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Overlapping Resilient H_2 Filtering for Uncertain Continuous-Time Systems

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Abstract— **The paper presents the expansion-contraction relations within the Inclusion Principle for a class of continuous-time uncertain systems when considering H_2 filtering with additive filter uncertainty. Norm bounded uncertainties are considered. The main contribution is the derivation of conditions under which a resilient filter designed in the expanded space is contracted into the initial system preserving simultaneously optimized upper bound for the error variance. An LMI procedure is supplied for resilient full order robust minimum variance filter design. The results are specialized into the overlapping decentralized filter setting. It enables to construct robust resilient H_2 filters with block tridiagonal gain matrices.**

I. INTRODUCTION

Real–world large scale complex systems collect usually data from geographically distributed sensors. For instance, the Kalman filtering approach in sensor networks, formation flying spacecraft, or multirobot localization lead to a high dimensional state vector. Central Kalman filtering requires to communicate large amount of data for such systems. With the rapid development in multiprocessor systems, an increasing interest has focused on obtaining high processing speed through parallelism in algorithms. It means that data are obtained by different subsystems. All such decentralized structures are based on local processing at the local processors or nodes taking the advantage of survivability, reduced computational complexity, scalability, and sharing of the sensing load when comparing it with the centralized fusion.

The Kalman filtering fails to provide a guaranteed performance in the sense of uncertain systems. This fact motivates an effort to design estimators guaranteeing upper bound on the performance in the sense of the error variance for any admissible uncertainty. These filters are referred as robust filters. While robustness relates to uncertainties in the plant, fragility relates to uncertainties or inaccuracies in the implementation of a designed filter.

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Standard assumption on designed controllers and filters is that they can be implemented exactly into real world systems. In practice, filters designed using theoretical methods and simulations are implemented imprecisely because of various reasons such as finite word length in any digital system, imprecision in analog–digital conversions, finite resolution instrumentation or the need for additional tuning of parameters in the final filter implementation. The controller designed for uncertain plants may be sufficiently robust against system parameters, but the controller parameters itself may be sensitive to relatively small perturbations and could even destabilize a closed-loop system. The importance of fragility, i.e. high sensitivity of filter parameters on its very small changes, is underlined when considering large-scale complex systems filtering using low cost local filters. While robustness relates to uncertainties in the plant, fragility relates to uncertainties or inaccuracies in the implementation of a designed filter. This situation naturally motivates the development of new effective filter design methods taking into account particular features of these systems including implementation aspects.

A. *Prior Work*

The decentralized structures for the Kalman filtering have been studied in [1], [2], [3]. The distributed Kalman filtering has been applied to sensor networks in [4], by Olfati-Saber in [5], the navigation problems [6], the multirobot localization [7], or the sensor fault detection [8]. All these filtering structures correspond with the BD form or the BBD form of sparse matrices used in large scale systems with non-overlapping subsystem patterns [9]. Overlapping subsystems approach by [10] leads to the filter design with block tridiagonal gain matrices (BTD). Overlapping decompositions have been systematically worked out within the concept of the Inclusion Principle [10]. The Inclusion Principle has been applied to different classes of systems and problems as illustrated for instance in [8], [11], [12].

Robustness against model parameter uncertainty has been intensively studied for many years. Various approaches to robust filtering problem have been developed [13], [14], [15], [16], [17].

To cope with fragility of observers, several types of observer uncertainties may be considered. H_2 resilient filter design within a multiplicative uncertainties considers [18]. H_2 nonfragile filter design for both additive and multiplicative filter uncertainties presents [19] for continuous time-time uncertain systems using the LMIs. Digital controller implementation and fragility issues studies [20].

The present paper extends the results in [19] to the overlapping H_2 resilient robust filter design for a class of uncertain systems with norm bounded uncertainties. The significance of this problem lies in scalability of the filter structure, its better performance and reliability of filters against local filter failures when comparing them with the BD structures and keeping simultaneously high processing speed.

To the authors knowledge, the expansion-contraction relations have not been extended up to now on the design of H_2 resilient filters for the considered class of uncertain systems.

B. Outline of the paper

The paper presents the design of overlapping H_2 resilient filter using the LMIs for a class of uncertain systems with norm bounded uncertainties. The expansion-contractions relations are derived for uncertain systems as well as the contractibility conditions for filters with uncertain gain matrix. Norm bounded filter uncertainties are considered. A constructive LMI-based algorithm is supplied. It is shown that to satisfy a freedom in the design of overlapping filter parameters, the input filter matrix must be given a priori in the original space and expanded for the LMI design.

II. Problem Formulation

A. Uncertain Systems and Robust Filters

Consider a class of uncertain systems described by:

$$\begin{aligned} \mathbf{S}: \ & \dot{x}(t) = [A + \Delta A(t)]\, x(t) + Bw(t),\ x(0) = x_0, \\ & y(t) = [C + \Delta C(t)]\, x(t) + Dw(t), \\ & z(t) = Lx(t), \end{aligned} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state, $y(t)\in\mathbb{R}^m$ the measurement output, $z(t)\in\mathbb{R}^p$ is the vector of state to be estimated, $w(t)\in\mathbb{R}^r$ is a zero-mean white noise input with unity power spectrum density matrix. The initial condition $x(0)$ is a zero-mean random variable uncorrelated with the input noise $w(t)$. The matrices A, B, C, D, L are known real constants of appropriate dimensions. Suppose that m=r. The notation for the dimensions m and r follows from the different usage of the corresponding vector components used later. $\Delta A(t)$ and $\Delta C(t)$ are real time-varying matrix functions of appropriate dimensions. Norm-bounded parameter uncertainties are supposed in the form

$$\Delta A(t) = M_a \Delta_a(t) H_a, \qquad \Delta C(t) = N_a \Delta_a(t) H_a, \tag{2}$$

where $M_a\in\mathbb{R}^{n\times\alpha}$, $N_a\in\mathbb{R}^{m\times\alpha}$, $H_a\in\mathbb{R}^{\beta\times n}$ are known real constant matrices of appropriate dimensions and $\Delta_a(t)\in\mathbb{R}^{\alpha\times\beta}$ is an unknown arbitrarily time-varying matrix with Lebesgue measurable elements satisfying $\Delta_a^T(t)\Delta_a(t)\leqslant I$ for all t.

The problem of robust linear filtering is to design an estimation of $z(t)$ given by $z_f(t)$=$\mathcal{F}\cdot y(t)$, where $\mathcal{F}$ is a linear operator such that as $t\to\infty$ it minimizes an upper bound $\sigma(\mathcal{F})$ of the estimation error variance for all admissible parameter uncertainty. Define the estimation error $e(t)$=$z(t) - z_f(t)$, then the problem of interest becomes

$$\min_{\mathcal{F}\in\mathcal{C}_f} \sigma(\mathcal{F}), \qquad \sup_{\|\Delta\|_2\leqslant I} \mathbb{E}\left[e^T(t)e(t)\right] \leqslant \sigma(\mathcal{F}), \tag{3}$$

where the feasible set $\mathcal{C}_f$ represents the set of all linear operators with minimum state-space realization of the form:

$$\begin{aligned} & \dot{x}_f(t) = A_f x_f(t) + B_f y(t), \quad x_f(0) = 0, \\ & z_f(t) = L_f x_f(t), \end{aligned} \tag{4}$$

where the matrices $A_f\in\mathbb{R}^{n_f\times n_f}$, $B_f\in\mathbb{R}^{n_f\times m}$, $L_f\in\mathbb{R}^{p\times n_f}$ and the scalar $n_f>0$ are the design parameters.

B. Resilient Filters

Now, consider the feasible set $\mathcal{C}_f$ represents the set of all linear operators with minimum state-space realization of the form:

$$\begin{aligned} \mathbf{S_f}: \ & \dot{x}_f(t) = [A_f + \Delta A_f(t)]\, x_f(t) + B_f\, y(t),\ x_f(0) = 0, \\ & z_f(t) = [L_f + \Delta L_f(t)]\, x_f(t), \end{aligned} \tag{5}$$

where the additive gain perturbations are represented by

$$\Delta A_f(t) = M_f \Delta_f(t) H_f, \qquad \Delta L_f(t) = N_f \Delta_f(t) H_f \tag{6}$$

such that $\Delta_f^T(t)\Delta_f(t)\leqslant I$ for all t.

Consider the case of full-order filtering n=n_f. Connecting the filter (5) to the system (1) leads to the relations

$$\begin{aligned} \hat{\mathbf{S}}: \ & \dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}w(t), \\ & e(t) = \hat{L}\hat{x}(t), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \hat{x}(t) &= \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \\ \hat{L} &= \begin{bmatrix} L & -[L_f + \Delta L_f(t)] \end{bmatrix} = \begin{bmatrix} L & -L_f - N_f\Delta_f(t)H_f \end{bmatrix}, \\ \hat{A} &= \begin{bmatrix} A + \Delta A(t) & 0 \\ B_f\,[C + \Delta C(t)] & A_f + \Delta A_f(t) \end{bmatrix} = \begin{bmatrix} A & 0 \\ B_f C & A_f \end{bmatrix} \\ &+ \begin{bmatrix} M_a & 0 \\ B_f N_a & M_f \end{bmatrix} \begin{bmatrix} \Delta_a(t) & 0 \\ 0 & \Delta_f(t) \end{bmatrix} \begin{bmatrix} H_a & 0 \\ 0 & H_f \end{bmatrix}. \end{aligned} \tag{8}$$

The estimation error variance as $t\to\infty$ satisfies

$$\mathbb{E}\left[e^T(t)e(t)\right] = \text{Tr}\left[\hat{L}P\hat{L}^T\right] \tag{9}$$

for all admissible uncertainties satisfying (2). $0<P$=P^T denotes the solution of the Lyapunov equation

$$P\hat{A}^T + \hat{A}P + \hat{B}\hat{B}^T = 0. \tag{10}$$

C. LMI design

Denote the partition of the matrix P in (9) as follows

$$P = \begin{bmatrix} P_x & U_c \\ U_c^T & P_f \end{bmatrix}, \quad P^{-1} \begin{bmatrix} \mathcal{Y} & R_c \\ R_c^T & Q_f \end{bmatrix} \tag{11}$$

where P_x, P_f, $\mathcal{Y}$, $Q_f \in \Re^{n\times n}$ are all symmetric positive definite matrices. Then U_c and R_c are nonsingular matrices [14]. The following theorem is used for the filter design [19].

Theorem 1: Consider a system $\mathbf{S}$ given by (1) subject to uncertainties (2) with nonfragile linear filter $\mathbf{S_f}$ given in (5) and additive gain perturbations (6). Then, the gain matrices of the minimum error variance filter $\mathcal{F} \in \mathcal{C}_f$ defined by

$$L_f = \mathcal{Z}[U_c^T \mathcal{X}]^{-1}, \;\; A_f = R_c^{-1}\Gamma[U_c^T \mathcal{X}]^{-1}, \;\; B_f = R_c^{-1}\mathcal{G}, \tag{12}$$

where $0<W=W^T$, $0<\mathcal{X}=\mathcal{X}^T=P_x^{-1}$, $0<\mathcal{Y}=\mathcal{Y}^T$, $\Gamma=R_c A_f U_c^T \mathcal{X}$, $\mathcal{G}=R_c B_f$, $\mathcal{R}=H_f U_c \mathcal{X}$, $\mathcal{Z}=L_f U_c^T \mathcal{X}$, ϵ, are the optimal feasible solution of the following convex programming problem over LMIs:

$$\min \text{Tr}\,[W]$$

$$\begin{bmatrix} \mathcal{X} & \mathcal{X} & -\mathcal{Z}+L^T & \mathcal{R}^T \\ \mathcal{X}^T & \mathcal{Y} & L^T & 0 \\ -\mathcal{Z}^T+L & L & W-\epsilon N_f N_f^T & 0 \\ \mathcal{R}^T & 0 & 0 & -\epsilon I \end{bmatrix} > 0$$

$$\begin{bmatrix} \mathcal{X}A+A^T\mathcal{X} & \Upsilon_a & \mathcal{X}B & \mathcal{X}M_a & 0 \\ \Upsilon_a^T & \Upsilon_b & \mathcal{Y}B+\mathcal{G}D & 0 & \mathcal{Y}M_a+\mathcal{G}N_a \\ B^T\mathcal{X}^T & B^T\mathcal{Y}^T+D^T\mathcal{G}^T & -I & 0 & 0 \\ M_a^T\mathcal{X}^T & 0 & 0 & -I & 0 \\ 0 & M_a^T\mathcal{Y}^T+N_a^T\mathcal{G} & 0 & 0 & -I \end{bmatrix} < 0 \tag{13}$$

where

$$\begin{aligned} \Upsilon_a &= \mathcal{X}A + A^T\mathcal{Y} + C^T\mathcal{G}^T + \Gamma^T + \delta(\mathcal{X}M_a M_a^T \mathcal{Y} \\ &\quad + \mathcal{X}M_a N_a^T \mathcal{G}^T) + \mu H_a^T H_a, \\ \Upsilon_b &= \mathcal{Y}A + A^T\mathcal{Y} + \mathcal{G}C + C^T\mathcal{G}^T + \mu H_a^T H_a. \end{aligned} \tag{14}$$

Remark 1: Notice only that the selection $U_c = \mathcal{X}^{-1} - \mathcal{Y}^{-1}$ leads to the case when $L = L_f$ which essentially simplifies the solution [15], [19].

D. Inclusion Principle

Consider now a new bigger system in the form

$$\begin{aligned} \tilde{\mathbf{S}}:\; \dot{\tilde{x}}(t) &= \left[\tilde{A} + \Delta\tilde{A}(t)\right]\tilde{x}(t) + \tilde{B}w(t), \\ \tilde{y}(t) &= \left[\tilde{C} + \Delta\tilde{C}(t)\right]\tilde{x}(t) + \tilde{D}w(t), \\ \tilde{z}(t) &= \tilde{L}\tilde{x}(t), \end{aligned} \tag{15}$$

where $\tilde{x}(t)\in\mathbb{R}^{\tilde{n}}$ is the state, $\tilde{y}(t)\in\mathbb{R}^{\tilde{m}}$ the measurement output, and $\tilde{z}(t)\in\mathbb{R}^{\tilde{p}}$. Suppose $n \le \tilde{n}$, $m \le \tilde{m}$, and $p \le \tilde{p}$. Norm bounded uncertainties $\Delta\tilde{A}(t)$, $\Delta\tilde{C}(t)$ satisfy

$$\Delta\tilde{A}(t) = \tilde{M}_a\tilde{\Delta}_a(t)\tilde{H}_a, \quad \Delta\tilde{C}(t) = \tilde{N}_a\tilde{\Delta}_a(t)\tilde{H}_a \tag{16}$$

with $\tilde{M}_a\in\mathbb{R}^{\tilde{n}\times\tilde{\alpha}}$, $\tilde{N}_a\in\mathbb{R}^{\tilde{m}\times\tilde{\alpha}}$, $\tilde{H}_a\in\mathbb{R}^{\tilde{\beta}\times\tilde{n}}$ known real constant matrices of appropriate dimensions and where $\tilde{\Delta}_a(t)\in\mathbb{R}^{\tilde{\alpha}\times\tilde{\beta}}$ is an unknown matrix with Lebesgue measurable elements satisfying $\tilde{\Delta}_a^T(t)\tilde{\Delta}_a(t) \leqslant I$ for all t. Associated to the system $\tilde{\mathbf{S}}$ given in (15) we have the following filter

$$\begin{aligned} \tilde{\mathbf{S}}_\mathbf{f}:\; \dot{\tilde{x}}_f(t) &= \left[\tilde{A}_f + \Delta\tilde{A}_f(t)\right]\tilde{x}_f(t) + \tilde{B}_f\,\tilde{y}(t),\; \tilde{x}_f(0) = 0, \\ \tilde{z}_f(t) &= \left[\tilde{L}_f + \Delta\tilde{L}_f(t)\right]\tilde{x}_f(t) \end{aligned} \tag{17}$$

such that $\tilde{A}_f\in\mathbb{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B}_f\in\mathbb{R}^{\tilde{n}\times\tilde{m}}$, $\tilde{L}_f\in\mathbb{R}^{\tilde{p}\times\tilde{n}}$. The additive gain perturbations are represented by

$$\Delta\tilde{A}_f(t) = \tilde{M}_f\tilde{\Delta}_f(t)\tilde{H}_f, \quad \Delta\tilde{L}_f(t) = \tilde{N}_f\tilde{\Delta}_f(t)\tilde{H}_f \tag{18}$$

of appropriate dimensions satisfying the standard assumption $\tilde{\Delta}_f^T(t)\tilde{\Delta}_f(t) \leqslant I$ for all t.

Consider the standard relations between the states and outputs within the Inclusion Principle. It means that the systems $\mathbf{S}$, $\tilde{\mathbf{S}}$ and $\mathbf{S_f}$, $\tilde{\mathbf{S}}_\mathbf{f}$ are related by the following linear transformations

$$\begin{aligned} \tilde{x}(t) &= Vx(t), & x(t) &= U\tilde{x}(t), \\ \tilde{y}(t) &= Ty(t), & y(t) &= S\tilde{y}(t), \\ \tilde{z}(t) &= Fz(t), & z(t) &= G\tilde{z}(t), \\ \tilde{x}_f(t) &= Vx_f(t), & x_f(t) &= U\tilde{x}_f(t), \\ \tilde{z}_f(t) &= Fz_f(t), & z_f(t) &= G\tilde{z}_f(t) \end{aligned} \tag{19}$$

for all t, where V, T, G and their pseudoinverse matrices $U=(V^TV)^{-1}V^T$, $S=(T^TT)^{-1}T^T$, $G=(F^TF)^{-1}F^T$, respectively, are constant full rank matrices of appropriate dimensions [10]. Suppose given a set of matrices (U,V,S,T,G,F). Then, the matrices $\tilde{A}$, $\Delta\tilde{A}(t)$, $\tilde{B}$, $\tilde{C}$, $\Delta\tilde{C}(t)$, $\tilde{D}$, $\tilde{L}$, A_f, $\Delta A_f(t)$, $\tilde{B}_f$, L_f, and $\Delta L_f(t)$ can be described in the following form

$$\begin{aligned} \tilde{A} &= VAU + M_1, & \Delta\tilde{A}(t) &= V\Delta A(t)U, \\ \tilde{B} &= VB + M_2, & \tilde{C} &= TCU + M_3, \\ \Delta\tilde{C}(t) &= T\Delta C(t)U, & \tilde{D} &= TD + M_4, \\ \tilde{L} &= FLU + M_5 & A_f &= U\tilde{A}_fV, \\ \Delta A_f(t) &= U\;\Delta\tilde{A}(t)V, & \tilde{B}_f &= VB_fS, \\ L_f &= G\tilde{L}_fV, & \Delta L_f(t) &= G\Delta\tilde{L}_f(t)V, \end{aligned} \tag{20}$$

where M_1, M_2, M_3, M_4 and M_5 are so called *complementary matrices*. Usually, the transformations (U,V,S,T,G,F) are selected a priori to define structural relations between the systems $\mathbf{S}$, $\tilde{\mathbf{S}}$ and $\mathbf{S_f}$, $\tilde{\mathbf{S}}_\mathbf{f}$. Given these transformations, the choice of the complementary matrices offers the degrees of freedom to obtain different expanded spaces with desirable properties [11], [21].

Definition 1: A system $\tilde{\mathbf{S}}$ *includes* a system $\mathbf{S}$, denoted as $\tilde{\mathbf{S}}\supset\mathbf{S}$, if there exists a set of constant matrices (U,V,S,T,G,F) such that $UV=I_n$, $ST=I_m$, $GF=I_p$ and for any initial condition $x(0)$ and any disturbance $w(t)$ of $\mathbf{S}$ the relations $x\,(t;x(0),w(t))=U\tilde{x}\,(t;Vx(0),w(t))$, $y[x(t)]=S\tilde{y}[\tilde{x}(t)]$ and $z[x(t)]=G\tilde{z}[\tilde{x}(t)]$ hold for all t.

Definition 2: A filter $\tilde{\mathbf{S}}_\mathbf{f}$ for the system $\tilde{\mathbf{S}}$ is *contractible* to the filter $\mathbf{S_f}$ of $\mathbf{S}$ if $\tilde{\mathbf{S}}\supset\mathbf{S}$ by Definition 1 and $x_f(t;y(t))=U\tilde{x}_f(t;\tilde{y}(t))$ and $z_f[x_f(t)]=G\tilde{z}_f[\tilde{x}_f(t)]$ hold for all t.

Denote $\hat{\tilde{\mathbf{S}}}$ the corresponding filter for the system $\tilde{\mathbf{S}}$ which has an analogous meaning as the filter $\hat{\mathbf{S}}$ for the initial system $\mathbf{S}$ given in (5).

Definition 3: A pair $\hat{\tilde{\mathbf{S}}}=(\tilde{\mathbf{S}}_\mathbf{f},\tilde{\mathbf{S}})$ *includes* the pair $\hat{\mathbf{S}}=(\mathbf{S_f},\mathbf{S})$, denoted as $\hat{\tilde{\mathbf{S}}}\supset\hat{\mathbf{S}}$, if $\tilde{\mathbf{S}}_\mathbf{f}$ is contractible to $\mathbf{S_f}$.

It is well known that the unique solution of the systems (1) and (15) are given by the following equations, respectively:

$$\begin{aligned} x\,(t;x_0,w(t)) &= \Phi_{\bar{A}}(t,0)x(0) + \int_0^t \Phi_{\bar{A}}(t,s)Bw(s)\,ds, \\ \tilde{x}\,(t;Vx_0,w(t)) &= \tilde{\Phi}_{\bar{\tilde{A}}}(t,0)Vx(0) + \int_0^t \tilde{\Phi}_{\bar{\tilde{A}}}(t,s)\tilde{B}w(s)\,ds, \end{aligned} \tag{21}$$

where $\bar{A}(t)=A+\Delta A(t)$, $\bar{\tilde{A}}(t)=\tilde{A}+\Delta\tilde{A}(t)$, and where $\Phi_{\bar{A}}$, $\tilde{\Phi}_{\bar{\tilde{A}}}$ are their corresponding *transition matrices*. Similarly, the solutions for the filters $\mathbf{S_f}$ and $\tilde{\mathbf{S}}_\mathbf{f}$ are given by

$$x_f(t; y(t)) = \int_0^t \Phi_{\bar{A}_f}(t,s) B_f\, y(s)\, ds, \qquad \tilde{x}_f(t; \tilde{y}(t)) = \int_0^t \tilde{\Phi}_{\bar{\tilde{A}}_f}(t,s) \tilde{B}_f\, \tilde{y}(s)\, ds \tag{22}$$

with $\bar{A}_f(t)$=A_f+$\Delta A_f(t)$, $\bar{\tilde{A}}_f(t)$=$\tilde{A}_f$+$\Delta\tilde{A}_f(t)$, respectively.

E. The Problem

Given a class of continuous-time uncertain systems $\mathbf{S}$ defined by (1). Consider an expanded system $\tilde{\mathbf{S}}$ represented by (15). Suppose that $\tilde{\mathbf{S}}\supset\mathbf{S}$ holds by Definition 1. Then, the specific goals are as follows

- Derive conditions under which $(\tilde{\mathbf{S}}_\mathbf{f}, \tilde{\mathbf{S}})\supset(\mathbf{S_f}, \mathbf{S})$.
- Specialize the global system results into overlapping decentralized filter design setting.
- Derive all the above results in terms of complementary matrices. Use the LMI approach to compute the required resilient filter gain matrices.

III. Solution

A. Expansion-Contraction Relations

Theorem 2: Consider the pairs $\hat{\tilde{\mathbf{S}}}$=$(\tilde{\mathbf{S}}_\mathbf{f}, \tilde{\mathbf{S}})$ and $\hat{\mathbf{S}}$=$(\mathbf{S_f}, \mathbf{S})$. Then, $\hat{\tilde{\mathbf{S}}}\supset\hat{\mathbf{S}}$ if and only if the following conditions

$$\begin{aligned}
&a) \quad U\tilde{\Phi}_{\bar{\tilde{A}}}(t,0)V = \Phi_{\bar{A}}(t,0),\\
&b) \quad U\tilde{\Phi}_{\bar{\tilde{A}}}(t,s)\tilde{B} = \Phi_{\bar{A}}(t,s)B,\\
&c) \quad S\bar{\tilde{C}}(t)\tilde{\Phi}_{\bar{\tilde{A}}}(t,0)V = \bar{C}(t)\Phi_{\bar{A}}(t,0),\\
&d) \quad S\bar{\tilde{C}}(t)\tilde{\Phi}_{\bar{\tilde{A}}}(t,s)\tilde{B} = \bar{C}(t)\Phi_{\bar{A}}(t,s)B,\\
&e) \quad S\tilde{D} = D,\\
&f) \quad G\tilde{L}\tilde{\Phi}_{\bar{\tilde{A}}}(t,0)V = L\Phi_{\bar{A}}(t,0),\\
&g) \quad G\tilde{L}\tilde{\Phi}_{\bar{\tilde{A}}}(t,s)\tilde{B} = L\Phi_{\bar{A}}(t,s)B,\\
&h) \quad U\tilde{\Phi}_{\bar{\tilde{A}}_f}(t,s)V = \Phi_{\bar{A}_f}(t,s),\\
&i) \quad G\bar{\tilde{L}}_f(t)\tilde{\Phi}_{\bar{\tilde{A}}_f}(t,s)VB_f = \bar{L}_f(t)\Phi_{\bar{A}_f}(t,s)B_f
\end{aligned} \tag{23}$$

are satisfied for all t and s, where $\bar{\tilde{C}}(t)$=C+$\Delta C(t)$.

Proof. By Definition 1, impose $x(t)$=$U\tilde{x}(t)$ for all t. Substitute (21) into this relation and compare both sides. We obtain the equalities *a)* and *b)* given in (23). Condition $y[x(t)]$=$S\tilde{y}[\tilde{x}(t)]$ is equivalent to relations *c)*, *d)* and *e)*. Finally, the condition $z[x(t)]$=$G\tilde{z}[\tilde{x}(t)]$ corresponds to the conditions *f)* and *g)* given in (23). By Definition 2, corresponding to the contractibility of the filter, $x_f(t)$=$U\tilde{x}_f(t)$ is equivalent to *h)* and $z_f[x_f(t)]$=$G\tilde{z}_f[\tilde{x}_f(t)]$ is equivalent to the condition *i)* in (23). □

Remark 2: To obtain the general solution of a time-varying system is very difficult. Because of this, an attempt has been made to approximate the solutions using transition matrices. However, even to compute such approximation via Peano-Baker series can be a complicated task excluding trivial cases [12], [22]. For this reason, next theorem, which is equivalent to Theorem 2, allows to obtain expanded systems only in terms of complementary matrices without any knowledge of the transition matrices.

Theorem 3: Consider the pairs $\hat{\tilde{\mathbf{S}}}$=$(\tilde{\mathbf{S}}_\mathbf{f}, \tilde{\mathbf{S}})$ and $\hat{\mathbf{S}}$=$(\mathbf{S_f}, \mathbf{S})$. Then, $\hat{\tilde{\mathbf{S}}}\supset\hat{\mathbf{S}}$ if and only if the following conditions

$$\begin{aligned}
&a) \quad UM_1^iV = 0, && b) \quad UM_1^{i-1}M_2 = 0,\\
&c) \quad SM_3M_1^{i-1}V = 0, && d) \quad SM_3M_1^{i-1}M_2 = 0,\\
&e) \quad SM_4 = 0, && f) \quad GM_5M_1^{i-1}V = 0,\\
&g) \quad GM_5M_1^{i-1}M_2 = 0 &&
\end{aligned} \tag{24}$$

hold for all i=$1, 2, ..., \tilde{n}$.

Proof. Consider the transition matrix $\tilde{\Phi}_{\bar{\tilde{A}}}$ of the expanded system $\tilde{\mathbf{S}}$ as a function of two variables defined by the Peano-Baker series

$$\begin{aligned}\tilde{\Phi}_{\bar{\tilde{A}}}(t,s) =& I + \int_s^t \bar{\tilde{A}}(\sigma_1)\, d\sigma_1 + \int_s^t \bar{\tilde{A}}(\sigma_1) \int_s^{\sigma_1} \bar{\tilde{A}}(\sigma_2)\, d\sigma_2 d\sigma_1\\
&+ \int_s^t \bar{\tilde{A}}(\sigma_1) \int_s^{\sigma_1} \bar{\tilde{A}}(\sigma_2) \int_s^{\sigma_2} \bar{\tilde{A}}(\sigma_3)\, d\sigma_3 d\sigma_2 d\sigma_1 + \cdots\end{aligned} \tag{25}$$

Similar expression can be written by the transition matrix $\Phi_{\bar{\tilde{A}}_f}$ for the filter $\tilde{\mathbf{S}}_f$. From Theorem 2 and by using relations (20), the conditions *a)–g)* given in (23) are equivalent to the conditions *a)–g)* given in (24), respectively. Moreover, conditions *h)* and *i)* in (23) are automatically satisfied in terms of complementary matrices. □

Although we have got necessary and sufficient conditions so that $\hat{\tilde{\mathbf{S}}}\supset\hat{\mathbf{S}}$, the requirements (24) are very difficult to verify due to the power of M_1 and the matrix products that appear. For this reason, we will work only with sufficient conditions which are given in the next proposition.

Proposition 1: Consider the pairs $\hat{\tilde{\mathbf{S}}}$=$(\tilde{\mathbf{S}}_\mathbf{f}, \tilde{\mathbf{S}})$ and $\hat{\mathbf{S}}$=$(\mathbf{S_f}, \mathbf{S})$. Then, $\hat{\tilde{\mathbf{S}}}\supset\hat{\mathbf{S}}$ if SM_4=0 and

$$\begin{aligned}
&a) \; M_1V = 0, \quad M_2 = 0, \quad M_3V = 0, \quad M_5V = 0, \quad \text{or}\\
&b) \; UM_1 = 0, \quad UM_2 = 0, \quad SM_3 = 0, \quad GM_5 = 0.
\end{aligned} \tag{26}$$

Proof. The proof is a direct consequence of Theorem 3. □

The condition SM_4=0 and the particular case *b)* in Proposition (1) under which the inclusion $\hat{\tilde{\mathbf{S}}}\supset\hat{\mathbf{S}}$ is used in the following text.

Now, suppose that a resilient filter $\tilde{\mathbf{S}}_\mathbf{f}$ is designed for the expanded system $\tilde{\mathbf{S}}$ using the LMIs by Theorem 1. It means that there is computed also the matrix $\tilde{W}$. Notice that the filter design is performed with a $\tilde{W}$ as a design parameter. Then, it is necessary to prove under which conditions the equality between upper bounds of minimized estimation error variance holds. Consider these bounds as follows

$$\sigma(\mathcal{F}) = \min_{P>0} \mathrm{Tr}[\hat{L}P\hat{L}^T], \qquad \tilde{\sigma}(\tilde{\mathcal{F}}) = \min_{\tilde{P}>0} \mathrm{Tr}[\hat{\tilde{L}}\tilde{P}\hat{\tilde{L}}^T]. \tag{27}$$

This condition presents the following theorem.

Theorem 4: Consider the pairs $\hat{\tilde{\mathbf{S}}}$=$(\tilde{\mathbf{S}}_\mathbf{f}, \tilde{\mathbf{S}})$ and $\hat{\mathbf{S}}$=$(\mathbf{S_f}, \mathbf{S})$. Suppose that F=G=I. Assume that UM_1=0, UM_2=0, SM_3=0, SM_4=0 and GM_5=0 hold. Suppose that the matrix P given in (11) satisfies $\tilde{P}_x$=VP_xV^T, $\tilde{U}_c$=VU_cV^T, $\tilde{P}_f$=VP_fV^T, where $\tilde{P}$>0 is the corresponding solution of the Lyapunov equation given in (10) for $\hat{\tilde{\mathbf{S}}}$. Then, $\hat{\tilde{\mathbf{S}}} \supset \hat{\mathbf{S}}$ and Tr $[\hat{L}P\hat{L}^T]$=Tr $[\hat{\tilde{L}}\tilde{P}\hat{\tilde{L}}^T]$.

Proof. Consider the systems $\hat{\tilde{\mathbf{S}}}$ and $\hat{\mathbf{S}}$. By comparing the matrices $\hat{L}P\hat{L}^T$ given by (11) with the matrices $\hat{\tilde{L}}\tilde{P}\hat{\tilde{L}}^T$, it is easy to prove that the condition Tr$[\hat{L}P\hat{L}^T]$=Tr$[\hat{\tilde{L}}\tilde{P}\hat{\tilde{L}}^T]$ hold if F=G=I is satisfied together with the relations $\tilde{P}_x$=VP_xV^T, $\tilde{U}_c$=VU_cV^T, $\tilde{P}_f$=VP_fV^T. □

Remark 3: The expansion of the vector z is generally possible within the presented expansion-contraction relations. This expansion is inhibiting when evaluating the upper bounds of the estimated error variance between the original and expanded systems. The comparison by Theorem 4 is possible only when no expansion of z is done. However, such restriction does not at all influence on the quality of the proposed estimation information.

Remark 4: The minimum asymptotic error variance satisfies the relation $\sigma(\mathcal{F}) + \varepsilon$=min Tr $[W]$, where $\varepsilon > 0$ is arbitrarily small number. The parameter ε represents the precision defined by the designer to compute the optimal numerical solution using the LMI problem by Theorem 1 so that ε can be considered arbitrarily small [14].

B. Overlapping Filters

Information structure constraints placed on the filter gain matrices includes several practically important structures corresponding with the forms of matrices used in the sparse matrices theory. Particularly, a block diagonal form (BD) serves for the decentralized filter design. Standard LMI-based computations perform such filter design directly. On the other hand, overlapping decompositions corresponding with a block tridiagonal form of gain matrices (BTD) necessitate to pass through the expansion-contraction process provided that the original system has no direct input into the overlapped parts [9].

A classical way of reasoning considers two overlapping subsystems with the structure of matrices A, $\Delta A(t)$, C, $\Delta C(t)$, and B, D respectively, in the form

$$\left[\begin{array}{c|c|c} * & * & * \\ \hline * & * & * \\ \hline * & * & * \end{array}\right], \qquad \left[\begin{array}{c|c} * & * \\ \hline * & * \\ \hline * & * \end{array}\right], \tag{28}$$

where A_{ii}, $\Delta A_{ii}(t)$ and C_{ii}, $\Delta C_{ii}(t)$ are $n_i \times n_i$ and $m_i \times n_i$ dimensional matrices for i=$1,2,3$, respectively. B_{ij} and D_{ij} for i=$1,2,3$ as well as j=$1,2$ are $n_i \times r_j$ and $m_i \times r_j$ dimensional matrices, respectively. The dimensions of the components of the vector $x^T(t)$=$\left[x_1^T(t), x_2^T(t), x_3^T(t)\right]$ are n_1, n_2, n_3, respectively, and satisfy n_1+n_2+n_3=n. The partition of $w^T(t)$=$\left[w_1^T(t), w_2^T(t)\right]$ has two components of dimensions r_1, r_2 such that r_1+r_2=r.

The matrix L has a generic decomposition structure given by the transposition of the second matrix in (28). The dimensions of the components of the vector $z^T(t)$=$\left[z_1^T(t), z_2^T(t)\right]$ are p_1, p_2 and satisfy p_1+p_2=p.

A standard particular selection of the matrices V and T has the form

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}, \qquad T = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix}. \tag{29}$$

The matrices U and S are pseudoinverses of V and T by Definition 1. The transformations (29) lead in a simple natural way to an expanded system where the state vector $x_2(t)$ appears repeated in $\tilde{x}^T(t)$=$\left[x_1^T(t), x_2^T(t), x_2^T(t), x_3^T(t)\right]$ [10]. The matrix $\tilde{A}$ is expanded accordingly using the matrices V, U, M_1 provided the matrix M_1 satisfies the condition of the case b) in Proposition (1). The same way of reasoning holds for the measurement output vector $y_2(t)$. The matrix $\tilde{C}$ is expanded in a completely analogous way as the matrix $\tilde{A}$. Moreover, notice that the equality F=G=I holds. Suppose the form of the matrix $\tilde{B}$ resulting from a generic form of the matrix B has two diagonal blocks of dimensions $(n_1+n_2) \times r_1$, $(n_2+n_3) \times r_2$ as indicated below

$$B = \left[\begin{array}{c|c} B_{11} & 0 \\ \hline B_{21} & B_{22} \\ \hline 0 & B_{32} \end{array}\right], \quad \tilde{B} = \left[\begin{array}{c|c} B_{11} & 0 \\ B_{21} & B_{22} \\ \hline B_{21} & B_{22} \\ 0 & B_{32} \end{array}\right]. \tag{30}$$

First, the original system is expanded. Then the filter design is performed in the expanded space. It is well known that the design of overlapping controllers/filters depends on the structure of matrices B. Type I corresponds with all nonzero element of all input matrices in (28), while Type II corresponds with all elements $(*)_{21}$=0 and $(*)_{22}$=0. The LMI filter design for Type I can be performed directly on the original system. Type II requires the LMI filter design in the expanded space because the direct design usually leads to infeasibility [9]. Suppose the structure of the matrices $D, \tilde{D}$ is fully analogous to the matrices $B, \tilde{B}$ in (30).

The expanded filter gain matrices $\tilde{A}_f$, $\Delta\tilde{A}_f(t)$ and $\tilde{L}_f$, $\Delta\tilde{L}_f(t)$ have a block diagonal form with two subblocks of dimensions $(n_1+n_2)\times(n_1+n_2)$, $(n_2+n_3)\times(n_2+n_3)$ and $p_1 \times (n_1+n_2)$, $p_2 \times (n_2+n_3)$, respectively.

$$\tilde{A}_f = \begin{bmatrix} \tilde{A}_{f11} & \tilde{A}_{f12} & 0 & 0 \\ \tilde{A}_{f21} & \tilde{A}_{f22} & 0 & 0 \\ 0 & 0 & \tilde{A}_{f33} & \tilde{A}_{f34} \\ 0 & 0 & \tilde{A}_{f43} & \tilde{A}_{f44} \end{bmatrix}, \; \tilde{L}_f = \begin{bmatrix} \tilde{L}_{f11} & \tilde{L}_{f12} & 0 & 0 \\ 0 & 0 & \tilde{L}_{f23} & \tilde{L}_{f24} \end{bmatrix}. \tag{31}$$

Consider $\tilde{C}_f$, $\Delta\tilde{C}_f(t)$, $\tilde{D}_f$ also as block diagonal matrices of appropriate dimensions.

Moreover, the design of overlapping dynamic controllers or filters includes the expansion–contraction relations between their dynamic parts. There appears a known problem. The dynamic controller or filter parameters cannot be arbitrarily selected for the expanded system. There is a crucial structural constraints on the input gain matrices. The constraints has the form B_f=$VU\tilde{B}_fTS$ when considering the filter input matrix contracted by the formula B_f=$U\tilde{B}_fT$ [23]. It is evident that to satisfy such constraints is very difficult. The way to overcome this problem leads to a modified design. A priori given matrix B_f is consequently expand according to the formula $\tilde{B}_f{=}VB_fS$ as it is still included in (20). This is an important step. It enables to use the standard LMIs resilient robust filter design procedure by Theorem 1 but with the restriction that $\tilde{B}_f$ is not a free parameter, but a given parameter. All other design parameters remain free within the LMI resilient filter design. It concerns the remaining gain matrices $\tilde{A}_f$ and $\tilde{L}_f$. Computations can be simply repeated with a new B_f if the current selection of B_f does not keep the design requirements. Therefore, consider simply a given matrix B_f and its expansion $\tilde{B}_f$ in the block diagonal the form

$$B_f = \begin{bmatrix} B_{f11} & 0 & 0 \\ 0 & B_{f22} & 0 \\ 0 & 0 & B_{f33} \end{bmatrix}, \ \tilde{B}_f = \begin{bmatrix} B_{f11} & 0 & 0 & 0 \\ 0 & B_{f22} & 0 & 0 \\ 0 & & B_{f22} & 0 \\ 0 & 0 & 0 & B_{f33} \end{bmatrix}. \tag{32}$$

The contracted gain matrices A_f and L_f have the form

$$\begin{aligned} A_f &= \begin{bmatrix} \tilde{A}_{f11} & \tilde{A}_{f12} & 0 \\ 0.5\tilde{A}_{f21} & 0.5(\tilde{A}_{f22}{+}\tilde{A}_{f33}) & 0.5\tilde{A}_{f34} \\ 0 & \tilde{A}_{f43} & \tilde{A}_{f44} \end{bmatrix}, \\ L_f &= \begin{bmatrix} \tilde{L}_{f11} & \tilde{L}_{f12} & 0 \\ 0 & \tilde{L}_{f23} & \tilde{L}_{f24} \end{bmatrix}. \end{aligned} \tag{33}$$

To simplify, summarize the resilient robust filter design for the Type II.

Proposition 2: Consider the pairs $\hat{\tilde{\mathbf{S}}}{=}(\tilde{\mathbf{S}}_{\mathbf{f}}, \tilde{\mathbf{S}})$ and $\hat{\mathbf{S}}{=}(\mathbf{S}_{\mathbf{f}}, \mathbf{S})$. Suppose that $\tilde{B}_f$ has a fixed structure given by (32) and that all other filter gain matrices and blocks of $\tilde{P}$ in the expanded space are BD matrices. Suppose that $F{=}G{=}I$, $UM_1{=}0$, $UM_2{=}0$, $SM_3{=}0$, $SM_4{=}0$ and $GM_5{=}0$ hold. Consider the matrix P given in the original space by (11) satisfying $\tilde{P}_x{=}VP_xV^T$, $\tilde{U}_c{=}VU_cV^T$, $\tilde{P}_f{=}VP_fV^T$, where $\tilde{P}{>}0$ is the corresponding solution of the Lyapunov equation given in (10) for $\hat{\tilde{\mathbf{S}}}$. Then, the contracted filter has a BTD structure of designed gain matrices satisfying $\hat{\tilde{\mathbf{S}}}{\supset}\hat{\mathbf{S}}$ and $\sigma(\mathcal{F}){=}\tilde{\sigma}(\tilde{\mathcal{F}})$.

Proof. It is straightforward because this proposition is a particular case of Theorem 4 $\square$.

IV. CONCLUSION

The paper presents the solution of the overlapping H_2 resilient robust filter design for a class of continuous-time uncertain systems. Conditions preserving the systems expansion-contraction relations for augmented systems and guaranteeing the quadratic cost bounds have been proved. They are derived in terms of conditions on complementary matrices. An LMI resilient filter design procedure has been supplied. The filter is designed in the expanded space and then it is contracted into the original system. The results have been specialized into overlapping decentralized filter setting. It has been shown a priori selection of input filter matrix is necessary to keep the freedom in designing the dynamic part of an overlapping filter. Moreover, to keep the equality on upper bounds of the estimation error variance between both augmented systems, no expansion of the state vector to be estimated is allowed. The method contributes to scalable multiple filter schemes by parallel redundancy in the increasing reliability against sensor failures, the reduction of computational complexity, and sharing the sensing load. The resulting filter has a block tridiagonal form.

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