array}{ll}
\text{NORMAL} & \\
\boldsymbol{p}^{ik}_{cN} = \dfrac{\psi^{ik}_{2N}}{2} \boldsymbol{N}^{ik}_{n+\frac{1}{2}} \boldsymbol{N}^{ik\,t}_{n+\frac{1}{2}} (\boldsymbol{p}^{i}_{n+1} - \boldsymbol{p}^{i}_{n}); & \boldsymbol{f}'^{ik}_{cN} = \dfrac{\psi^{ik}_{1N}}{2} \boldsymbol{N}^{ik}_{n+\frac{1}{2}} K_N \ (g^{ik}_{Nn+1} - g^{ik}_{Nn}) \\
\text{TANG. STICK} & \\
\boldsymbol{p}^{ik}_{cT} = \dfrac{\psi^{ik}_{2T}}{2} \boldsymbol{T}^{ik}_{n+\frac{1}{2}} \boldsymbol{T}^{ik\,t}_{n+\frac{1}{2}} (\boldsymbol{p}^{i}_{n+1} - \boldsymbol{p}^{i}_{n}); & \boldsymbol{f}'^{ik}_{cT} = \dfrac{\psi^{ik}_{1T}}{2} \ \boldsymbol{T}^{ik}_{n+\frac{1}{2}} \ K_T \ (g^{ik}_{Tn+1} - g^{ik}_{Tn}) \\
\text{TANG. SLIP} & \\
\boldsymbol{p}^{ik}_{cT} = 0 \ ; \qquad \boldsymbol{f}'^{ik}_{cT} = 0; & \boldsymbol{f}^{ik}_{cT} = -\mu \ \Psi \ \left| \boldsymbol{f}^{ik}_{cN} + \boldsymbol{f}'^{ik}_{cN} \right| \ \boldsymbol{T}^{ik}_{n+\frac{1}{2}}
\end{array}
\tag{7}
$$

where K_N, K_T are user–defined penalties for normal and tangential contact, g^{ik}_{Nn+1}, g^{ik}_{Nn}, g^{ik}_{Tn+1}, g^{ik}_{Tn} normal and tangential gaps at n and $n+1$, $\Psi = \pm 1$ the Slip direction, μ the friction coefficient and $\psi^{ik}_{1N}, \psi^{ik}_{1T}, \psi^{ik}_{2N}$ and ψ^{ik}_{2T} proportionality parameters of the updating variables. Notice that $\boldsymbol{p}^{ik}_{cN}$, $\boldsymbol{f}'^{ik}_{cN}$, $\boldsymbol{p}^{ik}_{cT}$, $\boldsymbol{f}'^{ik}_{cT}$ (at $n+1/2$) enforce the conservative response for normal and tangential Stick contacts. On the other hand, for Slip $\boldsymbol{p}^{ik}_{cT} = \boldsymbol{f}'^{ik}_{cT} = 0$ since no tangential penalty spring is present; the new Coulomb friction force $\boldsymbol{f}^{ik}_{cT}$ is computed with the absolute value of the total (contact plus updating) normal contact forces. The normal and tangential–Stick contact forces $\boldsymbol{f}^{ik}_{cN}$, $\boldsymbol{f}^{ik}_{cT}$ are defined in Eqs. 8 using the [4] derivative, providing a discrete expression that conserves the artificial penalty energy.

$$
\begin{aligned}
\boldsymbol{f}^{i}_{cN} = \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \boldsymbol{f}^{ik}_{cN} = \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \frac{V(g^{ik}_{Nn+1}) - V(g^{ik}_{Nn})}{g^{ik}_{Nn+1} - g^{ik}_{Nn}} \, \boldsymbol{N}^{ik}_{n+\frac{1}{2}} = \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} K_N \, \boldsymbol{N}^{ik}_{n+\frac{1}{2}} \ (g^{ik}_{Nn+1} + g^{ik}_{Nn}) \\
\boldsymbol{f}^{i}_{cT} = \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \boldsymbol{f}^{ik}_{cT} = \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \frac{V(g^{ik}_{Tn+1}) - V(g^{ik}_{Tn})}{g^{ik}_{Tn+1} - g^{ik}_{Tn}} \, \boldsymbol{T}^{ik}_{n+\frac{1}{2}} = \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} K_T \, \boldsymbol{T}^{ik}_{n+\frac{1}{2}} \ (g^{ik}_{Tn+1} + g^{ik}_{Tn})
\end{aligned}
$$

where $V(g^{ik}_{Nn+1}) = K_N(g^{ik}_{Nn+1})^2/2, V(g^{ik}_{Nn}) = K_N(g^{ik}_{Nn})^2/2$ are the normal and $V(g^{ik}_{Tn+1}) = K_T(g^{ik}_{Tn+1})^2/2$, $V(g^{ik}_{Tn}) = K_T(g^{ik}_{Tn})^2/2$, the tangential penalty potential contact energies.

4 DISCRETE LINEAR, ANGULAR MOMENTUM CONSERVATION AND CONSISTENT BODY ENERGY DISSIPATION

This section develops the discrete conserving framework of EFDA to obtain the body energy conservation for normal contact and conservation–dissipation for tangential.

4.1 Discrete linear momentum balance

The discrete variation of the linear momentum of EFDA is defined through the second of Eqs. 6, the discrete counterpart of second Newton's law. Therefore, for body i the resultant of the normal contact forces $\boldsymbol{f}^{i}_{cN}$, $\boldsymbol{f}'^{i}_{cN}$ plus the tangential $\boldsymbol{f}^{i}_{cT}, \boldsymbol{f}'^{i}_{cT}$ is equal to the discrete linear momentum balance between n and $n+1$. Also, the total linear momentum balance (Eq. 8) is the summation over that of each body and equals the resultant of the contact forces on n_{bd}.

$$\frac{\boldsymbol{p}^{tot}_{n+1} - \boldsymbol{p}^{tot}_{n}}{\Delta t} = \sum_{i=1}^{n_{bd}} \left(\boldsymbol{f}^{i}_{cN} + \boldsymbol{f}'^{i}_{cN} + \boldsymbol{f}^{i}_{cT} + \boldsymbol{f}'^{i}_{cT} \right) = \sum_{i=1}^{n_{bd}} \sum_{\substack{k=1 \\ k \neq i}}^{n_{bd}} \left(\boldsymbol{f}^{ik}_{cN} + \boldsymbol{f}'^{ik}_{cN} + \boldsymbol{f}^{ik}_{cT} + \boldsymbol{f}'^{ik}_{cT} \right) \quad (8)$$

Given two bodies i, k in contact, due to the AR principle: $\boldsymbol{f}^{ik}_{cN} = -\boldsymbol{f}^{ki}_{cN}$, $\boldsymbol{f}'^{ik}_{cN} = -\boldsymbol{f}'^{ki}_{cN}$, $\boldsymbol{f}^{ik}_{cT} = -\boldsymbol{f}^{ki}_{cT}$, $\boldsymbol{f}'^{ik}_{cT} = -\boldsymbol{f}'^{ki}_{cT}$ for Stick and $\boldsymbol{f}^{ik}_{cT} = -\boldsymbol{f}^{ki}_{cT}$, $\boldsymbol{f}'^{ik}_{cT} = \boldsymbol{f}'^{ki}_{cT} = 0$ for Slip. Then, the right term of Eq. 8 is zero in all situations and $\boldsymbol{p}^{tot}_{n+1} = \boldsymbol{p}^{tot}_{n}$.

4.2 Discrete angular momentum balance

We again redefine the variables $\boldsymbol{q}^{i}_{n+1}$, $\boldsymbol{q}^{i}_{n}$ as the positions of the contact point. From the second of Eqs. 6 and multiplying by the cross product $\times(\boldsymbol{q}^{i}_{n+1} - \boldsymbol{q}^{i}_{n})$, the discrete angular momentum balance for a body i is:

$$\frac{\boldsymbol{p}^{i}_{n+1} - \boldsymbol{p}^{i}_{n}}{\Delta t} \times (\boldsymbol{q}^{i}_{n+1} - \boldsymbol{q}^{i}_{n}) = \frac{\boldsymbol{J}^{i}_{n+1} - \boldsymbol{J}^{i}_{n}}{\Delta t} = \sum_{\substack{k=1 \\ k \neq i}}^{n_{bd}} \left(\boldsymbol{f}^{ik}_{cN} + \boldsymbol{f}'^{ik}_{cN} + \boldsymbol{f}^{ik}_{cT} + \boldsymbol{f}'^{ik}_{cT} \right) \times (\boldsymbol{q}^{i}_{n+1} - \boldsymbol{q}^{i}_{n}) \quad (9)$$

Invoking the AR principle and expressing the position's increments as function of the normal gap $(\boldsymbol{q}^{i}_{n+1} - \boldsymbol{q}^{i}_{n}) - (\boldsymbol{q}^{k}_{n+1} - \boldsymbol{q}^{k}_{n}) = g^{ik}_{Nn+1/2}\ (\boldsymbol{N}^{ki})^{t}$, the total angular momentum balance for n_{bd} is:

$$\frac{\boldsymbol{J}^{tot}_{n+1} - \boldsymbol{J}^{tot}_{n}}{\Delta t} = \sum_{i=1}^{n_{bd}} \sum_{k=i+1}^{n_{bd}} \left(\boldsymbol{f}^{ik}_{cN} + \boldsymbol{f}'^{ik}_{cN} + \boldsymbol{f}^{ik}_{cT} + \boldsymbol{f}'^{ik}_{cT} \right) \times g^{ik}_{Nn+\frac{1}{2}} (\boldsymbol{N}^{ki})^{t} \quad (10)$$

Since vectors $\boldsymbol{f}^{ik}_{cN} + \boldsymbol{f}'^{ik}_{cN}$, and the normal gap are collinear, their cross product is zero. The product between $\boldsymbol{f}^{ik}_{cT} + \boldsymbol{f}'^{ik}_{cT}$ and this gap is also zero since the tangential contact forces depend on the tangential gap: after some algebra we arrive to the triple product $(\boldsymbol{T}^{ik}_{n+1/2})^{t}\ \boldsymbol{T}^{ik}_{n+1/2} \times g^{ik}_{Nn+1/2} (\boldsymbol{N}^{ki})^{t}$ that is zero since the first vector is orthogonal to the cross product.

4.3 Discrete total bodies' energy balance

This equation is obtained by premultiplying both Eqs. 6 by $(\boldsymbol{p}^{i}_{n+1} - \boldsymbol{p}^{i}_{n})^{t}$ and $-(\boldsymbol{q}^{i}_{n+1} - \boldsymbol{q}^{i}_{n})^{t}$ respectively, then added for all contacting bodies n_{bd}. After some algebra:

$$
\begin{aligned}
\Delta E_{kin} = & \sum_{i=1}^{n_{bd}} \underbrace{(\boldsymbol{p}_{n+1}^{i} - \boldsymbol{p}_{n}^{i})^{t}\ \boldsymbol{M}_{i}^{-1}\ \boldsymbol{p}_{n+\frac{1}{2}}^{i}}_{\Delta E_{kin}^{i}} = -\sum_{i=1}^{n_{bd}} \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \Big[-\underbrace{(\boldsymbol{q}_{n+1}^{i} - \boldsymbol{q}_{n}^{i})^{t}\ \boldsymbol{f}_{cT}^{ik}}_{\Delta E_{\boldsymbol{f}_{cT}^{ik}}} \Big] \\
& + \sum_{i=1}^{n_{bd}} \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \Big[\underbrace{(\boldsymbol{q}_{n+1}^{i} - \boldsymbol{q}_{n}^{i})^{t}\ \boldsymbol{f}_{cN}^{ik}}_{\Delta E_{\boldsymbol{f}_{cN}^{ik}}} - \underbrace{(\boldsymbol{p}_{n+1}^{i} - \boldsymbol{p}_{n}^{i})^{t}\ \boldsymbol{M}_{i}^{-1}\ \boldsymbol{p}_{cN}^{ik}}_{\Delta E_{\boldsymbol{p}_{cN}^{ik}}(\psi_{2N}^{ik})} \\
& - \underbrace{(\boldsymbol{p}_{n+1}^{i} - \boldsymbol{p}_{n}^{i})^{t}\ \boldsymbol{M}_{i}^{-1}\ \boldsymbol{p}_{cT}^{ik}}_{\Delta E_{\boldsymbol{p}_{cT}^{ik}}(\psi_{2T}^{ik})} - \underbrace{(\boldsymbol{q}_{n+1}^{i} - \boldsymbol{q}_{n}^{i})^{t}\ \boldsymbol{f}_{cN}^{\prime ik}}_{\Delta E_{\boldsymbol{f}_{cN}^{\prime ik}}(\psi_{1N}^{ik})} - \underbrace{(\boldsymbol{q}_{n+1}^{i} - \boldsymbol{q}_{n}^{i})^{t}\ \boldsymbol{f}_{cT}^{\prime ik}}_{\Delta E_{\boldsymbol{f}_{cT}^{\prime ik}}(\psi_{1T}^{ik})} \Big]
\end{aligned}
\tag{11}
$$

where $\Delta E_{kin}^{i} = E_{n+1}^{i} - E_{n}^{i}$ is the kinetic body energy balance between n and $n+1$ when bodies are rigid and external forces are not applied. Then, $\Delta E_{\boldsymbol{f}_{cN}^{ik}}$, $\Delta E_{\boldsymbol{f}_{cT}^{ik}}$ are the contact forces energy balance and $\Delta E_{\boldsymbol{p}_{cN}^{ik}}(\psi_{2N}^{ik})$, $\Delta E_{\boldsymbol{p}_{cT}^{ik}}(\psi_{2T}^{ik})$, $\Delta E_{\boldsymbol{f}_{cN}^{\prime ik}}(\psi_{1N}^{ik})$, $\Delta E_{\boldsymbol{f}_{cT}^{\prime ik}}(\psi_{1T}^{ik})$, all functions of the proportionality parameters, are the updating variables energy balance.

This equation is the conserving framework that relates the total bodies' energy balance with that of the updating variables. The role of energy conservation for normal contact is included in the terms $\Delta E_{\boldsymbol{f}_{cN}^{ik}}$, $\Delta E_{\boldsymbol{f}_{cN}^{\prime ik}}(\psi_{1N}^{ik})$, $\Delta E_{\boldsymbol{p}_{cN}^{ik}}(\psi_{2N}^{ik})$, while dissipation–conservation for Slip and Stick is included in the terms $\Delta E_{\boldsymbol{f}_{cT}^{ik}}$, $\Delta E_{\boldsymbol{f}_{cT}^{\prime ik}}(\psi_{1T}^{ik})$, $\Delta E_{\boldsymbol{p}_{cT}^{ik}}(\psi_{2T}^{ik})$. Therefore, the energy loss is always consistent since the dissipation is included in the energy balance.

Notice that ΔE_{kin} is the total energy for all bodies. Since the energy related to normal contact is conserved, ψ_{1N}^{ik}, ψ_{2N}^{ik} may be positive or negative, and $\Delta E_{\boldsymbol{p}_{cN}^{ik}}$, $\Delta E_{\boldsymbol{f}_{cN}^{\prime ik}}$ add or subtract energy. The same can be said for ψ_{1T}^{ik}, ψ_{2T}^{ik} for the Stick and Slip cases:

STICK: the energy is conserved since the contact force does not create–dissipate tangential work. This condition is enforced by zeroing the right part of Eq. 11:

$$
0 = \sum_{i=1}^{n_{bd}} \sum_{\substack{k=1\\k\neq i}}^{n_{bd}} \Big[\Delta E_{\boldsymbol{f}_{cT}^{ik}} + \Delta E_{\boldsymbol{f}_{cN}^{ik}} + \Delta E_{\boldsymbol{p}_{cN}^{ik}} + \Delta E_{\boldsymbol{p}_{cT}^{ik}} + \Delta E_{\boldsymbol{f}_{cN}^{\prime ik}} + \Delta E_{\boldsymbol{f}_{cT}^{\prime ik}} \Big]
\tag{12}
$$

This equation provides infinite relationships that satisfy the total bodies' energy conservation for ψ_{1N}^{ik}, ψ_{2N}^{ik}, ψ_{1T}^{ik}, ψ_{2T}^{ik}. Using the AR principle and the reciprocities $\psi_{1N}^{ik} = \psi_{1N}^{ki}$, $\psi_{2N}^{ik} = \psi_{2N}^{ki}$, $\psi_{1T}^{ik} = \psi_{1T}^{ki}$ and $\psi_{2T}^{ik} = \psi_{2T}^{ki}$, Eq. 12 may be decoupled for the normal (Eq. 13) and tangential (Eq. 14) contacts between bodies i, k as:

$$
\begin{aligned}
& \underbrace{(\boldsymbol{p}_{n+1}^{i} - \boldsymbol{p}_{n}^{i})^{t}\ \boldsymbol{M}_{i}^{-1}\ \boldsymbol{p}_{cN}^{ik}}_{\Delta E_{\boldsymbol{p}_{cN}^{ik}}(\psi_{2N}^{ik})} + \underbrace{(\boldsymbol{p}_{n+1}^{k} - \boldsymbol{p}_{n}^{k})^{t}\ \boldsymbol{M}_{k}^{-1}\ \boldsymbol{p}_{cN}^{ki}}_{\Delta E_{\boldsymbol{p}_{cN}^{ki}}(\psi_{2N}^{ik})} + \underbrace{\left[(\boldsymbol{q}_{n+1}^{i} - \boldsymbol{q}_{n}^{i})^{t} - (\boldsymbol{q}_{n+1}^{k} - \boldsymbol{q}_{n}^{k})^{t}\right]\ \boldsymbol{f}_{cN}^{ik}}_{\Delta E_{\boldsymbol{f}_{cN}^{ik}}} \\
& \qquad + \underbrace{\left[(\boldsymbol{q}_{n+1}^{i} - \boldsymbol{q}_{n}^{i})^{t} - (\boldsymbol{q}_{n+1}^{k} - \boldsymbol{q}_{n}^{k})^{t}\right]\ \boldsymbol{f}_{cN}^{\prime ik}}_{\Delta E_{\boldsymbol{f}_{cN}^{\prime ik}}(\psi_{1N}^{ik})} = 0
\end{aligned}
\tag{13}
$$

$$\underbrace{(\boldsymbol{p}^i_{n+1}-\boldsymbol{p}^i_n)^t\ \boldsymbol{M}_i^{-1}\ \boldsymbol{p}^{ik}_{cT}}_{\Delta E_{\boldsymbol{p}^{ik}_{cT}}(\psi^{ik}_{2T})}+\underbrace{(\boldsymbol{p}^k_{n+1}-\boldsymbol{p}^k_n)^t\ \boldsymbol{M}_k^{-1}\ \boldsymbol{p}^{ki}_{cT}}_{\Delta E_{\boldsymbol{p}^{ki}_{cT}}(\psi^{ik}_{2T})}+\underbrace{\left[(\boldsymbol{q}^i_{n+1}-\boldsymbol{q}^i_n)^t\ -(\boldsymbol{q}^k_{n+1}-\boldsymbol{q}^k_n)^t\ \right]\ \boldsymbol{f}^{ik}_{cT}}_{\Delta E_{\boldsymbol{f}^{ik}_{cT}}}$$
$$+\underbrace{\left[(\boldsymbol{q}^i_{n+1}-\boldsymbol{q}^i_n)^t-(\boldsymbol{q}^k_{n+1}-\boldsymbol{q}^k_n)^t\right]\ \boldsymbol{f}'^{ik}_{cT}}_{\Delta E_{\boldsymbol{f}'^{ik}_{cT}}(\psi^{ik}_{1T})}=0 \qquad (14)$$

Both imply that the energy transferred to the normal and tangential penalty springs is recovered by the updating variables. The energies $\Delta E_{\boldsymbol{p}^{ik}_{cN}}$, $\Delta E_{\boldsymbol{p}^{ik}_{cT}}$ enforce the total bodies' energy conservation, while $\Delta E_{\boldsymbol{f}'^{ik}_{cN}}$, $\Delta E_{\boldsymbol{f}'^{ik}_{cT}}$ adjust the contact forces to the conservative solution.

SLIP: from physical considerations, the total energy dissipated by friction must be equal to the increment of total energy, $E_{n+1}-E_n=-\sum_{i=1}^{n_{bd}}\ \sum_{\substack{k=1\\k\neq i}}^{n_{bd}}\Delta E_{\boldsymbol{f}^{ik}_{cT}}$. This equality is enforced by EFDA zeroing the last summation of Eq. 11. Also, $\Delta E_{\boldsymbol{p}^{ik}_{cT}}(\psi^{ik}_{2T})=\Delta E_{\boldsymbol{f}'^{ik}_{cT}}(\psi^{ik}_{1T})=0$ and $\psi^{ik}_{1T}=\psi^{ik}_{2T}=0$ since there is no penalty spring in the tangential direction, see the Slip condition in Eq. 7. Therefore, the equation that provides the infinite (therefore undetermined) relations between ψ^{ik}_{1N}, ψ^{ik}_{2N} is:

$$\sum_{i=1}^{n_{bd}}\ \sum_{\substack{k=1\\k\neq i}}^{n_{bd}}\left[\underbrace{(\boldsymbol{p}^i_{n+1}-\boldsymbol{p}^i_n)^t\ \boldsymbol{M}_i^{-1}\ \boldsymbol{p}^{ik}_{cN}}_{\Delta E_{\boldsymbol{p}^{ik}_{cN}}(\psi^{ik}_{2N})}+\underbrace{(\boldsymbol{q}^i_{n+1}-\boldsymbol{q}^i_n)^t\ \boldsymbol{f}^{ik}_{cN}}_{\Delta E_{\boldsymbol{f}^{ik}_{cN}}}+\underbrace{(\boldsymbol{q}^i_{n+1}-\boldsymbol{q}^i_n)^t\ \boldsymbol{f}'^{ik}_{cN}}_{\Delta E_{\boldsymbol{f}'^{ik}_{cN}}(\psi^{ik}_{1N})}\right]=0 \qquad (15)$$

that represents a normal contact energy balance for all bodies. As in the previous case, this balance is enforced for every contact; using again AR, reciprocities $\psi^{ik}_{1N}=\psi^{ki}_{1N}$, $\psi^{ik}_{2N}=\psi^{ki}_{2N}$ and decoupling Eq. 15 for every contact, one arrives to relationships between ψ^{ik}_{1N}, ψ^{ik}_{2N}:

$$\underbrace{(\boldsymbol{p}^i_{n+1}-\boldsymbol{p}^i_n)^t\ \boldsymbol{M}_i^{-1}\ \boldsymbol{p}^{ik}_{cN}}_{\Delta E_{\boldsymbol{p}^{ik}_{cN}}(\psi^{ik}_{2N})}+\underbrace{(\boldsymbol{p}^k_{n+1}-\boldsymbol{p}^k_n)^t\ \boldsymbol{M}_k^{-1}\ \boldsymbol{p}^{ki}_{cN}}_{\Delta E_{\boldsymbol{p}^{ki}_{cN}}(\psi^{ik}_{2N})}+$$
$$\underbrace{\left[(\boldsymbol{q}^i_{n+1}-\boldsymbol{q}^i_n)^t-(\boldsymbol{q}^k_{n+1}-\boldsymbol{q}^k_n)^t\right]\ \boldsymbol{f}^{ik}_{cN}}_{\Delta E_{\boldsymbol{f}^{ik}_{cN}}}+\underbrace{\left[(\boldsymbol{q}^i_{n+1}-\boldsymbol{q}^i_n)^t-(\boldsymbol{q}^k_{n+1}-\boldsymbol{q}^k_n)^t\right]\ \boldsymbol{f}'^{ik}_{cN}}_{\Delta E_{\boldsymbol{f}'^{ik}_{cN}}(\psi^{ik}_{1N})}=0 \qquad (16)$$

The condition at the beginning of the SLIP item and the last equation, enforce both the energy conservation of the normal contact and the dissipation for tangential contact.

5 DYNAMIC CONTACT, ENHANCED PENALTY METHOD

For every contact, Eqs. 13, 14, 16 provide a non–unique relation between $\psi^{ik}_{1N},\psi^{ik}_{2N}$ and $\psi^{ik}_{1T},\psi^{ik}_{2T}$ respectively that automaticaly conserve or dissipate consistently the total energy for the tangential Stick and Slip cases. Using modal analysis decomposition, [2], it is possible to obtain the second order dynamic equation associated with the descriptive algorithm of Eqs. 6, defining an enhanced penalty contact model. The model described by Eqs. 17 consists of a spring and dashpot that control the gap and the penetration velocity, respectively.

$$\begin{aligned}
0 &= \ddot{q}_{Nn+1/2} + \overbrace{\omega_N^2\ \Delta t\ \frac{\psi_{1N}+\psi_{2N}}{2}}^{2\xi_N\omega_N} \dot{q}_{Nn+1/2} + \omega_N^2\ q_{Nn+1/2} \\
0 &= \underbrace{\ddot{q}_{Tn+1/2}}_{\text{Inertia}} + \underbrace{\overbrace{\omega_T^2\ \Delta t\ \frac{\psi_{1T}+\psi_{2T}}{2}}^{2\xi_T\omega_T} \dot{q}_{Tn+1/2}}_{\text{Dashpot}} + \underbrace{\omega_T^2\ q_{Tn+1/2}}_{\text{Spring}}
\end{aligned} \tag{17}$$

The variables $q_{Nn+1/2}$, $q_{Tn+1/2}$ represent the particle motion in normal and tangential direction. The dashpots are controlled by the user–defined parameters ξ_N, ξ_T, a penalization for penetration velocities that approximately enforce the consistency Kuhn–Tucker condition. Eqs. 17 provide the relations that can be easily generalized for any contact between rigid bodies i, k:

$$\psi_{1N}^{ik} + \psi_{2N}^{ik} = \frac{4\xi_N}{\Omega_N^{ik}}\ ; \qquad \psi_{1T}^{ik} + \psi_{2T}^{ik} = \frac{4\xi_T}{\Omega_T^{ik}} \tag{18}$$

where $\Omega_N^{ik} = \omega_N^{ik}\ \Delta t$, $\Omega_T^{ik} = \omega_T^{ik}\ \Delta t$, $\omega_N^{ik} = \sqrt{K_N/m}$, $\omega_T^{ik} = \sqrt{K_T/m}$, and m is the largest of the two contacting masses. The combination of Eqs. 13, 14, 16 with Eq. 18 provide the unique explicit expressions for $\psi_{1N}^{ik}, \psi_{2N}^{ik}$ and $\psi_{1T}^{ik}, \psi_{2T}^{ik}$. The insertion of these expressions in Eqs. 6 enforces a conservative response for Stick and consistent dissipative for Slip.

6 NUMERICAL SIMULATIONS

6.1 Elliptical particle Carom problem

In this subsection, the trajectory of the successive impacts of a rigid ellipse inside a one–meter square is simulated. The ellipse, of axes 15/6 cm is initially positioned at (0.45, 0.1) m, inclination $\alpha = 50°$ as seen in Fig. 1 top, and it is subjected to initial velocity $V_x = 1, V_y = -0.4$ m/s in direction $\theta = -22°$, without spin. The friction angle is $\phi = 15°$ and the rest of the numerical parameters are the same as those in the previous simulation. To visualize the rotation of the ellipse, the orientation is defined by the largest semiaxis.

Figs. 1 depict the evolution of trajectory (top), linear velocities (left), rotational velocity and energy (right) from EFDA and from the analytical solution reproduced in the

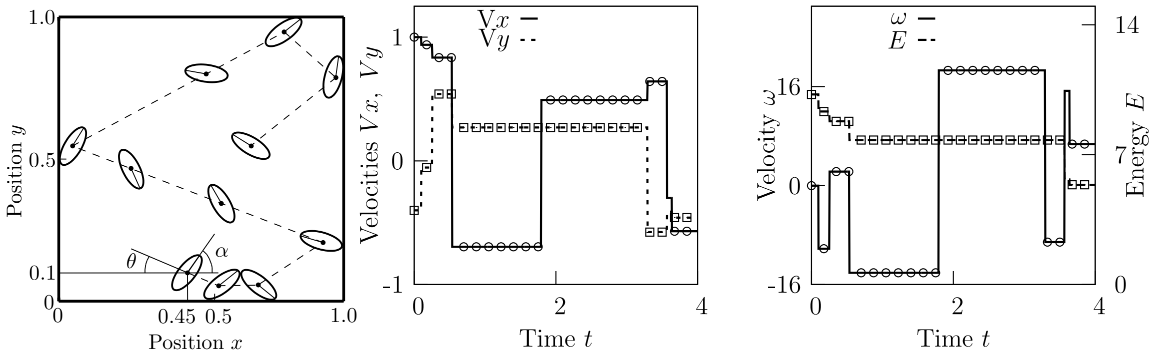

Figure 1: Carom problem with an elliptical particle: trajectory (top), linear velocities (bottom left), rotational velocity and total energy (bottom right). In the three, analytical (symbols) and numerical (lines) distributions coincide.

Appendix. First, the ellipse impacts against the bottom side of the box and consequently rotates since the line of action of the resultant contact force does not intersect the center of gravity, Figs. 1 top and bottom right. The initial V_x is larger than V_y, therefore, the contact point Slips along the horizontal side dissipating energy. Successive impacts decrease the tangential (with respect to any side) velocity, Fig. 1 bottom left. It is important to note that impacts may be conservative for one contact and dissipative for others: the velocity relation changes in every impact. This can be appreciated in Fig. 1 bottom right, where for impacts at $t \approx 1.8$, $t \approx 3.2$ s the energy is conserved, and at others dissipated, such as those before $t \approx 0.4$ and at $t \approx 3.5$, $t \approx 3.7$ s. Numerical and analytical results coincide perfectly since both EFDA and the analytical formulation are developed enforcing the energy conservation for normal contact and conservation–consistent dissipation for tangential.

6.2 The concave pendulum problem

The motion decay of two symmetrical positioned rigid disks resting on a semicircular rough surface under the action of gravity is analyzed, Fig. 2. In this problem, there are two sources of energy dissipation: friction between disk and surface, and frictional impact between disks. After every impact both disks periodically return to a lower height, until the motion stops. The disk and surface radii are 0.1, 0.5 m respectively, the friction angle is $\phi = 15°$, the initial position is defined by the angle $\gamma_{t0} = 45°$ and the initial velocity is zero.

In Figs. 3 left and right, results at the center of gravity are shown for only one of the disks, since the problem is symmetric. The left graphic depicts the polar position γ and velocity $\dot{\gamma}$, while the right one, the total energy E and the rotational velocity ω.

With the prescribed initial conditions, both disks slide from the beginning (dissipating energy) and eventually impact right at the bottom at $t \approx 0.3$ s. Before this first impact,

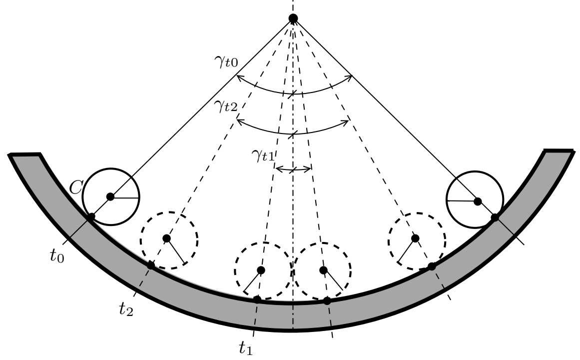

Figure 2: Pendulum problem. Initial, impact and maximum height after first impact defined by γ_{t_0}, γ_{t_1} and γ_{t_2}.

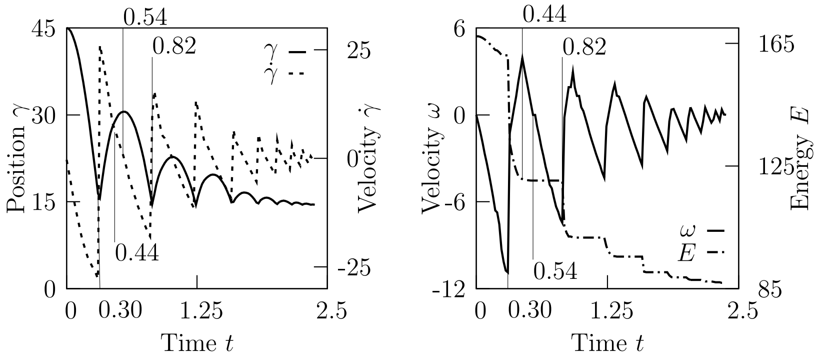

Figure 3: Pendulum problem. Evolution of center of gravity position γ, velocity $\dot{\gamma}$, rotational velocity ω and energy E with time t.

ω and $\dot{\gamma}$ increase (in modulus), while E decreases. At the moment of the first impact, $\dot{\gamma}$ and ω suddenly increase and reverse their values (abrupt change in the distributions) and the total energy E decreases also suddenly. As expected, the angular velocity ω reaches a maximum when the velocity of the contact point is zero $V_C = 0$ at $t_{roll} \approx 0.44$ s, then the disks start rolling and stop dissipating energy by friction. During rolling $\omega, \dot{\gamma}$ decrease to zero at $t \approx 0.54$ s the disks achieves its maximum height for the oscillation and therefore γ is maximum. Immediately, the disks continue rolling although downwards, increasing $\dot{\gamma}$ and ω until $t \approx 0.82$ s, when a new impact occurs. This sequence repeats continuously with a motion decay after every impact; the simulation stops when the energy loss is smaller than a fixed tolerance.

7 CONCLUSIONS

- The development of an energy frictional dissipating algorithm for contact problems (EFDA), that conserves momenta and dissipates energy according to the frictional Coulomb law, is presented. The key of EFDA is that conservation–dissipation is consistently enforced in a conserving framework through the modification of the contact kinematics with an additional linear momentum and a contact force.
- The energy of the normal contact is also conserved, therefore EFDA accurately obtains the real normal contact force and therefore, the real tangential contact force for the Slip case. This consistent energy conservation–dissipation ensure stability of the algorithm.
- For the more complex situation of the pendulum problem (two disks on a curved rough surface), EFDA also shows good stability simulating the impacts, transitions from sliding to rolling and motion decay until rest.

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