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Optimal complementary matrices in systems with overlapping decomposition: A computational approach

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Abstract— **The paper deals with linear quadratic (LQ) optimal control of linear time-invariant (LTI) systems which are decomposed into overlapped subsystems. A mathematical framework (inclusion principle) is available to formalize different structural properties and relations between the initial and the expanded systems, in which the so called *complementary matrices* play an important role. Up to now, only the structure and conditions on these matrices have been studied in the literature, but not the way to obtain their numerical values systematically. This paper presents a computational approach to select complementary matrices, which can be useful for a practical use of overlapping decompositions. The specific objective is to obtain the complementary matrices such that the quadratic performance for the expanded optimal control problem is minimum. An example is supplied to illustrate the use of the proposed algorithm.**

I. Introduction

In the context of large-scale and complex systems it is frequent to work with systems which share some components [14], [15], [16], [17], [18], [28]. This kind of systems can be treated as interconnected systems with overlapped subsystems (the subsystems share common parts). For this class of systems a mathematical framework, called Inclusion Principle, has been developed. This principle has been applied satisfactory in diverse areas as mechanical systems [4], [30], electric power systems [24], vehicles [19], [23], [25], [26], [27], [29], control of structures [3], applied mathematics [22], [31], [32], etc.

The main idea given by the Inclusion Principle is to expand an initial system, with shared components, into a higher dimensional space in which overlapped subsystems appear as disjoint. Under some conditions, the expanded space contains the essential information about the initial system. The relation between the initial and the expanded system is constructed on the basis of appropriate linear transformations. These transformations involve a set of so-called complementary matrices which have to satisfy well established necessary and sufficient conditions to ensure the Inclusion Principle. The selection of these allows to obtain different expanded systems satisfying different requirements. Some studies have illustrated the role and the influence of the choice of these matrices on properties like stability, controllability or observability. A contribution to this issue has been presented in [5], [6], [7], [9], [12] giving a new procedure for a flexible selection of complementary matrices.

In the context of the design of control systems, overlapping decompositions have been considered under the following conceptual framework. Suppose we have given a "real" dynamical system $\mathbf{S}$ to be controlled under some information or actuation constraints which can be modelled as a system decomposed into overlapped subsystems. Then, the initial system $\mathbf{S}$ can be expanded into a new "artificial" system $\tilde{\mathbf{S}}$ in such a manner that a control methodology can be advantageously designed for this system and transformed (contracted) to have a final control law which is implementable into the real system within the structural constraints. Linear quadratic control has been the methodology mostly considered in this framework [5], [9], [13], [16]. Other control methods have been adopted together with overlapping decomposition, mainly to cope with uncertainties, like sliding mode control [2], fuzzy control [1], guaranteed cost control [8], [11] and H_∞ control [10], [20].

The present paper lies in the context of overlapping optimal control. A previous paper [5] proposed a strategy for choosing the complementary matrices in an expansion-contraction process with state LQ optimal control for linear time-invariant systems. This strategy was based on the identification of a new block structure for the complementary matrices ensuring the inclusion principle and the contractibility for optimal controllers designed in the expanded system. This structure offers a significant degree of freedom for choosing different classes of complementary matrices, but this flexibility has not been yet exploited due to the lack of a computational scheme. Only specific examples have illustrated the selection of complementary matrices within this scheme up to now.

The motivation of this paper is to offer a computational algorithm to obtain numerical complementary matrices in an expansion-contraction process with LQ optimal control of linear-time invariant systems, thus extending the results of [5] with a practical implementable tool.

The paper is organized as follows. Section II presents necessary background results. Section III states the problem. The computational procedure is presented in Section IV, while Section V supplies a numerical example.

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II. Background Results

A. Inclusion Principle

Consider a pair of optimal control problems

$$\begin{aligned} &\min_u J(x_0,u) = \int_0^\infty \left[x^T(t)Q^*x(t)+u^T(t)R^*u(t)\right]\, dt, \\ &\text{s.t.} \quad \mathbf{S}: \dot{x}(t) = Ax(t)+Bu(t), \end{aligned} \tag{1}$$

$$\begin{aligned} &\min_u \tilde{J}(\tilde{x}_0,u) = \int_0^\infty \left[\tilde{x}^T(t)\tilde{Q}^*\tilde{x}(t)+u^T(t)\tilde{R}^*u(t)\right]\, dt, \\ &\text{s.t.} \quad \tilde{\mathbf{S}}: \dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t)+\tilde{B}u(t), \end{aligned} \tag{2}$$

where $x(t)\in\mathbb{R}^{\mathbf{n}}$ and $u(t)\in\mathbb{R}^{\mathbf{m}}$ are the state and the input of $\mathbf{S}$ and $\tilde{x}(t)\in\mathbb{R}^{\tilde{\mathbf{n}}}$ and $u(t)\in\mathbb{R}^{\mathbf{m}}$ are the corresponding to $\tilde{\mathbf{S}}$. The matrices A, B and $\tilde{A}$, $\tilde{B}$ are constant of dimensions $n\times n$, $n\times m$ and $\tilde{n}\times\tilde{n}$, $\tilde{n}\times m$, respectively. The weighting matrices Q^*, $\tilde{Q}^*$ are symmetric positive semi-definite and R^*, $\tilde{R}^*$ are symmetric positive definite. Suppose that the dimension of the state vector $x(t)$ of $\mathbf{S}$ is smaller than (or at most equal to) the vector $\tilde{x}(t)$ of $\tilde{\mathbf{S}}$. Let $x(t;x_0,u)$ denote the unique solution of $\mathbf{S}$ for a fixed input $u(t)$ and an initial state $x(0)$=x_0. Similar notation $\tilde{x}(t;\tilde{x}_0,u)$ is used for the system $\tilde{\mathbf{S}}$.

Remark. The value of the optimal cost depends on the initial state x_0. This dependence may be removed by using a standard well-known way as follows. Consider the initial state as a random vector with covariance matrix $E\{x_0x_0^T\}$=I. Thus, the expected performance index satisfies $E\{J\}\leqslant E\{x_0^TPx_0\}$=tr$(P)$, where tr$(P)$ denotes the trace of the matrix P, the unique solution of the corresponding Riccati equation.

Let us consider the following transformations:

$$V: \mathbb{R}^{\mathbf{n}} \longrightarrow \mathbb{R}^{\tilde{\mathbf{n}}}, \qquad U: \mathbb{R}^{\tilde{\mathbf{n}}} \longrightarrow \mathbb{R}^{\mathbf{n}}, \tag{3}$$

where rank$V$=n and such that UV=I_n, where I_n is the identity matrix of indicated dimension. Given a matrix V the pseudoinverse matrix U can be obtained by U=$(V^TV)^{-1}V^T$.

Definition 1: (*Inclusion Principle*) A system $\tilde{\mathbf{S}}$ *includes* the system $\mathbf{S}$, denoted by $\tilde{\mathbf{S}}\supset\mathbf{S}$, if there exists a pair of matrices (U,V) satisfying UV=I_n and such that for any initial state x_0 and any fixed input $u(t)$ of $\mathbf{S}$, the choice $\tilde{x}_0$=Vx_0 of the system $\tilde{\mathbf{S}}$ implies $x(t;x_0,u)$=$U\tilde{x}(t;Vx_0,u)$ for all $t\geq 0$. If $\tilde{\mathbf{S}}\supset\mathbf{S}$, then $\tilde{\mathbf{S}}$ is said to be an *expansion* of $\mathbf{S}$ and $\mathbf{S}$ is a *contraction* of $\tilde{\mathbf{S}}$.

There are two particular but important cases within the Inclusion Principle called *restrictions* and *aggregations*. These definitions are as follows.

Definition 2: A system $\mathbf{S}$ is a *restriction* of $\tilde{\mathbf{S}}$, if there exists a pair of matrices (U,V) satisfying UV=I and such that for any initial state x_0 and any fixed input $u(t)$ of $\mathbf{S}$, the choice $\tilde{x}_0$=Vx_0 implies $\tilde{x}(t;\tilde{x}_0,u)$=$Vx(t;x_0,u)$ for all $t\geq 0$.

Definition 3: A system $\mathbf{S}$ is an *aggregation* of $\tilde{\mathbf{S}}$ if there exists a pair of matrices (U,V) satisfying UV=I and such that for any initial state $\tilde{x}_0$ and any fixed input $u(t)$ of $\tilde{\mathbf{S}}$, the choice x_0=$U\tilde{x}_0$ implies $x(t;x_0,u)$=$U\tilde{x}(t;\tilde{x}_0,u)$ for all $t\geq 0$.

B. Complementary matrices

The expanded matrices $\tilde{A}$, $\tilde{B}$, $\tilde{Q}^*$ and $\tilde{R}^*$ of $\tilde{\mathbf{S}}$ can be expressed as

$$\begin{aligned} \tilde{A} &= VAU+M, & \tilde{B} &= VB+N, \\ \tilde{Q}^* &= U^TQ^*U+M_{Q^*}, & \tilde{R}^* &= R^*+N_{R^*}, \end{aligned} \tag{4}$$

where M, N, M_{Q^*} and N_{R^*} are the complementary matrices. The designer have to choose the matrices M_{Q^*} and N_{R^*} in such a way that the corresponding expanded weighting matrices $\tilde{Q}^*$ and $\tilde{R}^*$ are symmetric positive semi-definite and symmetric positive definite matrices, respectively.

For $\tilde{\mathbf{S}}$ to be an expansion of $\mathbf{S}$, a proper choice of M and N is required, [14], [15], [16], [17], [28]. In terms of complementary matrices, the previous definitions can be rewritten in the following form.

Theorem 1: A system $\tilde{\mathbf{S}}$ is an expansion of the system $\mathbf{S}$ if and only if UM^iV=0, $UM^{i-1}N$=0, for all $i=1,2,\ldots,\tilde{n}$.

Proposition 1: A system $\mathbf{S}$ is a restriction of the system $\tilde{\mathbf{S}}$ if and only if $MV=0$ and $N=0$.

Proposition 2: A system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ if and only if $UM=0$ and $UN=0$.

Moreover, by using complementary matrices different expanded systems $\tilde{\mathbf{S}}$ can be obtained, so we get some degrees of freedom in the expansion-contraction process.

C. System structures

Consider the structure of the matrices A, B given in (1) as follows:

$$A = \left[\begin{array}{cc|c} A_{11} & A_{12} & A_{13} \\ \hline A_{21} & A_{22} & A_{23} \\ \hline A_{31} & A_{32} & A_{33} \end{array}\right], \quad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{array}\right], \tag{5}$$

where A_{ii}, B_{ij} for i=1,2,3 and j=1,2 are $n_i\times n_i$, $n_i\times m_j$ dimensional matrices, respectively. In this paper we suppose that the state matrix A of $\mathbf{S}$ is composed of subsystems with one overlapped part, corresponding to the subsystem A_{22} in our case.

This structure has been extensively adopted as prototype in the literature within the Inclusion Principle [14], [15], [16], [17], [18], [21], [28]. The dimensions of the components x=$(x_1^T,x_2^T,x_3^T)^T$ are n_1, n_2, n_3 and satisfy the relation n_1+n_2+n_3=n. The partition of u=$(u_1^T,u_2^T)^T$ has two components of dimensions m_1, m_2 such that m_1+m_2=m.

Considering the overlapping structure of subsystem A_{22} in the original system, a standard particular selection of the transformation matrix V is given by

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix}. \tag{6}$$

This transformation leads in a simple natural way to an expanded system where the state vector x_2 appears repeated in $\tilde{x}$=$(x_1^T,x_2^T,x_2^T,x_3^T)^T$. The expanded matrices $\tilde{A}$=VAU and $\tilde{B}$=VB, without adding the complementary matrices M and N, respectively, have the form:

$$\tilde{A}=\left[\begin{array}{cc|cc} A_{11} & \frac{1}{2}A_{12} & \frac{1}{2}A_{12} & A_{13} \\ A_{21} & \frac{1}{2}A_{22} & \frac{1}{2}A_{22} & A_{23} \\ \hline A_{21} & \frac{1}{2}A_{22} & \frac{1}{2}A_{22} & A_{23} \\ A_{31} & \frac{1}{2}A_{32} & \frac{1}{2}A_{32} & A_{33} \end{array}\right], \quad \tilde{B}=\left[\begin{array}{c|c} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{array}\right]. \tag{7}$$

Theorem 2: Consider the system $\mathbf{S}$ given in (1) with the structure (5). Consider the transformation V given in (6). Then, $\tilde{\mathbf{S}} \supset \mathbf{S}$ if and only if the complementary matrices M and N have the following form:

$$M=\begin{bmatrix} 0 & M_{12} & -M_{12} & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -(M_{22}+M_{23}+M_{33}) & M_{33} & -M_{24} \\ 0 & M_{42} & -M_{42} & 0 \end{bmatrix}, \quad N=\begin{bmatrix} 0 & 0 \\ N_{21} & N_{22} \\ -N_{21} & -N_{22} \\ 0 & 0 \end{bmatrix} \tag{8}$$

and satisfy the conditions

$$\begin{bmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{bmatrix} \begin{bmatrix} M_{22}+M_{33} \end{bmatrix}^{i-1} \begin{bmatrix} M_{21} & M_{22}+M_{23} & M_{24} \end{bmatrix} = 0, \quad \begin{bmatrix} M_{12} \\ M_{23}+M_{33} \\ M_{42} \end{bmatrix} \begin{bmatrix} M_{22}+M_{33} \end{bmatrix}^{i-1} \begin{bmatrix} N_{21} & N_{22} \end{bmatrix} = 0 \tag{9}$$

for all $i=1,2,\cdots,\tilde{n}-1$.

Proposition 3: A system $\mathbf{S}$ is a restriction of the system $\tilde{\mathbf{S}}$ if and only if the matrices M and N have the following structure:

$$M=\begin{bmatrix} 0 & M_{12} & -M_{12} & 0 \\ 0 & M_{22} & -M_{22} & 0 \\ 0 & M_{32} & -M_{32} & 0 \\ 0 & M_{42} & -M_{42} & 0 \end{bmatrix}, \quad N=\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \tag{10}$$

Proposition 4: A system $\mathbf{S}$ is an aggregation of the system $\tilde{\mathbf{S}}$ if and only if the matrices M and N have the following structure:

$$M=\begin{bmatrix} 0 & 0 & 0 & 0 \\ M_{21} & M_{22} & M_{23} & M_{24} \\ -M_{21} & -M_{22} & -M_{23} & -M_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad N=\begin{bmatrix} 0 & 0 \\ N_{21} & N_{22} \\ -N_{21} & -N_{22} \\ 0 & 0 \end{bmatrix}. \tag{11}$$

Remark. By using the transformation V given in (6), Theorem 2 provides the most general structure of the complementary matrices M and N under which $\tilde{\mathbf{S}} \supset \mathbf{S}$, supposing that (9) is satisfied. Obviously, the structures of the matrices M and N given by Propositions 3 and 4 satisfy (9). However, it is well-known that an expanded system $\tilde{\mathbf{S}}$ is uncontrollable if N=0. This assertion has been proved in [7], [12], [21]. For this reason, we can not utilize a restriction to compute the cost function $\tilde{J}$. Although it is possible to apply the general Theorem 2, in this paper we will use Proposition 4 for simplicity.

III. Problem Statement

Consider the optimal control problems given in (1) and (2). We can observe that the minimization of $\tilde{J}$ depends on the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{Q}^*$, $\tilde{R}^*$. Consequently, for the given matrices A, B, Q^*, R^* and the transformations V, U, this minimization depends on the complementary matrices M, N, M_{Q^*} and N_{R^*}, according to (4). In this paper, we will consider that M_{Q^*} and N_{R^*} are fixed a priori, so that the expression of the expanded cost function $\tilde{J}$ in (2) is completely defined. The complementary matrices M and N remain "free".

The goal of this paper is to present a computational iterative algorithm for the effective selection of the variable matrices M and N such that the minimum value of $\tilde{J}$ is achieved. The following specific objectives are proposed:

- To present a guideline to select initial complementary matrices M_0 and N_0 in order that the algorithm can be initialized. These matrices have to verify the Inclusion Principle and simultaneously to guarantee the controllability to ensure the the optimal control for $\tilde{J}$.
- To give an algorithm to compute the complementary matrices M and N so that the value of the quadratic cost function $\tilde{J}$ reaches its minimum.
- To apply the algorithm on a numerical example.

IV. Computational Scheme

We know the structures and the conditions on the matrices M, N given in Section II-C, but it is necessary to select their numerical values. For this purpose we consider two stages:

(a) The selection of initial matrices M_0, N_0 so that:
   1) $\tilde{\mathbf{S}} \supset \mathbf{S}$ (Inclusion Principle) is satisfied.
   2) The corresponding expanded system $\tilde{\mathbf{S}}$ is controllable.

(b) The implementation of a Matlab-based iterative routine seeking for "optimal" complementary matrices M and N such that the associated quadratic cost $\tilde{J}$ in the expanded space $\tilde{\mathbf{S}}$ is minimum.

In our case, the point 1 is satisfied by using Proposition 4. Point 2 is required to ensure the solution of the corresponding optimal control problem. In [12] it has been proved that it is always possible to choose appropriate complementary matrices M_0, N_0 so that $\tilde{\mathbf{S}}$ is controllable. Thus, according to [12], taking the submatrices

$$M_{21}=A_{21}, \quad M_{22}=\frac{1}{2}A_{22}, \quad M_{24}=A_{23} \tag{12}$$

and choosing a submatrix M_{23} such that $\frac{1}{2}A_{22}-M_{23}$ has all their eigenvalues distinct and simultaneously different from the eigenvalues of A, the system $\tilde{\mathbf{S}}$ is controllable. This is a constructive method to get a controllable expanded space via complementary matrices.

The full computational procedure can be summarized in the following steps:

- **Let** (A,B,Q^*,R^*) **be** the given matrices for the system $\mathbf{S}$.
- **Choose** the matrices M_{R^*}, N_{R^*} to construct the expanded cost function $\tilde{J}$ so that $\tilde{Q}^*$ and $\tilde{R}^*$ are symmetric positive semi-definite and symmetric positive definite matrices, respectively.
- **Select** complementary matrices M_0 and N_0 with the structure given in (11), following the procedure presented in the previous point 2.

- **Obtain** the expanded matrices $(\tilde{A}_0, \tilde{B}_0, \tilde{Q}^*, \tilde{R}^*)$ for the initial matrices M_0 and N_0 by using relations (4).
- **Start the routine** with the initial complementary matrices M_0, N_0 and compute the matrices

$$\tilde{A}_n = VAU + M_n, \qquad \tilde{B}_n = VB + N_n$$

for each n=0,1,2,⋯. At each iteration, the algorithm verifies if the pair $(\tilde{A}_n, \tilde{B}_n)$ is controllable and then it minimizes the cost function $\tilde{J}$ given in (2) by using conveniently the **fmincon** function provided by Matlab, where the matrices M and N are the unknowns. Through the Matlab function **lqr** the matrix $\tilde{P}_n$ is obtained, which is the unique solution of the corresponding Riccati equation. Finally, $\tilde{J}_n$=tr$(\tilde{P}_n)$ gives the value of the cost function for each n. At the end of the process, we obtain the optimal complementary matrices M_{opt} and N_{opt} together with the optimal cost $\tilde{J}_{opt}$ for the system $\tilde{\mathbf{S}}$.

V. Example

A. Problem Statement

Consider the system $\mathbf{S}$ given in (1) and (5) and the associated cost function weighting matrices as follows:

$$A = \left[\begin{array}{c|ccc|c} -1 & 0 & 0 & 0 & 0 \\ \hline 1 & -2 & 2 & 2 & 1 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 \\ \hline 1 & 0 & 1 & 1 & -1 \end{array}\right], \quad B = \left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & 0 \\ 0 & 1 \\ -1 & 0 \\ \hline 0 & -1 \end{array}\right],$$

$$Q^* = \text{diag}\{1,2,2,2,1\}, \qquad R^* = I_2. \tag{13}$$

The pair (A,B) is controllable. The eigenvalues of the matrix A are $\{-2,-2,-2,-1,-1\}$. The overlapped subsystem A_{22} in the state matrix A corresponds to

$$\begin{bmatrix} -2 & 2 & 2 \\ 0 & -2 & 0 \\ 0 & 2 & -2 \end{bmatrix}.$$

The expanded matrices $\tilde{Q}^*$, $\tilde{R}^*$ are $\tilde{Q}^*$=I_8, $\tilde{R}^*$=I_2 by choosing the complementary matrix

$$M_{Q^*} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & -0.5 & 0 \\ 0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & -0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

and N_{R^*}=0, respectively.

According to the point 2 in Section IV, we choose the initial complementary matrix M_0 as follows:

$$M_0 = \left[\begin{array}{c|ccc|ccc|c} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 1 & -1 & 1 & 1 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 & -3 & 0 \\ \hline -1 & 1 & -1 & -1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & -1 & 3 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \tag{15}$$

and the initial matrix N_0 in the form:

$$N_0 = \left[\begin{array}{c|c} 0 & 0 \\ \hline 1 & 0 \\ 0 & 0 \\ 1 & 1 \\ \hline -1 & 0 \\ 0 & 0 \\ -1 & -1 \\ \hline 0 & 0 \end{array}\right]. \tag{16}$$

The submatrix

$$M_{23} = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

in (15) verifies that $\frac{1}{2}A_{22} - M_{23}$ has eigenvalues $\{0,1,2\}$, which are all distinct and different from the eigenvalues of matrix A.

With this selection, the initial expanded matrices are the following:

$$\tilde{A}_0 = VAU + M_0 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 2 & 2 & -2 & 2 & 2 & 2 \\ 0 & 0 & -2 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 2 & -4 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & 0.5 & 0.5 & 0 & 0.5 & 0.5 & -1 \end{bmatrix},$$

$$\tilde{B}_0 = VB + N_0 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ -1 & 0 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \end{bmatrix}. \tag{17}$$

It is easy to prove that the initial expanded pair $(\tilde{A}_0, \tilde{B}_0)$ is controllable.

For the complementary matrices M_0, N_0 the initial cost value is $\tilde{J}_0$=18.90.

Now, by applying the proposed algorithm, the obtained optimal complementary matrices M_{opt} and N_{opt} are the following:

$$M_{opt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1.81 & -2.00 & 0.95 & -0.42 & 2.00 & 1.34 & 0.63 & 0.40 \\ -1.09 & -0.55 & -2.00 & 0.03 & 0.64 & 2.00 & 0.62 & -0.17 \\ 1.80 & 2.00 & 0.53 & -2.00 & -1.75 & 1.77 & 2.00 & 0.26 \\ -1.81 & 2.00 & -0.95 & 0.42 & -2.00 & -1.34 & -0.63 & -0.40 \\ 1.09 & 0.55 & 2.00 & -0.03 & -0.64 & -2.00 & -0.62 & 0.17 \\ -1.80 & -2.00 & -0.53 & 2.00 & 1.75 & -1.77 & -2.00 & -0.26 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

and

$$N_{opt} = \begin{bmatrix} 0 & 0 \\ 2.00 & 2.00 \\ -2.00 & -0.21 \\ 2.00 & 2.00 \\ -2.00 & -2.00 \\ 2.00 & 0.21 \\ -2.00 & -2.00 \\ 0 & 0 \end{bmatrix}. \tag{19}$$

By using the optimal complementary matrices M_{opt} and N_{opt} given in (18) and (19), respectively, the minimum quadratic cost value for the expanded system $\tilde{\mathbf{S}}$ results to be $\tilde{J}$=3.56.

VI. Conclusion

This paper has dealt with LQ optimal control of a class of linear time invariant systems decomposed into overlapped subsystems. The main contribution is a computational procedure for the selection of complementary matrices based on a new block structure of these matrices which ensures the inclusion principle and system controllability. The paper has focussed on the computation of such complementary matrices that give the minimum cost function for the optimal control problem formulated for the expanded system. The computation scheme is simple in using advantage of available Matlab tools with an appropriate initialization according to the above mentioned new block structure. To the author's knowledge, this is the first attempt to come up with a tool for a systematic numerical computation of complementary matrices for systems with overlapping decompositions. Further work is planned to use this tool in overlapping decentralized control using LQ optimal control and also other control approaches, like guaranteed cost control or H_∞ control where theoretical results are available but computational tools are needed. The availability of numerical procedures may help to exploit the potential of overlapping decomposition in practical applications.

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