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Overlapping quadratic optimal control of linear time–varying commutative systems

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Abstract

Overlapping quadratic optimal control of linear time–invariant continuous-time systems by using generalized selection of complementary matrices has been recently developed as a powerful and effective mean for decentralized control design of linear time-invariant systems. In this paper, it is shown that similar generalizations exist for linear time-varying systems. The results presented here concern implicit conditions for a general form of the transition matrices and explicit conditions for a commutative class of linear time-varying systems.

1 Introduction

In a large variety of physical, natural and man-made systems, subsystems share common parts. It is useful to recognize this reality, which is usually determined by either system structure or computational reasons, to propose decentralized control schemes by using overlapping information sets. Decentralized control strategies offer satisfactory performance at minimum communication cost. The designer of overlapping decentralized control expands first the system into a larger space where the subsystems are disjoint, then designs decentralized controllers in the expanded space by using standard weak coupling disjoint control design methods and finally contracts the system and local control laws into the original space to implement such controllers.

This paper addresses the problem of overlapping decentralized control design via state linear quadratic optimal control (LQ) for a commutative class of continuous-time linear time-varying (LTV) systems.

1.1 Relevant references

The mathematical framework for expansion-contraction relations and conditions became known as the inclusion principle [8], [9], [10], [14]. This principle defines a framework for two dynamic systems with different dimensions, in which solutions of the system with larger dimension include solutions of the system with smaller dimension. Also, there exist another concept of Inclusion Principle called *extension* [7]. The relation between both systems is constructed usually on the base of appropriate linear transformations between the corresponding systems in the original and expanded spaces, where a key role in the selection of appropriate structure of all matrices in the expanded space is played by the so called *complementary matrices* [8], [14]. In fact, only two particular forms, called aggregations and restrictions, have been commonly adopted in the literature for numerical computations. A new characterization of the complementary matrices for linear time-invariant (LTI) systems has been presented in [2], [3], [4], [5], [6], which gives a more explicit way for their selection and includes aggregations and restrictions as particular cases. It relies on a new constructive way of approaching the concept of canonical form within the inclusion principle previously proposed in [10], [14].

One of the open research issues within the inclusion principle is the extension of the results available for LTI systems to LTV systems. To the authors knowledge, the only available results in this direction are in [11], where overlapping decentralized state LQ control of LTV systems is considered. However, these results are restricted to the use of aggregations and restrictions. The present paper extends the results both in [11] as well as those ones in [2], [3].

1.2 Outline of the paper

When abstracting the problem of quadratic optimal control, the influence of complementary matrices is an important issue. The strategy of selection of these matrices has been developed as an effective tool to find both structure and optimal values of free elements of complementary matrices for LTI systems. We devote the main part of this paper to an extension of this strategy for overlapping state LQ optimal control from LTI systems to a class of LTV systems with the commutativity property, including the contractibility conditions.

2 Problem formulation

2.1 Preliminaries

Consider the optimal control problems

$$\min_{u(t)} J(x_0,u) = x^T(t_f)\Pi\, x(t_f) + \\ + \int_{t_0}^{t_f} \left[x^T(t)Q^*(t)x(t) + u^T(t)R^*(t)u(t)\right] dt, \quad (1)$$

$$\text{s.t. } \mathbf{S}: \quad \dot{x}(t) = A(t)x(t) + B(t)u(t),$$

$$\min_{\tilde{u}(t)} \tilde{J}(\tilde{x}_0,\tilde{u}) = \tilde{x}^T(t_f)\tilde{\Pi}\, \tilde{x}(t_f) + \\ + \int_{t_0}^{t_f} \left[\tilde{x}^T(t)\tilde{Q}^*(t)\tilde{x}(t) + \tilde{u}^T(t)\tilde{R}^*(t)\tilde{u}(t)\right] dt, \quad (2)$$

$$\text{s.t. } \tilde{\mathbf{S}}: \quad \dot{\tilde{x}}(t) = \tilde{A}(t)\tilde{x}(t) + \tilde{B}(t)\tilde{u}(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and input of $\mathbf{S}$ at time t for $t \in [t_0, t_f]$; t_0 and t_f are the initial and the terminal time, respectively; $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ and $\tilde{u}(t) \in \mathbb{R}^{\tilde{m}}$ are those ones of $\tilde{\mathbf{S}}$. The matrices $A(t)$, $B(t)$ and $\tilde{A}(t)$, $\tilde{B}(t)$ are continuous in t of dimensions $n \times n$, $n \times m$ and $\tilde{n} \times \tilde{n}$, $\tilde{n} \times \tilde{m}$, respectively. $Q^*(t)$, $\tilde{Q}^*(t)$ are symmetric, nonnegative definite matrices, continuous in t, of dimensions $n \times n$, $\tilde{n} \times \tilde{n}$, respectively. $R^*(t)$, $\tilde{R}^*(t)$ are symmetric, positive definite matrices, continuous in t, of dimensions $m \times m$, $\tilde{m} \times \tilde{m}$, respectively. Π, $\tilde{\Pi}$ are constant, symmetric, nonnegative definite matrices of dimensions $n \times n$, $\tilde{n} \times \tilde{n}$, respectively. In problems (1) and (2) the final time t_f is fixed and $x(t_f)$ is free. The minimization of $J(x_0, u)$ searches for a control $u(t)$ without an excessive effort able to maintain the state vector $x(t)$ close to the zero required state at any time $t \in [t_0, t_f]$, with particular emphasis at the terminal time t_f as weighted by matrix Π. It is well known that the solution of the problem (1) exists, is unique and given in the form $u(t) = -K(t)x(t) = -(R^*)^{-1}(t)B^T(t)P(t)x(t)$, where $P(t)$ is the nonnegative definite symmetric solution of the corresponding Riccati equation [1]. If t_f is finite, this control law ensures a bounded state and the stability issues are absent. It t_f is infinite (with $\Pi = 0$), the question of stability becomes important. There are results ensuring that this control guarantees that the closed–loop system is exponentially stable under certain conditions related to controllability and observability [11]. We assume that the system $\mathbf{S}$ satisfies such conditions. Similar comments hold for problem (2). Suppose that the dimensions of the state and input vectors $x(t)$, $u(t)$ of $\mathbf{S}$ are smaller than (or at most equal to) those of $\tilde{x}(t)$, $\tilde{u}(t)$ of $\tilde{\mathbf{S}}$. Denote $x(t; x_0, u)$ the solution of $\mathbf{S}$ for a fixed input $u(t)$ and an initial state $x(0) = x_0$. Analogously, $\tilde{x}(t; \tilde{x}_0, \tilde{u})$ is used for the system $\tilde{\mathbf{S}}$. In order to simplify the notation denote $x(t; x_0, u) = x(t)$ and $\tilde{x}(t; \tilde{x}_0, \tilde{u}) = \tilde{x}(t)$. It is well known that

$$\begin{aligned} x(t) &= \Phi(t,t_0)\, x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\, d\tau, \\ \tilde{x}(t) &= \tilde{\Phi}(t,t_0)\, \tilde{x}_0 + \int_{t_0}^{t} \tilde{\Phi}(t,\tau)\tilde{B}(\tau)\tilde{u}(\tau)\, d\tau \end{aligned} \quad (3)$$

are the unique, continuously-differentiable solutions of the systems in (1) and (2), respectively. The *transition matrices* $\Phi(t, t_0)$, $\tilde{\Phi}(t, t_0)$ are given by the Peano-Baker series [13].

The systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ are related by the following transformations $\tilde{x}(t) = Vx(t)$, $x(t) = U\tilde{x}(t)$, $\tilde{u}(t) = Ru(t)$, $u(t) = Q\tilde{u}(t)$, where V, U, R and Q are constant matrices of appropriate dimensions and full ranks.

Definition 1 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively. We say that a system* $\tilde{\mathbf{S}}$ *includes the system* $\mathbf{S}$, *that is* $\tilde{\mathbf{S}} \supset \mathbf{S}$, *if there exist a quadruplet of constant matrices* (U, V, Q, R) *such that* $UV = I_n$, $QR = I_m$ *and for any initial state* x_0 *and any fixed input* $u(t)$ *of* $\mathbf{S}$, $x(t; x_0, u) = U\tilde{x}(t; Vx_0, Ru)$ *for all* $t \in [t_0, t_f]$.

Definition 2 *A pair* $(\tilde{\mathbf{S}}, \tilde{J})$ *includes a pair* $(\mathbf{S}, J)$ *if* $\tilde{\mathbf{S}} \supset \mathbf{S}$ *and* $J(x_0, u) = \tilde{J}(Vx_0, Ru)$. *In this case,* $(\tilde{\mathbf{S}}, \tilde{J})$ *is said to be an expansion of* $(\mathbf{S}, J)$ *and* $(\mathbf{S}, J)$ *is called a contraction of* $(\tilde{\mathbf{S}}, \tilde{J})$.

Definition 3 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, such that* $\tilde{\mathbf{S}} \supset \mathbf{S}$. *Then a control law* $\tilde{u}(t) = -\tilde{K}(t)\tilde{x}(t)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(t) = -K(t)x(t)$ *for* $\mathbf{S}$ *if the choice* $\tilde{x}_0 = Vx_0$ *and* $\tilde{u}(t) = Ru(t)$ *implies* $K(t)x(t; x_0, u) = Q\tilde{K}(t)\tilde{x}(t; Vx_0, Ru)$ *for all* $t \in [t_0, t_f]$, *for any initial state* x_0 *and any fixed input* $u(t)$ *of* $\mathbf{S}$.

In order to obtain conditions for expansions and contractions between the problems (1) and (2) and conditions for contractibility of control laws, the following matrix relations are introduced:

$$\tilde{A}(t) = VA(t)U + M(t), \quad \tilde{B}(t) = VB(t)Q + N(t),$$

$$\tilde{\Pi} = U^T\Pi U + M_\Pi, \quad \tilde{Q}^*(t) = U^TQ^*(t)U + M_{Q^*}(t),$$

$$\tilde{R}^*(t) = Q^TR^*(t)Q + N_{R^*}(t), \quad \tilde{K}(t) = RK(t)U + F(t),$$

where $M(t)$, $N(t)$, M_Π, $M_{Q^*}(t)$, $N_{R^*}(t)$ and $F(t)$ are called *complementary matrices.*

Usually, the transformations (U, V) and (Q, R) are selected a priori to define structural relations between the state and control variables in both systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$. Given these transformations, the choice of the complementary matrices gives degrees of freedom to complete the definition of the expansion-contraction framework involving problems $(\mathbf{S}, J)$ and $(\tilde{\mathbf{S}}, \tilde{J})$ to meet some design requirements.

3 Main results

3.1 General LTV systems

For $(\tilde{\mathbf{S}}, \tilde{J})$ to be an expansion of $(\mathbf{S}, J)$ and to ensure contractibility we must impose some conditions on the complementary matrices. This is provided by the following theorems.

Theorem 1 *Consider the problems given in (1) and (2).* $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$ *if and only if*

$$\begin{aligned} &U\tilde{\Phi}(t,t_0)V = \Phi(t,t_0), && U\tilde{\Phi}(t,\tau)N(\tau)R = 0, \\ &V^T M_\Pi V = 0, && V^T M_{Q^*}(t)V = 0, \\ &R^T N_{R^*}(t)R = 0 \end{aligned} \tag{4}$$

for all $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$.

Theorem 2 *Consider the problems given in (1) and (2).* $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$ *if* $V^T M_\Pi V = 0$, $V^T M_{Q^*}(t) V = 0$, $R^T N_{Q^*}(t) R = 0$ *and either*

$$\begin{aligned} &a) \;\; M(t)V = 0, && N(t)R = 0 \;\; or \\ &b) \;\; UM(t) = 0, && UN(t)R = 0 \end{aligned} \tag{5}$$

for all $t \in [t_0, t_f]$.

The conditions *a)* and *b)* are two independent sets of sufficient conditions for $(\tilde{\mathbf{S}}, \tilde{J})$ to be an expansion of $(\mathbf{S}, J)$.

Theorem 3 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, such that* $\tilde{\mathbf{S}} \supset \mathbf{S}$. *A control law* $\tilde{u}(t) = -\tilde{K}(t)\tilde{x}(t)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(t) = -K(t)x(t)$ *for* $\mathbf{S}$ *if and only if*

$$QF(t)\left[\tilde{\Phi}(t,t_0)Vx_0 + \int_{t_0}^{t} \tilde{\Phi}(t,\tau)\tilde{B}(\tau)\tilde{u}(\tau)\,d\tau\right] = 0$$

for all $t \in [t_0, t_f]$.

Theorem 4 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, such that* $\tilde{\mathbf{S}} \supset \mathbf{S}$. *A control law* $\tilde{u}(t) = -\tilde{K}(t)\tilde{x}(t)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(t) = -K(t)x(t)$ *for* $\mathbf{S}$ *if either*

$$\begin{aligned} &a) \; M(t)V = 0, && N(t)R = 0, && QF(t)V = 0, \;\; or \\ &b) \; UM(t) = 0, && UN(t)R = 0, && QF(t) = 0 \end{aligned} \tag{6}$$

for all $t \in [t_0, t_f]$.

Theorems 2 and 4 do not require to know the transition matrices. However, the selections of $M(t)$, $N(t)$, $F(t)$ are constrained only to restrictions and aggregations.

The transition matrix $\tilde{\Phi}(t,t_0)$ appears in the conditions given by Theorems 1 and 3. Since it depends on the system matrix $\tilde{A}(t)$, $\tilde{\Phi}(t,t_0)$ implicitly depends on the complementary matrix $M(t)$. On the other hand, it is very difficult, if not impossible, to obtain expressions for the transition matrices except for some particular classes of systems. Therefore, we focus our attention on a time-varying systems characterized by possessing the commutativity property.

3.2 Commutative systems

Let us start the presentation for this class of systems.

Definition 4 *A linear time-varying system* $\mathbf{S}$ *as (1) is a commutative system if and only if* $A(t)$ *satisfies* $A(t)\left(\int_{t_0}^{t} A(\tau)\,d\tau\right) = \left(\int_{t_0}^{t} A(\tau)\,d\tau\right) A(t)$ *for all* $t \in [t_0, t_f]$. *In such a case, the matrix* $\Phi(t,t_0)$ *is given by* $\Phi(t,t_0) = e^{\int_{t_0}^{t} A(\tau)\,d\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)^k$.

In particular, this is the case for any $A(t)$ given by $A(t) = \sum_{i=1}^{r} f_i(t)A_i$, where $f_i(t)$ are arbitrary real-valued functions of t and A_i are arbitrary constant $n \times n$ matrices which satisfy the commutativity conditions $A_iA_j = A_jA_i$, for all integers $1 \leq i, j \leq r$. A linear system is called *exponential* when its state-transition matrix can be written in the matrix exponential form $\Phi(t,t_0) = e^{\Gamma(t,t_0)}$, where $\Gamma(t,t_0)$ is an $n \times n$ matrix function of t and t_0. Any commutative system is exponential. In such a case, $\Gamma(t,t_0) = \int_{t_0}^{t} A(\tau)\,d\tau$. If $A(t)$ is a triangular matrix, then the solution can be reduced to solving readily a set of scalar differential equations. When $A(t)$ is a diagonal or a constant matrix, then it meets the commutative property and the results are well known. Summarizing, the class of systems for which $A(t)$ commutes with its integral is actually fairly large [12], [13], [15]. We need to know the conditions ensuring the commutativity property of an expanded system $\tilde{\mathbf{S}}$ when assuming the commutativity of the initial system $\mathbf{S}$. This result is given by the following proposition.

Proposition 1 *Consider* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, such that* $\tilde{\mathbf{S}} \supset \mathbf{S}$. *Suppose* $\mathbf{S}$ *a commutative system. Then* $\tilde{\mathbf{S}}$ *is a commutative system if and only if*

$$\begin{aligned} &VA(t)U\left(\int_{t_0}^{t} M(\tau)\,d\tau\right) + M(t)V\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)U + \\ &+ M(t)\left(\int_{t_0}^{t} M(\tau)\,d\tau\right) = V\left(\int_{t_0}^{t} A(\tau)\,d\tau\right)UM(t) + \\ &+ \left(\int_{t_0}^{t} M(\tau)\,d\tau\right)VA(t)U + \left(\int_{t_0}^{t} M(\tau)\,d\tau\right)M(t) \end{aligned} \tag{7}$$

for all $t \in [t_0, t_f]$.

The remaining of this subsection specifies Theorems 1 and 3 for the class of commutative systems.

Theorem 5 *Consider that* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, are commutative systems.* $(\tilde{\mathbf{S}}, \tilde{J}) \supset (\mathbf{S}, J)$ *if and only if*

$$U\left(\int_{t_0}^{t} M(\tau)\,d\tau\right)^{i} V = 0,\; U\left(\int_{\tau}^{t} M(\beta)\,d\beta\right)^{i-1} N(\tau)R = 0,$$
$$V^T M_\Pi V = 0,\; V^T M_{Q^*}(t) V = 0,$$
$$R^T N_{R^*}(t) R = 0$$

for $i = 1, ..., \tilde{n}$, *all* $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$.

Theorem 6 *Consider that* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, are commutative systems such that* $\tilde{\mathbf{S}} \supset \mathbf{S}$. *A control law* $\tilde{u}(t) = -\tilde{K}(t)\,\tilde{x}(t)$ *for* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(t) = -K(t)\,x(t)$ *for* $\mathbf{S}$ *if and only if*

$$\begin{aligned} QF(t)\left(\int_{t_0}^{t} M(\tau)\,d\tau\right)^{i-1} V = 0,\\ QF(t)\left(\int_{\tau}^{t} M(\beta)\,d\beta\right)^{i-1} N(\tau)R = 0 \end{aligned} \tag{9}$$

for $i = 1, ..., \tilde{n}$, *all* $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$.

3.3 Expansion–contraction process

Change of basis: The expansion-contraction process between systems $\mathbf{S}$ and $\tilde{\mathbf{S}}$ can be illustrated in the form

$$\begin{array}{ccccccccc} \mathbf{S} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \bar{\tilde{\mathbf{S}}} & \longrightarrow & \tilde{\mathbf{S}} & \longrightarrow & \mathbf{S} \\ \mathbb{R}^n & \xrightarrow{V} & \mathbb{R}^{\tilde{n}} & \xrightarrow{T_A^{-1}} & \mathbb{R}^{\tilde{n}} & \xrightarrow{T_A} & \mathbb{R}^{\tilde{n}} & \xrightarrow{U} & \mathbb{R}^n \\ \mathbb{R}^m & \xrightarrow{R} & \mathbb{R}^{\tilde{m}} & \xrightarrow{T_B^{-1}} & \mathbb{R}^{\tilde{m}} & \xrightarrow{T_B} & \mathbb{R}^{\tilde{m}} & \xrightarrow{Q} & \mathbb{R}^m, \end{array} \tag{10}$$

where $\bar{\tilde{\mathbf{S}}}$ denotes the expanded system with the new basis. The idea of using changes of basis in the expansion-contraction process was already introduced by Ikeda et al. [10] to represent $\tilde{\mathbf{S}}$ in a canonical form. Given V and R we define their pseudoinverses as $U = (V^TV)^{-1}V^T$ and $Q = (R^TR)^{-1}R^T$, respectively. Let us consider the changes of basis $T_A = (V\ W_A)$, $T_B = (R\ W_B)$, where W_A, W_B are chosen such that $\operatorname{Im} W_A = \operatorname{Ker} U$, $\operatorname{Im} W_B = \operatorname{Ker} Q$. Using these transformations it is easy to verify the conditions $\bar{U}\bar{V} = I_n$, $\bar{V}\bar{U} = \left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $\bar{Q}\bar{R} = I_m$, $\bar{R}\bar{Q} = \left(\begin{smallmatrix} I_m & 0 \\ 0 & 0 \end{smallmatrix}\right)$, where $\bar{V} = T_A^{-1}V = \left(\begin{smallmatrix} I_n \\ 0 \end{smallmatrix}\right)$, $\bar{U} = UT_A = (I_n\ 0)$ and $\bar{R} = T_B^{-1}R = \left(\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right)$, $\bar{Q} = QT_B = (I_m\ 0)$. In fact, obtaining these conditions is the motivating factor to define T_A and T_B.

Expansion-contraction in the new basis: For simplicity, we will consider the system $\mathbf{S}$ having the following structure:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{pmatrix} = \left(\begin{array}{c|c|c} A_{11}(t) & A_{12}(t) & A_{13}(t) \\ \hline A_{21}(t) & A_{22}(t) & A_{23}(t) \\ \hline A_{31}(t) & A_{32}(t) & A_{33}(t) \end{array}\right) \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} +$$
$$+ \left(\begin{array}{c|c|c} B_{11}(t) & B_{12}(t) & B_{13}(t) \\ \hline B_{21}(t) & B_{22}(t) & B_{23}(t) \\ \hline B_{31}(t) & B_{32}(t) & B_{33}(t) \end{array}\right) \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix},$$

where $A_{ii}(t)$, $B_{ii}(t)$, $i = 1, 2, 3$, are $n_i \times n_i$, $n_i \times m_i$ matrices, respectively. This system is composed of two subsystems with one overlapped part, but it is well known that it can be easily generalized for any number of interconnected overlapped subsystems. This structure has been extensively adopted as a prototype structure in the literature [9], [11], [14].

Consider $(\bar{\tilde{\mathbf{S}}}, \bar{\tilde{J}})$ defined by the problem

$$\begin{aligned} \min_{\bar{\tilde{u}}(t)}\ \bar{\tilde{J}}(\bar{\tilde{x}}_0, \bar{\tilde{u}}(t)) = \bar{\tilde{x}}^T(t_f)\bar{\tilde{\Pi}}\,\bar{\tilde{x}}(t_f) + \\ + \int_{t_0}^{t_f} \left[\bar{\tilde{x}}^T(t)\bar{\tilde{Q}}^*(t)\bar{\tilde{x}}(t) + \bar{\tilde{u}}^T(t)\bar{\tilde{R}}^*(t)\bar{\tilde{u}}(t)\right] dt, \end{aligned} \tag{12}$$

$$\text{s.t. } \bar{\tilde{\mathbf{S}}}: \quad \dot{\bar{\tilde{x}}}(t) = \bar{\tilde{A}}(t)\,\bar{\tilde{x}}(t) + \bar{\tilde{B}}(t)\,\bar{\tilde{u}}(t),$$

where $\bar{\tilde{A}}(t)$, $\bar{\tilde{B}}(t)$, $\bar{\tilde{\Pi}}$, $\bar{\tilde{Q}}^*(t)$ and $\bar{\tilde{R}}^*(t)$ denote the matrices in the system $\bar{\tilde{\mathbf{S}}}$ of appropriate dimensions. The vectors $\bar{\tilde{x}}(t)$ and $\bar{\tilde{u}}(t)$ are defined as $\bar{\tilde{x}}(t) = T_A^{-1}Vx(t) = \bar{V}x(t)$, $\bar{\tilde{u}}(t) = T_B^{-1}Ru(t) = \bar{R}u(t)$. Now, analogously to $\tilde{\mathbf{S}}$, denote the relations for the system $\bar{\tilde{\mathbf{S}}}$ as

$$\bar{\tilde{A}}(t) = \bar{V}A(t)\bar{U} + \bar{M}(t), \quad \bar{\tilde{B}}(t) = \bar{V}B(t)\bar{Q} + \bar{N}(t),$$
$$\bar{\tilde{\Pi}} = \bar{U}^T\Pi\bar{U} + \bar{M}_\Pi, \quad \bar{\tilde{Q}}^*(t) = \bar{U}^TQ^*(t)\bar{U} + \bar{M}_{Q^*}(t),$$
$$\bar{\tilde{R}}^*(t) = \bar{Q}^TR^*(t)\bar{Q} + \bar{N}_{R^*}(t),$$

where the new complementary matrices are $\bar{M}(t) = T_A^{-1}M(t)T_A$, $\bar{N}(t) = T_A^{-1}N(t)T_B$, $\bar{M}_\Pi = T_A^TM_\Pi T_A$, $\bar{M}_{Q^*}(t) = T_A^TM_{Q^*}(t)T_A$, $\bar{N}_{R^*}(t) = T_B^TN_{R^*}(t)T_B$.

Note. Since changes of basis do not affect the commutativity property, the system $\bar{\tilde{\mathbf{S}}}$ is commutative if $\tilde{\mathbf{S}}$ is commutative.

Consider in $\tilde{\mathbf{S}}$, $M(t) = (M_{ij}(t))$, $N(t) = (N_{ij}(t))$, $M_\Pi = (M_{\Pi_{ij}})$, $M_{Q^*}(t) = (M_{Q^*_{ij}}(t))$, $N_{R^*}(t) = (N_{R^*_{ij}}(t))$ for $i, j = 1, ..., 4$, with $M_{\Pi_{ij}} = M^T_{\Pi_{ji}}$, $M_{Q^*_{ij}}(t) = M^T_{Q^*_{ji}}(t)$, $N_{R^*_{ij}}(t) = N^T_{R^*_{ji}}(t)$, where each matrix has appropriate dimensions corresponding to initial structure given in (11). Suppose the matrices $\bar{M}(t) = \left(\begin{smallmatrix} \bar{M}_{11}(t) & \bar{M}_{12}(t) \\ \bar{M}_{21}(t) & \bar{M}_{22}(t) \end{smallmatrix}\right)$, $\bar{N}(t) = \left(\begin{smallmatrix} \bar{N}_{11}(t) & \bar{N}_{12}(t) \\ \bar{N}_{21}(t) & \bar{N}_{22}(t) \end{smallmatrix}\right)$,

$\bar{M}_\Pi = \begin{pmatrix} \bar{M}_{\Pi_{11}} & \bar{M}_{\Pi_{12}} \\ \bar{M}_{\Pi_{12}}^T & \bar{M}_{\Pi_{22}} \end{pmatrix}$, $\bar{M}_{Q^*}(t) = \begin{pmatrix} \bar{M}_{Q^*_{11}}(t) & \bar{M}_{Q^*_{12}}(t) \\ \bar{M}^T_{Q^*_{12}}(t) & \bar{M}_{Q^*_{22}}(t) \end{pmatrix}$, $\bar{N}_{R^*}(t) = \begin{pmatrix} \bar{N}_{R^*_{11}}(t) & \bar{N}_{R^*_{12}}(t) \\ \bar{N}^T_{R^*_{12}}(t) & \bar{N}_{R^*_{22}}(t) \end{pmatrix}$, where each submatrix has appropriate dimension. We need to know the form of the submatrices $\bar{M}_{ij}(t)$, $\bar{N}_{ij}(t)$, $\bar{M}_{\Pi_{ij}}$, $\bar{M}_{Q^*_{ij}}(t)$ and $\bar{N}_{R^*_{ij}}(t)$ for $i,j = 1,2$. This is given in the following propositions.

Proposition 2 *Consider that* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given in (1) and (12), respectively, are commutative systems such that* $\tilde{\tilde{\mathbf{S}}} \supset \mathbf{S}$. *Then* $\bar{M}(t) = \begin{pmatrix} 0 & \bar{M}_{12}(t) \\ \bar{M}_{21}(t) & \bar{M}_{22}(t) \end{pmatrix}$, *where* (0) *denotes a matrix of order* n *and the other blocks satisfy* $\int_{t_0}^t \bar{M}_{12}(\tau)\,d\tau \left(\int_{t_0}^t \bar{M}_{22}(\tau)\,d\tau\right)^{i-2} \int_{t_0}^t \bar{M}_{21}(\tau)\,d\tau = 0$ *for* $i = 2,\dots,\tilde{n}$ *and all* $t \in [t_0, t_f]$.

Proposition 3 *Consider that* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given in (1) and (12), respectively, are commutative systems such that* $\tilde{\tilde{\mathbf{S}}} \supset \mathbf{S}$. *Then* $\bar{N}(t) = \begin{pmatrix} 0 & \bar{N}_{12}(t) \\ \bar{N}_{21}(t) & \bar{N}_{22}(t) \end{pmatrix}$, *where* (0) *is an* $n \times m$ *matrix and the other blocks satisfy* $\int_{t_0}^t \bar{M}_{12}(\tau)\,d\tau \left(\int_{\tau}^t \bar{M}_{22}(\beta)\,d\beta\right)^{i-2} \bar{N}_{21}(\tau) = 0$ *for* $i = 2,\dots,\tilde{n}$, *all* $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$.

Theorem 7 *Consider that* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given in (1) and (12), respectively, are commutative systems.* $(\tilde{\tilde{\mathbf{S}}}, \tilde{\tilde{J}}) \supset (\mathbf{S}, J)$ *if* $\bar{M}_\Pi = \begin{pmatrix} 0 & \bar{M}_{\Pi_{12}} \\ \bar{M}^T_{\Pi_{12}} & \bar{M}_{\Pi_{22}} \end{pmatrix}$, $\bar{M}_{Q^*}(t) = \begin{pmatrix} 0 & \bar{M}_{Q^*_{12}}(t) \\ \bar{M}^T_{Q^*_{12}}(t) & \bar{M}_{Q^*_{22}}(t) \end{pmatrix}$, $\bar{N}_{R^*}(t) = \begin{pmatrix} 0 & \bar{N}_{R^*_{12}}(t) \\ \bar{N}^T_{R^*_{12}}(t) & \bar{N}_{R^*_{22}}(t) \end{pmatrix}$ *and either*

$$a)\quad \bar{M}(t) = \begin{pmatrix} 0 & \bar{M}_{12}(t) \\ 0 & \bar{M}_{22}(t) \end{pmatrix}, \quad \bar{N}(t) = \begin{pmatrix} 0 & \bar{N}_{12}(t) \\ 0 & \bar{N}_{22}(t) \end{pmatrix} \quad or$$

$$b)\quad \bar{M}(t) = \begin{pmatrix} 0 & 0 \\ \bar{M}_{21}(t) & \bar{M}_{22}(t) \end{pmatrix}, \quad \bar{N}(t) = \begin{pmatrix} 0 & \bar{N}_{12}(t) \\ \bar{N}_{21}(t) & \bar{N}_{22}(t) \end{pmatrix}$$

for all $t \in [t_0, t_f]$.

Contractibility: Suppose the complementary matrix $F(t) = (F_{ij}(t))$, $i,j = 1,\dots,4$, where $F_{11}(t)$, $F_{22}(t)$, $F_{33}(t)$ and $F_{44}(t)$ are $m_1 \times n_1$, $m_2 \times n_2$, $m_2 \times n_2$ and $m_3 \times n_3$ matrices, respectively. Define $\bar{F}(t) = \begin{pmatrix} \bar{F}_{11}(t) & \bar{F}_{12}(t) \\ \bar{F}_{21}(t) & \bar{F}_{22}(t) \end{pmatrix}$ where $\bar{F}_{11}(t)$ and $\bar{F}_{22}(t)$ are $m \times n$ and $m_2 \times n_2$ matrices, respectively. Similarly, denote $K(t) = (K_{ij}(t))$, $i,j = 1,\dots,3$, where $K_{11}(t)$, $K_{22}(t)$, $K_{33}(t)$ are $m_i \times n_i$ matrices, $i = 1,\dots,3$, respectively. The gain matrix $\tilde{\tilde{K}}(t)$ for the system $\tilde{\tilde{\mathbf{S}}}$ has the form $\tilde{\tilde{K}}(t) = \bar{R}K(t)\bar{U} + \bar{F}(t)$, where $\tilde{\tilde{K}}(t) = T_B^{-1}\tilde{K}(t)T_A$ and $\bar{F}(t) = T_B^{-1}F(t)T_A$. By Definition 3, $\tilde{\tilde{u}}(t) = -\tilde{\tilde{K}}(t)\tilde{\tilde{x}}(t)$ of $\tilde{\tilde{\mathbf{S}}}$ is contractible to the control law $u(t) = -K(t)x(t)$ of $\mathbf{S}$ whenever $K(t)x(t;x_0,u) = \bar{Q}\tilde{\tilde{K}}(t)\tilde{\tilde{x}}(t;\bar{V}x_0,\bar{R}u)$ for all $t \in [t_0, t_f]$.

Theorem 8 *Consider that* $\mathbf{S}$ *and* $\tilde{\tilde{\mathbf{S}}}$ *given in (1) and (12), respectively, are commutative systems such that* $\tilde{\tilde{\mathbf{S}}} \supset \mathbf{S}$. *A control law* $\tilde{\tilde{u}}(t) = -\tilde{\tilde{K}}(t)\tilde{\tilde{x}}(t)$ *in the expanded system* $\tilde{\tilde{\mathbf{S}}}$ *is contractible to the control law* $u(t) = -K(t)x(t)$ *of* $\mathbf{S}$ *if and only if* $\bar{F} = \begin{pmatrix} 0 & \bar{F}_{12}(t) \\ \bar{F}_{21}(t) & \bar{F}_{22}(t) \end{pmatrix}$ *and satisfies*

$$\begin{aligned} \bar{F}_{12}(t)\left(\int_{t_0}^t \bar{M}_{22}(\tau)\,d\tau\right)^{i-1} \int_{t_0}^t \bar{M}_{21}(\tau)\,d\tau = 0, \\ \bar{F}_{12}(t)\left(\int_{\tau}^t \bar{M}_{22}(\beta)\,d\beta\right)^{j-1} \bar{N}_{21}(\tau) = 0 \end{aligned} \tag{13}$$

for $i = 1,\dots,\tilde{n}-1$, $j = 1,\dots,\tilde{n}$, *all* $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$.

3.4 Selection of complementary matrices

Up to now, the above results do not depend on the selection of the matrices V and R, so that they can be applied for any expansion-contraction process. To use these results in a practical scheme, we start by defining specific transformations V and R to expand a given problem (1). Here we consider the following expansion transformation matrices $V = \begin{pmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{pmatrix}$, $R = \begin{pmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{pmatrix}$. The changes of basis to define the system $\tilde{\tilde{\mathbf{S}}}$ for the above transformations are given by $T_A = \begin{pmatrix} I_{n_1} & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & I_{n_2} \\ 0 & I_{n_2} & 0 & -I_{n_2} \\ 0 & 0 & I_{n_3} & 0 \end{pmatrix}$ and the corresponding inverse T_A^{-1}. Analogously for the matrices T_B, T_B^{-1}.

Theorem 9 *Consider that* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, are commutative systems.* $\tilde{\mathbf{S}} \supset \mathbf{S}$ *if and only if*

$$\begin{pmatrix} \int_{t_0}^t M_{12}(\tau)\,d\tau \\ \int_{t_0}^t (M_{23}(\tau)+M_{33}(\tau))\,d\tau \\ \int_{t_0}^t M_{42}(\tau)\,d\tau \end{pmatrix} \left(\int_{t_0}^t \big(M_{22}(\tau)+M_{33}(\tau) \big)\,d\tau\right)^{i-2} \times$$
$$\times \int_{t_0}^t \big(M_{21}(\tau) \;\; M_{22}(\tau)+M_{23}(\tau) \;\; M_{24}(\tau) \big)\,d\tau = 0,$$
$$\begin{pmatrix} \int_{t_0}^t M_{12}(\tau)\,d\tau \\ \int_{t_0}^t (M_{23}(\tau)+M_{33}(\tau))\,d\tau \\ \int_{t_0}^t M_{42}(\tau)\,d\tau \end{pmatrix} \left(\int_{\tau}^t \big(M_{22}(\beta)+M_{33}(\beta) \big)\,d\beta\right)^{i-2} \times$$
$$\times \big(N_{21}(\tau) \;\; N_{22}(\tau)+N_{23}(\tau) \;\; N_{24}(\tau) \big) = 0$$

for $i = 2,\dots,\tilde{n}$, *all* $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$, *where*

$$M(t) = \begin{pmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{pmatrix}(t).$$

The matrix $N(t)$ *has the same structure as* $M(t)$.

Theorem 10 *Consider that* $\mathbf{S}$ *and* $\tilde{\mathbf{S}}$ *given in (1) and (2), respectively, are commutative systems such that* $\tilde{\mathbf{S}} \supset \mathbf{S}$. *A control law* $\tilde{u}(t) = -\tilde{K}(t)\tilde{x}(t)$ *in the expanded system* $\tilde{\mathbf{S}}$ *is contractible to the control law* $u(t) = -K(t)x(t)$ *of the system* $\mathbf{S}$ *if and only if*

$$\begin{gathered}
\begin{pmatrix} F_{12}(t) \\ F_{23}(t)+F_{33}(t) \\ F_{42}(t) \end{pmatrix} \left(\int_{t_0}^{t} \left(M_{22}(\tau)+M_{33}(\tau) \right) d\tau \right)^{i-1} \times \\
\times \int_{t_0}^{t} \left(M_{21}(\tau) \; M_{22}(\tau)+M_{23}(\tau) \; M_{24}(\tau) \right) d\tau = 0, \\
\begin{pmatrix} F_{12}(t) \\ F_{23}(t)+F_{33}(t) \\ F_{42}(t) \end{pmatrix} \left(\int_{\tau}^{t} \left(M_{22}(\beta)+M_{33}(\beta) \right) d\beta \right)^{j-1} \times \\
\times \left(N_{21}(\tau) \; N_{22}(\tau)+N_{23}(\tau) \; N_{24}(\tau) \right) = 0
\end{gathered} \tag{14}$$

for $i = 1, ..., \tilde{n}-1$, $j = 1, ..., \tilde{n}$, *all* $t \in [t_0, t_f]$ *and all* $\tau \in [t_0, t]$, *where the matrix* $F(t)$ *has the form*

$$F(t) = \begin{pmatrix} 0 & F_{12} & -F_{12} & 0 \\ F_{21} & F_{22} & F_{23} & F_{24} \\ -F_{21} & -(F_{22}+F_{23}+F_{33}) & F_{33} & -F_{24} \\ 0 & F_{42} & -F_{42} & 0 \end{pmatrix}(t). \tag{15}$$

Note. It is important to recognize that it is not necessary to know the transition matrices explicitly in order to select the complementary matrices satisfying the required conditions.

4 Conclusion

The inclusion principle has been specialized for a quadratic optimal control design for both general and commutative continuous-time linear time-varying systems. The strategy of generalized selection of complementary matrices has been developed for commutative continuous-time linear time-varying systems. It includes the presentation of a general structure of complementary matrices. This structure offers flexibility in selection of complementary matrices resulting in more appropriate costs when designing quadratic optimal control via overlapping decompositions for this class of systems.

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