Observer Design for Takagi-Sugeno Lipschitz Systems Affected by Disturbances using Quadratic Boundedness

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***Abstract*—In this paper, a proportional observer design using quadratic boundedness is proposed in order to estimate the state of a system described by a Takagi-Sugeno model with a Lipschitz nonlinearity term, and affected by unknown disturbances. The conditions for ensuring that the error between the real and the estimated state converge within an ellipsoidal region about zero, are provided in the form of a linear matrix inequality (LMI) formulation. Then, the simulation results of this approach applied to a four-wheeled omni-directional mobile robot will be shown.**

***Index Terms*—Takagi-Sugeno, Proportional Observer, Quadratic Boundedness, Mobile Robot.**

I. Introduction

The observer design for nonlinear systems satisfying a Lipschitz continuity conditions is a topic which has been investigated for many years. It was first considered by [1], and many researchers have studied observer design for Lipschitz systems using various approaches. For instance, [2] presents some fundamental insights into observer design for the class of Lipschitz nonlinear systems. On the other hand, [3] have introduced an approach for robust H_∞ observer design for a class of Lipschitz nonlinear systems with time-varying uncertainties.

Unlike the approaches above, we will consider systems where the part of the model outside the Lipschitz nonlinearity can be described by a Takagi-Sugeno representation. The TS approach introduced by [4] provides a useful tool to represent with a good precision a large class of nonlinear systems, by merging together multiple local affine dynamic linear models. This approach, which is closely related to the linear parameter varying one [5], [6], allows converting a nonlinear system into a linear-like representation by embedding the system's non-linearities inside some varying parameters. The problem of designing observer for TS systems has been investigated by several researchers. For example, [7] has introduced the analysis and design of two different sliding mode observers for dynamic TS systems. In [8], an observer design method for TS systems with unmeasurable premise variables was proposed. However, for systems affected by disturbances, alternative techniques should be applied. This is the case of [9] that investigated robust fault diagnosis of proton exchange membrane (PEM) fuel cells by introducing the TS interval observers that consider uncertainty in a bounded context.

The notion of quadratic boundedness was introduced by [10], [11]. Roughly speaking, a system is said to be quadratically bounded if all its solutions are bounded and this behaviour can be guaranteed with a quadratic Lyapunov function. By requiring quadratic boundedness, observers for systems affected by bounded disturbance can be designed. [12] have shown the design of state observers for a general class of nonlinear discrete-time systems that satisfy a one-sided Lipschitz condition. [13] gives a brief controller design by using quadratic boundedness for a TS system with input or state constraint and bounded noise. In 2019, [14] has presented a generic development devoted to generate state observers for linear uncertain systems and meanwhile developed a state observer for UAV system with disturbance.

In our work, we will bound the disturbance in a certain range. Then, we will require the estimation error to be quadratically bounded. In this paper, analysis and design conditions for a proportional observer using quadratic boundedness are proposed in order to estimate the state of the Lipschitz systems described by TS model. The aim of this observer is to estimate the internal state of a given system with disturbance. The proposed approach is illustrated in simulation applied to a four-wheeled omni-directional mobile robot.

This paper is structured as follow. In Section 2, the continuous-time TS Lipschitz systems considered will be introduced. In Section 3, the procedure for proportional observer design for the TS Lipschitz system using quadratic boundedness will be presented. In Section 4, the proposed methodology is applied to a four-wheeled omni-directional mobile robot. Section 5 presents the simulation results. Finally, Section 6 summarizes the main conclusions and discusses possible future work.

II. Takagi-Sugeno Lipschitz Systems

Consider the following continuous-time nonlinear system:

$$\begin{aligned} \dot{x}(t) &= g(x(t), u(t), d(t,x)) \\ y(t) &= h(x(t), u(t)) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $u \in \mathbb{R}^m$ is the input, $d \in \mathbb{R}^s$ is a disturbance affecting the state, $y \in \mathbb{R}^q$ is the system output and g and h are some nonlinear functions. In the following, we assume that (1) can be represented or approximated by a

TS fuzzy system consisting of a set of fuzzy rules, where each rule i takes the form [15]

Rule i: IF $\xi_1(t)$ is M_{i1} and ... and $\xi_p(t)$ is M_{ip}.

$$\text{THEN: } \begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) + G_i d(t,x) + f(x(t)) \\ y(t) = Cx(t) + Du(t) \end{cases} \tag{2}$$

where $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{n\times m}$ and $G_i \in \mathbb{R}^{q\times s}$ are known matrices. M_{ij} is the fuzzy set and r is the number of model rules. $\xi(t)$ denotes the vector containing all the individual elements $\xi_1(t)$,..., $\xi_p(t)$. Note that d is the disturbance affecting the system which we assume to be in some known set Ω which is closed and bounded [11], that is,

$$d(t,x) \in \Omega \qquad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n \tag{3}$$

In particular, we will consider Ω to be the set of m-vectors whose Euclidean norm is less than or equal to 1, such that $||d(t,x)|| \leq 1$. Note that it is always possible to fit into this case by a proper rescaling of the matrices G_i.

On the other hand, $f(x(t))$ is a nonlinear function assumed to be Lipschitz with respect to the state x, which means there exists $\alpha > 0$ such that:

$$||f(x) - f(\hat{x})|| \leq \alpha ||x - \hat{x}||$$

The system (2) can be described equivalently by:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\big(A_i x(t) + B_i u(t) + G_i d(t,x)\big) + f(x(t)) \\ y(t) &= Cx(t) + Du(t) \quad i = 1,...,r \end{aligned} \tag{4}$$

The weighing functions μ_i are nonlinear, depend on the decision variable $\xi(t)$, and satisfy the following properties:

$$h_i(\xi(t)) = \prod_{j=1}^{p} M_{ij}(\xi_j(t)) \tag{5}$$

$$\mu_i(\xi(t)) = \frac{h_i(\xi(t))}{\sum_{i=1}^{r} h_i(\xi(t))} \tag{6}$$

$$0 \leq \mu_i(\xi(t)) \leq 1 \tag{7}$$

$$\sum_{i=1}^{r} \mu_i(\xi(t)) = 1 \tag{8}$$

where $M_{ij}(\xi_j(t))$ is the grade of membership of $\xi_j(t)$ in M_{ij}.

In the next section, a methodology for designing a proportional observer for a system modelled by (4) using the quadratic boundedness approach will be presented.

III. Observer Design using quadratic boundedness

For the system (4), let us consider the state observer:

$$\begin{aligned} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))\big(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))\big) + f(\hat{x}(t)) \\ \hat{y}(t) &= C\hat{x}(t) + Du(t) \end{aligned} \tag{9}$$

where $L_i \in \mathbb{R}^{n\times m}$ is the observer gain matrix to be designed. Let us define the observation error

$$e(t) = x(t) - \hat{x}(t)$$

Then, the estimation error dynamics is given by:

$$\dot{e}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\big((A_i - L_i C)e(t) + G_i d(t,x)\big) + f_e(x,\hat{x}) \tag{10}$$

where $f_e(x,\hat{x}) = f(x(t)) - f(\hat{x}(t))$

Due to the presence of disturbances, the proportional observer will not converge to zero. For this reason, a quadratic boundedness design approach will be applied.

Let us define:

$$\hat{A}_i = A_i - L_i C$$

then (10) can be written as

$$\dot{e}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))\big(\hat{A}_i e(t) + G_i d(t,x)\big) + f_e(x,\hat{x}) \tag{11}$$

Before introducing the theorem for the design of the proportional observer using quadratic boundedness which is the main theoretical result of this paper, let us recall some Definitions and Lemmas which will be used in the theorem's proof.

Definition 1 [11]: *A dynamical system described by*

$$\dot{x} = Ax(t) + Gd(t,x) \quad d(t,x) \in \Omega \quad \forall t \in \mathbb{R}, \forall x \in \mathbb{R}^n \tag{12}$$

is quadratically bounded with Lyapunov matrix P if P is a positive definite symmetric matrix and:

$$x^T P x > 1 \Rightarrow x^T P \dot{x} < 0 \quad \forall d \in \Omega \tag{13}$$

Note that rate of change of the function $V = x^T P x$, along any trajectory $x(t)$ of system (12) is given by:

$$\frac{dV(x)}{dt} = 2x^T P\big(Ax + Gd(t,x)\big)$$

Lemma 1 [8]: *For any matrices X and Y with appropriate dimensions, and any positive-definite matrix Λ, the following property holds:*

$$X^T Y + Y^T X \leq X^T \Lambda X + Y^T \Lambda^{-1} Y$$

Lemma 2 [11]: *Suppose P and B are symmetric positive semidefinite matrices and Q is a symmetric positive definite matrix. Then:*

$$x^T Q x - 2(x^T B x)^{1/2} > 0 \qquad \text{for } x^T P x > 1$$

if and only if there exists a scalar $\beta > 0$ such that

$$Q - \beta P - \beta^{-1} B \geq 0$$

For the system (11), we can obtain the following theorem for observer design.

Theorem 1: Suppose P is a symmetric, positive definite matrix. Then, system (11) is quadratically bounded with P as a Lyapunov matrix if there exist scalars $\beta, \gamma, \varepsilon > 0$ such that

$$\begin{bmatrix} P\hat{A}_i + \hat{A}_i^T P + \gamma\alpha^T\alpha + \varepsilon P + \beta P & 0 & PG_i \\ 0 & \varepsilon^{-1}P - \gamma I & 0 \\ G_i^T P & 0 & -\beta I \end{bmatrix} \leq 0 \qquad \text{for} \quad 1 \leq i \leq r \tag{14}$$

Proof Let us use as a quadratic Lyapunov candidate $V = e^T P e$. Applying *Definition 1*, the system is quadratically bounded with Lyapunov matrix P if P is a positive definite symmetric matrix and

$$e^T P e > 1 \Rightarrow e^T P \left[\sum_{i=1}^{r} \mu_i(\xi(t))(\hat{A}_i e + G_i d(t,x)) + f_e(x,\hat{x})\right] < 0 \tag{15}$$

Using the same procedure in the proof of Theorem 1 in [11], we have that (15) is equivalent to:

$$\begin{aligned} & max\Big(2e^T P \sum_{i=1}^{r} \mu_i(\xi(t))\hat{A}_i e + 2e^T P f_e(x,\hat{x}) \\ & +2e^T P \sum_{i=1}^{r} \mu_i(\xi(t)) G_i d : ||d|| \leq 1\Big) < 0 \quad \text{for} \quad e^T P e > 1 \end{aligned} \tag{16}$$

The above condition is equivalent to

$$\begin{aligned} & e^T \sum_{i=1}^{r} \mu_i(\xi(t))\big(P\hat{A}_i + \hat{A}_i^T P\big)e + e^T P f_e(x,\hat{x}) + f_e(x,\hat{x})^T P e \\ & +2\big(e^T P \sum_{i=1}^{r} \mu_i(\xi(t)) G_i \sum_{i=1}^{r} \mu_i(\xi(t)) G_i^T P e\big)^{1/2} < 0 \quad \text{for } e^T P e > 1 \end{aligned} \tag{17}$$

Applying *Lemma 1*, the term $e^T P f_e(x,\hat{x}) + f_e(x,\hat{x})^T P e$ can be replaced as follows (note that a positive definite matrix has a unique positive definite square root):

$$\begin{aligned} & e^T P^{1/2} P^{1/2} f_e(x,\hat{x}) + f_e(x,\hat{x})^T P^{1/2} P^{1/2} e \leq \\ & e^T P^{1/2} \Lambda P^{1/2} e + f_e(x,\hat{x})^T P^{1/2} \Lambda^{-1} P^{1/2} f_e(x,\hat{x}) \end{aligned} \tag{18}$$

On the other hand, due to the Lipschitz property, the following condition holds

$$\gamma e^T \alpha^T \alpha e - \gamma f_e(x,\hat{x})^T f_e(x,\hat{x}) \leq 0 \quad \forall \gamma > 0$$

Let us choose $\Lambda = \varepsilon I$, then we have that (17) holds if:

$$\begin{aligned} & e^T P \sum_{i=1}^{r} \mu_i(\xi(t))\hat{A}_i e + e^T \sum_{i=1}^{r} \mu_i(\xi(t))\hat{A}_i^T P + \gamma e^T \alpha^T \alpha e \\ & -\gamma f_e(x,\hat{x})^T f_e(x,\hat{x}) + e^T \varepsilon P e + \varepsilon^{-1} f_e(x,\hat{x})^T f_e(x,\hat{x}) \\ & +2\big(e^T P \sum_{i=1}^{r} \mu_i(\xi(t)) G_i \sum_{i=1}^{r} \mu_i(\xi(t)) G_i^T P e\big)^{1/2} < 0 \end{aligned} \tag{19}$$

which can be written in a compact form

$$\begin{aligned} & \begin{bmatrix} e \\ f_e(x,\hat{x}) \end{bmatrix}^T \sum_{i=1}^{r} \mu_i(\xi(t)) \begin{bmatrix} P\hat{A}_i + \hat{A}_i^T P + \gamma\alpha^T\alpha + \varepsilon P & 0 \\ 0 & \varepsilon^{-1} P - \gamma I \end{bmatrix} \begin{bmatrix} e \\ f_e(x,\hat{x}) \end{bmatrix} \\ & +2\left(\begin{bmatrix} e \\ f_e(x,\hat{x}) \end{bmatrix}^T \begin{bmatrix} P\sum_{i=1}^{r} \mu_i(\xi(t)) G_i \sum_{i=1}^{r} \mu_i(\xi(t)) G_i^T P & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e \\ f_e(x,\hat{x}) \end{bmatrix}\right)^{1/2} < 0 \end{aligned} \tag{20}$$

Applying *Lemma 2*, and exploiting a basic property of matrices [16], which says that given two negative definite matrices, their linear combination with non-negative coefficients (of which at least one is different from zero) is negative definite, we obtain that (20) is equivalent to

$$\begin{aligned} & -\begin{bmatrix} P\hat{A}_i + \hat{A}_i^T P + \gamma\alpha^T\alpha + \varepsilon P & \\ 0 & \varepsilon^{-1} P - \gamma I \end{bmatrix} - \beta \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix} \\ & -\beta^{-1} \begin{bmatrix} PG_iG_i^T P & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad \text{for} \quad 1 \leq i \leq r \end{aligned} \tag{21}$$

Taking Schur's complement, we obtain (14), which completes the proof. □

It is interesting to highlight that since ε and β are free variables, it is necessary to find a way to determine the optimal one. This procedure will be shown in the simulation section.

Note that (14) is a BMI due to the product of matrix variables: L and P, ε and P, β and P. Using a change of variables $W_i = PL_i$, we transform the BMI to into (22). Then, if ε and β are considered as constants we obtain LMIs [17], as stated by the following corollary.

Corollary: The state estimation error between the real state and the observed one is quadratically bounded if there exist a positive definite matrix P, a matrix W, and β, γ, $\varepsilon > 0$ such that the following LMI holds:

$$\begin{bmatrix} T_{11} & 0 & PG_i \\ 0 & \varepsilon^{-1} P - \gamma I & 0 \\ G_i^T P & 0 & -\beta I \end{bmatrix} \leq 0 \quad \text{for} \quad 1 \leq i \leq r \tag{22}$$

where

$$T_{11} = PA_i - W_i C - C^T W_i^T + A_i^T P + (\varepsilon + \beta)P + \gamma(\alpha^T \alpha)$$

The observer gains are given by

$$L_i = P^{-1} W_i, \quad 1 \leq i \leq r \tag{23}$$

such that using the parallel decomposition approach leads to

$$L = \sum_{i=1}^{r} \mu_i(\xi(t)) L_i \tag{24}$$

IV. Mobile Robot case study

A. Mobile robot model description

For testing the proposed methodology, we will consider a four wheeled omni-directional mobile robot (shown as in Fig. 1) for which the dynamic model, that relates the motors' voltage with the robot's accelerations is given by [18]:

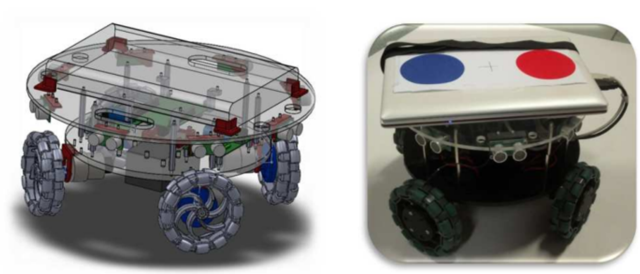

Fig. 1. Four-wheeled omni-directional mobile robot [19].

$$\dot{x} = v_x \quad (25)$$

$$\dot{v}_x = (A_{11}c_\theta^2 + A_{22}s_\theta^2)v_x + ((A_{11} - A_{22})s_\theta c_\theta - \omega)v_y \quad (26)$$
$$+ K_{11}c_\theta sign(v_x c_\theta + v_y s_\theta) - K_{22}s_\theta sign(-v_x s_\theta + v_y c_\theta)$$
$$- B_{21}s_\theta u_1 + B_{12}c_\theta u_2 - B_{23}s_\theta u_3 + B_{14}c_\theta u_4$$

$$\dot{y} = v_y \quad (27)$$

$$\dot{v}_y = ((A_{11} - A_{22})s_\theta c_\theta + \omega)v_x + (A_{11}s_\theta^2 + A_{22}c_\theta^2)v_y \quad (28)$$
$$+ K_{11}s_\theta sign(v_x c_\theta + v_y s_\theta) + K_{22}c_\theta sign(-v_x s_\theta + v_y c_\theta)$$
$$- B_{21}c_\theta u_1 + B_{12}s_\theta u_2 - B_{23}c_\theta u_3 + B_{14}s_\theta u_4$$

$$\dot{\theta} = \omega \quad (29)$$

$$\dot{\omega} = A_{33}\omega + B_{31}u_1 + B_{32}u_2 + B_{33}u_3 + B_{34}u_4 + K_{33}sign(\omega) \quad (30)$$

where (x, y) is the robot position, θ is the angle with respect to the defined front of the robot ($s_\theta = \sin\theta$ and $c_\theta = \cos\theta$), v_x, v_y and ω are the corresponding linear/angular velocities, and u_1, u_2, u_3 and u_4 are the motor voltages applied to the wheels 1, 2, 3, and 4, respectively.

We will consider that the parameters A_{ii}, B_{ij}, $i = 1,2,3$ are known since they contain physical constants (or values) that can be estimated experimentally with reasonable accuracy. On the other hand, the coefficients K_{ii}, $i = 1,2,3$ contain Coulomb frictions coefficients, which act as an unknown disturbance.

The nonlinear system (25)-(30) can be rewritten in a compact matrix form as (dependence of θ and ω on t is omitted):

$$\dot{\zeta}(t) = A(\theta)\zeta(t) + B(\theta)u(t) + G(\theta)d(\zeta) + f(\zeta) \quad (31)$$
$$\psi(t) = C\zeta(t) \quad (32)$$

where $\zeta = \begin{bmatrix} x & v_x & y & v_y & \theta & \omega \end{bmatrix}^T \in \mathbb{R}^6$ is the system state, $u = \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \end{bmatrix}^T \in \mathbb{R}^4$ is the input and $\psi = \begin{bmatrix} x & y & \theta \end{bmatrix}^T \in \mathbb{R}^3$ is the output provided by the sensors.

$$A(\theta) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & q_1 & 0 & q_2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & q_3 & 0 & q_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & A_{33} \end{bmatrix} \quad (33)$$

$$q_1 = A_{11}c_\theta^2 + A_{22}s_\theta^2 \qquad q_2 = (A_{11} - A_{22})s_\theta c_\theta$$
$$q_3 = (A_{11} - A_{22})s_\theta c_\theta \qquad q_4 = A_{11}s_\theta^2 + A_{22}c_\theta^2$$

$$B(\theta) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ -B_{21}s_\theta & B_{12}c_\theta & -B_{23}s_\theta & B_{14}c_\theta \\ 0 & 0 & 0 & 0 \\ -B_{21}c_\theta & B_{12}s_\theta & -B_{23}c_\theta & B_{14}s_\theta \\ 0 & 0 & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} \end{bmatrix} \quad (34)$$

$$f(\zeta) = \begin{bmatrix} 0 & -\omega v_y & 0 & \omega v_x & 0 & 0 \end{bmatrix}^T \quad (35)$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (36)$$

$$G(\theta) = \begin{bmatrix} 0 & 0 & 0 \\ K_{11}c_\theta & -K_{22}s_\theta & 0 \\ 0 & 0 & 0 \\ K_{11}s_\theta & K_{22}c_\theta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \quad (37)$$

$$d(\zeta) = \begin{bmatrix} sign(v_x c_\theta + v_y s_\theta) \\ sign(-v_x s_\theta + v_y c_\theta) \\ sign(\omega) \end{bmatrix} \quad (38)$$

Since we do not measure the whole state vector, we are interested in designing an observer which provides estimates $\hat{v}_x$, $\hat{v}_y$, $\hat{\omega}$ of v_x, v_y, ω. In order to take into account the presence of the uncertain term $G(\theta)d(\zeta)$ in (31) a proportional observer using quadratic boundedness is designed in the following subsection.

B. Takagi-Sugeno model of the mobile robot

Let us define:

$$\xi_1 = c_\theta^2 \qquad \xi_2 = c_\theta \qquad \xi_3 = s_\theta$$

we can easily obtain their minimum and maximum value.

$$\xi_1 \in [0, 1] \qquad \xi_2 \in [-1, 1] \qquad \xi_3 \in [-1, 1]$$

The matrix A defined in (33) becomes:

$$A(\xi) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & q_{1TS} & 0 & q_{2TS} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & q_{3TS} & 0 & q_{4TS} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & A_{33} \end{bmatrix} \quad (39)$$

$$q_{1TS} = A_{11}\xi_1 + A_{22}(1 - \xi_1) \qquad q_{2TS} = (A_{11} - A_{22})\xi_2\xi_3$$
$$q_{3TS} = (A_{11} - A_{22})\xi_2\xi_3 \qquad q_{4TS} = A_{11}(1 - \xi_1) + A_{22}\xi_1$$

and matrix G becomes

$$G(\xi) = \begin{bmatrix} 0 & 0 & 0 \\ K_{11}\xi_2 & -K_{22}\xi_3 & 0 \\ 0 & 0 & 0 \\ K_{11}\xi_3 & K_{22}\xi_2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & K_{33} \end{bmatrix} \quad (40)$$

Then, we can obtain a multiple model with the following structure

$$\dot{\zeta}(t) = \sum_{i=1}^{8} \mu_i(\xi(t))\big(A_i(\xi)\zeta(t) + B_i u(t) + G_i(\xi_k)d(t)\big) + f(\zeta)$$
$$\psi(t) = C\zeta(t) \quad (41)$$

C. Observer design

Let us consider the following observer for the system (41):

$$\dot{\hat{\zeta}} = \sum_{i=1}^{8} \mu_i(\xi(t))\big(A_i(\xi)\hat{\zeta}(t) + B_i(\theta)u(t) + L_i(\theta)(\psi - \hat{\psi})\big) + f(\hat{\zeta}) \tag{42}$$

where $\hat{\zeta} = \begin{bmatrix}\hat{x} & \hat{v}_x & \hat{y} & \hat{v}_y & \hat{\theta} & \hat{\omega}\end{bmatrix}^T \in \mathbb{R}^6$ is the estimated state, and the matrices $L(\theta)$ denote the proportional gain.

Let us define the state estimation error:

$$e = \zeta - \hat{\zeta}$$

Then, its dynamics is described by:

$$\dot{e}(t) = \sum_{i=1}^{8} \mu_i(\xi(t))\Big(\big(A_i(\theta) - L_i(\theta)C\big)e(t) + G_i(\xi)d(\zeta)\Big) + f_e(\zeta, \hat{\zeta}) \tag{43}$$

where $f_e(\zeta, \hat{\zeta})$ is a nonlinear function assumed to be Lipschitz with respect to the state $(\zeta, \hat{\zeta})$:

$$f_e(\zeta, \hat{\zeta}) = \begin{bmatrix}0 & -\omega v_y + \hat{\omega}\hat{v}_y & 0 & \omega v_x - \hat{\omega}\hat{v}_x & 0 & 0\end{bmatrix}^T \tag{44}$$

such that

$$||f_e(\zeta, \hat{\zeta})|| \leq \alpha||\zeta - \hat{\zeta}|| = \alpha||e|| \tag{45}$$

Note that, by assuming that due to the limited available power in the robot's motors, bounds for the elements of the matrix α can be calculated such in the following

$$\alpha = \begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & max(|\omega|,|\hat{\omega}|) & 0 & max(|v_y|,|\hat{v}_y|)\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & max(|\omega|,|\hat{\omega}|) & 0 & 0 & 0 & max(|v_x|,|\hat{v}_x|)\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

With this model structure, we can apply the LMI (22) to obtain the gains of the observer (42).

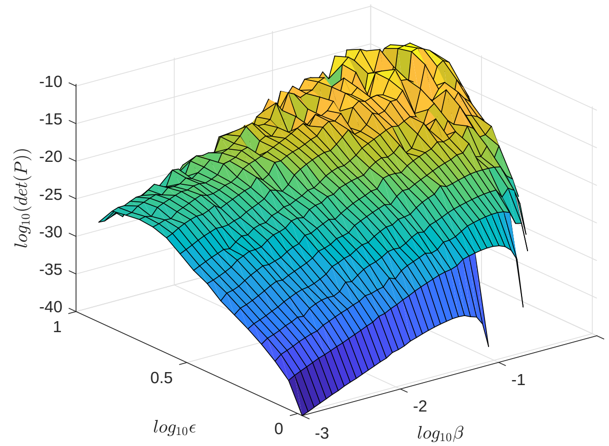

Fig. 2. Determinant of matrix P

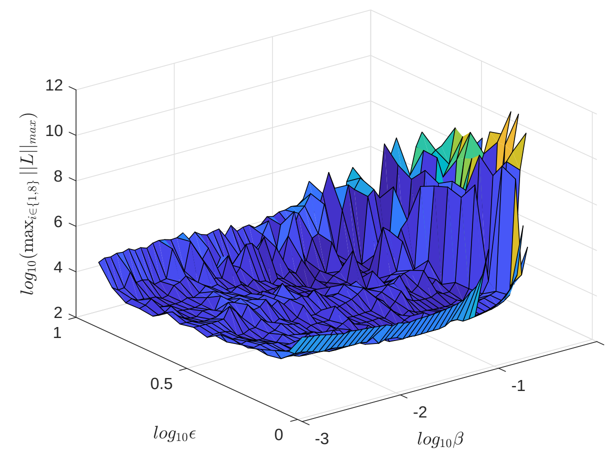

Fig. 3. Maximum value of the term in L

V. SIMULATION RESULTS

A. Simulation Conditions

For designing the observer providing the smallest state estimation error, we have to adjust adequately the value of ε and β. To do so, we will find the extreme of determinant of matrix P and maximum value of the term in L for the feasible region of ε and β. Figure 2 shows how the determinant logarithm of matrix P varies in terms of ε and β. On the other hand, Figure 3 allows to determine the maximum value of L in function of the same parameters. With a high observer gain, the observer converges to the system states very quickly. However, high observer gain leads to a peaking phenomenon in which initial estimator error can be prohibitively large [20], for this reason, we choose the optimal value of β and ε from determinant of matrix P.

From the figure above, we could use $\beta = 0.4942$, and $\varepsilon = 4.4984$ as the choice which forces the error to converge close to zero.

The initial condition and input are chosen as: $x(0) = \begin{bmatrix}2 & 0.1 & 1 & 0.2 & 1 & 0\end{bmatrix}^T$, $u = \begin{bmatrix}50 & 10 & -50 & -10\end{bmatrix}^T$ and the maximum velocities are set as:

$$\begin{bmatrix}max(|v_x|)\\ max(|v_y|)\\ max(|\omega|)\end{bmatrix} = \begin{bmatrix}max(|\hat{v}_x|)\\ max(|\hat{v}_y|)\\ max(|\hat{\omega}|)\end{bmatrix} = \begin{bmatrix}0.5\\ 0.5\\ 1\end{bmatrix}$$

B. Simulation results

Figure 4 shows respectively the estimation results of v_x, v_y and the angular speed ω. In the figure, the blue line depicts the results obtained proportional observer using the quadratic boundedness approach, which is the method introduced in this paper. The green line shows the observer designed using a standard Lyapunov asymptotical stability condition. On the other hand, the red line shows the real trajectory of the system.

From these figures, we can see that the estimation error obtained using the observer designed with quadratic boundedness increased the performance of the observer (in terms

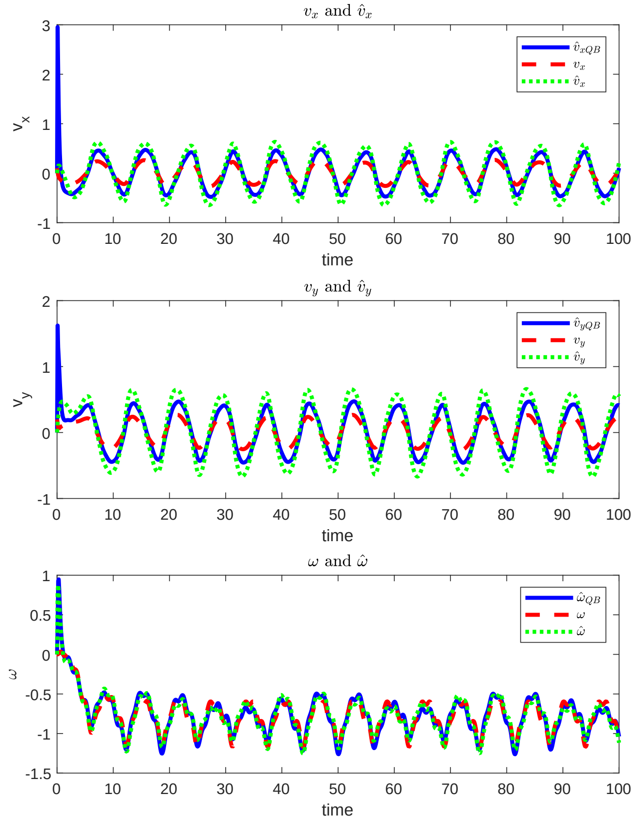

Fig. 4. Simulation results for state variable v_x, v_y and ω

of a smaller error) compared with the one without quadratic boundedness.

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we have proposed a method for a proportional observer design for a Takagi-Sugeno model with Lipschitz nonlinearities affected by a disturbance term. Simulation results using a four-wheeled mobile robot have been used to show the application of the proposed method to a real case study. The obtained results can be deemed as satisfactory, when compared to the ones obtained without applying the proposed quadratic boundedness design approach. Future work will aim at extending the proposed method to use the observed state to feed a controller for trajectory tracking in presence of disturbances. Additionally the systematic tuning for ε and β will be investigated.

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