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MODELING FLOWS IN PERIODICALLY HETEROGENEOUS POROUS MEDIA WITH DEFORMATION-DEPENDENT PERMEABILITY

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Abstract. The paper proposes a non-linear model of the Biot continuum. The non-lienarity is introduced in terms of the material coefficients which are expressed as linear functions of the macroscopic response. These functions are obtained by the sensitivity analysis of the homogenized coefficients computed for a given geometry of the porous structure which transforms due to the local deformation. Linear kinematics is assumed, however, the approach can be extended to large deforming porous materials.

1 INTRODUCTION

We adapt the classical Biot model of poroelastic media for situations when the deformation has a significant influence on the permeability tensor controlling the seepage flow and on the other poroelastic coefficients. Under the small deformation assumption and the first order gradient theory of the continuum, the constitutive laws are considered usually in linearized forms involving material constants independent on the field variables, like deformation, or stress. In this context our treatment of the material coefficients depending on the stress and deformation state can be viewed as an extension of the first order theory, whereby the linear strain kinematics still holds and the initial domain is taken as the reference. The Biot model is derived using the homogenization of the fluid-structure interaction problem for a periodic medium involving elastic skeleton with pore fluid [6], whereby the Darcy flow law is obtained by homogenizing the Stokes flow [1]. In this paper, to extend the theory to solution-dependent coefficients, the sensitivity analysis well known from the shape optimization is adopted [4], cf. [8]. The sensitivity is performed w.r.t. the microscopic field of displacements associated with the solid skeleton. By virtue of the homogenization scheme, this field can be expressed as a linear function of the macroscopic strain and pore pressure. As the result, the homogenized model of the poroelastic medium is non-linear, since the effective coefficients are linear functions of the macroscopic response. We present a linearization scheme consisting of the predictor and corrector problems to be solved at each time level.

2 HOMOGENIZED POROELASTIC MEDIUM

First we introduce the decomposition of the porous material into the solid and fluid parts. At this level, the two phases constituting the poroelastic medium are separated: the solid part is labelled by the subscript m, while the fluid part is referred to by the subscript c. Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain occupied by the poroelastic medium; we consider decomposition Ω into the matrix and channel parts $\Omega = \Omega^\varepsilon_m \cup \Omega^\varepsilon_c \cup \Gamma^\varepsilon$, $\quad \Omega^\varepsilon_m \cap \Omega^\varepsilon_c = \emptyset$, $\Gamma^\varepsilon = \overline{\Omega^\varepsilon_m} \cap \overline{\Omega^\varepsilon_c}$. As usually in the homogenization theory, see [3, 5], by ε we denote the small parameter expressing the scale, i.e. the ratio between the micro- and the macroscopic lengths; all variables dependent on the above decomposition are labelled by ε. Assuming very slow flows only in the pores, we can treat the upscaling problem separately for the solid and fluid parts. In order to ensure relevance of such a decoupled homogenization, we are taking a specific type of the porous medium subject to slow deformation processes possibly inducing moderate pressure gradients and flow rates characterized by negligible inertia effects. As a consequence of relatively large permeability and low pressure gradients, slow flow rates and small viscosity have negligible influence on the fluid-structure interaction. Therefore, we consider the following upscaling:

- Homogenization of the fluid-structure interaction problem while the locally constant pore pressure at the level of the representative periodic cell is considered;

- Homogenization of the Stokes flow of a slightly viscous fluid in the pore geometry defined by the undeformed configuration of reference periodic cell.

Then we can combine the homogenized models of the static poroelastic medium and the Darcy flow to obtain an approximate macroscopic description of the fluid structure interaction. In Section 4, this model will be modified to allow for nonlinear effects induced by the deforming microscopic configuration.

2.1 Porous solid saturated by static fluid

In this section we consider the static (or a steady state) problem of deformed elastic porous structure saturated by a fluid under a constant pressure. This last simplifying assumption introduces no significant modelling error in the fluid-structure interaction as far as the flow is slow.

2.1.1 Model at the microscopic scale

The deformation of the matrix is governed by the following problem for $(\boldsymbol{u}^\varepsilon, p^\varepsilon)$ such that

$$\begin{aligned} \nabla \cdot (\mathbb{D}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon)) &= \boldsymbol{f}^\varepsilon \ , \quad \text{in } \Omega_m^\varepsilon \ , \\ \boldsymbol{n}^{[m]} \cdot \mathbb{D}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) &= \boldsymbol{g}^\varepsilon \ , \quad \text{on } \partial_{\text{ext}}\Omega_m^\varepsilon \ , \\ \boldsymbol{n}^{[m]} \cdot \mathbb{D}^\varepsilon \boldsymbol{e}(\boldsymbol{u}^\varepsilon) &= -p^\varepsilon \boldsymbol{n}^{[m]} \ , \quad \text{on } \Gamma^\varepsilon \ , \end{aligned} \tag{1}$$

and

$$\int_{\partial\Omega_c^\varepsilon} \boldsymbol{u}^\varepsilon \cdot \boldsymbol{n}^{[c]} \, \mathrm{dS}_x + \gamma p^\varepsilon |\Omega_c^\varepsilon| = -J^\varepsilon \ , \tag{2}$$

where $\boldsymbol{u}^\varepsilon$ is the displacement vector of the matrix, p^ε is the fluid pressure, $\mathbb{D}^\varepsilon$ is the elasticity fourth-order tensor of the matrix and γ is the fluid compressibility. The applied surface-force and volume-force fields are denoted respectively by $\boldsymbol{g}^\varepsilon$ and $\boldsymbol{f}^\varepsilon$. The outer unit normal vector of the boundary Ω_m^ε is denoted by $\boldsymbol{n}^{[m]}$. Condition (2) expresses the change of the porosity (represented by volume $|\Omega_c^\varepsilon|$), is compensated by fluid compression and by the fluid out-flow $J^{\varepsilon,\alpha}$ through external boundary $\partial_{\text{ext}}\Omega_c^\varepsilon = \partial\Omega_c^\varepsilon \cup \partial\Omega$, *i.e.* outwards to Ω. We assume that $\boldsymbol{f}^\varepsilon$ and $\boldsymbol{g}^\varepsilon$ are defined in such a way that the solvability conditions associated with (1)-(2) are satisfied. The weak formulation of this boundary value problem was given in [6].

2.1.2 Homogenization result

We assume that the domain Ω is obtained from a periodic microstructure generated by a representative unit cell Y decomposed as follows

$$Y = Y_m \cup Y_c \cup \Gamma_Y \ , \quad Y_c = Y \setminus Y_m \ , \quad \Gamma_Y = \overline{Y_m} \cap \overline{Y_c} \ . \tag{3}$$

Without loss of generality we can define $Y = (]0,1[)^3$ to be the unit cube, so $|Y| = 1$. The upscaling procedure of the heterogeneous continuum consists in the limit analysis with respect to $\varepsilon \to 0$. For this we use the periodic unfolding method, see [3, 1]. The analogous notation is employed when upscaling from the mesoscopic-to-macroscopic scale. By $\frac{1}{|D|}\!\!\!\!\!\!\int_D = |Y|^{-1} \int_D$ with $D \subset \overline{Y}$ we denote the local average, although $|Y| = 1$.

We assume weak convergence of the external forces; denoting by χ_m^ε the characteristic function of the matrix, $\chi_m^\varepsilon \boldsymbol{f}^\varepsilon$ converge towards $(1-\phi)\boldsymbol{f}$ where $\boldsymbol{f}$ is a local averaged volume-force acting on the matrix. The volume fraction of pores is defined by $\phi = |Y_c|/|Y|$.

When $\varepsilon \to 0$, the strain is a two-scale function defined from its macroscopic part $\boldsymbol{e}(\boldsymbol{u}(x))$ and its fluctuating part $\boldsymbol{e}_y(\boldsymbol{u}^1(x,y))$, $x \in \Omega$ and $y \in Y$, where the fluctuations are proportional to macroscopic strains. There are so called characteristic displacements $\boldsymbol{\omega}^{ij}(y)$ and $\boldsymbol{\omega}^P(y)$ such that $\boldsymbol{u}^1(x,y) = \boldsymbol{\omega}^{ij}(y)\partial_j u_i(x) - \boldsymbol{\omega}^P(y)p$, where p is the constant fluid pressure in Ω. Functions $\boldsymbol{\omega}^{ij}(y)$ and $\boldsymbol{\omega}^P(y)$ are obtained as solutions of the following problems: find $(\boldsymbol{\omega}^{ij}, \boldsymbol{\omega}^P) \in \mathbf{H}^1_\#(Y_m) \times \mathbf{H}^1_\#(Y_m)$ satisfying

$$\begin{aligned} a_Y^m\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij},\, \boldsymbol{v}\right) &= 0\,, \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y_m)\,, \\ a_Y^m\left(\boldsymbol{\omega}^P,\, \boldsymbol{v}\right) &= \fint_{\Gamma_Y} \boldsymbol{v} \cdot \boldsymbol{n}^{[m]}\, \mathrm{dS}_y\,, \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y_m)\,, \end{aligned} \tag{4}$$

where $a_Y^m(\boldsymbol{w},\, \boldsymbol{v}) = \fint_{Y_m}(\mathbb{D}\boldsymbol{e}_y(\boldsymbol{w})) : \boldsymbol{e}_y(\boldsymbol{v})$ and $\boldsymbol{\Pi}^{ij} = (\Pi_k^{ij})$, $i,j,k = 1,2,3$ with $\Pi_k^{ij} = y_j\delta_{ik}$. Above $\mathbf{H}^1_\#(Y_m)$ is the Sobolev space $\boldsymbol{W}^{1,2}(Y_m)$ of vector-valued Y-periodic functions (the subscript $\#$).

Using the characteristic responses (4) obtained at the microscopic scale the effective properties of the deformable porous medium are given by

$$A_{ijkl} = a_Y^m\left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij},\, \boldsymbol{\omega}^{kl} + \boldsymbol{\Pi}^{kl}\right)\,, \quad C_{ij} = -\fint_{Y_m} \mathrm{div}_y \boldsymbol{\omega}^{ij}\,, \quad N = a_Y^m\left(\boldsymbol{\omega}^P,\, \boldsymbol{\omega}^P\right)\,. \tag{5}$$

Obviously, the tensors $\mathbb{A} = (A_{ijkl})$ and $\boldsymbol{C} = (C_{ij})$ are symmetric; moreover $\mathbb{A}$ is positive definite and $N > 0$.

At this first-level of the homogenization process, we obtain a model of the poroelasticity involving the skeleton displacements $\boldsymbol{u} \in \mathbf{H}^1(\Omega)/\mathrm{RBM}(\Omega)$ and fluid pressure $p \in \mathbb{R}$ which is constant due to our assumptions. These state variables verify the following equations:

$$\begin{aligned} &\int_\Omega (\mathbb{A}\boldsymbol{e}(\boldsymbol{u}) - p\boldsymbol{B}) : \boldsymbol{e}(\boldsymbol{v}) = \int_\Omega (1-\phi)\boldsymbol{f}\cdot\boldsymbol{v} + \int_{\partial\Omega}(1-\phi)\boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{dS}_x\,, \quad \forall\, \boldsymbol{v} \in \mathbf{H}^1(\Omega)\,, \\ &\int_\Omega \boldsymbol{B} : \boldsymbol{e}(\boldsymbol{u}) + pM|\Omega| = -J, \quad \text{with } \boldsymbol{B} := \boldsymbol{C} + \phi\boldsymbol{I}, \quad M = N + \phi\gamma\,, \end{aligned} \tag{6}$$

where J is the limit of the total flux J^ε outwards Ω.

2.2 Homogenization of the Stokes problem

We consider the Stokes flow through the channel network constituting domain Ω_c^ε. For a while, we disregard any deformation of the skeleton, so that the homogenization result reported e.g. in [1] can simply be recorded. The steady flow problem is defined in terms of the flow velocity $\boldsymbol{w}^\varepsilon$ and pressure p which satisfy the following relations:

$$\begin{aligned} -\eta^\varepsilon\nabla^2\boldsymbol{w}^\varepsilon + \nabla p^\varepsilon &= \boldsymbol{f}^\varepsilon\,, \quad \text{in } \Omega_c^\varepsilon\,, \\ \nabla\cdot\boldsymbol{w}^\varepsilon &= 0\,, \quad \text{in } \Omega_c^\varepsilon\,, \\ \boldsymbol{w}^\varepsilon &= 0;\,, \quad \text{on } \Gamma^\varepsilon\,, \\ -p^\varepsilon\boldsymbol{n}^{\beta,\varepsilon} + \eta^\varepsilon\boldsymbol{n}^{[c]}\cdot\nabla\boldsymbol{w}^\varepsilon) &= \boldsymbol{g}^\varepsilon\,, \quad \text{on } \partial_{\mathrm{ext}}\Omega_c^\varepsilon\,, \end{aligned} \tag{7}$$

where $\boldsymbol{g}^\varepsilon$ is given on the exterior boundary of the channels. By virtue of the small viscosity ansatz, see [5, 1], we defined $\eta^\varepsilon = \varepsilon^2\bar{\eta}$ which decreases with the scale. The homogenization result give rise the Darcy law involving the permeability $\boldsymbol{K} = (K_{ij})$ which associates the the macroscopic pressure gradient $\nabla_x p^0$ with the mean flow velocity $\boldsymbol{w}$. Thus, we obtain

$$K_{ij} = \fint_{Y_c} \psi_i^j = \fint_{Y_c} \nabla_y \boldsymbol{\psi}^i : \nabla_y \boldsymbol{\psi}^i \;, \quad \text{and} \quad \boldsymbol{w} = -\frac{\boldsymbol{K}}{\bar{\eta}} \left(\nabla_x p^0 - \boldsymbol{f} \right) \;, \tag{8}$$

where the local microscopic response functions $\boldsymbol{\psi}^i$ are solutions of the *microscopic problem*: Find $(\boldsymbol{\psi}^i, \pi^i) \in \mathbf{H}^1_\#(Y_c) \times L^2(Y_c)$, $i = 1, 2, 3$ such that

$$\begin{aligned} \int_{Y_c} \nabla_y \boldsymbol{\psi}^k : \nabla_y \boldsymbol{v} - \int_{Y_c} \pi^k \nabla \cdot \boldsymbol{v} = \int_{Y_c} v_k \quad \forall \boldsymbol{v} \in \mathbf{H}^1_\#(Y_c) \;, \\ \int_{Y_c} q \nabla_y \cdot \boldsymbol{\psi}^k = 0 \quad \forall q \in L^2(Y_c) \;. \end{aligned} \tag{9}$$

2.3 Coupled flow deformation problem

Although the homogenized poroelasticity coefficients were obtained for the constant pressure distribution, the result can be generalized to account for slow flows. Such a treatment is coherent with the Biot model, in spite of the fact that the genuine fluid-structure interaction which admits pressure gradients at the microscopic pore level leads to additional new terms and homogenized coefficients.

The Biot model of poroelastic media for quasi-static problems is constituted by the following equations involving the homogenized coefficients:

$$\begin{aligned} -\nabla \cdot \boldsymbol{\sigma} = \boldsymbol{f}^s \qquad & \text{in } \Omega \;, \\ \boldsymbol{B} : \boldsymbol{e}(\dot{\boldsymbol{u}}) + M \dot{p} = -\nabla \cdot \boldsymbol{w} \qquad & \text{in } \Omega \;, \\ \boldsymbol{\sigma} = \mathbb{A} \boldsymbol{e}(\boldsymbol{u}) - \boldsymbol{B} p) \qquad & \text{in } \Omega \;, \\ \boldsymbol{w} = -\frac{\boldsymbol{K}}{\bar{\eta}} \left(\nabla p - \boldsymbol{f}^f \right) \qquad & \text{in } \Omega \;. \end{aligned} \tag{10}$$

These equations will be considered in Section 4 where a nonlinear problem is introduced. For this, boundary conditions must be prescribed for the displacement and the pressure fields.

$$\begin{aligned} \boldsymbol{u} = 0 \;, \quad \text{on } \partial_u \Omega \;, \qquad \boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{g}^s \;, \quad \text{on } \partial_\sigma \Omega \;, \\ p = p_\partial \;, \quad \text{on } \partial_p \Omega \;, \qquad \boldsymbol{n} \cdot \boldsymbol{w} = w_n \;, \quad \text{on } \partial_w \Omega \;, \end{aligned} \tag{11}$$

whereby the decomposition of $\partial\Omega$ into disjoint parts is considered

$$\partial\Omega = \partial_\sigma \Omega \cup \partial_u \Omega \;, \quad \partial_\sigma \Omega \cap \partial_u \Omega = \emptyset \;, \quad \partial\Omega = \partial_w \Omega \cup \partial_p \Omega \;, \quad \partial_w \Omega \cap \partial_p \Omega = \emptyset \;. \tag{12}$$

In Section 4 we shall use the following spaces and admissibility sets:

$$\begin{aligned} \boldsymbol{U}(\Omega) = \{ \boldsymbol{u} \in \mathbf{H}^1(\Omega) | \; \boldsymbol{u} = 0 \text{ on } \partial_u \Omega \} \;, \\ P(\Omega) = \{ p \in H^1(\Omega) | \; p = p_\partial \text{ on } \partial_p \Omega \} \;, \end{aligned} \tag{13}$$

whereby the space $P_0(\Omega)$ is defined according to $(13)_2$ with $p_\partial \equiv 0$. Formally we shall use $\boldsymbol{U}_0(\Omega)$ which is identified with $\boldsymbol{U}(\Omega)$ due to (11).

3 DEFORMATION SENSITIVITY OF THE HOMOGENIZED COEFFICIENTS

We use the shape sensitivity technique and the material derivative approach (see a.g. [4]) to obtain the sensitivity of homogenized coefficients involved in the Biot continuum with respect to the configuration transformation corresponding to the microscopic deformation.

We consider a "flux" of material points which is given in terms of a vectorial (design velocity) differentiable and Y-periodic field $\vec{\mathcal{V}}(y)$, $y \in Y$ so that for $y \in \Gamma$ it describes the "flux" of points on the design boundary. Construction of $\vec{\mathcal{V}} : \overline{Y} \longrightarrow \mathbb{R}^3$ is based of the microscopic displacement and deformation fields recovery related to the corrector result of the homogenization. We shall discuss details below; for now let us consider the design velocity field $\vec{\mathcal{V}}$ defined in Y, so that we can parametrize the "material point" position in Y by $z_i(y,\tau) = y_i + \tau \mathcal{V}_i(y)$, $y \in Y$, $i = 1,2$, where τ is the "time-like" variable. Throughout the text below we shall use the notion of the following derivatives: $\delta(\cdot)$ is the total (*material*) derivative, $\delta_\tau(\cdot)$ is the partial (*local*) derivative *w.r.t.* τ. These derivatives are computed as the directional derivatives in the direction of $\vec{\mathcal{V}}(y)$, $y \in Y$, see e.g. [9] for details.

3.1 Shape sensitivity analysis of the permeability

We are interested in the influence of variation of the shape of the interface Γ on the homogenized permeability defined in (8). Let us consider a general functional $\Phi(\phi) = \int_Y F(\phi)$, where $\phi(y)$ corresponds to a microscopic corrector field, F is a sufficiently regular operator. Using the chain rule differentiation,

$$\delta\Phi(\phi) = \delta_\phi\Phi(\phi) \circ \delta\phi + \delta_\tau\Phi(\phi) = \delta_\phi\Phi(\phi) \circ \delta\phi + \int_Y F(\phi)\mathrm{div}\mathcal{V} + \int_Y \delta_\tau F(\phi) \;, \tag{14}$$

where $\delta_\tau F(\phi)$ is the shape derivative of $F(\phi)$ for a fixed argument ϕ. We shall see that sensitivity $\delta\phi$ can be eliminated from the sensitivity formula.

We shall apply (14) to differentiate K_{ij} in (8). Thus, we get

$$\delta K_{ij} = \fint_{Y_c} \psi^i_j \nabla \cdot \mathcal{V} - K_{ij} \fint_Y \nabla \cdot \mathcal{V} + \fint_{Y_c} \delta\psi^i_j \;. \tag{15}$$

To eliminate the dependence on $\delta\psi^i_j$ in the last integral of the above expression, we differentiate $(9)_1$ which yields

$$\fint_{Y_c} \nabla_y \delta\boldsymbol{\psi}^i : \nabla_y \boldsymbol{v} - \fint_{Y_c} \delta\pi^i \nabla \cdot \boldsymbol{v} + \delta_\tau \left(\fint_{Y_c} \nabla_y \boldsymbol{\psi}^i : \nabla_y \boldsymbol{v} - \fint_{Y_c} \pi^i \nabla \cdot \boldsymbol{v} \right) = \delta_\tau \fint_{Y_c} v_i \;. \tag{16}$$

Using the substitution $\boldsymbol{v} = \boldsymbol{\psi}^j$ in $(9)_1$ evaluated for $k = i$, due to the incompressibility constraint $(9)_2$ one obtains

$$\fint_{Y_c} \delta\psi^j_i = \fint_{Y_c} \nabla_y \delta\boldsymbol{\psi}^j : \nabla_y \boldsymbol{\psi}^i \;. \tag{17}$$

Now combining (15)-(17)

$$\delta K_{ij} = \delta_\tau \left(\fint_{Y_c} \psi_j^i + \fint_{Y_c} \psi_i^j + \fint_{Y_c} \pi^i \nabla_y \cdot \boldsymbol{\psi}^j - \fint_{Y_c} \nabla_y \boldsymbol{\psi}^j : \nabla_y \boldsymbol{\psi}^i \right) . \tag{18}$$

Finally we need the partial shape derivatives, as follows

$$\begin{aligned} \delta_\tau K_{ij} &= \delta_\tau \fint_{Y_c} \psi_j^i = \fint_{Y_c} \psi_j^i \nabla \cdot \mathcal{V} - K_{ij} \fint_Y \nabla \cdot \mathcal{V} , \\ \delta_\tau \fint_{Y_c} \nabla_y \boldsymbol{\psi}^j : \nabla_y \boldsymbol{\psi}^i &= \fint_{Y_c} \left(\nabla_y \boldsymbol{\psi}^j : \nabla_y \boldsymbol{\psi}^i \nabla \cdot \mathcal{V} - \partial_l^y \mathcal{V}_r \partial_r^y \psi_k^i \partial_l^y \psi_k^j - \partial_l^y \mathcal{V}_r \partial_r^y \psi_k^j \partial_l^y \psi_k^i \right) \\ &\quad - \fint_{Y_c} \nabla_y \boldsymbol{\psi}^j : \nabla_y \boldsymbol{\psi}^i \fint_Y \nabla_y \cdot \mathcal{V} . \end{aligned} \tag{19}$$

3.2 Deformation sensitivity analysis of the poroelasticity coefficients

In contrast with the sensitivity of the permeability coefficients which depend on the shape of ∂Y_c only, the poroelasticity may depend on the strain associated with $\vec{\mathcal{V}}(y)$, $y \in Y_m$.

By virtue of the shape sensitivity based on the domain parametrization [4], the following formulae hold,

$$\begin{aligned} \delta_\tau a_Y^m (\boldsymbol{u}, \boldsymbol{v}) = & \fint_{Y_m} D_{irks} \left(\delta_{rj} \delta_{sl} \nabla_y \cdot \mathcal{V} - \delta_{jr} \partial_s^y \mathcal{V}_l - \delta_{ls} \partial_r^y \mathcal{V}_j \right) e_{kl}^y(\boldsymbol{u}) e_{ij}^y(\boldsymbol{v}) \\ & - a_Y^m (\boldsymbol{u}, \boldsymbol{v}) \fint_Y \nabla_y \cdot \mathcal{V} , \end{aligned} \tag{20}$$

$$\delta_\tau \fint_{Y_m} \nabla_y \cdot \boldsymbol{v} = \fint_{Y_m} \left(\nabla_y \cdot \mathcal{V} \nabla_y \cdot \boldsymbol{v} - \partial_i^y \mathcal{V}_k \partial_k^y \boldsymbol{v}_i \right) - \fint_{Y_m} \nabla_y \cdot \boldsymbol{v} \fint_Y \nabla_y \cdot \mathcal{V} , \tag{21}$$

and $\delta_\tau \Pi_k^{ij} = \mathcal{V}_j \delta_{ik}$. We shall need the sensitivity identity obtained by differentiation in $(4)_2$ which yields

$$a_Y^m \left(\delta \boldsymbol{\omega}^P, \boldsymbol{v} \right) = \delta_\tau \left(\fint_{Y_m} \nabla_y \cdot \boldsymbol{v} - a_Y^m \left(\boldsymbol{\omega}^P, \boldsymbol{v} \right) \right) . \tag{22}$$

We can now differentiate the expressions for the homogenized coefficients given in (5). First we obtain the sensitivity of A_{ijkl}, whereby $(4)_1$ is employed:

$$\delta A_{ijkl} = \delta_\tau a_Y^m \left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}, \boldsymbol{\omega}^{kl} + \boldsymbol{\Pi}^{kl} \right) + a_Y^m \left(\delta_\tau \boldsymbol{\Pi}^{ij}, \boldsymbol{\omega}^{kl} + \boldsymbol{\Pi}^{kl} \right) + a_Y^m \left(\boldsymbol{\omega}^{ij} + \boldsymbol{\Pi}^{ij}, \delta_\tau \boldsymbol{\Pi}^{kl} \right) . \tag{23}$$

The sensitivity of C_{ij} is obtained, as follows:

$$\begin{aligned} \delta C_{ij} &= a_Y^m \left(\delta \boldsymbol{\omega}^P, \boldsymbol{\Pi}^{ij} \right) + \delta_\tau a_Y^m \left(\boldsymbol{\omega}^P, \boldsymbol{\Pi}^{ij} \right) + a_Y^m \left(\boldsymbol{\omega}^P, \delta_\tau \boldsymbol{\Pi}^{ij} \right) \\ &= \delta_\tau a_Y^m \left(\boldsymbol{\omega}^P, \boldsymbol{\omega}^{ij} \right) - \delta_\tau \fint_{Y_m} \nabla_y \cdot \boldsymbol{\omega}^{ij} + \delta_\tau a_Y^m \left(\boldsymbol{\omega}^P, \boldsymbol{\Pi}^{ij} \right) + a_Y^m \left(\boldsymbol{\omega}^P, \delta_\tau \boldsymbol{\Pi}^{ij} \right) , \end{aligned} \tag{24}$$

where we used (22) with $\boldsymbol{v}$ substituted by $\boldsymbol{\omega}^{ij}$ and the first r.h.s. integral in $(24)_1$ was rewritten using $(4)_1$.

The sensitivity of N is derived using (22) with $\boldsymbol{v}$ substituted by $\boldsymbol{\omega}^P$; thus, we obtain

$$\delta N = a_Y^m\left(\delta\boldsymbol{\omega}^P,\,\boldsymbol{\omega}^P\right) + \delta_\tau \fint_{Y_m} \nabla_y\cdot\boldsymbol{\omega}^P = 2\delta_\tau \fint_{Y_m} \nabla_y\cdot\boldsymbol{\omega}^P - \delta_\tau a_Y^m\left(\boldsymbol{\omega}^P,\,\boldsymbol{\omega}^P\right)\;. \tag{25}$$

Finally, the sensitivity of the volume fraction is

$$\delta\phi = \fint_{Y_c} \nabla_y\cdot\mathcal{V} - \phi\fint_{Y} \nabla_y\cdot\mathcal{V}\;. \tag{26}$$

To summarize, the sensitivity formulae (23)-(25) can be evaluated using expressions (20) and (21). Using (24), (25) and (26) we obtain the sensitivities of $\boldsymbol{B}$ and M, i.e.

$$\delta\boldsymbol{B} = \delta\boldsymbol{C} + \delta\phi\boldsymbol{I}\;,\quad \delta M = \delta N + \gamma\delta\phi\;. \tag{27}$$

4 NONLINEAR MODEL

We establish a nonlinear model based on the Biot-type continuum for which the constitutive laws are derived using homogenization of the fluid-structure interaction problem, as reported above. Still we assume a range of small deformations and linear kinematics, so that the linear Cauchy strain is the deformation measure with all its consequences.

4.1 Model with solution-dependent effective material coefficients

The nonlinear model is introduced by the following steps:

1. By $\mathcal{M}(Y)$ we denote a reference microscopic configuration featured by the domain decomposition (3) and by the elasticity $\mathbb{D}(y)$ distributed in Y_m. We assume that the fluid properties are independent of the microscopic configuration geometry. For $\mathcal{M}(Y)$ the homogenized coefficients $(\mathbb{A}^0, \boldsymbol{B}^0, M^0, \boldsymbol{K}^0)$ and their sensitivities $\delta(\mathbb{A}^0, \boldsymbol{B}^0, M^0, \boldsymbol{K}^0)$ are computed, as explained in sections 3.1 and 3.2.

2. We employ the corrected microscopic response which can be introduced using the scale decoupling ansatz and the corrector basis functions. By $\bar{\boldsymbol{u}}$ and $\tilde{\boldsymbol{u}}$ we denote the affine and fluctuating parts of the microscopic displacement fields relevant to the level of Y, respectively. They are introduced, as follows

$$\begin{aligned}\bar{\boldsymbol{u}}(x,y) &= \boldsymbol{\Pi}^{ij}(y)e_{ij}^y(\boldsymbol{u}(x))\;,\\ \boldsymbol{u}^1(x,y) &= \boldsymbol{\omega}^{ij}(y)e_{ij}^y(\boldsymbol{u}(x)) - \boldsymbol{\omega}^P(y)p(x)\;,\end{aligned} \tag{28}$$

   where $x\in\Omega$ and $y\in Y_m$. Note that the affine part $\bar{\boldsymbol{u}}$ is generated by the (locally) homogeneous strain field which is observed at the macroscopic level. Based on (28), the convective displacement $\tilde{\boldsymbol{u}}$ is established and can be associated with the "design velocity" field employed in the sensitivity formulae:

$$\tilde{\boldsymbol{u}}(x,y) = \bar{\boldsymbol{u}}(x,y) + \boldsymbol{u}^1(x,y) = \tilde{\tau}\vec{\mathcal{V}}(x,y)\;. \tag{29}$$

3. Using $\tilde{\boldsymbol{u}}(x,y)$ the initial microscopic configuration can be transformed to a spatial deformed one denoted by $\tilde{\mathcal{M}}(\tilde{\boldsymbol{u}}(x,\cdot),Y)$ and associated with domain $\tilde{Y}(x) = Y + \{\tilde{\boldsymbol{u}}(x,\cdot)\}_{y\in Y}$ for $x \in \Omega$. Since $\tilde{Y}(x)$ depends on the macroscopic coordinate, the microstructure is perturbed from its periodic structure, however, the homogenization procedure can still be applied and the homogenized coefficients $H(\tilde{\mathcal{M}}(\tilde{\boldsymbol{u}},Y))$ can be computed directly for the new geometry. These coefficients describe linear constitutive laws relevant to the actual deformed configuration. Due to the sensitivity analysis explained above, the perturbed coefficients $H(\tilde{\mathcal{M}}(\tilde{\boldsymbol{u}},Y))$ can be approximated using the first order expansion formulae which have the generic form

$$H(\tilde{\mathcal{M}}(\tilde{\boldsymbol{u}}(x,\cdot),Y)) \approx \tilde{H}(\mathcal{M}(Y),\tilde{\boldsymbol{u}}) = H^0(\mathcal{M}(Y)) + \delta H^0(\mathcal{M}(Y)) \circ \tilde{\tau}\vec{\mathcal{V}}(x,\cdot)\;. \quad (30)$$

As the main advantage of using this approximation, (30) allows to avoid direct computations of $H(\tilde{\mathcal{M}}(\tilde{\boldsymbol{u}}(x,\cdot),Y))$ for all perturbed configurations. Instead, the sensitivities $\delta H^0(\mathcal{M}(Y))$ are computed only once along with $H^0(\mathcal{M}(Y))$ for the reference periodic cell Y. Moreover, by virtue of (28) and (29) we obtain the generic formula for homogenized coefficients depending on the macroscopic response $(\boldsymbol{e}(\boldsymbol{u}),p)$, thus

$$\begin{aligned} \tilde{H}(\boldsymbol{e}(\boldsymbol{u}),p) &= H^0 + \delta_e H^0 : \boldsymbol{e}(\boldsymbol{u}) + \delta_p H^0 p\;, \\ (\delta_e H^0)_{ij} &:= (\partial_e \delta H^0 \circ \tilde{\boldsymbol{u}})_{ij} = \delta H^0 \circ \boldsymbol{\omega}^{ij}\;, \\ \delta_p H^0 &:= \partial_p \delta H^0 \circ \tilde{\boldsymbol{u}} = \delta H^0 \circ (-\boldsymbol{\omega}^P)\;. \end{aligned} \quad (31)$$

4. Using the generic form approximation (31) of the perturbed homogenized coefficients we define the nonlinear model by replacing constants $(\mathbb{A}^0, \boldsymbol{B}^0, M^0, \boldsymbol{K}^0)$ employed in (6) and (8) by linear extensions depending on the macroscopic response. Thus we get the following system to be satisfied by the couple $(\boldsymbol{u},p) \in \boldsymbol{U}(\Omega) \times P(\Omega)$

$$\begin{aligned} &\int_\Omega \left(\tilde{\mathbb{A}}\boldsymbol{e}(\boldsymbol{u}) - p\tilde{\boldsymbol{B}}\right) : \boldsymbol{e}(\boldsymbol{v}) = \int_\Omega \tilde{\boldsymbol{f}}^s \cdot \boldsymbol{v} + \int_{\partial\Omega} \tilde{\boldsymbol{g}}^s \cdot \boldsymbol{v}\, \mathrm{dS}_x\;, \quad \forall \boldsymbol{v} \in \boldsymbol{U}(\Omega)\;, \\ &\int_\Omega q\left(\tilde{\boldsymbol{B}} : \boldsymbol{e}(\dot{\boldsymbol{u}}) + \dot{p}\tilde{M}\right) + \int_\Omega \frac{\tilde{\boldsymbol{K}}}{\bar{\eta}}\,(\nabla_x p - \boldsymbol{f}) \cdot \nabla_x q = 0 \quad \forall q \in P_0(\Omega)\;, \end{aligned} \quad (32)$$

where $\tilde{\boldsymbol{f}}^s$ and $\tilde{\boldsymbol{g}}^s$ attain the form (31) due to their dependence on the volume fraction ϕ.

Nonlinear problem (32) can be discretized in time with a given time step Δt. In the next section we explain a simplified linearized model, to obtain a solution at each time level by a non-iterative computation.

4.2 Linearization scheme

We propose a linearization scheme for (32) presented in the form of a two-step predictor–corrector solver. The unknown fields $(\boldsymbol{u},p)$ associated with the actual time level $t =$

$t^0 + \Delta t$, where t^0 is the previous time level, can be decomposed, as follows

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \delta \boldsymbol{u} \;, \quad p = \bar{p} + \delta p \;, \tag{33}$$

where $(\bar{\boldsymbol{u}}, \bar{p})$ is the response of the linear model, whereas $(\delta \boldsymbol{u}, \delta p)$ is the correction. The linearization scheme is based on the first order expansion of all the integrals involved in (32), using (31) and (33).

Notation employed in the linearization scheme. In what follows we shall need some further notation: let $\boldsymbol{W} = \partial_e(\boldsymbol{X} : \bar{\boldsymbol{e}}) \circ \langle\ \rangle_e$, then $\boldsymbol{W}\delta\boldsymbol{e} = \partial_e(\boldsymbol{X} : \bar{\boldsymbol{e}}) \circ \delta\boldsymbol{e}$; in analogy, let $Z = \partial_p(\boldsymbol{X}\,\bar{\boldsymbol{e}}) \circ \langle\ \rangle_p$, then $Z\delta p = \partial_p(\boldsymbol{X}\,\bar{\boldsymbol{e}}) \circ \delta p$. Also the following abbreviations will be used:

$$\begin{aligned} \overline{\partial}\mathbb{A}^0 &= \partial_e \mathbb{A}^0 \circ \boldsymbol{e}(\bar{\boldsymbol{u}}) + \partial_p \mathbb{A}^0 \circ \bar{p} \;, \quad \overline{\partial}\boldsymbol{B}^0 = \partial_e \boldsymbol{B}^0 \circ \boldsymbol{e}(\bar{\boldsymbol{u}}) + \partial_p \boldsymbol{B}^0 \circ \bar{p} \;, \\ \overline{\partial}M^0 &= \partial_e M^0 \circ \boldsymbol{e}(\bar{\boldsymbol{u}}) + \partial_p M^0 \circ \bar{p} \;, \quad \overline{\partial}\boldsymbol{K}^0 = \partial_e \boldsymbol{K}^0 \circ \boldsymbol{e}(\bar{\boldsymbol{u}}) + \partial_p \boldsymbol{K}^0 \circ \bar{p} \;, \end{aligned} \tag{34}$$

With this notation in hand we can introduce coefficients depending on the "predictor solution", $(\bar{\boldsymbol{u}}, \bar{p})$, and on the response $(\boldsymbol{u}^0, p^0)$ at the previous time step t^0, which obeys the same decomposition (33),

$$\begin{aligned} \overline{\mathbb{A}}(\bar{\boldsymbol{u}}, \bar{p}) &= \mathbb{A}^0 + \overline{\partial}\mathbb{A}^0 + \partial_e(\mathbb{A}^0 \boldsymbol{e}(\bar{\boldsymbol{u}})) \circ \langle\ \rangle_e - \partial_e(\boldsymbol{B}^0 \bar{p}) \circ \langle\ \rangle_e \\ \overline{\boldsymbol{B}}(\bar{\boldsymbol{u}}, \bar{p}) &= \boldsymbol{B}^0 + \overline{\partial}\boldsymbol{B}^0 + \partial_p(\boldsymbol{B}^0 \bar{p}) \circ \langle\ \rangle_p - \partial_p(\mathbb{A}^0 \boldsymbol{e}(\bar{\boldsymbol{u}})) \circ \langle\ \rangle_p \\ \overline{\boldsymbol{D}}(\bar{\boldsymbol{u}}, \bar{p}, \boldsymbol{u}^0, p^0) &= \boldsymbol{B}^0 + \overline{\partial}\boldsymbol{B}^0 + \partial_e \boldsymbol{B}^0 : (\boldsymbol{e}(\bar{\boldsymbol{u}}) - \boldsymbol{e}(\boldsymbol{u}^0)) \circ \langle\ \rangle_e + \partial_e M^0 (\bar{p} - p^0) \circ \langle\ \rangle_e \\ \overline{M}(\bar{\boldsymbol{u}}, \bar{p}, \boldsymbol{u}^0, p^0) &= M^0 + \overline{\partial}M^0 + \partial_p(M^0 \bar{p}) \circ \langle\ \rangle_p + \partial_p \boldsymbol{B}^0 : (\boldsymbol{e}(\bar{\boldsymbol{u}}) - \boldsymbol{e}(\boldsymbol{u}^0)) \circ \langle\ \rangle_p \;, \\ \overline{\boldsymbol{K}}(\bar{\boldsymbol{u}}, \bar{p}) &= \boldsymbol{K}^0 + \overline{\partial}\boldsymbol{K}^0 \;, \\ \overline{\boldsymbol{G}} &= \partial_e \boldsymbol{K}^0 (\nabla \bar{p} - \boldsymbol{f}) \circ \langle\ \rangle_e \;, \\ \overline{\boldsymbol{Q}} &= \partial_p \boldsymbol{K}^0 (\nabla \bar{p} - \boldsymbol{f})) \circ \langle\ \rangle_p \;, \end{aligned} \tag{35}$$

$$\begin{aligned} &\overline{\boldsymbol{F}}(\bar{\boldsymbol{u}}, \bar{p}) = \overline{\partial}\boldsymbol{K}^0 (\boldsymbol{f} - \nabla \bar{p}) \;, \quad \overline{\boldsymbol{f}^s}(\bar{\boldsymbol{u}}, \bar{p}) = \partial_e \boldsymbol{f}^s \circ \boldsymbol{e}(\bar{\boldsymbol{u}}) + \partial_p \boldsymbol{f}^s \circ \bar{p} \;, \\ &\overline{S}(\bar{\boldsymbol{u}}, \bar{p}) = \overline{\partial}M^0 (\bar{p} - p^0) + \overline{\partial}\boldsymbol{B}^0 : (\boldsymbol{e}(\bar{\boldsymbol{u}}) - \boldsymbol{e}(\boldsymbol{u}^0)) \;, \quad \overline{\boldsymbol{g}^s}(\bar{\boldsymbol{u}}, \bar{p}) = \partial_e \boldsymbol{g}^s \circ \boldsymbol{e}(\bar{\boldsymbol{u}}) + \partial_p \boldsymbol{g}^s \circ \bar{p} \;, \\ &\overline{\boldsymbol{R}}(\bar{\boldsymbol{u}}, \bar{p}) = \overline{\partial}\mathbb{A}^0 \boldsymbol{e}(\bar{\boldsymbol{u}}) - \overline{\partial}\boldsymbol{B}^0 \bar{p} \;. \end{aligned} \tag{36}$$

Using the first-order approximations in (32), the following linearized scheme can be introduced to compute a solution $(\boldsymbol{u}, p)$ decomposed in the predictor–corrector parts according to (33). For a given previous time level response $(\boldsymbol{u}^0, p^0)$ and given external loads at the actual time level, we proceed, as follows.

1. Compute $(\bar{\boldsymbol{u}}, \bar{p}) \in \boldsymbol{U}(\Omega) \times P(\Omega)$ by solving the linear problem obtained using (32) where all coefficients are constant being replaced by $(\mathbb{A}^0, \boldsymbol{B}^0, M^0, \boldsymbol{K}^0)$,

$$\begin{aligned}
&\int_\Omega \left(\mathbb{A}^0 \boldsymbol{e}(\bar{\boldsymbol{u}}) - \boldsymbol{B}^0 \bar{p}\right) : \boldsymbol{e}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}^s \cdot \boldsymbol{v} + \int_{\partial\Omega} \boldsymbol{g}^s \cdot \boldsymbol{v} \, \mathrm{dS}_x \;, \\
&\int_\Omega q \left(\boldsymbol{B}^0 : (\boldsymbol{e}(\bar{\boldsymbol{u}}) - \boldsymbol{e}(\boldsymbol{u}^0)) + M^0(\bar{p} - p^0)\right) + \Delta t \int_\Omega \frac{\boldsymbol{K}^0}{\bar{\eta}} (\nabla \bar{p} - \boldsymbol{f}) \cdot \nabla q = 0 \;,
\end{aligned} \tag{37}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{U}_0(\Omega) \times P_0(\Omega)$.

2. Compute corrections $(\delta \boldsymbol{u}, \delta p) \in \boldsymbol{U}_0(\Omega) \times P_0(\Omega)$ by solving

$$\begin{aligned}
&\int_\Omega \left(\overline{\mathbb{A}} \boldsymbol{e}(\delta \boldsymbol{u}) - \delta p \overline{\boldsymbol{B}}\right) : \boldsymbol{e}(\boldsymbol{v}) = \int_\Omega \left(\overline{\boldsymbol{f}^s} \cdot \boldsymbol{v} - \overline{\boldsymbol{R}} : \boldsymbol{e}(\boldsymbol{v})\right) + \int_{\partial\Omega} \overline{\boldsymbol{g}^s} \cdot \boldsymbol{v} \;, \\
&\int_\Omega q \left(\overline{\boldsymbol{D}} : \boldsymbol{e}(\delta \boldsymbol{u}) + \overline{M} \delta p\right) \\
&+ \Delta t \int_\Omega \nabla q \cdot \left(\overline{\boldsymbol{K}} \nabla \delta p + \overline{\boldsymbol{G}} : \boldsymbol{e}(\delta \boldsymbol{u}) + \overline{\boldsymbol{Q}} \delta p\right) = \Delta t \int_\Omega \left(\nabla q \cdot \overline{\boldsymbol{F}} - q \overline{S}\right) \;,
\end{aligned} \tag{38}$$

for all $(\boldsymbol{v}, q) \in \boldsymbol{U}_0(\Omega) \times P_0(\Omega)$.

5 CONCLUSIONS

The proposed macroscopic model of the fluid-saturated porous medium (37) and (38), is weakly nonlinear, cf. [2]; it allows to respect how the material parameters at the microscopic scale depend on the strain and pressure, although, for periodic structures the microproblems are solved just once together with computing the sensitivity of the homogenized coefficients. The same strategy can be pursued to develop a two-scale model of poroelastic media undergoing large deformation, cf. [7]. As an advantage, quite similar considerations lead to an efficient procedures of calculating homogenized coefficients for locally periodic functionally graded porous media. The model is implemented in our in-house code SfePy, see http://sfepy.org.

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