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THE RATIONAL HOMOTOPY TYPE OF $(n-1)$-CONNECTED MANIFOLDS OF DIMENSION UP TO $5n-3$

DIARMUID CROWLEY AND JOHANNES NORDSTRÖM

Abstract. We define the Bianchi-Massey tensor of a topological space X to be a linear map $\mathcal{B} \to H^*(X)$, where $\mathcal{B}$ is a subquotient of $H^*(X)^{\otimes 4}$ determined by the algebra $H^*(X)$. We then prove that if M is a closed $(n-1)$-connected manifold of dimension at most $5n-3$ (and $n \geq 2$) then its rational homotopy type is determined by its cohomology algebra and Bianchi-Massey tensor, and that M is formal if and only if the Bianchi-Massey tensor vanishes.

We use the Bianchi-Massey tensor to show that there are many $(n-1)$-connected $(4n-1)$-manifolds that are not formal but have no non-zero Massey products, and to present a classification of simply-connected 7-manifolds up to finite ambiguity.

1. Introduction

This paper is concerned with the rational homotopy theory of closed $(n-1)$-connected manifolds of dimension up to $5n-3$ for $n \geq 2$. A continuous map $f : X \to Y$ is a rational homotopy equivalence if the induced maps $f_* : \pi_k(X) \otimes \mathbb{Q} \to \pi_k(Y) \otimes \mathbb{Q}$ are isomorphisms. If the spaces are simply-connected then this condition is equivalent to $f^* : H^*(Y) \to H^*(X)$ being an isomorphism of the cohomology algebras (throughout the paper, we use cohomology with rational coefficients unless explicitly stated otherwise). A further fundamental rational homotopy invariant is the Massey product structure on $H^*(X)$. In particular, non-trivial Massey products are an obstruction to X being *formal* in the sense of Sullivan [30] (see §3.4).

Miller [25] proved that, for $n \geq 2$, any closed $(n-1)$-connected manifold of dimension $\leq 4n-2$ is formal. On the other hand, it was well known that there are examples of non-formal closed $(n-1)$-connected manifolds of dimension $\geq 4n-1$ [23, 14]. Closed $(n-1)$-connected $(4n-1)$-manifolds therefore represents a borderline situation, the simplest non-trivial case from the point of view of rational homotopy. In this paper we are concerned with closed $(n-1)$-connected manifolds whose dimensions range from $4n-1$ to $5n-3$. In this range the only possible non-trivial Massey products are triple products, but these do not in general suffice for determining the rational homotopy type.

The starting point of this paper is the definition of what we term the *Bianchi-Massey tensor*. It is similar in style to the definition of Massey triple products, but unlike Massey products it is completely independent of auxiliary choices. The Bianchi-Massey tensor captures precisely the information needed to determine the rational homotopy type of a closed $(n-1)$-connected manifold M of dimension $m \leq 5n-3$, and in particular it is a *complete* obstruction to formality of such manifolds. Moreover, the Bianchi-Massey tensor can be computed directly from the cohomology ring of a coboundary W for M such that the restriction map $H^*(W) \to H^*(M)$ is surjective in degree $\leq m+1-3n$. This makes the determination of the rational homotopy type tractable for many examples.

Throughout the paper, all manifolds will be assumed to be connected. All graded algebras and rings will also be assumed to be connected, in the sense that the degree 0 part has rank 1, and moreover graded commutative.

1.1. The Bianchi-Massey tensor.

We will define the Bianchi-Massey tensor on the cohomology of a differential graded algebra. Let us first summarise some notation for various spaces of tensors, set up in further detail in §2.1. For a graded vector space V let $\mathcal{G}^k V$ denote the quotient of $V^{\otimes k}$ by relations of graded symmetry, *e.g.* if V is concentrated in odd degree then $\mathcal{G}^k V$ is the kth exterior power $\Lambda^k V$, while if V is concentrated in even degree then $\mathcal{G}^k V = P^k V$, the space of homogeneous degree k polynomials. $\mathcal{G}^k V$ is itself a graded vector space. There is a linear map $\mathcal{G}^2\mathcal{G}^2 V \to \mathcal{G}^4 V$ given by full graded symmetrisation, and we denote its kernel by $K[\mathcal{G}^2\mathcal{G}^2 V]$.

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Remark. If V is concentrated in odd degree, then $K[\mathcal{G}^2\mathcal{G}^2V] = K[P^2\Lambda^2V]$ can be identified with the subspace of $V^{\otimes 4}$ consisting of tensors that satisfy the symmetries of the Riemann curvature tensor, in particular the first Bianchi identity.

Given a (graded commutative) graded algebra H^*, the product $H^* \otimes H^* \to H^*$ by definition factors through a map $\mathcal{G}^2H^* \to H^*$. We denote its kernel by $E^* \subseteq \mathcal{G}^2H^*$, and let

$$\mathcal{B}^*(H^*) := \mathcal{G}^2E^* \cap K[\mathcal{G}^2\mathcal{G}^2H^*]. \tag{1}$$

When H^* is the cohomology algebra of a topological space or (graded commutative) differential graded algebra $\bullet$, we will use $\mathcal{B}^*(\bullet)$ as a short-hand for $\mathcal{B}^*(H^*(\bullet))$.

Given a differential graded algebra $(\mathcal{A}, d)$ over $\mathbb{Q}$, let $\mathcal{Z}^k := \ker d \subseteq \mathcal{A}^k$, the space of closed elements of degree k. Pick a right inverse $\alpha\colon H^*(\mathcal{A}) \to \mathcal{Z}^*$ for the projection to cohomology. This induces a map $\alpha^2\colon \mathcal{G}^2H^*(\mathcal{A}) \to \mathcal{Z}^*$, taking exact values precisely on E^*; there is a degree -1 linear map $\gamma : E^* \to \mathcal{A}^{*-1}$ such that $\alpha^2(e) = d\gamma(e)$ for $e \in E^*$. Now observe that the map $\mathcal{G}^2E^* \to \mathcal{A}^{*-1}$ induced by the graded symmetrisation of

$$E^* \otimes E^* \to \mathcal{A}^{*-1},\ e \otimes e' \mapsto \gamma(e)\alpha^2(e')$$

takes closed values on $\mathcal{B}^*(\mathcal{A}) \subseteq \mathcal{G}^2E^*$. It is easy to see that the induced degree -1 map

$$\mathcal{F} : \mathcal{B}^*(\mathcal{A}) \to H^{*-1}(\mathcal{A}) \tag{2}$$

is independent of the choice of γ. It is not as obvious, but nevertheless true, that $\mathcal{F}$ is also independent of the choice of α (Lemma 2.5).

Definition 1.1. The *Bianchi-Massey tensor* of the DGA $(\mathcal{A}, d)$ is the linear map defined in (2).

Any algebra homomorphism $H^* \to H'^*$ maps $E^* \to E'^*$ and hence $\mathcal{B}^*(H^*) \to \mathcal{B}^*(H'^*)$. In particular, if $\phi : \mathcal{A} \to \mathcal{A}'$ is a DGA homomorphism then the induced map $\phi_\#$ on cohomology maps $\mathcal{B}^*(\mathcal{A})$ to $\mathcal{B}^*(\mathcal{A}')$, and the Bianchi-Massey tensor is clearly functorial in the sense that the diagram below commutes:

$$\begin{array}{ccc} \mathcal{B}^*(\mathcal{A}) & \xrightarrow{\phi_\#} & \mathcal{B}^*(\mathcal{A}') \\ \downarrow{\scriptstyle\mathcal{F}} & & \downarrow{\scriptstyle\mathcal{F}'} \\ H^{*-1}(\mathcal{A}) & \xrightarrow{\phi_\#} & H^{*-1}(\mathcal{A}') \end{array}$$

The definition of formality therefore immediately implies that the Bianchi-Massey tensor of $\mathcal{A}$ must be trivial if $\mathcal{A}$ is formal.

1.2. **Determining the rational homotopy type.** Any simply-connected topological space X has a rationalisation $(X_\mathbb{Q}, f)$ (unique up to homotopy, see *e.g.* [15, Theorem 9.7]), which is a simply-connected space $X_\mathbb{Q}$ together with a map $f : X \to X_\mathbb{Q}$ such that $f_* : \pi_k(X) \otimes \mathbb{Q} \to \pi_k(X_\mathbb{Q})$ are isomorphisms. Two spaces X and X' are rationally homotopy equivalent if and only if their rationalisations are homotopy equivalent.

For any CW complex X, we can define the Bianchi-Massey tensor $\mathcal{F}_X\colon \mathcal{B}^*(X) \to H^{*-1}(X)$ in terms of the algebra $\Omega_{\mathrm{PL}}(X)$ of piecewise linear forms on X (see Sullivan [30, §7] or Félix–Halperin–Thomas [15, II 10(c)] for the definition of $\Omega_{\mathrm{PL}}(X)$). The theorem below identifies the Bianchi-Massey tensor as a complete obstruction to realising an isomorphism of the cohomology algebras of closed $(n{-}1)$-connected manifolds of dimension up to $5n{-}3$ by a rational homotopy equivalence. Such obstructions are studied more generally by Halperin and Stasheff [17].

Note that if M is $(n{-}1)$-connected, then $\mathcal{B}^*(M)$ is trivial in degrees $\leq 4n{-}1$. If $m \leq 5n{-}3$, then by Poincaré duality the only non-trivial part of $H^*(M)$ in degree $\geq 4n{-}2$ is $H^m(M)$, so the only non-trivial component of $\mathcal{F}$ is $\mathcal{B}^{m+1}(M) \to H^m(M)$. (However, even without any connectedness hypothesis it is still the case that the $\mathcal{B}^{m+1}(M) \to H^m(M)$ component of $\mathcal{F}$ determines $\mathcal{F}$ completely using Poincaré duality, see Lemma 2.8.)

Theorem 1.2. *For $n \geq 2$, the rational homotopy type of a closed simply-connected, rationally $(n{-}1)$-connected manifold M of dimension $m \leq 5n{-}3$ is determined by its cohomology algebra $H^*(M)$ and the top component of the Bianchi-Massey tensor, $\mathcal{F}_M\colon \mathcal{B}^{m+1}(M) \to H^m(M)$. More* *precisely, if M and M' are closed $(n-1)$-connected m-manifolds and $G\colon H^*(M) \to H^*(M')$ is an isomorphism then there exists a homotopy equivalence $g\colon M'_{\mathbb{Q}} \to M_{\mathbb{Q}}$ of the rationalisations such that $G = g^*$ if and only if the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{B}^{m+1}(M) & \xrightarrow{G} & \mathcal{B}^{m+1}(M') \\ \downarrow{\scriptstyle \mathcal{F}_M} & & \downarrow{\scriptstyle \mathcal{F}_{M'}} \\ H^m(M) & \xrightarrow{G} & H^m(M') \end{array}$$

We deduce Theorem 1.2 from Corollary 3.9, which characterises the minimal model of M in terms of $H^*(M)$ and the Bianchi-Massey tensor. In Corollary 3.10, that picture of the minimal model also lets us understand when M is formal.

Theorem 1.3. *For $n \geq 2$, a closed $(n-1)$-connected M of dimension $m \leq 5n-3$ is formal if and only if its Bianchi-Massey tensor $\mathcal{F}_M\colon \mathcal{B}^{m+1}(M) \to H^m(M)$ is trivial.*

In the special case of simply-connected 7-manifolds whose product $H^2(M) \times H^2(M) \to H^4(M)$ is trivial, this was already argued by Muñoz and Tralle [26, Theorem 14]. More generally, on the other hand, Kadeishvili [20] proved that one can define an A_∞-algebra structure on the cohomology of any topological space, whose equivalence class determines the rational homotopy type of the space. In particular, by [32, Proposition 7] the space is formal if the auxiliary choices in the definition can be made so that all the higher-order products vanish (a precise interpretation of the slogan that "a space is formal if and only if the Massey products vanish uniformly".)

One perspective on the Bianchi-Massey tensor is that it identifies the components of the A_∞-structure that are significant in the context of $(n-1)$-connected manifolds of dimension $\leq 5n-3$, see §2.5. Discarding the components that depend on choices is useful for understanding examples, and in the applications discussed below.

Remark 1.4. For manifolds that are not simply-connected, the Bianchi-Massey tensor is still a well-defined invariant, but the relation between the rational homotopy type and the minimal model is less straight-forward. When we say we consider $(n-1)$-connected manifolds, we will therefore always require $n \geq 2$.

1.3. **Realisation.** We next turn to the question of realisation of invariants of the above type. In §3.5 we apply a modification of Sullivan's methods for the realisation of rational models by closed simply-connected manifolds [30, Theorem 13.2] to obtain Theorem 1.5 below on realisation by $(n-1)$-connected manifolds. The proof of Theorem 1.5 relies on work Su [29].

Note that when the dimension is divisible by 4 there are additional *a priori* necessary conditions. If H^* is a $4k$-dimensional Poincaré duality algebra then H^{2k} inherits the structure of a non-degenerate quadratic form over $\mathbb{Q}$ once we choose a generator $\alpha \in (H^{4k})^\vee$. To realise the algebra as the rational cohomology of a closed oriented manifold, it is clearly necessary that we can choose α so that H^{2k} is isometric over $\mathbb{Q}$ to a non-singular integral form (equivalently to a diagonal form $\Sigma \pm x_i^2$). If we also want to prescribe rational Pontrjagin classes, then we must be able to choose α so that in addition the associated Pontrjagin numbers are integers which satisfy Hirzebruch's signature theorem and which are the Pontrjagin numbers of some closed $(n-1)$-connected manifold.

Theorem 1.5. *Let $n \geq 2$ and $m \leq 5n-2$. Let H^* be an m-dimensional rational Poincaré duality algebra that is $(n-1)$-connected,* i.e. *$H^0 = \mathbb{Q}$ and $H^k = 0$ for $0 < k < n$. Let $p_i \in H^{4i}$ for $4i \leq m$, and let $F\colon \mathcal{B}^{m+1}(H^*) \to H^m$ be a linear map. If $m = 4k$ assume in addition that there is an $\alpha \in (H^{4k})^\vee$ such that the following hold:*

(i) *the induced quadratic form on H^{2k} is isometric over $\mathbb{Q}$ to a sum of squares $\Sigma \pm x_i^2$;*
(ii) *(H^*, p_*, α) satisfies the signature theorem;*
(iii) *the numbers $\alpha(p_{i_1} \dots p_{i_r})$ for $\sum_j i_j = k$ are integers realised as the Pontrjagin numbers of some closed smooth $(n-1)$-connected $4k$-manifold.*

Then there is a closed smooth $(n-1)$-connected m-dimensional manifold M with rational Pontrjagin classes $p_*(M)$ and Bianchi-Massey tensor $\mathcal{F}_M$ such that

$$(H^*(M), p_*(M), \mathcal{F}_M) \cong (H^*, p_*, F).$$

Note that the above realisation statement applies in a wider dimension range than the preceding classification statements. However, for an $(n-1)$-connected Poincaré algebra H^* of dimension $m \geq 5n-1$ there can be obstructions to realising a map $\mathcal{B}^{m+1}(H^*) \to H^m$ as the Bianchi-Massey tensor of a rational Poincaré space.

Example 1.6. Let X be a connected rational Poincaré space of dimension $5n-1$ with $H^n(X) \cong \mathbb{Q}^3$ and $H^{2n}(X) \cong \mathbb{Q}^2$, with bases $\{x_1, x_2, x_3\} \subset H^n(X)$ and $\{y_1, y_2\} \subset H^{2n}(X)$ such that $x_1x_3 = y_1$, $x_2x_3 = y_2$ and $x_1^2 = x_2^2 = x_3^2 = x_1x_2 = 0$. Then $(x_1y_1)x_2^2 - (y_1x_2)(x_2x_1) - (x_2y_2)x_1^2 + (x_1y_2)(x_1x_2)$ is a non-zero element of $\mathcal{B}^{5n}(X)$, and it is explained in Example 3.7 that $\mathcal{F}$ must vanish on this element.

Thus as we increase the dimension of M, the rational homotopy type is harder to describe not only because we need to add data to capture the higher-order Massey products, but also because the constraints on realising $\mathcal{F}$ become more opaque. We do not explore these issues further in this paper.

We can also consider the problem of *integral* realisation of Bianchi-Massey tensors. For a torsion-free ring F^*, we can define $\mathcal{B}^*(F^*)$ entirely analogously to (1). When F^* is the free part of $H^*(M;\mathbb{Z})$, we abbreviate this to $\mathcal{B}^*(M;\mathbb{Z})$. We henceforth implicitly assume that all manifolds are oriented and define the "integral restriction" $\overline{\mathcal{F}}_M : \mathcal{B}^{m+1}(M;\mathbb{Z}) \to \mathbb{Q}$ as the composition $\mathcal{B}^{m+1}(M;\mathbb{Z}) \to \mathcal{B}^{m+1}(M) \to H^m(M) \to \mathbb{Q}$, where the second map is $\mathcal{F}_M$ and the third is integration over the fundamental class.

Remark 1.7. Treating the Bianchi-Massey tensor of a closed oriented m-manifold M as an element of $\mathcal{B}^*(M)^\vee$ is not natural in the context of Theorem 1.2—since the fundamental class is not invariant under rational homotopy equivalence—but it is in the context of diffeomorphism classification and/or cohomology with integer coefficients, where $\mathcal{F}_M$ and $\overline{\mathcal{F}}_M$ are equivalent.

Even under our simplifying connectivity assumptions, the question of the realisation of integral Poincaré duality rings by simply-connected manifolds is a hard problem. The questions that motivated our work really concern the lowest-dimensional non-trivial case, *i.e.* simply-connected 7-manifolds. However, the arguments say something at least in the more general critical case of $(n-1)$-connected $(4n-1)$-manifolds, so we state our integral realisation claim in that context.

We focus on the minimal integral data required to support the Bianchi-Massey tensor of an $(n-1)$-connected $(4n-1)$-manifold M. Let $TH^*(M;\mathbb{Z}) \subset H^*(M;\mathbb{Z})$ be the torsion subgroup, $FH^*(M;\mathbb{Z}) := H^*(M;\mathbb{Z})/TH^*(M;\mathbb{Z})$, and $FH^{n*} \subset FH^*$ the subring of elements in degree divisible by n. For an $(n-1)$-connected $(4n-1)$-manifold we have $FH^{n*} = H^0 \oplus H^n \oplus FH^{2n}$, and $\mathcal{B}^{4n}(FH^{n*}) = \mathcal{B}^{4n}(M;\mathbb{Z})$.

Theorem 1.8. *Let F^{n*} be a connected torsion-free graded ring concentrated in degrees 0, n and $2n$, and $\overline{\mathcal{F}} : \mathcal{B}^{4n}(F^{n*}) \to \mathbb{Q}$ a homomorphism. Then there exists some closed $(n-1)$-connected M^{4n-1} with an isomorphism $FH^{n*}(M;\mathbb{Z}) \cong F^{n*}$ that identifies $\overline{\mathcal{F}}_M$ with $\overline{\mathcal{F}}$.*

Moreover, $H^n(M;\mathbb{Z})$ can be taken to be torsion-free, i.e. *$H^n(M;\mathbb{Z}) \cong F^n$. In addition, if $\overline{\mathcal{F}}_M$ is integer-valued then $H^{2n}(M;\mathbb{Z})$ can be taken to be torsion-free too.*

In §4 we first describe how the Bianchi-Massey tensor of the boundary of a suitable compact manifold W can be computed in terms of the cohomology ring of W, and then construct the required M as a boundary of a $4n$-manifold. See Theorem 4.4 for more details on the cohomology ring of the $(4n-1)$-manifold produced by this argument.

1.4. **Classification up to finite ambiguity.** One of the motivations of Sullivan's work on rational homotopy theory is the principle that the rational homotopy type of a simply-connected manifold together with some characteristic class and integral data determines the diffeomorphism type up to finite ambiguity, *e.g.* [30, Theorem 13.1] classifies smooth manifolds up to finite ambiguity in terms of their rational homotopy type, rational Pontrjagin classes, bounds on torsion and certain integral lattice invariants.

Kreck and Triantafillou [23] work with less than the full rational homotopy type and present stronger results, *e.g.* [23, Theorem 2.2], where less of the lattice data is required explicitly, or can be replaced by parts of the integral cohomology ring. In the first instance, we are not too concerned about how little of the integral cohomology ring $H^*(M;\mathbb{Z})$ one needs to remember; in Proposition 3.12 we explain how to deduce the following result.

Theorem 1.9. *For $n \geq 2$ and $m \leq 5n-3$, the diffeomorphism type of a closed $(n-1)$-connected m-manifold M is determined up to finite ambiguity by its integral cohomology ring $H^*(M;\mathbb{Z})$, Pontrjagin classes $p_k(M) \in H^{4k}(M;\mathbb{Z})$ and Bianchi-Massey tensor $\overline{\mathcal{F}}_M : \mathcal{B}^{m+1}(M;\mathbb{Z}) \to \mathbb{Q}$;* i.e. *given such an M, the set of $(n-1)$-connected m-manifolds M' with a ring isomorphism $G : H^*(M';\mathbb{Z}) \to H^*(M;\mathbb{Z})$ such that $G(p_k(M')) = p_k(M)$, and $G^\# \overline{\mathcal{F}}_M = \overline{\mathcal{F}}_{M'}$ contains only finitely many diffeomorphism classes.*

For simply-connected 7-manifolds, we can simplify the invariants from Theorem 1.9 to align them more closely with the realisation statement Theorem 1.8. In this case Poincaré duality means that $H^*(M)$ is determined up to isomorphism by the rational cup square $P^2H^2(M) \to H^4(M)$. Hence given a bound on the size of $TH^*(M;\mathbb{Z})$, the full cohomology ring $H^*(M;\mathbb{Z})$ is determined up to finite ambiguity by the truncated ring $FH^{2*}(M;\mathbb{Z}) = H^0(M;\mathbb{Z}) \oplus H^2(M;\mathbb{Z}) \oplus FH^4(M;\mathbb{Z})$. Similarly $p_1(M) \in H^4(M;\mathbb{Z})$ is determined up to finite ambiguity by its image $\widetilde{p}_1(M) \in FH^4(M;\mathbb{Z})$, and by Theorem 1.9 we have

Corollary 1.10. *For all $N \geq 0$, closed* 1*-connected* 7*-manifolds M with $|TH^*(M;\mathbb{Z})| < N$ are classified up to finite ambiguity by $FH^{2*}(M;\mathbb{Z})$, $\widetilde{p}_1(M) \in FH^4(M;\mathbb{Z})$ and the Bianchi-Massey tensor $\overline{\mathcal{F}}_M : \mathcal{B}^8(M;\mathbb{Z}) \to \mathbb{Q}$.*

In §5.3 we build on Theorem 1.8 to also study which $\widetilde{p}_1$ are realised. Proposition 5.6 gives a satisfactory understanding of which integral invariants are realised by simply-connected spin 7-manifolds, and we then discuss directions for a complete classification of such manifolds.

1.5. **Non-formal manifolds with only trivial Massey products.** We will elaborate on the relationship between the Bianchi-Massey tensor and the Massey triple products of elements of $H^*(M)$ in §2.4. To simplify the notation, let us limit the present summary to the critical case of $(n-1)$-connected $(4n-1)$-manifolds, where the triple products only involve classes in $H^n(M)$. The triple product $\langle x, z, y\rangle$ of $x, y, z \in H^n(M)$ is defined if and only if $xz = yz = 0 \in H^{2n}(M)$. Considered as an element of $H^{3n-1}(M)$, the triple product in general depends on auxiliary choices of cocycle representatives, but $\langle x, z, y\rangle$ is well-defined considered as an element of the quotient space $H^{3n-1}(M)/(xH^{2n-1}(M) + yH^{2n-1}(M))$. Therefore if $w \in H^n(M)$ has $xw = yw = 0$ then we get a well-defined

$$\langle x, z, y\rangle w \in H^{4n-1}(M) \cong \mathbb{Q}. \tag{3}$$

If the cup product $c : \mathcal{G}^2H^n(M) \to H^{2n}(M)$ is trivial then (3) defines an element $q \in (H^n(M)^\vee)^{\otimes 4}$. It is graded anti-symmetric under swapping $x \leftrightarrow y$ or $z \leftrightarrow w$, symmetric under swapping both $x \leftrightarrow z$ and $y \leftrightarrow w$, and also satisfies the Bianchi identity. In §2.1, we explain that the space of such tensors ($\not{K}[\mathrm{Sym}^2\mathrm{Anti}^2H^n(M)^\vee]$ in the notation there) is naturally dual to $K[P^2\mathcal{G}^2H^n(M)]$. When c is trivial, the latter space equals $\mathcal{B}^{4n}(M)$, and $\mathcal{F}$ and q correspond under this duality. If c is not trivial, then the duality can still be used to determine all defined values of (3) from $\mathcal{F}$. In turn, we can use Poincaré duality to determine all Massey triple products from (3).

A basic point of this paper is that the $\mathcal{F}$ side of this duality is more useful when c is non-trivial: while the Bianchi-Massey tensor still determines the Massey triple products, the converse is not true in general. In §5.2 we study how the domain $\mathcal{B}^{4n}(M)$ of the Bianchi-Massey tensor of an $(n-1)$-connected $(4n-1)$-manifold, and the set of defined Massey triple products, depend on the kernel $E^{2n} \subseteq \mathcal{G}^2H^n(M)$ of c. Combined with the above realisation results, one finds that there are many examples of closed $(4n-1)$-manifolds with non-trivial Bianchi-Massey tensor but no non-trivial Massey products. The examples that are simplest to describe have $n = 2k+1$ odd.

Example 1.11. Combining Example 5.4 and Theorem 1.8, there exists for each $k \geq 1$ a non-formal $2k$-connected $(8k+3)$-manifold M with $H^{2k+1}(M;\mathbb{Z}) \cong \mathbb{Z}^4$ and $H^{4k+2}(M;\mathbb{Z}) \cong \mathbb{Z}^3$ such that $c : \Lambda^2H^{2k+1}(M;\mathbb{Z}) \to H^{4k+2}(M;\mathbb{Z})$ is equivalent to

$$(f^1 \wedge f^2 - f^3 \wedge f^4,\ f^1 \wedge f^3 + f^2 \wedge f^4,\ f^1 \wedge f^4 - f^2 \wedge f^3)$$

(where f^i is a basis for $H^{2k+1}(M;\mathbb{Z})^\vee$) even though all Massey products of a space with that cohomology ring are trivial—indeed, if $x, y \in H^{2k+1}(M)$ have $x \cup y = 0$ then x and y are linearly dependent.

Example 1.12. Combining Example 5.3 and Theorem 1.8, there exists for $k \geq 1$ a non-formal $(2k-1)$-connected $(8k-1)$-manifold M with $H^{2k}(M;\mathbb{Z}) \cong \mathbb{Z}^5$ and $H^{4k}(M;\mathbb{Z}) \cong \mathbb{Z}^3$ such that $c : P^2H^{2k}(M;\mathbb{Z}) \to H^{4k}(M;\mathbb{Z})$ is equivalent to

$$2\,(x_1x_4 + x_3x_5,\ x_2x_5 + x_3x_4,\ x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_4^2 + x_5^2),$$

(describing a homomorphism $P^2\mathbb{Z}^r \to \mathbb{Z}$ as a homogeneous quadratic polynomial with even cross terms) even though all Massey products of a space with that cohomology ring are trivial.

1.6. **Intrinsic formality.** A space X is said to be *intrinsically formal* if any space with cohomology algebra isomorphic to $H^*(X)$ is rationally homotopy equivalent to X. In this case the only defined triple Massey products are ones such as $\langle x, y, x\rangle$, which are trivial a priori. It is immediate from Theorem 1.3 that any $(n-1)$-connected manifold M of dimension $m \leq 5n-3$ whose cohomology algebra has $\mathcal{B}^{m+1}(M) = 0$ is intrinsically formal, while if $\mathcal{B}^{m+1}(M) \neq 0$ then Theorem 1.5 lets us realise $H^*(M)$ as the cohomology algebra of some non-formal space (or indeed of a closed manifold).

Corollary 1.13. *A closed $(n-1)$-connected manifold M of dimension $m \leq 5n-3$ is intrinsically formal if and only if $\mathcal{B}^{m+1}(M) = 0$.*

Cavalcanti [9, Theorem 4] showed that if M is a closed $(n-1)$-connected $(4n-1)$-manifold and there is an element $\varphi \in H^{2n-1}(M)$ such that $H^n(M) \to H^{3n-1}(M)$, $x \mapsto \varphi x$ is an isomorphism ("M has a hard Lefschetz property") then M is formal if $b_n(M) \leq 2$ and its Massey products vanish uniformly if $b_n(M) \leq 3$. As an illustration of our results we can make the following improvement.

Theorem 1.14. *Let M be a closed $(n-1)$-connected $(4n-1)$-manifold. If $b_n(M) \leq 3$ and there is a $\varphi \in H^{2n-1}(M)$ such that $H^n(M) \to H^{3n-1}(M)$, $x \mapsto \varphi x$ is an isomorphism then M is intrinsically formal.*

The algebraic claim that this relies on, Proposition 5.1, is essentially the same as the claim that the Ricci curvature of a manifold of dimension ≤ 3 determines the full Riemann curvature tensor.

Closed Kähler manifolds (which covers holonomy groups $U(n)$ and $SU(n)$) are formal due to the seminal work of Deligne, Griffiths, Morgan and Sullivan [13], while Amann and Kapovitch [3] proved that positive quaternionic-Kähler manifolds (which have holonomy $Sp(n)Sp(1)$) are always formal. In contrast, it is an open problem whether Riemannian manifolds with exceptional holonomy (*i.e.* G_2 or $Spin(7)$) need to be formal. The hard Lefschetz property holds in particular for closed Riemannian 7-manifolds with holonomy G_2, so Theorem 1.14 shows that if one wants to find G_2-manifolds that are not formal then one needs to look for ones with b_2 at least 4. One of our motivations for studying the Bianchi-Massey tensor has been to provide a tool for testing whether examples of G_2-manifolds are formal.

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2. Massey products and the Bianchi-Massey tensor

The main result of this section that will be used in the other parts of the paper is that the Bianchi-Massey tensor is well-defined, as claimed in the introduction. We then put the Bianchi-Massey tensor in context by discussing its relationship with various notions of triple products: classical Massey, A_∞, and a certain interpretation of uniform Massey triple products. In particular, we show that for a DGA whose cohomology satisfies m-dimensional Poincaré duality, the top degree part of the Bianchi-Massey tensor determines the rest of the Bianchi-Massey tensor, as well as the other notions of triple products. While this context is hopefully helpful for understanding the Bianchi-Massey tensor, it plays a limited role in the rest of the paper.

2.1. **Some trilinear and quadrilinear algebra.** Let us set up some notation for various tensor powers of a finite dimensional graded vector space V over $\mathbb{Q}$ and its dual $V^\vee$.

As in the introduction, we let $\mathcal{G}^k V$ denote the kth graded symmetric power of V, *i.e.* the quotient of $V^{\otimes k}$ by relations of graded commutativity. Dually, we let $\mathrm{Grad}^k V^\vee \subseteq (V^\vee)^{\otimes k}$ denote the subspace of graded symmetric tensors. If V is concentrated in even degree then $\mathrm{Grad}^k V^\vee$ is the same as $\mathrm{Sym}^k V^\vee$, the space of symmetric tensors, while if V is concentrated in odd degree it coincides with the alternating tensors $\mathrm{Alt}^k V^\vee$.

Similarly, let $\mathrm{Anti}^k V^\vee \subseteq (V^\vee)^{\otimes k}$ denote the subspace of graded antisymmetric tensors. By this we mean that under a transposition of two arguments, the sign change is the opposite of that for a graded symmetric tensor. Let $\not{\mathcal{G}}^k V$ denote the quotient of $V^{\otimes k}$ by the analogous relations of graded anticommutativity. So if V is concentrated in even degree then $\not{\mathcal{G}}^k V = \Lambda^k V$ and $\mathrm{Anti}^k V^\vee = \mathrm{Alt}^k V^\vee$, while if V is concentrated in odd degree then $\not{\mathcal{G}}^k V = P^k V$ and $\mathrm{Anti}^k V^\vee = \mathrm{Sym}^k V^\vee$. There are natural dualities between $\mathrm{Grad}^k V^\vee$ and $\mathcal{G}^k V$, and between $\mathrm{Anti}^k V^\vee$ and $\not{\mathcal{G}}^k V$.

Remark 2.1. The projections $\mathrm{Grad}^k V \to \mathcal{G}^k V$ and $\mathrm{Anti}^k V \to \not{\mathcal{G}}^k V$ are isomorphisms. However, that is not true if we replace V by a graded vector space over a field of arbitrary characteristic or by an abelian group. Because we will also consider tensors on cohomology groups with integer coefficients in §4, we are therefore somewhat fussy about distinguishing between subspaces and quotients.

For any space of tensors where a notion of graded symmetrisation or antisymmetrisation makes sense, let

- $K[\bullet]$ denote the kernel of full graded symmetrisation,
- $\not{K}[\bullet]$ denote the kernel of full graded antisymmetrisation and
- $C[\bullet]$ denote the kernel of graded symmetrisation under *even* permutations.

For example, $K[\mathrm{Grad}^2 V^\vee \otimes V^\vee]$ is the kernel of the symmetrisation map $\mathrm{Grad}^2 V^\vee \otimes V^\vee \to \mathrm{Grad}^3 V^\vee$, while $K[\mathcal{G}^2\mathcal{G}^2 V]$ is the kernel of $\mathcal{G}^2\mathcal{G}^2 V \to \mathcal{G}^4 V$. Similarly, $C[(V^\vee)^{\otimes 3}]$ consists of those $\alpha \in (V^\vee)^{\otimes 3}$ such that

$$\alpha(x,y,z) + (-1)^{k(i+j)}\alpha(z,x,y) + (-1)^{i(j+k)}\alpha(y,z,x) = 0$$

for any $x, y, z \in V$ of degrees i, j and k respectively. Note that $C[\bullet]$ contains both $K[\bullet]$ and $\not{K}[\bullet]$.

In the rest of this subsection we study some further features of tensors with various symmetries, which will be used primarily in §2.3–2.5. Consider the map

$$\begin{gathered} \phi : (V^\vee)^{\otimes 3} \to (V^\vee)^{\otimes 3} \\ (\phi\alpha)(x,y,z) := (-1)^{jk}\alpha(x,z,y) - (-1)^{ij+jk+ik}\alpha(z,y,x), \end{gathered} \tag{4}$$

for x, y, z of degrees i, j, k respectively. Here are some elementary observations.

Lemma 2.2.

(i) *The image of ϕ is contained in $C[(V^\vee)^{\otimes 3}]$, while $\ker\phi$ consists precisely of those tensors that are graded-invariant under cyclic permutations.*

(ii) *$\frac{1}{3}\phi^2$ is a projection onto $C[(V^\vee)^{\otimes 3}]$.*

(iii) *ϕ maps $\mathrm{Grad}^2 V^\vee \otimes V^\vee$ to $\mathrm{Anti}^2 V^\vee \otimes V^\vee$ and vice versa, with kernels $\mathrm{Grad}^3 V^\vee$ and $\mathrm{Anti}^3 V^\vee$ respectively.*

(iv) *ϕ maps $K[\mathrm{Grad}^2 V^\vee \otimes V^\vee]$ isomorphically to $\not{K}[\mathrm{Anti}^2 V^\vee \otimes V^\vee]$ and vice versa.*

Proof.

(i) Trivial.

(ii) Note that

$$(\phi^2\alpha)(x,y,z) = 2\alpha(x,y,z) - (-1)^{k(i+j)}\alpha(z,x,y) - (-1)^{i(j+k)}\alpha(y,z,x).$$

Therefore $\phi^2_{|C} = 3\mathrm{Id}$. Meanwhile (i) implies that $\ker\phi$ is a direct complement to $C[(V^\vee)^{\otimes 3}]$.

(iii) If $\alpha(y,x,z) = \epsilon(-1)^{ij}\alpha(x,y,z)$ (with $\epsilon = \pm 1$), then

$$\begin{aligned}(\phi\alpha)(y,x,z) &= (-1)^{ik}\alpha(y,z,x) - (-1)^{ij+ik+jk}\alpha(z,x,y)\\ &= \epsilon(-1)^{ik+jk}\alpha(z,y,x) - \epsilon(-1)^{ij+jk}\alpha(x,z,y) = -\epsilon(-1)^{ij}(\phi\alpha)(x,y,z).\end{aligned}$$

(iv) Follows from (ii) and (iii) and noting that $K[\mathrm{Grad}^2 V^\vee \otimes V^\vee] = C[\mathrm{Grad}^2 V^\vee \otimes V^\vee]$ and $\not K[\mathrm{Anti}^2 V^\vee \otimes V^\vee] = C[\mathrm{Anti}^2 V^\vee \otimes V^\vee]$. □

We can consider $\mathrm{Grad}^2\mathrm{Grad}^2 V^\vee$ as the subspace of $(V^\vee)^{\otimes 4}$ consisting of quadrilinear functions $\alpha(x,y,z,w)$ that are graded symmetric under swapping $x \leftrightarrow y$ or $z \leftrightarrow w$, and under swapping both $x \leftrightarrow z$ and $y \leftrightarrow w$. Note that

$$\begin{aligned}(\mathrm{Grad}^2 V^\vee \otimes (V^\vee)^{\otimes 2}) \,\cap\, \mathrm{Grad}^2((V^\vee)^{\otimes 2}) \;&=\; \mathrm{Grad}^2\mathrm{Grad}^2 V^\vee,\\ (\mathrm{Anti}^2 V^\vee \otimes (V^\vee)^{\otimes 2}) \,\cap\, \mathrm{Grad}^2((V^\vee)^{\otimes 2}) \;&=\; \mathrm{Grad}^2\mathrm{Anti}^2 V^\vee.\end{aligned}$$

Therefore, if we set

$$\psi = \phi \otimes \mathrm{Id} : (V^\vee)^{\otimes 4} \to (V^\vee)^{\otimes 4}, \tag{5}$$

then Lemma 2.2 implies that ψ maps $\mathrm{Grad}^2\mathrm{Grad}^2 V^\vee$ to $\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee$ and vice versa. The images are precisely $\not K[\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee]$ and $K[\mathrm{Grad}^2\mathrm{Grad}^2 V^\vee]$ respectively, as can be seen from Lemma 2.2 and that

$$\begin{aligned}K[\mathrm{Grad}^2\mathrm{Grad}^2 V^\vee] \;&=\; \mathrm{Grad}^2\mathrm{Grad}^2 V^\vee \,\cap\, K[\mathrm{Grad}^2 V^\vee \otimes V^\vee] \otimes V^\vee,\\ \not K[\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee] \;&=\; \mathrm{Grad}^2\mathrm{Anti}^2 V^\vee \,\cap\, K[\mathrm{Anti}^2 V^\vee \otimes V^\vee] \otimes V^\vee.\end{aligned}$$

We obtain a pair of naturally dual exact sequences:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \mathrm{Grad}^4 V^\vee & \longrightarrow & \mathrm{Grad}^2\mathrm{Grad}^2 V^\vee & \xrightarrow{\psi} & \mathrm{Grad}^2\mathrm{Anti}^2 V^\vee & \longrightarrow & \mathrm{Anti}^4 V^\vee & \longrightarrow 0\\ 0 \longleftarrow & \mathcal{G}^4 V & \longleftarrow & \mathcal{G}^2\mathcal{G}^2 V & \xleftarrow{\psi^\vee} & \mathcal{G}^2\not{\mathcal{G}}^2 V & \longleftarrow & \not{\mathcal{G}}^4 V & \longleftarrow 0\end{array} \tag{6}$$

Remark 2.3. This shows in particular that there is a natural perfect pairing

$$K[\mathcal{G}^2\mathcal{G}^2 V] \times \not K[\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee] \to \mathbb{Q},$$

and $K[\mathcal{G}^2\mathcal{G}^2 V]$ and $\not K[\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee]$ can both be regarded as measuring a "symmetry defect" of elements of $\mathrm{Grad}^2\mathrm{Grad}^2 V^\vee$.

Remark 2.4. For $\alpha, \beta \in \mathrm{Grad}^2 V^\vee$, we can define $\alpha \owedge \beta \in \not K[\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee]$ by

$$\begin{aligned}(\alpha \owedge \beta)(x,y,z,w) \;:=\; & \quad (-1)^{jk}\alpha(x,z)\beta(y,w) - (-1)^{ij+jk+ik}\alpha(z,y)\beta(x,w)\\ & -(-1)^{ij+j\ell+k\ell}\alpha(x,w)\beta(z,y) + (-1)^{ij+j\ell+i\ell}\alpha(y,w)\beta(x,z).\end{aligned}$$

This induces a linear map $\mathrm{Grad}^2\mathrm{Grad}^2 V^\vee \to \not K[\mathrm{Grad}^2\mathrm{Anti}^2 V^\vee]$, clearly equal to the restriction of ψ. When V is concentrated in even degree, so that this is a map $\mathrm{Sym}^2\mathrm{Sym}^2 V^\vee \to \not K[\mathrm{Sym}^2\mathrm{Alt}^2 V^\vee]$, Besse [6, Definition 1.110] calls $\owedge$ the *Kulkarni-Nomizu product*. Some features of this algebra are familiar from the context of Riemannian geometry, *e.g.* that $\not K[\mathrm{Sym}^2\mathrm{Alt}^2 V^\vee]$ has dimension $\frac{1}{12}(\dim V)^2((\dim V)^2 - 1)$.

2.2. **Well-definedness of the Bianchi-Massey tensor.** As in the introduction, for a graded ring H^*, we let $\mathcal{B}^*(H^*) := K[\mathcal{G}^2 E^*] \subseteq K[\mathcal{G}^2\mathcal{G}^2 H^*]$ where E^* the kernel of the product map $c \colon \mathcal{G}^2 H^* \to H^*$, and abbreviate $\mathcal{B}^*(H^*(\bullet))$ to $\mathcal{B}^*(\bullet)$ when $\bullet$ is a DGA or a topological space.

In the proof that the Bianchi-Massey tensor is well-defined, we will use the following notation. For a graded vector space V^*, a DGA $\mathcal{A}$ and linear maps $\rho : V^* \to \mathcal{A}^{*+r}$, $\sigma : V^* \to \mathcal{A}^{*+s}$, let $\rho{\cdot}\sigma : \mathcal{G}^2 V^* \to \mathcal{A}$ be the degree $r{+}s$ linear map induced by the graded commutative (with respect to the grading on V^*) bilinear map

$$\begin{aligned}V^p \times V^q \to \mathcal{A}^{p+q+r+s},\ (v,w) \mapsto\ & \tfrac{1}{2}\big((-1)^{ps}\rho(v)\sigma(w) + (-1)^{(p+s)q}\rho(w)\sigma(v)\big)\\ =\ & \tfrac{1}{2}\big((-1)^{ps}\rho(v)\sigma(w) + (-1)^{r(p+s)}\sigma(v)\rho(w)\big).\end{aligned} \tag{7}$$

Note that $\rho\cdot\sigma = (-1)^{rs}\sigma\cdot\rho$, and the Leibniz rule holds in the sense that

$$d(\rho\cdot\sigma) = (d\rho)\cdot\sigma + (-1)^r\rho\cdot d\sigma$$

as degree $r+s+1$ maps $\mathcal{G}^2V \to \mathcal{A}$. Further, for ρ_i of degree r_i and $v_i \in V^{q_i}$

$$\begin{aligned}\big((\rho_1\cdot\rho_2)\cdot(\rho_3\cdot\rho_4)\big)\big((v_1v_2)(v_3v_4)\big) &= \tfrac{1}{8}\sum_{g\in G}(-1)^{a(g)}\rho_1(v_{g(1)})\rho_2(v_{g(2)})\rho_3(v_{g(3)})\rho_4(v_{g(4)})\\ &= \tfrac{1}{8}\sum_{g\in G}(-1)^{b(g)}\rho_{g(1)}(v_1)\rho_{g(2)}(v_2)\rho_{g(3)}(v_3)\rho_{g(4)}(v_4),\end{aligned}$$

where $G \subset S_4$ is the wreath product S_2 wr S_2 (an order 8 subgroup of S_4), and

$$\begin{aligned} a(g) &= \sum q_{g(i)}r_j \ + \sum_{g(i)>g(j)} q_iq_j,\\ b(g) &= \sum q_ir_{g(j)} \ + \sum_{g(i)>g(j)} r_ir_j,\end{aligned}$$

summing over $1 \leq i < j \leq 4$. That implies in particular that as long as at most one of the r_i is odd,

$$(\rho_1\cdot\rho_2)\cdot(\rho_3\cdot\rho_4) + (\rho_1\cdot\rho_3)\cdot(\rho_2\cdot\rho_4) + (\rho_1\cdot\rho_4)\cdot(\rho_2\cdot\rho_3) \ : \ \mathcal{G}^2\mathcal{G}^2V \to \mathcal{A} \tag{8}$$

is in fact fully graded symmetric, *i.e.* it factors through the graded symmetrisation $\mathcal{G}^2\mathcal{G}^2V \to \mathcal{G}^4V$.

Lemma 2.5. *Let $(\mathcal{A}, d)$ be a DGA, and $\mathcal{Z}^* := \ker d$. Choose a right inverse $\alpha\colon H^*(\mathcal{A}) \to \mathcal{Z}^*$ for the projection to cohomology, and a linear map $\gamma\colon E^* \to \mathcal{A}^{*-1}$ such that $d\gamma = \alpha^2$ on E^*. Then the linear map*

$$\gamma\cdot\alpha^2 : \mathcal{G}^2E^* \to \mathcal{A}^{*-1}$$

takes values in $\mathcal{Z}^{*-1}$ on $\mathcal{B}^*(\mathcal{A}) := \mathcal{B}^*(H^*(\mathcal{A}))$, and the induced map

$$\mathcal{F} : \mathcal{B}^*(\mathcal{A}) \to H^{*-1}(\mathcal{A})$$

is independent of the choice of γ and α.

Proof. Note that $d(\gamma\cdot\alpha^2) : \mathcal{G}^2E^* \to \mathcal{A}^*$ is the restriction of $(\alpha^2)^2 : \mathcal{G}^2\mathcal{G}^2H^*(\mathcal{A}) \to \mathcal{A}^*$, which factors through $\mathcal{G}^4H^*(\mathcal{A})$. Therefore $d(\gamma\cdot\alpha^2)$ vanishes on the intersection of $\mathcal{G}^2E^*$ with the kernel of $\mathcal{G}^2\mathcal{G}^2H^*(\mathcal{A}) \to \mathcal{G}^4H^*(\mathcal{A})$, which is $\mathcal{B}^*(\mathcal{A})$ by definition. Hence $\gamma\cdot\alpha^2$ maps $\mathcal{B}^*(\mathcal{A}) \to \mathcal{Z}^{*-1}$ as claimed.

Now consider replacing γ by $\gamma_\bullet = \gamma+\eta$, for some $\eta : E^* \to \mathcal{Z}^{*-1}$. Then $\gamma_\bullet\cdot\alpha^2 - \gamma\cdot\alpha^2 = \eta\cdot\alpha^2 = -d(\eta\cdot\gamma)$ takes exact values on all of $\mathcal{G}^2E^*$, so certainly $\mathcal{F}$ is independent of the choice of γ, given the choice of α.

Now consider replacing α by $\alpha_\bullet := \alpha + d\beta$ for some arbitrary linear map $\beta : H^*(\mathcal{A}) \to \mathcal{A}^{*-1}$. Then

$$\alpha_\bullet^2 - \alpha^2 = d\beta\cdot(2\alpha + d\beta)$$

on $\mathcal{G}^2H^*(\mathcal{A})$, so a possible choice of $\gamma_\bullet$ such that $d\gamma_\bullet = \alpha_\bullet^2$ is

$$\gamma_\bullet := \gamma + \beta\cdot(2\alpha + d\beta).$$

To make the next equation equation unambiguous, we suppress the $\cdot$ in the products that are to be evaluated first. On $\mathcal{G}^2E^*$ we have

$$\begin{aligned} &\gamma_\bullet\cdot\alpha_\bullet^2 - \gamma\cdot\alpha^2 + d\big(\gamma\cdot\beta(2\alpha+d\beta) + \tfrac{2}{3}\beta\alpha\cdot\beta d\beta\big)\\ &= \beta(2\alpha+d\beta)\cdot\alpha^2 + \gamma\cdot(d\beta)(2\alpha+d\beta) + \beta(2\alpha+d\beta)\cdot(d\beta)(2\alpha+d\beta)\\ &\quad + d\gamma\cdot\beta(2\alpha+d\beta) - \gamma\cdot(d\beta)(2\alpha+d\beta) + \tfrac{2}{3}(d\beta)\alpha\cdot\beta d\beta - \tfrac{2}{3}\beta\alpha\cdot(d\beta)^2\\ &= 2\alpha^2\cdot\beta(2\alpha+d\beta) + \beta(2\alpha+d\beta)\cdot(d\beta)(2\alpha+d\beta) + \tfrac{2}{3}\beta d\beta\cdot(d\beta)\alpha - \tfrac{2}{3}\beta\alpha\cdot(d\beta)^2\\ &= 4\alpha^2\cdot\beta\alpha + 2\alpha^2\cdot\beta d\beta + 4\beta\alpha\cdot(d\beta)\alpha + \tfrac{8}{3}\beta d\beta\cdot(d\beta)\alpha + \tfrac{4}{3}\beta\alpha\cdot(d\beta)^2 + \beta d\beta\cdot(d\beta)^2.\end{aligned}$$

The right hand side factors through a map $\mathcal{G}^4H^*(\mathcal{A}) \to \mathcal{A}^{*-1}$: the first term, the sum of the second and third terms, the sum of the fourth and fifth terms, and the sixth term are each of the form

(8), and hence fully graded-symmetric. Thus the restriction of $\gamma_\bullet\alpha_\bullet^2 - \gamma\alpha^2$ to $\mathcal{B}^*(\mathcal{A})$ takes exact values, so $\mathcal{F}$ is independent of the choice of α too. □

Remark 2.6. An alternative way to describe the Bianchi-Massey tensor is to argue that the (unsymmetrised) map $\gamma\alpha^2 : E^* \otimes E^* \to \mathcal{A}^{*-1},\ e \otimes e' \mapsto \gamma(e)\alpha^2(e')$ induces a well-defined map $K[E^* \otimes E^*] \to H^{*-1}(\mathcal{A})$. But $\mathrm{Anti}^2 E^* \subseteq K[E^* \otimes E^*]$, and the restriction of $\gamma\alpha^2$ to $\mathrm{Anti}^2 E^*$ takes exact values. Thus the induced map descends to $K[E^* \otimes E^*]/\mathrm{Anti}^2 E^*$, which is naturally isomorphic to $\mathcal{B}^*(\mathcal{A})$.

2.3. **Uniform Massey triple products.** As before, let $\mathcal{A}^*$ be a DGA, and $\mathcal{Z}^*$ the subspace of closed elements. Let H^* denote the cohomology of $\mathcal{A}^*$, and $E^* \subseteq \mathcal{G}^2 H^*$ the kernel of the product $\mathcal{G}^2 H^* \to H^*$.

Consider $K[E^* \otimes H^*]$, *i.e.* the kernel of the full symmetrisation $E^* \otimes H^* \to \mathcal{G}^3 H^*$. Given choices of $\alpha : H^* \to \mathcal{Z}^*$ and $\gamma : E^* \to \mathcal{A}^{*-1}$ such that $d\gamma = \alpha^2$ as before, we get an induced map $\gamma\alpha : E^* \otimes H^* \to \mathcal{A}^{*-1}$ whose restriction to $K[E^* \otimes H^*]$ takes closed values. We call the resulting map

$$\mathcal{T} : K[E^* \otimes H^*]^* \to H^{*-1} \tag{9}$$

the *uniform Massey triple product* of $\mathcal{A}$. Unlike the Bianchi-Massey tensor, it does depend on the choices of α and γ, in the following way.

Given different choices α', γ', pick $\beta : H^* \to \mathcal{A}^{*-1}$ such that $d\beta = \alpha' - \alpha$. Then $\gamma' - \gamma - \beta(\alpha' + \alpha)$ maps $E^* \to \mathcal{Z}^{*-1}$. Let δ_β be the induced map $E^* \to H^{*-1}$ (which does depend on the choice of β, but in a way that is not significant). Now

$$\mathcal{T}' - \mathcal{T} : K[E^* \otimes H^*] \to H^{*-1}$$

is the restriction of the map $\delta_\beta \mathrm{Id} : E^* \otimes H^* \to H^{*-1},\ e \otimes x \mapsto \delta_\beta(e)x$ (for any choice of β). Thus $\mathcal{T}$ is well-defined precisely modulo restrictions to $K[E^* \otimes H^*]$ of maps of the form $\delta\mathrm{Id} : E^* \times H^* \to H^{*-1}$ for $\delta : E^* \to H^{*-1}$.

Next we prove that $\mathcal{T}$ is equivalent to $\mathcal{F}$ for Poincaré DGAs.

Definition 2.7. Let H^* be a finite dimensional graded commutative algebra over $\mathbb{Q}$. We call $\alpha \in (H^m)^\vee$ a *Poincaré class* if the linear map

$$\alpha\cap : H^i \to (H^{m-i})^\vee,\ x \mapsto (y \mapsto \alpha(xy))$$

is an isomorphism for all i. We say that H^* is m*-dimensional Poincaré* if such an α exists.

A DGA $\mathcal{A}$ is m-dimensional Poincaré if its cohomology is.

Lemma 2.8. (i) *For any DGA $\mathcal{A}$, $\mathcal{T}$ determines $\mathcal{F}$.*

(ii) *If $\mathcal{A}$ is m-dimensional Poincaré, then the top component $\mathcal{F} : \mathcal{B}^{m+1}(\mathcal{A}) \to H^m(\mathcal{A})$ determines $\mathcal{T}$ (and hence also the other components of $\mathcal{F}$).*

Proof. We can control all the relevant maps in terms of

$$\mu : K[E^* \otimes H^* \otimes H^*] \to H^{*-1}$$

defined by restricting $E^* \otimes H^* \otimes H^* \to \mathcal{A}^{*-1},\ e \otimes x \otimes y \mapsto \gamma(e)\alpha(x)\alpha(y)$ to $K[E^* \otimes H^* \otimes H^*]$ and observing that the restriction takes values in $\mathcal{Z}^{*-1}$. $K[E \otimes H^* \otimes H^*]$ contains $K[E^* \otimes H^*] \otimes H^*$, and the restriction of μ to that equals $\mathcal{T}\mathrm{Id}$. This is determined by $\mathcal{T}$, and if $\mathcal{A}$ is m-dimensional Poincaré then the degree m component of $\mathcal{T}\mathrm{Id}$ also determines $\mathcal{T}$.

The map μ factors through the projection $K[E^* \otimes H^* \otimes H^*] \to K[E^* \otimes \mathcal{G}^2 H^*]$. The restriction to $K[E^* \otimes E^*]$ in turn factors through the projection to $\mathcal{B}^*(H^*)$, where it induces $\mathcal{F}$ (see Remark 2.6).

Now observe that the Bianchi identity for elements of $K[\mathrm{Grad}^2 \mathcal{G}^2 H^*] \subset K[\mathcal{G}^2 H^* \otimes \mathcal{G}^2 H^*]$ means that this space is contained in the image of $K[\mathcal{G}^2 H^* \otimes H^*] \otimes H^*$. Hence the image of $K[E^* \otimes H^*] \otimes H^*$ in $K[E^* \otimes \mathcal{G}^2 H^*]$ contains $K[\mathrm{Grad}^2 E^*]$, which maps onto $\mathcal{B}^*(H^*)$. This proves (i).

For (ii), we need to prove that if $\mathcal{F} : \mathcal{B}^{m+1}(\mathcal{A}) \to H^m(\mathcal{A})$ is trivial, then the choices in the definition of $\mathcal{T}$ can be made so that $\mathcal{T}$ vanishes too. It suffices to show that if the restriction of μ to the degree $m+1$ part of $K[E^* \otimes E^*]$ vanishes, then we can make the choices so that μ vanishes on all of the degree $m+1$ part of $K[E^* \otimes \mathcal{G}^2 H^*]$. Now let $D^* \subset \mathcal{G}^2 H^*$ be a direct complement to E^*. Then $E^* \otimes \mathcal{G}^2 H^* = E^* \otimes E^* \ \oplus\ E^* \otimes D^*$. Let $p : E^* \otimes \mathcal{G}^2 H^* \to E^* \otimes D^*$ be the corresponding projection.

Now, if we change the choice of γ to $\gamma' = \gamma + \eta$ for some $\eta : E^* \to \mathcal{Z}^{*-1}$, then the map $\mu' - \mu : K[E^* \otimes \mathcal{G}^2 H^*] \to H^{*-1}$ is the restriction of the degree -1 map $E^* \otimes \mathcal{G}^2 H^* \to H^{*-1}$ induced by $[\eta] \times c$. The latter vanishes on $E^* \otimes E^*$, while by Poincaré duality any map from the degree $m{+}1$ part of $E^* \otimes D^*$ to $H^m(\mathcal{A})$ can realised this way for some choice of η.

Since the kernel of the restriction $p : K[E^* \otimes \mathcal{G}^2 H^*] \to E^* \otimes D^*$ is precisely $K[E^* \otimes E^*]$, it follows that by changing the choice of γ, we can ensure that μ is zero on the degree $m+1$ part of $K[E^* \otimes \mathcal{G}^2 H^*]$ provided that the restriction to $K[E^* \otimes E^*]$ vanishes. □

2.4. **Massey triple products.** Let us recall the definition of Massey triple products. Let $(\mathcal{A}, d)$ be a DGA, and $\mathcal{Z} := \ker d$. Suppose $x \in H^i(\mathcal{A})$, $y \in H^j(\mathcal{A})$ and $z \in H^k(\mathcal{A})$, such that $xy = yz = 0$. Choose representatives $\alpha_x, \alpha_y, \alpha_z \in \mathcal{Z}^*$. Then $\alpha_x\alpha_y$ and $\alpha_y\alpha_z$ are exact, say $d\gamma_{xy}$ and $d\gamma_{yz}$ respectively. Then $\gamma_{xy}\alpha_z - (-1)^i\alpha_x\gamma_{yz} \in \mathcal{A}^{i+j+k-1}$ is closed, so represents a class in $H^{i+j+k-1}(\mathcal{A})$. The choices of αs and γs can change that class by elements of $xH^{j+k-1}(\mathcal{A}) + zH^{i+j-1}(\mathcal{A})$, but the image

$$\langle x, y, z\rangle \ \in \ \frac{H^{i+j+k-1}(\mathcal{A})}{xH^{j+k-1}(\mathcal{A}) + zH^{i+j-1}(\mathcal{A})}$$

is well-defined, and that is the Massey triple product.

Note that $(xy)z - (-1)^{i(j+k)}(yz)x \in K[E^* \otimes H^*]$, and is mapped to $\langle x, y, z\rangle$ by the uniform triple product $\mathcal{T}$ from (9). Thus the Massey triple products are determined by $\mathcal{T}$, as one would expect. By Lemma 2.8, the Massey triple products of a Poincaré duality DGA are thus also determined by the top component of the Bianchi-Massey tensor $\mathcal{F}$. However, it may be illuminating to consider how to recover the Massey triple products from $\mathcal{F}$ more directly.

If $x, y, z, w \in H^*(\mathcal{A})$ of degrees i, j, k and ℓ respectively and $xz = yz = xw = yw = 0 \in H^*(\mathcal{A})$, then

$$q(x, y, z, w) := (-1)^{jk}\langle x, z, y\rangle w \in H^{i+j+k+\ell-1}(\mathcal{A}) \tag{10}$$

is well-defined. This approach to eliminating the indeterminacy of the triple products has been exploited by Taylor [31].

The next elementary lemma lets us recover the defined values of (10) from the Bianchi-Massey tensor. Write the product $H^* \times H^* \to \mathcal{G}^2 H^*$ as $(x, y) \mapsto xy$.

Lemma 2.9. *If $x, y, z, w \in H^*(\mathcal{A})$ such that $xz = xw = yz = yw \in E^*$ (*i.e.* the products in $H^*(\mathcal{A})$ vanish) then*

$$v := (xz)(yw) - (-1)^{jk+j\ell+k\ell}(xw)(yz) \in \mathcal{B}^*(\mathcal{A}), \tag{11}$$

and

$$\mathcal{F}(v) = \langle x, z, y\rangle w. \tag{12}$$

Definition 2.10. We call elements of $\mathcal{B}^*(\mathcal{A})$ of the form (11) *ordinary.*

Lemma 2.11 (*cf.* Hepworth [19, Lemma 3.1.4])**.**

(i) *If $x, y, z, w \in H^*(\mathcal{A})$ such that $xz = yz = xw = yw = 0 \in H^*(\mathcal{A})$, then*

$$\begin{aligned} q(x, y, z, w) \ &= \ -(-1)^{ij}q(y, x, z, w) \\ = -(-1)^{k\ell}q(x, y, w, z) \ &= \ (-1)^{(i+j)(k+\ell)}q(z, w, x, y) \end{aligned}$$

(ii) *If in addition $xy = zw = 0 \in H^*(\mathcal{A})$ then*

$$q(x, y, z, w) + (-1)^{k(i+j)}q(z, x, y, w) + (-1)^{i(j+k)}q(y, z, x, w) = 0.$$

If the product $H^*(\mathcal{A}) \times H^*(\mathcal{A}) \to H^*(\mathcal{A})$ is trivial in non-zero degrees then q induces a linear map $H^{>0}(\mathcal{A})^{\otimes 4} \to H^{*-1}(\mathcal{A})$. Equivalently, if $\alpha \in H^m(\mathcal{A})^\vee$, then $\alpha \circ q$ is in the degree $m{+}1$ part of $(H^{>0}(\mathcal{A})^\vee)^{\otimes 4}$. Lemma 2.11(i) means that $\alpha \circ q$ is graded anti-symmetric under swapping $x \leftrightarrow y$ or $z \leftrightarrow w$, and also symmetric under swapping both $x \leftrightarrow z$ and $y \leftrightarrow w$, so $\alpha \circ q \in \mathrm{Grad}^2\mathrm{Anti}^2 H^*(\mathcal{A})^\vee$. Moreover, (ii) says that the Bianchi identity holds, so $\alpha \circ q$ in fact belongs to the degree $m{+}1$ part of $K\!\!\!/[\mathrm{Grad}^2\mathrm{Anti}^2 H^{>0}(\mathcal{A})^\vee]$.

Now suppose that $\mathcal{A}$ is m-dimensional Poincaré (*e.g.* that $\mathcal{A}$ is the DGA of piecewise linear forms on a closed oriented m-manifold M). Then for $x \in H^i(\mathcal{A})$, $y \in H^j(\mathcal{A})$ the annihilator of $xH^{j+k-1}(\mathcal{A}) + yH^{i+k-1}(\mathcal{A}) \subseteq H^{i+j+k-1}(\mathcal{A})$ is precisely

$$\{w \in H^{m+1-i-j-k}(\mathcal{A}) : xw = yw = 0 \in H^*(\mathcal{A})\}.$$

Hence for $x \in H^i(\mathcal{A})$, $y \in H^j(\mathcal{A})$, $z \in H^k(\mathcal{A})$ such that $xz = yz = 0$ the triple product $\langle x, z, y\rangle$ is completely determined by the values of $q(x, y, z, w) \in H^m(\mathcal{A}) \cong \mathbb{Q}$ for $w \in H^{m+1-i-j-k}(\mathcal{A})$ such that $xw = yw = 0$, and hence by the Bianchi-Massey tensor.

On the other hand, suppose that $N \subset H^*(\mathcal{A})$ is a subspace such that the product $N \times N \to H^*(\mathcal{A})$ is trivial (so $\mathcal{G}^2 N \subseteq E$) and that $\mathcal{B}^{m+1}(\mathcal{A})$ is the degree $m+1$ part of $K[\mathcal{G}^2\mathcal{G}^2N]$; *e.g.* if $\mathcal{A}$ is $(n-1)$-connected with $H^n(\mathcal{A}) \times H^n(\mathcal{A}) \to H^{2n}(\mathcal{A})$ trivial and $m = 4n-1$ then we could take $N = H^n(\mathcal{A})$. Then (12) means that $\alpha \circ \mathcal{F}$ can be recovered from $\alpha \circ q$ using the duality $\not{K}[\mathrm{Grad}^2\mathrm{Anti}^2 N^\vee] \cong K[\mathcal{G}^2\mathcal{G}^2N]^\vee$ from Remark 2.3. In particular, if $\mathcal{A}$ is in addition m-dimensional Poincaré, then q determines the top component of $\mathcal{F}$, and hence the rest of $\mathcal{F}$ too. Put differently, the surjectivity of the map $\psi^\vee : \mathcal{G}^2\not{\mathcal{G}}^2 N \to K[\mathcal{G}^2\mathcal{G}^2N]$ implies that in this situation $\mathcal{B}^{m+1}(\mathcal{A})$ is spanned by ordinary elements; meanwhile (12) implies that the top component of $\mathcal{F}$ can be recovered from the Massey triple products whenever $\mathcal{B}^{m+1}(\mathcal{A})$ is generated by ordinary elements.

We will see in §5.2 that when the product structure of $H^*(\mathcal{A})$ is non-trivial, then it is often *not* the case that $\mathcal{B}^{m+1}(\mathcal{A})$ is generated by ordinary elements. Then $\mathcal{F}$ is not determined by Massey triple products.

2.5. **Relationship with A_∞-structures.** For any DGA $\mathcal{A}$, one may define an A_∞-structure on $H^*(\mathcal{A})$, which consists of a sequence of linear maps $\mu_k \colon H^*(\mathcal{A})^{\otimes k} \to H^*(\mathcal{A})$ of degree $2-k$, for $k \geq 2$, see *e.g.* Amann [2, §8.5] or Vallette [32]. μ_2 is simply the product on the cohomology algebra. The definition of the higher products relies on arbitrary choices, but the structure is well-defined up to a suitable notion of A_∞-isomorphism. Moreover, $H^*(\mathcal{A})$ with this A_∞-structure is quasi-isomorphic to $\mathcal{A}$ itself, so in particular determines the homotopy type of $\mathcal{A}$. There is also a notion of homotopy equivalence of A_∞ algebras, and two simply-connected spaces are rationally homotopy equivalent if and only if their cohomology rings are equivalent in the sense of Kadeishvili [21, Theorem 9.1]; see also Vallette [32, Theorem 8] or Amann [2, §8.5].

We shall only be concerned with $\mu_3 \colon H^*(\mathcal{A})^{\otimes 3} \to H^{*-1}(\mathcal{A})$, which may be defined as follows. Let $\mathcal{Z}^* \subseteq \mathcal{A}^*$ denote the subspace of closed elements as before. Pick a right inverse $\alpha \colon H^*(\mathcal{A}) \to \mathcal{Z}^*$ of the projection $\mathcal{Z}^* \to H^*(\mathcal{A})$, and a $\gamma \colon \mathcal{Z}^* \to \mathcal{A}^{*-1}$ such that $d\gamma \colon \mathcal{Z}^* \to \mathcal{Z}^*$ is a projection to the exact part. Further pick a map $p : \mathcal{A}^* \to H^*(\mathcal{A})$ such that $p(\beta) = [\beta]$ for $\beta \in \mathcal{Z}^*$, and set

$$\mu_3(x, z, y) := p\big(\gamma(\alpha(x)\alpha(z))\alpha(y) - (-1)^i \alpha(x)\gamma(\alpha(z)\alpha(y))\big) \in H^*(\mathcal{A}).$$

If $xz = yz = 0$ then clearly

$$\mu_3(x, z, y) = \langle x, z, y\rangle \mod xH^{j+k-1}(\mathcal{A}) + yH^{i+k-1}(\mathcal{A}),$$

cf. [24, Lemma 9.4.6]. (For $k \geq 4$, the precise relationship between k-fold Massey products and the higher products μ_k is more subtle: see Buijs, Moreno-Fernández and Murillo [8].)

We can also relate μ_3 to the uniform Massey triple product. If we define $\widehat{\mu}_3 : \mathcal{G}^2 H^* \otimes H^* \to H^{*-1}$ to be induced by $(x, y, z) \mapsto p(\gamma(\alpha(x)\alpha(y))\alpha(z)$, then

$$(\phi^\vee \circ \widehat{\mu}_3)(x, y, z) = \widehat{\mu}_3((-1)^{jk}(xz)y - (-1)^{ij+ik+jk}(zy)x) = (-1)^{jk}\mu_3(x, z, y),$$

so $\widehat{\mu}_3$ determines μ_3. Meanwhile, the restriction of $\widehat{\mu}_3$ to $K[E^* \otimes H^*]$ is the uniform triple product $\mathcal{T}$. Because μ_3 determines the restriction of $\widehat{\mu}_3$ to $K[\mathcal{G}^2 H^* \otimes H^*]$ by the duality of Lemma 2.2, it also determines $\mathcal{T}$.

The next lemma is essentially a converse statement.

Lemma 2.12. *Suppose $\mathcal{A}$ and $\mathcal{A}'$ are DGAs with a chosen isomorphism $H^*(\mathcal{A}) \cong H^*(\mathcal{A}')$, such that $\mathcal{T} = \mathcal{T}'$. Then there is a DGA $\mathcal{E}$ with $H^*(\mathcal{E}) = 0$ such that the choices in the definition of the A_∞ triple products μ_3 of $\mathcal{A} \times \mathcal{E}$ and μ_3' of $\mathcal{A}'$ can be made so that $\mu_3 = \mu_3'$.*

Proof. The hypothesis means that, having made the choices of α', γ' and p' in the definition of μ_3', we can choose α and γ such that the restrictions of $\widehat{\mu}_3$ and $\widehat{\mu}_3'$ to $K[E^* \otimes H^*]$ agree.

Now, take $\mathcal{E}$ to be the free DGA (*i.e.* the differential of any generator is never contained in the multiplicative algebra of the generators) whose generators in degree k is the degree $k+1$ part of the image of $c : \mathcal{G}^2H^* \to H^*$. Choose $\gamma : \mathcal{G}^2H^* \to (\mathcal{A} \times \mathcal{E})^{*-1}$ to be the sum of the $\gamma : \mathcal{G}^2H^* \to \mathcal{A}^{*-1}$ above and $c : \mathcal{G}^2H^* \to \mathcal{E}^{*-1}$. Then $d(\alpha\gamma) : \mathcal{G}^2H^* \otimes H^* \to \mathcal{A}^* \times \mathcal{E}^*$ has no kernel outside $K[\mathcal{G}^2H^* \otimes H^*]$. Therefore, by choosing p we can adjust the map $\widehat{\mu}_3 : K[\mathcal{G}^2H^* \otimes H^*] \to H^{*-1}$ arbitrarily subject only to its restriction to $K[E^* \otimes H^*]$ equalling $\mathcal{T}$. □

In summary, μ_3 determines $\mathcal{T}$, and hence also $\mathcal{F}$. On the other hand, $\mathcal{T}$ essentially captures all the information of μ_3, but in a way that makes the dependence on choices more transparent. When $\mathcal{A}$ satisfies Poincaré duality, $\mathcal{F}$ in turn captures all the information of $\mathcal{T}$ in a way that eliminates dependence on choices altogether.

3. The rational homotopy type

In this section we prove our main theorems on the significance of the Bianchi-Massey tensor for the rational homotopy type—and hence formality and diffeomorphism classification up to finite ambiguity—of $(n-1)$-connected manifolds of dimension $m \leq 5n-3$.

3.1. Minimal Sullivan algebras.

We first classify $(n-1)$-connected m-dimensional Poincaré minimal Sullivan algebras via their cohomology algebras and Bianchi-Massey tensors. Recall that a *minimal Sullivan algebra* is a DGA $(\mathcal{M}, d)$ that is free as a graded algebra, $\mathcal{M} \cong \Lambda V$, and has a well-ordered basis $\{v_\alpha\} \subset V$ such that dv_α lies in the subalgebra generated by $\{v_\beta : \beta < \alpha\}$, and $\alpha \leq \beta \Rightarrow \deg v_\alpha \leq \deg v_\beta$. We are only interested in the case when $\mathcal{M}$ is simply-connected. In this case, the minimality condition reduces to saying that $\mathcal{M}$ is free, and

$$\text{for any } v \in \mathcal{M}^i,\ dv \text{ is a linear combination of products of elements of degree} < i. \tag{13}$$

Let $\mathcal{A}$ be a finite-dimensional graded commutative algebra or DGA over $\mathbb{Q}$. We call $\mathcal{A}$ *j-connected* if $\mathcal{A}^0 = \mathbb{Q}$ and $\mathcal{A}^k = 0$ for $1 \leq k \leq j$. Recall also from Definition 2.7 that a DGA $\mathcal{A}$ is m-dimensional Poincaré if and only if $H^*(\mathcal{A})$ is.

The key to the role of the Bianchi-Massey tensor in this paper is the following existence and uniqueness result for minimal Sullivan algebras with prescribed Bianchi-Massey tensor.

Theorem 3.1. *Let $n \geq 2$.*

(i) *Let $m \leq 5n-2$. For every $(n-1)$-connected m-dimensional Poincaré algebra H^* (in the sense of Definition 2.7) and linear map $\mathcal{F}\colon \mathcal{B}^{m+1}(H^*) \to H^m$, there exists an $(n-1)$-connected minimal Sullivan algebra $\mathcal{M}$ with $H^*(\mathcal{M}) = H^*$ and Bianchi-Massey tensor $\mathcal{F}$.*

(ii) *Let $m \leq 5n-3$. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are $(n-1)$-connected m-dimensional Poincaré minimal Sullivan algebras and $G\colon H^*(\mathcal{M}_1) \to H^*(\mathcal{M}_2)$ is an isomorphism of the cohomology algebras then there is a DGA isomorphism $\phi\colon \mathcal{M}_1 \to \mathcal{M}_2$ such that $\phi_\# = G$ if and only if the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{B}^{m+1}(\mathcal{M}_1) & \xrightarrow{G} & \mathcal{B}^{m+1}(\mathcal{M}_2) \\ \downarrow{\scriptstyle \mathcal{F}_1} & & \downarrow{\scriptstyle \mathcal{F}_2} \\ H^m(\mathcal{M}_1) & \xrightarrow{G} & H^m(\mathcal{M}_2) \end{array}$$

Let us first outline the essence of the proof. The standard technique is to describe the degree i part V^i of the generating set of the minimal algebra $\mathcal{M} = \Lambda V$ recursively in terms of the cohomology algebra and any further data (*i.e.* the Bianchi-Massey tensor in this case), using the property (13). Let $\mathcal{M}_{[i]}$ denote the sub-DGA generated by elements of degree $\leq i$, or equivalently $\mathcal{M}_{[i]} = \Lambda V^{\leq i}$. Presenting V^i as the sum of its closed subspace C^i and a direct complement N^i, in the ith step of the recursion one identifies N^{i-1} and C^i with the kernel and cokernel of $H^i(\mathcal{M}_{[i-2]}) \to H^i(\mathcal{M})$ respectively (this relies on our algebras being simply-connected, so that $V^1 = 0$). In the setting of Theorem 3.1, this map is essentially determined by the cohomology algebra except for $3n-1 \leq i \leq m-n$ and $i = m$, where the Bianchi-Massey tensor appears (actually, it might be more accurate to say that the uniform triple product $\mathcal{T}$ appears, and that this is controlled by the Bianchi-Massey tensor via Lemma 2.8). To prove the uniqueness statement (ii), one can argue that the generating set of any minimal algebra with the prescribed cohomology and Bianchi-Massey tensor can be described this way—while the description involves some arbitrary choices, those can be expressed in terms of the cohomology algebra. (If $m = 5n-2$, then some information about fourfold Massey products would be needed to capture the essence of $H^i(\mathcal{M}_{[i-2]}) \to H^i(\mathcal{M})$ for $i = m-n$ and m, which is why (ii) requires $m \leq 5n-3$.)

We will split the argument into two parts. First we explain in Proposition 3.4 that more generally, a minimal algebra $\mathcal{M}$ that is m-dimensional Poincaré is essentially characterised by $\mathcal{M}_{[k]}$ for any k with $2k \geq m-1$, together with the map $H^m(\mathcal{M}_{[k]}) \to H^m(\mathcal{M})$. This is implicit in Kreck-Triantafillou [23, Theorem 1.2]. Proposition 3.8 then shows that for $(n-1)$-connected Poincaré minimal algebras of dimension $m \leq 5n-3$, that data is characterised by the cohomology algebra and the Bianchi-Massey tensor.

Proving Theorem 3.1 in one go would be shorter in total (though longer than the proofs of Propositions 3.4 and 3.8 individually), in part because splitting up the proof as we do involves first encoding much of the algebra structure of $H^*(\mathcal{M})$ in terms of the map $H^m(\mathcal{M}_{[k]}) \to H^m(\mathcal{M})$, and then reconstructing the algebra again. Our reason for organising the proof as we do is that we hope that it will make is easier to identify what further invariants are needed to determine the rational homotopy type if we weaken the connectedness hypothesis, and also that it clarifies the relation to the results of Kreck and Triantafillou.

3.2. **Reconstructing a minimal DGA from a partial Poincaré class.** We employ the following terminology from Kreck and Triantafillou [23, §1].

Definition 3.2. Let H^* be a finite dimensional graded commutative algebra over the rationals. For $2k+1 \geq m$, we call $\alpha \in (H^m)^\vee$ a *k-partial Poincaré class* if the map

$$\alpha\cap : H^i \to (H^{m-i})^\vee, \; x \mapsto (y \mapsto \alpha(xy))$$

is an isomorphism for $m - k \leq i \leq k$ and injective for $i = k+1$ (and hence surjective for $i = m-k+1$).

We aim to prove in Proposition 3.4 that a minimal DGA $\mathcal{M}$ that is m-dimensional Poincaré can essentially be reconstructed from its truncation $\mathcal{M}_{[k]}$ and a k-partial Poincaré class $\alpha \in H^m(\mathcal{M}_{[k]})$, provided $2 \leq k \leq 2m+1$. Let us first consider the following easier problem of reconstructing a Poincaré algebra $\bar{H}^*$ from a truncation $\bar{H}^{\leq k+1}$ together with a k-partial Poincaré class in the degree m part of $\Lambda\bar{H}^{\leq k+1}$ (by which we mean the algebra generated by $\bar{H}^{\leq k+1}$, with no relations imposed beyond those generated by the relations of $\bar{H}$ in degree $\leq k+1$).

Lemma 3.3. *Let $k \geq 2$ and $m \leq 2k+1$.*

(i) *Let $\phi : H^* \to \bar{H}^*$ be an algebra homomorphism that is an isomorphism in degree $\leq k$, and let $\bar{\alpha} \in \bar{H}^*$ be a Poincaré class. Then $\phi^\vee\bar{\alpha} \in H^m$ is a k-partial Poincaré class if and only if ϕ is injective.*

(ii) *Let H^* be an algebra generated by elements of degree $\leq k+1$, and let $\alpha \in H^m$ be a k-partial Poincaré class. Then there exists a unique (up to isomorphism) m-dimensional Poincaré duality algebra $\bar{H}^*$ with an algebra homomorphism $\phi : H^* \to \bar{H}^*$ and Poincaré class $\bar{\alpha} \in (\bar{H}^m)^\vee$ such that*
- *$\phi : H^i \cong \bar{H}^i$ is an isomorphism for $i \leq k$ and $\phi : H^{k+1} \hookrightarrow \bar{H}^{k+1}$ is injective;*
- *$\phi^\vee\bar{\alpha} = \alpha$.*

Proof. (i) Immediate from the commutativity of the diagram

$$\begin{array}{ccc} H^i & \xrightarrow{\;\phi\;} & \bar{H}^i \\ {\scriptstyle(\phi^\vee\bar{\alpha})\cap}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}\cap} \\ (H^{m-i})^\vee & \xleftarrow{\;\phi^\vee\;} & (\bar{H}^{m-i})^\vee. \end{array}$$

(ii) Let $\bar{H}^i := H^i$ for $i \leq k$, and $\bar{H}^i := (H^{m-i})^\vee$ for $i > k$. Define the product of $x \in \bar{H}^i$ and $y \in \bar{H}^j$ using the product structure of H^* if $i + j \leq k$, and as $\bar{\alpha} \cap (xy) \in (H^{m-i-j})^\vee = \bar{H}^{i+j}$ if

$i, j \leq k$ and $i+j \geq k$. If $i \leq k$ and $j > k$, so that $y \in (H^{m-j})^\vee$, let $xy = y \cap x \in (H^{m-i-j})^\vee$, *i.e.* the map $H^{m-i-j} \to \mathbb{Q}, z \mapsto y(xz)$. If $i, j > k$ then $xy = 0$. □

Given an m-dimensional Poincaré algebra $(\bar{H}^*, \bar{\alpha})$, the multiplication induces a natural map $\phi : \Lambda \bar{H}^{\leq k+1} \to \bar{H}^*$ that is an isomorphism in degrees $\leq k+1$. Applying Lemma 3.3(ii) to this ϕ and $\alpha := \phi^\vee \bar{\alpha}$ one recovers the original $\bar{H}^*$.

On the other hand, given an algebra H^*, let $H^*_{[k]}$ denote the subalgebra generated by classes of degree $\leq k$ (the image of the natural map $\Lambda H^{\leq k} \to H^*$). Note that whether $\alpha \in (H^m)^\vee$ is a k-partial Poincaré class depends only on the restriction of α to $H^m_{[k+1]}$. For any k-partial Poincaré class $\alpha \in (H^m)^\vee$, we can thus apply Lemma 3.3(ii) to $\big(H^*_{[k+1]}, \alpha_{|H^m_{[k+1]}}\big)$ to canonically construct a Poincaré algebra $\bar{H}^*$ (though there need not be a canonical way to extend the homomorphism $H^*_{[k+1]} \to \bar{H}^*$ to H^*).

In this sense, in Proposition 3.4(i) the restriction of α to $H^m_{[k+1]}(\mathcal{N})$ is responsible for reconstructing the cohomology algebra of $\mathcal{M}$, while the remaining components of α encode Massey products etc.

Note that Lemma 3.3(i) implies in particular that if $\mathcal{M}$ is a DGA and $\alpha \in H^m(\mathcal{M})^\vee$ is a Poincaré class then the image of α in $H^m(\mathcal{M}_{[k]})^\vee$ is a k-partial Poincaré class (where as before $\mathcal{M}_{[k]} \subseteq \mathcal{M}$ denotes the sub-DGA generated by elements of degree $\leq k$).

Proposition 3.4. *Let $k \geq 2$ and $m \leq 2k+1$.*

(i) *Let $\mathcal{N}$ be a simply-connected minimal Sullivan algebra generated in degree $\leq k$, and let $\alpha \in H^m(\mathcal{N})^\vee$ be a k-partial Poincaré class. Then there is a minimal Sullivan algebra $\mathcal{M}$ with Poincaré class $\alpha_\mathcal{M} \in H^m(\mathcal{M})^*$ and an isomorphism $\tau \colon \mathcal{N} \to \mathcal{M}_{[k]}$ such that $\tau^\vee_\# \alpha_\mathcal{M} = \alpha$.*

(ii) *Let $\mathcal{M}_1, \mathcal{M}_2$ be simply-connected minimal Sullivan algebras that are m-dimensional Poincaré. Let $\tau \colon \mathcal{M}_{1[k]} \to \mathcal{M}_{2[k]}$ and $G \colon H^m(\mathcal{M}_1) \to H^m(\mathcal{M}_2)$ be isomorphisms, such that the diagram below commutes.*

$$\begin{array}{ccc} H^m(\mathcal{M}_{1[k]}) & \xrightarrow{\tau_\#} & H^m(\mathcal{M}_{2[k]}) \\ \downarrow & & \downarrow \\ H^m(\mathcal{M}_1) & \xrightarrow{G} & H^m(\mathcal{M}_2) \end{array}$$

Then there is an isomorphism $\phi \colon \mathcal{M}_1 \to \mathcal{M}_2$ such that the restriction $\phi_{[k]} \colon \mathcal{M}_{1[k]} \to \mathcal{M}_{2[k]}$ equals τ and $\phi_\# \colon H^m(\mathcal{M}_1) \to H^m(\mathcal{M}_2)$ equals G.

Proof. (i) If $k \geq m$, then constructing the generators V^i in degree $i > k$ for $\mathcal{M}$ is a trivial recursion: d maps $V^i = N^i$ isomorphically to the closed subspace of $\mathcal{M}^{i+1}_{[i]}$. The following claim, which lets us increase k inductively until we reach $k = m$, therefore proves the result.

There exists a minimal Sullivan algebra $\mathcal{E}$ generated in degree $\leq k+1$ with a $(k+1)$-partial Poincaré class $\alpha_\mathcal{E} \in H^m(\mathcal{E})^\vee$ and an isomorphism $\phi \colon \mathcal{N} \to \mathcal{E}_{[k]}$ such that $\phi^\vee_\# \alpha_\mathcal{E} = \alpha$.

Let us first construct the generating space V for $\mathcal{E}$. We take the degree $\leq k$ parts to equal those of $\mathcal{N}$ (and define ϕ to be the inclusion). Choose a direct complement C^{k+1} of the image of $\alpha\cap : H^{k+1}(\mathcal{N}) \to H^{m-k-1}(\mathcal{N})^\vee$, and set $V^{k+1} := C^{k+1} \oplus N^{k+1}$, where $d \colon V^{k+1} \to \mathcal{N}^{k+2}$ has kernel C^{k+1} and maps N^{k+1} isomorphically to a subspace of $\mathcal{Z}^{k+2}$ (the closed subspace of $\mathcal{N}^{k+2}$) representing the kernel of $\alpha\cap : H^{k+2}(\mathcal{N}) \to H^{m-k-2}(\mathcal{N})^\vee$.

We need to study $H^m(\mathcal{E})$. Note that $\mathcal{E}^m = \mathcal{N}^m \oplus V^{k+1} \otimes \mathcal{N}^{m-k-1}$, and the closed subspace is contained in $\mathcal{N}^m \oplus V^{k+1} \otimes \mathcal{Z}^{m-k-1}$. Therefore $H^m(\mathcal{E})$ can be written as a direct sum of the images of $H^m(\mathcal{N})$ and $C^{k+1} \otimes H^{m-k-1}(\mathcal{N})$ and a direct complement W.

Note that $\mathcal{E}^{m-1} = \mathcal{N}^{m-1} \oplus V^{k+1} \otimes \mathcal{N}^{m-k-2}$. Therefore we have $d\mathcal{E}^{m-1} \cap C^{k+1} \otimes \mathcal{Z}^{m-k-1} = C^{k+1} \otimes d\mathcal{N}^{m-k-2}$, so the map $C^{k+1} \otimes H^{m-k-1}(\mathcal{N}) \to H^m(\mathcal{E})$ is injective. On the other hand, the kernel of $H^m(\mathcal{N}) \to H^m(\mathcal{E})$ consists of classes represented by elements of $\mathcal{N}^m \cap d(V^{k+1} \otimes \mathcal{N}^{m-k-2}) = dN^{k+1} \otimes \mathcal{Z}^{m-k-2}$. Since this is contained in the kernel of α by construction, α factors through $\phi_\#$.

We can therefore define the restriction of $\alpha_\mathcal{E}$ to the image of $H^m(\mathcal{N})$ by requiring that $\alpha = \phi^\vee_\# \alpha_\mathcal{E}$. Further we define the restriction to $C^{k+1} \otimes H^{m-k-1}(\mathcal{N})$ to be the natural map arising from C^{k+1}

being a subspace of $H^{m-k-1}(\mathcal{N})^\vee$, and choose the restriction to W to be 0. It remains to prove that this $\alpha_\mathcal{E} \in H^m(\mathcal{E})^\vee$ is a $(k{+}1)$-partial Poincaré class.

For $m-k \leq i \leq k$, $H^i(\mathcal{E}) \cong H^i(\mathcal{N})$, and it is easy to see that $\alpha_\mathcal{E}\cap$ is equivalent to the isomorphism $\alpha\cap$. Meanwhile $H^{k+1}(\mathcal{E}) \cong H^{k+1}(\mathcal{N}) \oplus C^{k+1}$, and $\alpha_\mathcal{E}\cap : H^{k+1}(\mathcal{E}) \to H^{m-k-1}(\mathcal{E})^\vee \cong H^{m-k-1}(\mathcal{N})^\vee$ equals the injective map $\alpha\cap$ on the $H^{k+1}(\mathcal{N})$ summand, and the inclusion $C^{k+1} \hookrightarrow H^{m-k-1}(\mathcal{N})^\vee$ on C^{k+1}. Since we chose C^{k+1} to be a direct complement of the image of $\alpha\cap$ that means that $\alpha_\mathcal{E}\cap$ is an isomorphism on $H^{k+1}(\mathcal{E})$ too. Finally, $H^{k+2}(\mathcal{E}) = \mathcal{Z}^{k+2}/dN^{k+1} \cong H^{k+2}(\mathcal{N})/\ker(\alpha\cap)$, and $\alpha_\mathcal{E}\cap : H^{k+2}(\mathcal{N})/\ker(\alpha\cap) \to H^{m-k-2}(\mathcal{E})^\vee \cong H^{m-k-2}(\mathcal{N})^\vee$ is the map induced by $\alpha\cap$, so injective.

(ii) follows by induction from the following claim.

> Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be minimal Sullivan algebras generated in degree $\leq k{+}1$, with $(k{+}1)$-partial Poincaré classes $\alpha_j \in H^m(\mathcal{E}_j)^\vee$. Suppose $\tau\colon \mathcal{E}_{1[k]} \to \mathcal{E}_{2[k]}$ is an isomorphism such that the class $\tau^\vee_\#\alpha_2 \in H^m(\mathcal{E}_{1[k]})$ equals the restriction of α_1. Then there exists an isomorphism $\phi\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that $\phi_{[k]} = \tau$ and $\phi^\vee_\#\alpha_2 = \alpha_1$.

Setting $\mathcal{N}_j := \mathcal{E}_{j[k]}$, we can use the argument in (ii) to describe the generating spaces of $\mathcal{E}_j$. This involves choices of $C^{k+1}_j \subseteq H^{m-k-1}(\mathcal{N}_j)^*$ and $dN^{k+1}_j \subseteq \mathcal{N}^{k+1}_j$, and we choose $C^{k+1}_1 = \tau^\vee_\#(C^{k+1}_2)$ and $dN^{k+1}_2 = \tau(dN^{k+1}_1)$. For any linear map $\kappa\colon N^{k+1}_1 \to \mathcal{Z}^{k+1}_2$, we can define an isomorphism $\phi_\kappa\colon \mathcal{E}_1 \to \mathcal{E}_2$ by setting it to equal τ on $\mathcal{E}_{1[k]}$, $\tau_\#$ on C^{k+1}_1, and $\kappa + d^{-1}{\circ}\tau{\circ}d$ on N^{k+1}_1 (taking the inverse of $d : N^{k+1}_2 \to dN^{k+1}_2$)—indeed any isomorphism $\phi\colon \mathcal{E}_1 \to \mathcal{E}_2$ such that $\phi_{[k]} = \tau$ is of this form.

It remains to understand $\phi^\vee_\kappa\alpha_2$. In the decomposition from (ii) of $H^m(\mathcal{E}_1)$ as the direct sum of the image of $H^m(\mathcal{E}_{1[k]})$, $C^{k+1}_1 \otimes H^{m-k-1}(\mathcal{E}_{1[k]})$ and W, the restrictions of α_1 and $\phi^\vee_\kappa\alpha_2$ to the first two summands agree for any κ.

Let $\mathcal{W} \subseteq \mathcal{E}^m_1$ be a subspace of closed representatives of W. The projection $p\colon \mathcal{E}^m_1 \to N^{k+1} \otimes \mathcal{N}^{m-k-1}_1$ (with kernel $\mathcal{N}^m_1 \oplus C^{k+1}_1 \otimes \mathcal{N}^{m-k-1}_1$) maps $\mathcal{W} \hookrightarrow N^{k+1}_1 \otimes \mathcal{Z}^{m-k-1}_1$. Let us now explain that the induced map $p\colon W \to N^{n+1}_1 \otimes H^{m-k-1}(\mathcal{N}_1)$ is injective too.

Suppose that for some $w \in \mathcal{W}$, $p(w) = \sum n_j \otimes dx_j$, for $n_j \in N^{k+1}_1$ and $x_j \in \mathcal{N}^{m-k-2}_1$. Note that $\sum dn_j \otimes x_j \in \mathcal{N}^{k+2}_1\mathcal{N}^{m-k-2}_1 \subseteq \mathcal{N}^m_1$. Therefore $w - d(\sum n_j \otimes x_j)$, which represents the same class in $H^m(\mathcal{N}_1)$ as w, lies in the kernel of p. But W is by definition a direct complement to the space of classes represented by elements of the kernel of p. Hence $W \hookrightarrow N^{n+1}_1 \otimes H^{m-k-1}(\mathcal{N}_1)$.

The restriction of $(\phi_\kappa)^\vee_\#\alpha_2 - (\phi_0)^\vee_\#\alpha_2$ to W equals the composition of p with the linear map $N^{k+1}_1 \otimes H^{m-k-1}(\mathcal{N}_1) \to \mathbb{Q}$, $n \otimes x \mapsto \alpha_2(\kappa(n)\tau_\# x)$. Because $\tau_\# : H^{m-k-1}(\mathcal{N}_1) \to H^{m-k-1}(\mathcal{N}_2)$ and $\alpha_2\cap\colon H^{m-k-1}(\mathcal{N}_2) \to H^{k+1}(\mathcal{N}_2)^\vee$ are isomorphisms, any linear functional on $N^{k+1}_1 \otimes H^{m-k-1}(\mathcal{N}_1)$ is realised this way for some $\kappa\colon N^{k+1}_1 \to \mathcal{Z}^{k+1}_2$. Thus by adjusting the choice of κ, the restriction of $(\phi_\kappa)^\vee_\#\alpha_2$ to W can be made to equal α_1. □

3.3. **Partial Poincaré classes and the Bianchi-Massey tensor.** We pointed out above that in Proposition 3.4(i) the restriction of α to $H^m_{[k+1]}(\mathcal{N})$ is responsible for reconstructing the cohomology algebra of $\mathcal{M}$. If we now require that $\mathcal{N}$ is $(n{-}1)$-connected (in addition to being minimal and generated in degree $\leq k$), we consider to what extent the remaining components of α are determined by the Bianchi-Massey tensor.

In the setting of primary interest in this paper, *i.e.* when $m \leq 5n{-}3$, and for suitable k we can show that essentially all of those remaining components are captured by the Bianchi-Massey tensor. For $m = 5n{-}2$ we can at least show that the components of the Bianchi-Massey tensor are independent of the components responsible for reconstructing the Poincaré cohomology algebra.

In the statement of the next lemma, we use the following notation: given k and $\mathcal{N}$, let $\mathcal{B}^*_k(\mathcal{N})$ be the intersection of $\mathcal{B}^*(\mathcal{N})$ with the image in $\mathcal{G}^2\mathcal{G}^2H^*(\mathcal{N})$ of $((H^*)^{\otimes 3})^{\leq k+2} \otimes H^*$. In other words, $\mathcal{B}^*_k(\mathcal{N})$ is the part of $\mathcal{B}^*(\mathcal{N})$ that involves only quadruples of classes where the sums of the three lowest degrees is at most $k{+}2$. Note that if $\mathcal{N}$ is $(n{-}1)$-connected then certainly $\mathcal{B}^*(\mathcal{N})$ involves only classes of degree $\geq n$, so if $k \leq 3n{-}3$ then $\mathcal{B}^*_k(\mathcal{N}) = 0$. The proof of (i) is tidier in this case, but to prove the realisation result Theorem 3.1(i) when $m = 5n{-}2$ we need to allow $k = 3n{-}2$.

Lemma 3.5. *Let $n \geq 2$ and $m \geq 4n{-}1$.*

(i) *Let $k \leq 4n-3$, and let $\mathcal{N}$ be an $(n-1)$-connected minimal Sullivan algebra generated in degree $\leq k$.*

(a) *If $m \leq 6n-3$ then the intersection of $H^m_{[k+1]}(\mathcal{N})$ (*i.e. *the subspace of $H^m(\mathcal{N})$ generated by products of classes of degree $\leq k+1$) with the image of $\mathcal{F}\colon \mathcal{B}^{m+1}(\mathcal{N}) \to H^m(\mathcal{N})$ is contained in $\mathcal{F}(\mathcal{B}^{m+1}_k(\mathcal{N}))$.*

(b) *If $m \leq 5n-2$ then the kernel of $\mathcal{F}\colon \mathcal{B}^{m+1}(\mathcal{N}) \to H^m(\mathcal{N})$ is contained in $\mathcal{B}^{m+1}_k(\mathcal{N})$.*

(ii) *Suppose $m \leq 5n-3$ and $k \leq 3n-3$. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be $(n-1)$-connected minimal Sullivan algebras generated in degree $\leq k$. Given an isomorphism $G\colon H^{\leq k}(\mathcal{N}_1) \to H^{\leq k}(\mathcal{N}_2)$ of the truncated cohomology rings, and k-partial Poincaré classes $\alpha_j \in H^m(\mathcal{N}_j)^\vee$, there is an isomorphism $\tau\colon \mathcal{N}_1 \to \mathcal{N}_2$ such that $\tau_\# = G$ on $H^{\leq k}(\mathcal{N}_1)$ and $\tau^\vee_\# \alpha_2 = \alpha_1$ if and only if the diagram below commutes.*

$$\begin{array}{ccccc}
(\Lambda H^{\leq k}(\mathcal{N}_1))^m \oplus \mathcal{B}^{m+1}(\mathcal{N}_1) & \longrightarrow & H^m(\mathcal{N}_1) & \xrightarrow{\alpha_1} & \mathbb{Q} \\
\big\downarrow G & & & & \big\downarrow \\
(\Lambda H^{\leq k}(\mathcal{N}_2))^m \oplus \mathcal{B}^{m+1}(\mathcal{N}_2) & \longrightarrow & H^m(\mathcal{N}_2) & \xrightarrow{\alpha_2} & \mathbb{Q}
\end{array}$$

Remark 3.6. If $\mathcal{N}$ is generated in degree $\leq k$, then $\Lambda H^{\leq j}(\mathcal{N}) \hookrightarrow H^*(\mathcal{N})$ for any $j > k$. If $\mathcal{N}$ is $(n-1)$-connected, then the image $H^*_{[j]}(\mathcal{N})$ is the same for all $k \leq j < 3n-2$; however it is *not* necessarily isomorphic to $\Lambda H^{\leq k}(\mathcal{N})$. In (ii), $G : H^{\leq k}(\mathcal{N}_1) \to H^{\leq k}(\mathcal{N}_2)$ would therefore not automatically induce a map $H^*_{[k]}(\mathcal{N}_1) \to H^*_{[k]}(\mathcal{N}_2)$ (never mind extend to $H^*(\mathcal{N}_1) \to H^*(\mathcal{N}_2)$) if we did not also assume the commutativity of the diagram.

Since $\mathcal{B}^*_k(\mathcal{N}) = 0$ if $k \leq 3n-3$, (i) implies that the maps $\mathcal{F}\colon \mathcal{B}^{m+1}(\mathcal{N}_i) \to H^m(\mathcal{N}_i)$ in (ii) are injective, with image transverse to $H^m_{[k]}(\mathcal{N}_i) = H^m_{[k+1]}(\mathcal{N}_i)$.

Insisting that $k \geq m-2n$ in (ii) ensures that if $\mathcal{M}$ is $(n-1)$-connected then $\mathcal{B}^{m+1}(\mathcal{M}_{[k]}) = \mathcal{B}^{m+1}(\mathcal{M})$. The statement would not in general be true if we instead set $k := [m/2]$.

Proof. The argument is similar to the induction steps in the proof of Proposition 3.4.

(i) We begin by describing a generating space V for $\mathcal{N} = \Lambda V$, as before writing V^i as a direct sum of the closed subspace C^i and a direct complement N^i.

By a trivial recursion we find that $V^i = 0$ for $0 < i < n$, and $V^i = C^i$ for $n \leq i \leq 2n-2$ (*i.e.* V^i consists of only closed elements). Further $C^i \cong H^i(\mathcal{N})$ for $i \leq 2n-1$.

For $2n \leq i \leq 3n-2$, the closed subspace of $\mathcal{N}^i_{[i-1]}$ is precisely $(\mathcal{G}^2 C^*)^i$. Thus C^i is identified with a direct complement to the image of $(\mathcal{G}^2 C^*)^i \to H^i(\mathcal{N})$, while d maps N^{i-1} isomorphically to its kernel.

For $3n-1 \leq i \leq k+1$, the closed subspace of $\mathcal{N}^i_{[i-1]}$ is a direct sum of $(\mathcal{G}^2 C^* \oplus \mathcal{G}^3 C^*)^i$ and the closed subspace of $(C^* \otimes N^*)^i$; we let T^i denote the latter term. Thus C^i is identified with a direct complement to the image of $(\mathcal{G}^2 C^* \oplus \mathcal{G}^3 C^* \oplus T)^i \to H^i(\mathcal{N})$ (for $i \leq k$), and N^{i-1} is mapped isomorphically to the kernel.

Let $\widetilde{\mathcal{C}} := \Lambda C \subseteq \mathcal{N}$. Then we can decompose $\mathcal{N}^*$ as a direct sum of $\widetilde{\mathcal{C}}^*$, $\widetilde{\mathcal{C}}^* \otimes N^*$, $\widetilde{\mathcal{C}}^* \otimes \mathcal{G}^2 N^*$ etc. The first term consists of only closed forms, while the closed subspace of the second term contains $C^* \otimes T^*$.

(a) $d(\widetilde{\mathcal{C}} \otimes N^*) \subseteq \widetilde{\mathcal{C}} \oplus \widetilde{\mathcal{C}} \otimes T^*$, while any element of $\widetilde{\mathcal{C}} \oplus \widetilde{\mathcal{C}} \otimes N^*$ that is the differential of something in $\widetilde{\mathcal{C}} \otimes \mathcal{G}^{\geq 2} N^*$ belongs to $\widetilde{\mathcal{C}} \otimes N^*$.

Note that for $i \leq 3n-1$, $E^i \cong \ker\big((\mathcal{G}^2 C^*)^i \to H^{i+1}(\mathcal{N})\big)$, which is the injective image under d of a subspace $\overline{N}^{i-1} \subseteq N^{i-1}$ (in fact equality holds for $i \leq 3n-2$). Because $\mathcal{B}^{m+1}(\mathcal{N})$ only involves E^i for $i \leq m+1-2n \leq 3n-1$, we can therefore choose the map $\gamma : E^i \to \mathcal{N}^{i-1}$ in the definition of $\mathcal{F}$ to take values in $\overline{N}^{i-1}$, so that $d\gamma$ takes values in $\mathcal{G}^2 C^* \subseteq \widetilde{\mathcal{C}}^*$. In particular, the image of $\mathcal{F} : \mathcal{B}^{m+1}(\mathcal{N}) \to H^m(\mathcal{N})$ is represented by elements of $N^* \otimes \widetilde{\mathcal{C}}^*$.

On the other hand, $H^m_{[k+1]}(\mathcal{N})$ is the subspace represented by the degree m part of $\widetilde{\mathcal{C}}^* \oplus T^* \otimes \widetilde{\mathcal{C}}^*$. Therefore the intersection of the images in $H^m(\mathcal{N})$ can be represented by elements in the intersection $\widetilde{\mathcal{C}}^* \otimes T^*$, corresponding to $\mathcal{B}^{m+1}_k(\mathcal{N})$.

(b) Using once more that $E^i \cong \ker\big((\mathcal{G}^2C^*)^i \to H^{i+1}(\mathcal{N})\big) = d\overline{N}^{i-1}$ in the degrees that contribute to the degree $m+1$ part of $E^* \otimes E^*$, the degree $m+1$ part of $K[E^* \otimes E^*]$ is isomorphic to the closed subspace $\mathcal{K}^m$ of the degree m part of $\overline{N}^* \otimes d\overline{N}^*$. Further $\mathrm{Anti}^2 E^*$ is mapped to $d(\overline{N}^* \otimes \overline{N}^*)$. We therefore obtain an isomorphism $\mathcal{B}^{m+1}(\mathcal{N}) \to (K[E^* \otimes E^*]/\mathrm{Anti}^2 E^*)^{m+1} \to (\mathcal{K}^*/d(\overline{N}^* \otimes \overline{N}^*))^m$ (*cf.* Remark 2.6).

We see that $\mathcal{F} : \mathcal{B}^{m+1}(\mathcal{N}) \to H^m(\mathcal{N})$ is the composition of that isomorphism with the projection to $H^m(\mathcal{N})$. The kernel of the latter map is represented by elements of $d(\widetilde{\mathcal{C}} \otimes N^{\geq 3n-2})$, whose preimage in $\mathcal{B}^{m+1}(\mathcal{N})$ is contained in $\mathcal{B}^{m+1}_k(\mathcal{N})$.

(ii) To describe the homomorphism $\tau : \mathcal{N}_1 \to \mathcal{N}_2$, we need to specify its values on the generating spaces $V_1^i \subset \mathcal{N}_1$. The values on $C_1^i \subseteq V_1^i$ are determined by G. Thus the only flexibility that remains for adjusting $\tau^\vee_\# \alpha_2$ is the restriction of τ to N_1^*.

We now claim that any closed element of $\mathcal{N}^m$ can be written as a sum of products of generators such that at most one factor in each term is not closed. The hypothesis that $m \leq 5n-3$ ensures that we can decompose $\mathcal{N}^m$ as a direct sum of the degree m parts of $\widetilde{\mathcal{C}}^m$, $N^* \otimes \widetilde{\mathcal{C}}^*$ and $\mathcal{G}^2 N^*$ (in particular we do not need $\widetilde{\mathcal{C}}^* \otimes \mathcal{G}^2 N^*$). If $x \in \mathcal{N}^m$ has component $\sum m_j \otimes n_j$ in $N^i \otimes N^{m-i}$, then the $N^i \otimes \widetilde{\mathcal{C}}^{m-i+1}$ component of dx is $\sum m_j \otimes dn_j$. That vanishes only if $\sum m_j \otimes n_j$ does, proving the claim.

For $2n-2 \leq j \leq k$, let L_j be the closed subspace of

$$N^{2n-1} \otimes \mathcal{Z}^{m-2n+1} \ \oplus \cdots \oplus \ N^j \otimes \mathcal{Z}^{m-j} \ \oplus \ N^{j+1} \otimes dN^{m-j-2} \ \oplus \cdots \oplus \ N^k \otimes dN^{m-k-1},$$

where as usual $\mathcal{Z}^* \subseteq \mathcal{N}^*$ is the closed subalgebra. So

$$\mathcal{K}^m = L_{2n-2} \subseteq L_{2n-1} \subseteq \cdots \subseteq L_k,$$

and $\widetilde{\mathcal{C}}^m \oplus L_k = \mathcal{Z}^m$. Since the hypothesis means that $\tau^\vee_\# \alpha_2$ and α_1 agree on $\widetilde{\mathcal{C}}^m \oplus \mathcal{K}$ (the first term represents elements of $H^m_{[k+1]}(\mathcal{N})$, and the second the image of $\mathcal{F}$), the desired conclusion follows by induction from the following claim.

> Let $2n-1 \leq j \leq k$, and let $\tau : \mathcal{N}_1 \to \mathcal{N}_2$ be an isomorphism. Then there exists another isomorphism ϕ with $\phi_{[j-1]} = \tau_{[j-1]}$ such that $\phi^*_\# \alpha_2$ and α_1 agree on $\widetilde{\mathcal{C}}^m \oplus L_j$ if and only if $\tau^\vee_\# \alpha_2$ and α_1 agree on $\widetilde{\mathcal{C}}^m \oplus L_{j-1}$.

Set $\phi_\kappa := \tau + \kappa$, for some linear map $\kappa : N^j \to \mathcal{Z}^j$. Let W be a direct complement to L_{j-1} in L_j. Then $p \colon W \hookrightarrow N^j \otimes H^{m-j}(\mathcal{N})$. Now $(\phi_\kappa)^\vee_\# \alpha_2 = \tau^\vee_\# \alpha_2$ on L_{j-1}, while on W the difference equals the composition of p with the linear map $N_1^j \otimes H^{m-j}(\mathcal{N}_1) \to \mathbb{Q}$, $n \otimes x \mapsto \alpha_2(\kappa(n)\tau_\# x)$. Because $\tau_\# : H^{m-j}(\mathcal{N}_1) \to H^{m-j}(\mathcal{N}_2)$ and $\alpha_2 \cap \colon H^{m-j}(\mathcal{N}_2) \to H^j(\mathcal{N}_2)^\vee$ are isomorphisms, any linear functional on $N_1^j \otimes H^{m-j}(\mathcal{N}_1)$ is realised this way for some $\kappa \colon N_1^j \to \mathcal{Z}_2^j$. Thus by adjusting the choice of κ, the restriction of $(\phi_\kappa)^\vee_\# \alpha_2$ to W can be made to equal α_1. □

Example 3.7. Suppose H^* is an algebra with $x_1,\ x_2,\ x_3 \in H^n$ such that the products x_1^2, x_2^2, and x_1x_2 vanish in H^{2n}, and $y_1 = x_1x_3$ and $y_2 = x_2x_3 \in H^{2n}$ are linearly independent. Then $b := (x_1y_1)x_2^2 - (y_1x_2)(x_2x_1) - (x_2y_2)x_1^2 + (x_1y_2)(x_1x_2)$ is a non-zero element of $\mathcal{B}^{5n}_{3n-2}(H^*)$.

If a DGA $\mathcal{N}$ has $H^*(\mathcal{N})$ of this form, then $\mathcal{F}(b) = 0$. For in the definition of $\mathcal{F}$, choose the linear map $\alpha : H^*(\mathcal{N}) \to \mathcal{N}^*$ such that $\alpha(y_1) = \alpha(x_1)\alpha(x_3)$ and $\alpha(y_2) = \alpha(x_2)\alpha(x_3)$. We can then choose $\gamma : E^* \to \mathcal{N}^{*-1}$ such that $\gamma(x_1y_1) = \gamma(x_1^2)\alpha(x_3)$ and $\gamma(x_2y_1) = \gamma(x_1x_2)\alpha(x_3)$. Then $\mathcal{F}(b) \in H^{5n-1}(\mathcal{N})$ is represented by

$$\gamma(x_1y_1)\alpha(x_2)^2 - \gamma(y_1x_2)\alpha(x_2)\alpha(x_1) - \gamma(x_1^2)\alpha(x_2)\alpha(y_2) + \alpha(x_1)\alpha(y_2)\gamma(x_1x_2) = 0.$$

There exist $(n-1)$-connected $(5n-1)$-dimensional Poincaré algebras H^* with the above property, as can be seen for instance by applying Lemma 3.3(ii) to an algebra supported in degree 0, n and $2n$, with $\alpha = 0$ as a $(3n-2)$-partial Poincaré class of degree $5n-1$. Thus the realisation statement of Theorem 3.1(i) does not extend beyond $m \leq 5n-2$.

Similarly, there exist $(n-1)$-connected minimal DGAs $\mathcal{N}$ generated in degree $\leq 3n-1$ such that $H^*(\mathcal{N})$ has the above property. This demonstrates that if $m \geq 5n-1$, then for all $k \geq m-2n$ there can be a minimal DGA $\mathcal{N}$ generated in degree $\leq k$ such that $\mathcal{F}$ fails to be injective on $\mathcal{B}^{m+1}(\mathcal{N})$. That is where our proof of Theorem 3.1(i) breaks down for $m \geq 5n-1$.

From Lemma 3.5 we can now deduce a result that links up with Proposition 3.4.

Proposition 3.8. *Let $4n-1 \leq m \leq 5n-2$ and $k = m-2n$. Then given an $(n-1)$-connected m-dimensional Poincaré duality algebra $(\bar{H}^*, \bar{\alpha})$ and a linear map $F : \mathcal{B}^{m+1}(\bar{H}^*) \to \mathbb{Q}$, there exists a minimal Sullivan algebra $\mathcal{N}$ generated in degree $\leq k$, with*

- *an injection $\phi : H^{\leq k+1}(\mathcal{N}) \hookrightarrow \bar{H}^{\leq k+1}$ that is an isomorphism in degree $\leq k$ and*
- *a k-partial Poincaré duality class $\alpha \in H^m(\mathcal{N})$*

such that

- *the restriction of α to $H^m_{[k+1]}(\mathcal{N})$ is $\phi^\vee \bar{\alpha}$ (note that ϕ induces a well-defined map $H^*_{[k+1]}(\mathcal{N}) \to \bar{H}^*$, because $\mathcal{N}$ is generated in degree $\leq k$), and*
- *the Bianchi-Massey tensor $\mathcal{F} : \mathcal{B}^{m+1}(\mathcal{N}) \to H^m(\mathcal{N})$ satisfies $\alpha \circ \mathcal{F} = F \circ \phi'$, where the map $\phi' : \mathcal{B}^{m+1}(\mathcal{N}) \to \mathcal{B}^{m+1}(\bar{H})$ is the isomorphism induced by ϕ.*

Moreover, if $m \leq 5n-3$ then $\mathcal{N}$ is unique.

Proof. If $m \leq 5n-3$, so that $k \leq 3n-3$, then it is easy construct the unique minimal $\mathcal{N}$ generated in degree $\leq k$ with an injection $\phi : H^{\leq k+1}(\mathcal{N}) \hookrightarrow \bar{H}^{\leq k+1}$ that is an isomorphism in degree $\leq k$. Then Lemma 3.5(i) shows that $\mathcal{F} : \mathcal{B}^{m+1}(\mathcal{N}) \to H^m(\mathcal{N})$ is injective, with image transverse to $H^m_{[k+1]}(\mathcal{N})$. Therefore we can define $\alpha \in H^m(\mathcal{N})^\vee$ by setting $\alpha = \bar{\alpha} \circ \phi$ on $H^m_{[k+1]}(\mathcal{N})$, and $\alpha = F \circ \mathcal{F}^{-1}$ on the image of $\mathcal{F}$, and extending arbitrarily to the rest of $H^m(\mathcal{N})$. That α is a k-partial Poincaré class follows from Lemma 3.3(i). The relevant uniqueness follows immediately from Lemma 3.5(ii).

It remains to prove the existence claim in the borderline case $m = 5n-2$, and this requires a little more work. While the generating spaces V^i of $\mathcal{N}$ are determined by $\bar{H}^*$ in degree $i < k = 3n-2$ (as in the proof of Lemma 3.5(i)), we must also describe V^{3n-2}. C^{3n-2} is just a direct complement to the image of $(\mathcal{G}^2 C^*)^{3n-2} \to H^{3n-2}$ like before.

The closed subspace of $\mathcal{N}^{3n-1}_{[3n-2]}$ is $(\mathcal{G}^2 C^*)^{3n-1} \oplus T^{3n-1}$, where T^{3n-1} is the closed subspace of $C^n \otimes N^{2n-1}$. Then $T^{3n-1} \cong K[H^n \otimes E^{2n}]$, and

$$C^{2n-1} \otimes T^{3n-1} \cong H^{2n-1} \otimes K[H^n \otimes E^{2n}] \supseteq H^{2n-1} \otimes (H^n)^{\otimes 3} \cap K[E^{3n-1} \otimes E^{2n}].$$

The RHS maps surjectively onto $\mathcal{B}^{5n-1}_k(H^*) \overset{\phi'}{\cong} \mathcal{B}^{5n-1}_k(\bar{H}^*)$. We can this restrict the given F to $\mathcal{B}^{5n-1}_k(\bar{H}^*) \to \mathbb{Q}$, pull back to $H^{2n-1} \otimes (H^n)^{\otimes 3} \cap K[E^{3n-1} \otimes E^{2n}]$ and then extend arbitrarily to a map $\widetilde{F} : C^{2n-1} \otimes T^{3n-1} \to \mathbb{Q}$. That is equivalent to an $F' : T^{3n-1} \to (C^{2n-1})^\vee \cong \bar{H}^{3n-1}$, characterised by $\widetilde{F}(c \otimes t) = \bar{\alpha}(\phi(c) F'(t))$. On the other hand, the algebra induces a map $a : \mathcal{G}^2 C^* \to \bar{H}^*$. We set $d : N^{3n-2} \to \mathcal{N}^{3n-1}_{[3n-2]}$ to map isomorphically to the kernel of

$$a + F' : (\mathcal{G}^2 C^*)^{3n-1} \oplus T^{3n-1} \to \bar{H}^{3n-1}.$$

Having thus defined $\mathcal{N}$, set $\phi : H^{\leq 3n-1}(\mathcal{N}) \to \bar{H}^{\leq 3n-1}$ to be induced by a in degrees $\leq 3n-2$, and by $a + F'$ in degree $3n-1$.

Note that $\mathcal{F}(\mathcal{B}^{5n-1}_k(\mathcal{N}))$ can be represented by elements of $C^{2n-1} \otimes T^{3n-1}$. The choices of $d : N^{3n-2} \to \mathcal{N}^{3n-1}_{[3n-2]}$ and ϕ ensure that the restriction of $\bar{\alpha} \circ \phi$ to $\mathcal{F}(\mathcal{B}^{5n-1}_k(\mathcal{N}))$ equals the restriction of $\widetilde{F}$, which in turn equals $F \circ \mathcal{F}^{-1}$ by construction. By Lemma 3.5(i) the intersection of $H^m_{[k+1]}(\mathcal{N})$ and the image of $\mathcal{F}$ is contained in $\mathcal{F}(\mathcal{B}^{5n-1}_k(\mathcal{N}))$. We can therefore once more define $\alpha \in H^m(\mathcal{N})^\vee$ by setting $\alpha = \bar{\alpha} \circ \phi$ on $H^m_{[k+1]}(\mathcal{N})$, and $\alpha = F \circ \mathcal{F}^{-1}$ on the image of $\mathcal{F}$ (and extending arbitrarily). □

Combining Proposition 3.8 and Proposition 3.4 completes the proof of Theorem 3.1.

3.4. **Minimal models and manifolds.** A *minimal model* of a DGA $\mathcal{A}$ is a minimal Sullivan algebra $\mathcal{M}$ together with a quasi-isomorphism $q : \mathcal{M} \to \mathcal{A}$, *i.e.* a DGA homomorphism whose induced map $q_\# : H^*(\mathcal{M}) \to H^*(\mathcal{A})$ is an isomorphism.

Recall that a minimal model of a CW complex X is a minimal model of $\Omega_{\mathrm{PL}}(X)$ and that for simply-connected CW complexes with rational cohomology of finite type every quasi-isomorphism of minimal models is realised by a rational homotopy equivalence of spaces, see Sullivan [30, §8] and Félix-Halperin [15, Proposition 17.13]. Theorem 1.2 now follows directly from the following claim.

Corollary 3.9. *Let $\mathcal{A}$ and $\mathcal{A}'$ be $(n{-}1)$-connected Poincaré DGAs of dimension $m \leq 5n{-}3$ $(n \geq 2)$, with minimal models $q : \mathcal{M} \to \mathcal{A}$ and $q' : \mathcal{M}' \to \mathcal{A}'$. If $G : H^*(\mathcal{A}) \to H^*(\mathcal{A}')$ is an isomorphism of the cohomology rings then there exists a DGA isomorphism $\phi : \mathcal{M} \to \mathcal{M}'$ such that $q_\# \circ \phi_\# = G \circ q'_\#$ if and only if the diagram below commutes.*

$$\begin{array}{ccc} \mathcal{B}^{m+1}(\mathcal{A}) & \xrightarrow{G} & \mathcal{B}^{m+1}(\mathcal{A}') \\ \downarrow{\scriptstyle \mathcal{F}} & & \downarrow{\scriptstyle \mathcal{F}'} \\ H^m(\mathcal{A}) & \xrightarrow{G} & H^m(\mathcal{A}') \end{array}$$

Proof. Apply Theorem 3.1(ii) to $(q'_\#)^{-1} \circ G \circ q_\#$. □

Next recall that a DGA $\mathcal{A}$ is said to be *formal* if there is a quasi-isomorphism $\widehat{q} : \mathcal{M} \to (H^*(\mathcal{A}), 0)$ from its minimal model $\mathcal{M}$—in other words, if $\mathcal{A}$ and $(H^*(\mathcal{A}), 0)$ have the same minimal model. A space X is formal if its DGA of piecewise linear de Rham forms is. The following proposition is therefore the algebraic formulation of Theorem 1.3.

Corollary 3.10. *An $(n{-}1)$-connected Poincaré DGA $\mathcal{A}$ of dimension $m \leq 5n{-}3$ is formal if and only if the Bianchi-Massey tensor $\mathcal{F} \colon \mathcal{B}^{m+1}(\mathcal{A}) \to H^m(\mathcal{A})$ is trivial.*

Proof. Since the DGA $(H^*(\mathcal{A}), 0)$ has $\mathcal{F} = 0$, the functoriality of the Bianchi-Massey tensor implies that if $\mathcal{A}$ is formal then its minimal model, and hence $\mathcal{A}$ itself, also have $\mathcal{F} = 0$.

Conversely if $\mathcal{F} = 0$ then we can let $\mathcal{A}' := (H^*(\mathcal{A}), 0)$, set $G \colon H^*(\mathcal{A}) \to H^*(\mathcal{A}')$ to be the tautological isomorphism and apply Corollary 3.9 to deduce that $\mathcal{A}$ is formal. □

Remark 3.11. Our reasoning has been guided by the notion of k-formality of Fernández and Muñoz [16]. They define M to be *k-formal* if one can choose the generating set V for a minimal model $\mathcal{M}$ and direct complements N^i to $C^i \subseteq V^i$ for $i \leq k$ so that the cohomology of the ideal $I_k := N^{\leq k}\mathcal{M}_{[k]} \subseteq \mathcal{M}_{[k]}$ maps trivially to $H^*(M)$. According to [16, Theorem 3.1], a closed orientable manifold of dimension $\leq 2k{+}1$ is formal if and only if it is k-formal—this can also be deduced from Proposition 3.4 (under the simplifying assumption of simple-connectedness).

For $m \leq 5n{-}3$ and $m{-}2n \leq k \leq 3n{-}3$, the algebraic considerations in Lemma 3.5 essentially identify the Bianchi-Massey tensor as a complete obstruction to k-formality for closed $(n{-}1)$-connected m-manifolds. One could thus prove Theorem 1.3 more briefly by appealing to the results of Fernández and Muñoz, but we have set up the argument to prove the more general Theorem 1.2 too.

We now move from rational classification to the classification of manifolds up to finite ambiguity. Specialising the results of Kreck and Triantafillou [23] to the context of this paper, we can interpret their results in terms of the Bianchi-Massey tensor, and deduce the following superficially stronger version of Theorem 1.9.

Proposition 3.12. *For $n \geq 2$ and $5 \leq m \leq 5n{-}3$, closed $(n{-}1)$-connected m-manifolds M are classified up to finite ambiguity by the truncated integral cohomology ring $H^{\leq \frac{m}{2}+1}(M;\mathbb{Z})$, the cohomology algebra $H^*(M)$, rational Pontrjagin classes $p_k(M) \in H^{4k}(M)$ and the Bianchi-Massey tensor $\mathcal{B}^{m+1}(M) \to H^m(M)$.*

Proof. Kreck and Triantafillou [23, Theorem 2.2] prove that the diffeomorphism type of a closed simply connected M of dimension $m \geq 5$ with formal $([\frac{m}{2}]{+}1)$-skeleton is determined up to finite ambiguity by the truncated cohomology ring $H^{\leq \frac{m}{2}+1}(M;\mathbb{Z})$, the rational Pontrjagin classes, and $\alpha_M \in H^m(\mathcal{M}_{[\frac{m}{2}]})^\vee$; here $\mathcal{M}_{[\frac{m}{2}]}$ is the subalgebra of the minimal model of M generated by elements of degree $\leq \frac{m}{2}$, and α_M is the pull-back of $\int_M \in H^m(M)^\vee$ under $H^m(\mathcal{M}_{[\frac{m}{2}]}) \to H^m(M)$. If M is $(n-1)$-connected and $m \leq 5n-3$ then the $([\frac{m}{2}]+1)$-skeleton is certainly formal, and Proposition 3.8 implies that α_M is determined up to isomorphism by the cohomology algebra and the Bianchi-Massey tensor. □

3.5. **Rational realisation.** In this subsection we prove Theorem 1.5 using Theorem 3.1(i) and rational surgery, adapted to the setting of $(n-1)$-connected manifolds. When the dimension m is not divisible by 4, we can proceed by making some minor adjustments to Sullivan's proof of [30, Theorem 13.2]. When $m = 4k$ the most convenient statement of rational surgery for generalisation to $(n-1)$-connected manifolds is Barge's [4, Theorem 1] and the best proof for these purposes is found in the PhD thesis of Su [29].

We are given $(H^*, p_*, \mathcal{F})$, an $(n-1)$-connected m-dimensional rational Poincaré duality algebra H^* ($m \leq 5n-2$), together with candidate Pontrjagin classes $p_* \in H^{4*}$ and a linear map $\mathcal{F}\colon \mathcal{B}^{m+1}(H^*) \to H^m$ that is a candidate for the Bianchi-Massey tensor. By Theorem 3.1(i), there exists a Sullivan minimal algebra $\mathcal{M}$ with $H^*(\mathcal{M}) = H^*$ and Bianchi-Massey tensor $\mathcal{F}$. By [30, §8] (see also [15, Theorem 17.10]), $\mathcal{M}$ is realised by a rational space X which is $(n-1)$-connected since $\mathcal{M}$ is $(n-1)$-connected. The cohomology classes $p_* \in H^{4*}(\mathcal{M}) = H^{4*}(X)$ define a map $p\colon X \to \Pi_{4i\geq n} K(\mathbb{Q}, 4i)$ to the indicated product of rational Eilenberg-MacLane spaces. If $BO\langle n\rangle$ denotes the $(n-1)$-connected cover of BO, then the universal Pontrjagin classes on $BO\langle n\rangle$ define a rational equivalence $p\langle n\rangle\colon BO\langle n\rangle \to \Pi_{4i\geq n} K(\mathbb{Q}, 4i)$ and we let Y be the space in the following pullback square:

$$\begin{array}{ccc} Y & \longrightarrow & BO\langle n\rangle \\ \downarrow & & \downarrow{\scriptstyle p\langle n\rangle} \\ X & \xrightarrow{p} & \Pi_{4i\geq n} K(\mathbb{Q}, 4i) \end{array}$$

We note that $Y \to X$ is a rational equivalence since $p\langle n\rangle$ is a rational equivalence. If we set T to be the Thom space of the stable bundle over Y induced by the map $Y \to BO\langle n\rangle$, then the stable homotopy groups of T satisfy $\pi_m^s(T) \otimes \mathbb{Q} \cong H_m(Y) \cong H_m(X) = \mathbb{Q}$, since $Y \to X$ is a rational equivalence. We wish to find a closed smooth m-manifold M together with a bundle map,

$$\begin{array}{ccc} \nu_M & \xrightarrow{\overline{f}} & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & Y, \end{array}$$

where ν_M denotes the stable normal bundle of M, ξ is the stable bundle over Y induced from the map $Y \to BO\langle n\rangle$ and $f\colon M \to Y$ is of non-zero degree. The existence of $(M, f, \overline{f})$ is automatic when $m \neq 4k$ and when $m = 4k$, it is proven by Su [29, Lemma 3.2.2] for the case when $BO\langle n\rangle = BSO$. Specifically, Su shows that there is class in $x \in \pi_{4k}^s(T)$ such that normal maps $(M, f, \overline{f})$ obtained from x via the Pontrjagin-Thom isomorphism satisfy $f_*([M]) = \alpha$. The argument there works just as well in our case as we have the correctly adapted assumption (iii) that the Pontrjagin numbers defined by α and p_* are those of a $(n-1)$-connected manifold.[1] Hence we have the desired normal map $(M, f, \overline{f})$. Since Y is $(n-1)$-connected, we may perform surgery below the middle dimension [36, §1] on $f\colon M \to Y$ to make M $(n-1)$-connected. We continue further with rational surgery as in the proof of [30, Theorem 13.2] to achieve that $f\colon M \to Y$ is a rational equivalence with M still $(n-1)$-connected using the assumption when $m = 4k$, that the intersection form defined by α is equivalent to a sum or squares. Then $(H^*(M), p_*(M), \mathcal{F}(M)) = (H^*(X), p_*, \mathcal{F})$, proving Theorem 1.5.

[1]We thank Jim Davis and Zhixu Su for explaining this point to us.

4. Coboundaries and integrality

In this section we explain how to compute the Bianchi-Massey tensor of a closed $(n-1)$-connected m-manifold M if M has a coboundary W such that the restriction homomorphism $H^*(W) \to H^*(M)$ is surjective in degree $\leq m-3n+1$. We can use this to construct compact manifolds W whose boundaries M realise a given cup-product structure and Bianchi-Massey tensor.

4.1. **Computing the Bianchi-Massey tensor via a coboundary.** Let W be a compact $(m+1)$-manifold with boundary M. We call W a *coboundary of M over $H^{\leq s}$* if the restriction map $j : H^*(W) \to H^*(M)$ is surjective in degrees $\leq s$. Then we can pick a right inverse $r : H^{\leq s}(M) \to H^{\leq s}(W)$ of j. We will denote r by $x \mapsto \hat{x}$.

In what follows we will assume that M is $(n-1)$-connected, and that W is a coboundary over $H^{\leq s}$ for $s := m-3n+1$. Then $E^{\leq n+s}$ is contained in $\mathcal{G}^2 H^{\leq s}$. Therefore composing r with the cup product of W induces a map $E^{\leq n+s} \to H^{\leq n+s}(W)$, $e \mapsto \hat{e}$. In fact, this takes values in $H^*_0(W) \subseteq H^*(W)$, the image of the natural map $H^*(W, M) \to H^*(W)$.

The condition that M is $(n-1)$-connected ensures in turn that $\mathcal{G}^2 E^{\leq n+s}$ and $\mathcal{G}^2 E^*$ are equal in degree $\leq m+1$, so in particular we obtain an induced map $(\mathcal{G}^2 E)^{m+1} \to (\mathcal{G}^2 H^*_0(W))^{m+1}$. The intersection form of W is a well-defined map $\lambda_W : (\mathcal{G}^2 H^*_0(W))^{m+1} \to \mathbb{Q}$. Let

$$A_W : (\mathcal{G}^2 E)^{m+1} \to \mathbb{Q} \tag{14}$$

be the composition. On the other hand, we identify the top degree part $\mathcal{B}^{m+1}(M) \to H^m(M)$ of the Bianchi-Massey tensor with a linear map $\mathcal{F}_M : \mathcal{B}^{m+1}(M) \to \mathbb{Q}$ by composition with the integration map of M.

Lemma 4.1. *Let M be a closed $(n-1)$-connected m-manifold, and let W be a coboundary of M over $H^{\leq s}$ for $s := m-3n+1$. Then the restriction of A_W to $\mathcal{B}^{m+1}(M)$ equals $\mathcal{F}_M$.*

Proof. Choose $\widehat{\alpha} : H^{\leq s}(M) \to \Omega^{\leq s}_{\mathrm{PL}}(W)$ so that

$$\begin{array}{ccc} H^{\leq s}(M) & \xhookrightarrow{\;r\;} & H^{\leq s}(W)\,. \\ \alpha\big\downarrow & \overset{\widehat{\alpha}}{\searrow} & \big\uparrow \\ \Omega^{\leq s}_{\mathrm{PL}}(M) & \twoheadleftarrow & \Omega^{\leq s}_{\mathrm{PL}}(W) \end{array}$$

commutes. For $ee' \in (\mathcal{G}^2 E^{\leq s+n})^{m+1}$, we then have that $\widehat{\alpha}^2(e), \widehat{\alpha}^2(e') \in \Omega^*_{\mathrm{PL}}(W)$ are representatives of $\hat{e}$ and $\hat{e}' \in H^*_0(W)$. If $d\gamma(e) = \alpha^2(e)$ as in the definition of $\mathcal{F}$ and $\rho : W \to [0,1]$ is a cut-off function supported on a collar neighbourhood of M then $\widehat{\alpha}^2(e) - d(\rho\gamma(e)) \in \Omega^*_{\mathrm{PL}}(W)$ represents a pre-image of $u \in H^*(W, M)$ of $\hat{e}$, and by definition $\lambda_W(\hat{e}\hat{e}') = u\,\hat{w}$. In the notation of (7), we can write this as

$$-A_W \;=\; \int_W (\widehat{\alpha}^2 - d(\rho\gamma))\cdot\widehat{\alpha}.$$

Hence, as maps $(\mathcal{G}^2 E)^{m+1} \to \mathbb{Q}$,

$$\int_M \gamma\cdot\alpha^2 \;=\; \int_W d\big(\rho\gamma\cdot\widehat{\alpha}^2\big) \;=\; A_W \;-\; \int_W \widehat{\alpha}^2\cdot\widehat{\alpha}^2.$$

The last term factors through $(\mathcal{G}^2 E)^{m+1} \to \mathcal{G}^4 H^*(M)$, so vanishes on $\mathcal{B}^{m+1}(M)$, while the restriction of the left hand side to $\mathcal{B}^{m+1}(M)$ equals $\mathcal{F}_M$ by definition. □

By (12) this also lets us compute Massey triple products using the coboundary W, so Lemma 4.1 is a generalisation of [19, Proposition 3.2.6].

If the cup product $H^*(M)\times H^*(M) \to H^*(M)$ is trivial in positive degrees (so $E^* = \mathcal{G}^2 H^{>0}(M)$), then the cup-square $\mathcal{G}^2 H^{>0}(W) \to H^*_0(W)$ and the intersection form (together with r) define a degree $m+1$ element of $\mathrm{Grad}^2\mathrm{Grad}^2 H^{>0}(M)^\vee$. In view of Remark 2.3, Lemma 4.1 means that the Bianchi-Massey tensor measures the failure of that 4-tensor to be fully graded symmetric. More generally, when M bounds over $H^{\leq s}$ we could use the lemma as the definition of the Bianchi-Massey tensor, and deduce that it is independent of the choice of coboundary from the full graded symmetry of the quadruple cup product on H^* of closed oriented manifolds.

The coboundary perspective is useful for understanding the relation between the Bianchi-Massey tensor and cohomology with integer coefficients. Recall that $\bar{c} : \mathcal{G}^2 H^*(M;\mathbb{Z}) \to H^*(M;\mathbb{Z})$ is the integral cup-square map. Let $\overline{E}^*$ denote the kernel of $\bar{c}$ modulo torsion (or equivalently, the pre-image of E^* under the map $\mathcal{G}^2 H^*(M;\mathbb{Z}) \to \mathcal{G}^2 H^*(M)$), and $\mathcal{B}^*(M;\mathbb{Z}) = K[\overline{E}^*]$, the kernel of $\mathcal{G}^2\overline{E}^* \to \mathcal{G}^4 H^n(M;\mathbb{Z})$. While it is hard to see how to define an integral version of the Bianchi-Massey tensor in terms of singular cochains, we may obviously define the "integral restriction" $\overline{\mathcal{F}}_M : \mathcal{B}^{m+1}(M;\mathbb{Z}) \to \mathbb{Q}$ as the composition of $\mathcal{F}_M$ with $\mathcal{B}^{m+1}(M;\mathbb{Z}) \to \mathcal{B}^{m+1}(M)$, as we did in the introduction.

If $M = \partial W$ and $H^{\leq s}(W;\mathbb{Z}) \to H^{\leq s}(M;\mathbb{Z})$ is onto, we shall say that W is a coboundary over $H^{\leq s}(M;\mathbb{Z})$. If M is $(n-1)$-connected of dimension m and $s = m-3n+1$, then we can relate the Bianchi-Massey tensor of M to the torsion linking form of M,

$$b_M \colon (\mathcal{G}^2 TH^*(M;\mathbb{Z}))^{m+1} \to \mathbb{Q}/\mathbb{Z},$$

by a straight-forward adaptation of Hepworth's argument relating b_M to Massey triple products [19, Proposition 3.1.6]. To describe this relationship we let $B_M : (\mathcal{G}^2\overline{E}^*)^{m+1} \to \mathbb{Q}/\mathbb{Z}$ be the homomorphism induced by

$$\overline{E}^i \times \overline{E}^{m+1-i} \to \mathbb{Q}/\mathbb{Z},\ (e, e') \mapsto b_M(\bar{c}(e), \bar{c}(e')). \tag{15}$$

Corollary 4.2. *Let W be a closed $(m+1)$-manifold with $(n-1)$-connected boundary M, which is a coboundary over $H^{\leq s}(M;\mathbb{Z})$ (for $s = m+1-3n$). Pick a right inverse $\bar{r} : H^{\leq s}(M;\mathbb{Z}) \to H^{\leq s}(W;\mathbb{Z})$, and define $\bar{A}_W : (\mathcal{G}^2\overline{E})^{m+1} \to \mathbb{Q}$ analogously to A_W in* (14). *Then*

$$\bar{A}_W = B_M \mod \mathbb{Z}$$

and

$$\overline{\mathcal{F}}_M = B_M|_{\mathcal{B}^{m+1}(M;\mathbb{Z})} \mod \mathbb{Z}.$$

Proof. For the first equality we recall that if $\bar{x} \in H^i(W;\mathbb{Z})$ and $\bar{y} \in H^{m+1-i}(W;\mathbb{Z})$ restrict to torsion classes $x, y \in H^*(M;\mathbb{Z})$, then by [1, Theorem 2.1]

$$b_M(x, y) = -\lambda_W(\bar{x}, \bar{y}). \tag{16}$$

The second equality follows immediately from the first together with Lemma 4.1. □

Remark 4.3. If an $(n-1)$-connected M admits any coboundary over $H^{\leq s}(M;\mathbb{Z})$ then the mod $\mathbb{Z}$ reduction of $\overline{\mathcal{F}}_M : \mathcal{B}^{m+1}(M;\mathbb{Z}) \to \mathbb{Q}$ is determined by the torsion linking form—in particular, if $H^*(M;\mathbb{Z})$ is torsion-free then $\overline{\mathcal{F}}$ takes integer values. Because of the non-commutativity of the cup product on singular cochains, we do not see a reason for this claim to be true in the absence of such a coboundary.

4.2. **Integral realisation.** We now turn to the problem of realising a prescribed integral restriction of the Bianchi-Massey tensor. This problem seems quite complicated in general, but by restricting attention to the critical case of $(n-1)$-connected $(4n-1)$-manifolds we can realise a large class of Bianchi-Massey tensors by boundaries of $4n$-manifolds.

We focus on the following basic invariants of an $(n-1)$-connected $(4n-1)$-manifold M:

(i) $H^{n*}(M;\mathbb{Z}) = H^0(M;\mathbb{Z}) \oplus H^n(M;\mathbb{Z}) \oplus H^{2n}(M;\mathbb{Z})$, the part of the cohomology ring supported in degree divisible by n (capturing in particular the product $\bar{c} \colon \mathcal{G}^2 H^n(M;\mathbb{Z}) \to H^{2n}(M;\mathbb{Z})$);
(ii) the linking form, $b_M \colon TH^{2n}(M;\mathbb{Z}) \times TH^{2n}(M;\mathbb{Z}) \to \mathbb{Q}/\mathbb{Z}$;
(iii) the Bianchi-Massey tensor, $\overline{\mathcal{F}}_M \colon \mathcal{B}^{4n}(M;\mathbb{Z}) \to \mathbb{Q}$.

Note that the domain of $\overline{\mathcal{F}}_M$ depends only on $H^{n*}(M;\mathbb{Z})$: it equals $\mathcal{B}^{4n}(H^{n*}(M;\mathbb{Z})) := K[P^2\overline{E}^{2n}]$, where $\overline{E}^{2n} := \ker(\rho \circ \bar{c})$ for $\rho \colon H^{2n}(M;\mathbb{Z}) \to H^{2n}(M;\mathbb{Z})/T$ the projection.

Hence we define a *linking model* as an algebraic model of M, which is a triple

$$(H^{n*}, b, \overline{\mathcal{F}})$$

where $H^{n*} = H^0 \oplus H^n \oplus H^{2n}$ is a graded ring with H^n torsion-free, $b \colon T \times T \to \mathbb{Q}/\mathbb{Z}$ is a nonsingular symmetric torsion form on the torsion subgroup $T \subseteq H^{2n}$, and $\overline{\mathcal{F}} \colon \mathcal{B}^{4n}(H^{n*}) \to \mathbb{Q}$ is a homomorphism. Recalling Corollary 4.2, we say that $\overline{\mathcal{F}}$ and b are *compatible* if

$$\overline{\mathcal{F}} = B|_{\mathcal{B}^{4n}(H^{n*})} \mod \mathbb{Z}, \tag{17}$$

where $B\colon P^2\overline{E}^{2n} \to \mathbb{Q}/\mathbb{Z}$ is defined from b as in (15). Our main realisation result is the following

Theorem 4.4. *Let $n \geq 2$, and let $(H^{n*}, b, \overline{\mathcal{F}})$ be a linking model with compatible b and $\mathcal{F}$. Then there exists some $(n{-}1)$-connected closed M^{4n-1} with an isomorphism $H^{n*}(M;\mathbb{Z}) \cong H^{n*}$ that identifies $(b_M, \overline{\mathcal{F}}_M) = (b, \overline{\mathcal{F}})$. In addition, we may assume that $H^*(M)$ is concentrated in degrees $* = 0, n, 2n{-}1, 2n, 3n{-}1$ and $4n{-}1$ and the same holds for $H^*(M;\mathbb{Z})$ when $n = 2, 4$ or n is odd.*

Before proving Theorem 4.4 we show how it implies Theorem 1.8 of the introduction, where we used simpler algebraic models for M. A *torsion free model* is a pair

$$(F^{n*}, \overline{\mathcal{F}}),$$

where $F^{n*} = F^0 \oplus F^n \oplus F^{2n}$ is a torsion-free ring and $\overline{\mathcal{F}}\colon \mathcal{B}^{4n}(F^{n*}) \to \mathbb{Q}$ is a homomorphism. With this terminology, Theorem 1.8 states that any torsion free model can be realised by an $(n{-}1)$-connected $(4n{-}1)$-manifold. Given a linking model $(H^{n*}, b, \overline{\mathcal{F}})$, we obtain a torsion free model $(H^{n*}/T, \overline{\mathcal{F}})$, so Theorem 1.8 follows immediately from Theorem 4.4 and the following

Lemma 4.5. *Given any torsion free model $(F^{n*}, \overline{\mathcal{F}})$ there is a linking model $(H^{n*}, b, \overline{\mathcal{F}})$ with compatible b and $\overline{\mathcal{F}}$ such that $F^{n*} = H^{n*}/T$.*

Proof. Choose any extension of $\overline{\mathcal{F}}\colon \mathcal{B}^{4n}(F^{n*}) \to \mathbb{Q}$ to $\overline{\mathcal{F}}'\colon P^2\overline{E}^{2n} \to \mathbb{Q}$ and let $B_0\colon P^2\overline{E}^{2n} \to \mathbb{Q}/\mathbb{Z}$ be the composition of $\overline{\mathcal{F}}'$ with the canonical surjection $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$. We define $R \subseteq \overline{E}^{2n}$ to be the radical of B_0:

$$R := \{r \in \overline{E}^{2n} \mid B_0(re) = 0 \ \forall e \in \overline{E}^{2n}\}.$$

We let $S \subseteq \overline{E}^{2n}/R$ be the torsion subgroup and fix a projection $\overline{E}^{2n}/R \to S$, which in turn defines a surjection $\pi\colon \overline{E}^{2n} \to S$. The map π induces a surjection $P^2(\pi)\colon P^2\overline{E}^{2n} \to P^2S$ such that B_0 factors over $P^2(\pi)$; *i.e.* B_0 induces a (possibly singular) symmetric torsion form

$$b_0\colon P^2S \to \mathbb{Q}/\mathbb{Z}.$$

Set $\widehat{S} := \mathrm{Hom}(S, \mathbb{Q}/\mathbb{Z})$ and define a nonsingular torsion form b by

$$b\colon (S \times \widehat{S}) \times (S \times \widehat{S}) \to \mathbb{Q}/\mathbb{Z}, \quad b\big((s_1, \sigma_1), (s_2, \sigma_2)\big) = b_0(s_1, s_2) + \sigma_1(s_2) + \sigma_2(s_1).$$

To finish the proof, let $p\colon \mathcal{G}^2F^n \to \overline{E}^{2n}$ be any projection, set $(T, b) := (S \oplus \widehat{S}, b)$, and let $H^n = F^n$ and $H^{2n} = F^{2n} \oplus T$, with product

$$\mathcal{G}^2F^n \to H^{2n} \oplus T, \quad ff' \mapsto (\overline{c}(ff'), \pi \circ p(ff')).$$

It is clear that $F^{n*} = H^{n*}/T$ as rings and that $\overline{\mathcal{F}}$ and b are compatible; *i.e.* satisfy (17). □

To prove Theorem 4.4, we note that it follows directly from the next two lemmas. Lemma 4.6 is a generalisation of results of Schmitt [27] and we defer its proof to the next subsection.

Lemma 4.6. *Let $n \geq 2$, and let a connected torsion-free ring $G^{n*} = G^0 \oplus G^n \oplus G^{2n}$ and an even symmetric bilinear form $\lambda\colon G^{2n} \times G^{2n} \to \mathbb{Z}$ be given. Then there is a compact $(n{-}1)$-connected $4n$-manifold $W = W(G^{n*}, \lambda)$ with $(n{-}1)$-connected boundary such that the following hold:*

(i) *W has the homotopy type of an $(n{-}1)$-connected $4n$-dimensional CW complex;*
(ii) *$H^*(W)$ is concentrated in dimensions $0, n$ and $2n$;*
(iii) *$H^*(W;\mathbb{Z})$ is concentrated in dimensions $0, n$ and $2n$ when $n = 2, 4$ or n is odd;*
(iv) *$H^{n*}(W;\mathbb{Z}) = G^{n*}$;*
(v) *$\lambda_W = \lambda$.*

Lemma 4.7. *Given a linking model $(F^{n*}, b, \overline{\mathcal{F}})$, we can find (G^{n*}, λ) with $G^n = F^n$ such that the manifold $W(G^{n*}, \lambda)$ of Lemma 4.6 has boundary M with*

$$(H^{n*}(M;\mathbb{Z}), b_M, \overline{\mathcal{F}}_M) \cong (F^{n*}, b, \overline{\mathcal{F}}).$$

Proof. By [34, Theorem 6] there is a nondegenerate symmetric bilinear form on a free abelian group G_2 such that $\lambda_2 \colon G_2 \times G_2$ presents $-b$. That is, there is a surjection $\pi_2 \colon G_2 \to T$ such that

$$\lambda_2(y_1, y_2) = -b\big(\pi_2(y_1), \pi_2(y_2)\big) \mod \mathbb{Z},$$

for all $(y_1, y_2) \in G_2 \times G_2$. We now turn to $\bar{c}\colon \mathcal{G}^2 F^n \to F^{2n}$ and recall that $\overline{E}^{2n} = \ker(\bar{c} \circ \rho) \subseteq \mathcal{G}^2 F^n$ is a summand. We fix a projection $p\colon \mathcal{G}^2 F^n \to \overline{E}^{2n}$ and note that $\bar{c}|_{\overline{E}^{2n}} \colon \overline{E}^{2n} \to T \subseteq F^{2n}$. Since $\overline{E}^{2n}$ is free, we can choose a homomorphism $q\colon \overline{E}^{2n} \to G_2$ such that $\pi_2 \circ q = \rho \circ \bar{c}|_{\overline{E}^{2n}} \colon \overline{E}^{2n} \to T$.

To apply Lemma 4.6, we choose (G^{n*}, λ_1) as follows:

(i) Set $G^n = F^n$ and $G^{2n} = F^{2n}/T \oplus G_2 \oplus \overline{E}^{2n} \oplus (\overline{E}^{2n})^\vee$,
(ii) Define the product $\mathcal{G}^2 G^n \to G^{2n}$ by $(\rho \circ \bar{c}, q \circ p, p, 0)$,
(iii) Define the symmetric bilinear form λ by $(G^{2n}, \lambda) = (G/T, 0) \oplus (G_2, \lambda_2) \oplus (\overline{E}^{2n} \oplus (\overline{E}^{2n})^\vee, \lambda_3)$,

where $\lambda_3\big((e_1, \alpha_1), (e_2, \alpha_2)\big) = \lambda_{\overline{E}}(e_1, e_2) + \alpha_1(e_2) + \alpha_2(e_1)$, for $(\lambda_{\overline{E}})$ an even symmetric bilinear form on $\overline{E}^{2n}$ which we shall vary as needed. From the exact sequence

$$\cdots \to H^{4n}(W, M; \mathbb{Z}) \to H^{4n}(W; \mathbb{Z}) \to H^{4n}(M; \mathbb{Z}) \to H^{4n+1}(W; \mathbb{Z}) \to \ldots$$

and the fact that $H^{4n+1}(W;\mathbb{Z}) = 0$, we have $H^{2n}(M;\mathbb{Z}) = F^{2n}/T \oplus T = F^{2n}$. By (16) $b_M = b$. By Lemma 4.6 $H^{n*}(W;\mathbb{Z}) = G^{n*}$, and so $\ker(\mathcal{G}^2 H^n(W;\mathbb{Z}) \to H^{2n}(W;\mathbb{Z})) = \overline{E}$ since it is the intersection of the kernels of $\bar{c}$, p and $q \circ p$. It follows that $\bar{c}_M \colon \mathcal{G}^2 H^n(M;\mathbb{Z}) \to H^{2n}(M;\mathbb{Z})$ is identified with $\bar{c}$.

It remains to determine $\overline{\mathcal{F}}_M$ and we do this using Lemma 4.1. Note that the map A that computes $\mathcal{F}_M$ in Lemma 4.1 is simply the map $P^2 E \to \mathbb{Q}$ induced by λ. It follows that $\overline{\mathcal{F}}_M = \overline{\mathcal{F}}_2 + \overline{\mathcal{F}}_{\overline{E}}$ where $\overline{\mathcal{F}}_2$ is induced by λ_2 and $\overline{\mathcal{F}}_{\overline{E}}$ is induced by $\lambda_{\overline{E}}$. By construction $\overline{\mathcal{F}}_M = \overline{\mathcal{F}}_2 \mod \mathbb{Z}$. Hence it remains to show that $\lambda_{\overline{E}}$ can be chosen to realise any integer-valued homomorphism $\overline{\mathcal{F}}\colon \mathcal{B}^{4n}(M;\mathbb{Z}) \to \mathbb{Z}$, and we do this in the following paragraphs.

Letting Sym^2_0 denote even symmetric bilinear forms, we thus want to prove that the composition of $\mathrm{Sym}^2_0(\overline{E}^{2n})^\vee \to (P^2 \overline{E}^{2n})^\vee$ with restriction to $\mathcal{B}^{4n}(H^*(M;\mathbb{Z}))$ maps onto $\mathcal{B}^{4n}(H^*(M;\mathbb{Z}))^\vee$. Given that there is an isomorphism $\mathrm{Sym}^2 \mathrm{Grad}^2 H^n(M;\mathbb{Z})^\vee \cong (P^2 \mathcal{G}^2 H^n(M;\mathbb{Z}))^\vee$, it suffices to prove the surjectivity mod 2.

Now, the annihilator of $\mathrm{Sym}^2_0 \mathrm{Grad}^2 H^n(M;\mathbb{Z}_2)^\vee$ in $P^2 \mathcal{G}^2 H^n(M;\mathbb{Z}_2)$ is the $\mathbb{Z}_2$-vector space of squares of elements of $\mathcal{G}^2 H^n(M;\mathbb{Z}_2)$. That clearly intersects trivially with $\mathcal{B}^{4n}(M;\mathbb{Z}_2)$, since expanding the square of a non-zero element $\mathcal{G}^2 H^n(M;\mathbb{Z}_2)$ to an element of $\mathcal{G}^4 H^n(M;\mathbb{Z}_2)$ can never give 0. □

4.3. **Proof of Lemma 4.6.** Our first step is to identify a finite $(n-1)$-connected $2n$-dimensional CW complex $K(G^{n*})$ such that $H^0(K) \oplus H^n(K) \oplus H^{2n}(K)$ realises the prescribed ring G^{n*}. Let $F = G^n$, a free abelian group, and let r be its rank. Let $K(F^\vee, n)$ be the indicated Eilenberg-MacLane space, and $K(F^\vee, n)^{(2n-2)}$ a $(2n-2)$-skeleton of $K(F^\vee, n)$. Attach $b_{2n-2}(K(F^\vee, n)^{(2n-2)})$ $(2n-1)$-cells to kill $H^{2n-2}(K(F^\vee, n)^{(2n-2)})$, calling the resulting complex K_0'. The space K_0' has the rational homotopy type of $\vee_{i=1}^r S^n$ and by the standard inductive construction of $K(F^\vee, n)$ as in [18, Example 4.17] and Serre's Theorem on the homotopy groups of simply-connected finite CW complexes [28], we may assume that K_0' is a finite CW-complex. We then set

$$K_0 := \begin{cases} \vee_{i=1}^r S^n & n = 2, 4 \text{ or } n \text{ odd}, \\ K_0' & \text{otherwise}. \end{cases}$$

Lemma 4.8. *Let G^{n*} be a torsion-free graded ring concentrated in degree 0, n and $2n$, with G^{2n} of rank s. Then for $i = 1, \ldots, s$, there are maps $\phi_i \colon S^{2n-1} \to K_0$, such for $\phi := \sqcup_{i=1}^s \phi_i$, the CW-complex*

$$K(G^{n*}) := K_0 \cup_\phi (\cup_{i=1}^s e^{2n})$$

has $H^{n*}(K;\mathbb{Z}) = G^{n*}$.

Proof. We give a proof that applies for both definitions of K_0. We recall the ith-Γ-group of a finite simply-connected CW-complex K, which is the group

$$\Gamma_i(K) := \mathrm{Im}\big(\pi_i(K^{(i-1)}) \to \pi_i(K^{(i)})\big),$$

where $K^{(i-1)} \to K^{(i)}$ is the inclusion of the $(i-1)$-skeleton of K into the i-skeleton of K. The Γ groups lie in Whitehead's long exact sequence

$$\cdots \to H_{i+1}(K;\mathbb{Z}) \xrightarrow{b} \Gamma_i(K) \xrightarrow{i_*} \pi_i(K) \xrightarrow{\rho} H_i(K;\mathbb{Z}) \to \dots,$$

where i_* is the obvious inclusion, ρ is the Hurewicz homomorphism and b is a certain "boundary homomorphism": see [5, Ch. 2]. Hence for $K = K(F^\vee, n)$ we have

$$b\colon H_{2n}(K(F^\vee, n);\mathbb{Z}) \cong \Gamma_{2n-1}(K(F^\vee, n)),$$

and by [12, Theorem 3.4.3] there is a natural surjective homomorphism

$$Q\colon H_{2n}(K(F^\vee, n);\mathbb{Z}) \to (\mathcal{G}^2 F)^\vee,$$

which is given by taking the cup squares of elements in $F = H^n(K(F^\vee, n);\mathbb{Z})$ and evaluating against $H_{2n}(K(F^\vee, n);\mathbb{Z})$.

Now consider $i_{0*}\colon \pi_{2n-1}(K_0) \to \Gamma_{2n-1}(K(F^\vee, n))$. We claim that $Q \circ b^{-1} \circ i_{0*}$ is onto. When $K_0 = K(F^\vee, n)^{(2n-2)}$, we have that i_{0*} is onto by definition and so $Q \circ b^{-1} \circ i_{0*}$ is onto. When $K_0 = \vee_{i=1}^r S^n$, we let $\{x_1, \dots, x_r\}$ be a basis for F. For $i \neq j$, the element $[x_i x_j] \in \mathcal{G}^2 F$ can be realised by Whitehead products $[\iota_i, \iota_j]$ where $\iota_k\colon S^n \to \vee_{i=1}^r S^n$ is the inclusion of the kth summand. When $n = 2, 4$, elements of the form $[x_i^2] \in \mathcal{G}^2 F$ can be realised by maps $\iota_i \circ h$, where $h\colon S^{2n-1} \to S^n$ has Hopf-invariant 1. For more details see [27, Prosition 3.11] in the case $n = 2$, the case $n = 4$ is analogous. Hence we choose $\phi_i \in \pi_{2n-1}(K_0)$ such that

$$Q \circ b^{-1} \circ i_{0*}(\phi_i) = \bar{c}^\vee(y_i^\vee),$$

where $\bar{c}^\vee\colon (G^{2n})^\vee \to (\mathcal{G}^2 F)^\vee$ is the dual to the product map $\bar{c}\colon \mathcal{G}^2 G^n \to G^{2n}$ and $\{y_1^\vee, \dots, y_s^\vee\}$ is a basis for $(G^{2n})^\vee$. By construction, we may then identify $H^{2n}(K(G^{n*});\mathbb{Z}) = G^{2n}$ and the cup product structure on $H^{n*}(K(G^{n*});\mathbb{Z})$ is given by the ring structure of G^{n*}. □

When $n = 2$, the construction of the manifolds $W = W(G^{n*}, \lambda)$ follows easily from the results of Schmitt [27, §3] which build on handlebody theory and classical embedding results of Haefliger. When $K_0 = K(F^\vee, n)^{(2n-2)}$, we may have many layers of handles to attach, and it is convenient to use the theory of thickenings as developed by Wall [35]. We briefly recall the notion of a thickening: Let K be a simply-connected finite connected CW-complex. An m-thickening of K is a pair (W, ϕ) where W is a compact m-manifold with simply-connected boundary ∂W and $\phi\colon K \to W$ is a homotopy equivalence. Since the map ϕ will be clear in our arguments from the discussion, we suppress it and from the notation and call W a thickening of K.

Now let (G^{n*}, λ) be as in the hypotheses of Lemma 4.6 and apply Lemma 4.8 to obtain the $2n$-complex $K = K(G^{n*})$. By [35, §3 Trivial thickening], there are unique $4n$-thickenings $W_0(K)$ of K and $W_0(K_0)$ of K_0 which are compact submanifolds of $\mathbb{R}^{4n}$ and which are called *trivial thickenings*. Moreover, by [35, Suspension Theorem], $W_0(K) \cong W_0'(K) \times D^1$, where $W_0'(K) \subset \mathbb{R}^{4n-1}$ also thickens K. It follows that $\lambda_{W_0(K)} = 0$ is the trivial form. By assumption, the required form λ on $W(G^{n*}, \lambda)$ is an even form, so it suffices to show how to modify the intersection form of $W_0(K)$ by any even form, without changing the cup-product structure.

By construction, $W_0(K) = W_0(K_0) \cup_\phi (\cup_{i=1}^s h_i^{2n})$ is obtained from the trivial thickening of K_0 by attaching s $2n$-handles $h_i^{2n} \cong D^{2n} \times D^{2n}$ along a framed embedding

$$\phi\colon \coprod_{i=1}^s (D^{2n} \times S^{2n-1}) \hookrightarrow \partial W_0(K_0),$$

where we attach one handle for each element of a basis of G^{2n}, which we assume has rank s. Now by [33, Lemma 1], every even symmetric bilinear form l is realised as the intersection form of handlebody $W_l = D^{4n} \cup_{\phi_0} (\cup_{i=1}^s h_{0j}^{2n})$ which is obtained by attaching $2n$-handles h_{0i}^{2n} along a framed embedding

$$\phi_0\colon \coprod_{i=1}^s (D^{2n} \times S^{2n-1}) \hookrightarrow D^{4n-1} \subset S^{4n-1}.$$

We take W_l to have the intersection form (G^{2n}, λ). Fixing an embedding $D^{4n-1} \hookrightarrow \partial W_0(K_0)$ disjoint from $\mathrm{Im}(\phi)$, we then form the framed embedding $\phi' = \phi + \phi_0$ by tubing together the components of ϕ and ϕ_0. We define

$$W(G^{n*}, \lambda) := W_0(K_0) \cup_{\phi'} (\cup_{i=1}^s h_i^{2n})$$

to be the manifold obtained by attaching $2n$-handles to $W_0(K_0)$ along ϕ'. Since $\mathrm{Im}(\phi_0) \subset D^{4n-1}$, $W = W(G^{n*}, \lambda)$ has that same homotopy type as $W_0(K)$ and hence the same cup-product structure. On the other hand, the intersection form of W is identified with the intersection form of W_l which is the intersection form required for Lemma 4.6. This completes the proof of Lemma 4.6 (and hence of Theorem 4.4).

Remark 4.9. If we put aside the Bianchi-Massey tensor and focus on realising cohomology algebras with certain features, then the ideas in the proof of Lemma 4.6 extend to higher dimensions. For instance, suppose that $n \geq 2$ and $G^0 \oplus G^n \oplus G^{2n}$ is a torsion-free graded ring, and that we wish to realise $G^{n*} \otimes \mathbb{Q}$ as $H^{\leq 2n}(M)$ of a closed $(n-1)$-connected m-manifold M with $m \geq 4n+1$. Then we can take K as in Lemma 4.8, let $W_0^m(K)$ be the trivial m-dimensional thickening of K, and let $M := \partial(W_0^m(K) \times I) = W_0^m(K) \cup_{\mathrm{Id}} W_0^m(K)$. Then $H^*(K;\mathbb{Z}) \cong H^*(W;\mathbb{Z}) \cong H^*(M;\mathbb{Z})$ for $* \leq m-2n-1$, so $H^{\leq 2n}(M) = G^{n*} \otimes \mathbb{Q}$.

5. Applications to $(4n-1)$-manifolds

In this section we discuss applications of the Bianchi-Massey tensor. We begin with the proof of Theorem 1.14 from the introduction, and then give examples where the Bianchi-Massey tensor is non-trivial despite all Massey products vanishing (or there not being any defined triple products at all).

We restrict our attention to $(n-1)$-connected $(4n-1)$-manifolds. Then the space $\mathcal{B}^{4n}(M)$, on which the significant components of the Bianchi-Massey tensor are defined, involves only $H^n(M)$ and the kernel E^{2n} of the cup product $\mathcal{G}^2 H^n(M) \to H^{2n}(M)$ (to make sense of the graded power $\mathcal{G}^2$, we interpret $H^n(M)$ as a graded vector space concentrated in degree n). The applications essentially reduce to understanding the details of how $\mathcal{B}^{4n}(M)$ depends on E^{2n}.

In the final section we briefly discuss the role of Bianchi-Massey tensor in the classification of simply-connected spin 7-manifolds.

5.1. **Intrinsic formality and the hard Lefschetz property.**

We now prove Theorem 1.14, on the intrinsic formality of closed $(n-1)$-connected $(4n-1)$-manifolds with $b_3 \leq 3$ and a hard Lefschetz property. In view of Corollary 1.13 it suffices to prove that $\mathcal{B}^{4n}(M) = 0$. By Poincaré duality, the hard Lefschetz property is equivalent to equivalent to $\mathrm{Ann}\, E^{2n} \subseteq \mathcal{G}^2 H^n(M)^\vee$ containing a non-degenerate bilinear form q. Hence Theorem 1.14 is a consequence of the following algebraic result.

Proposition 5.1. *Let V be a graded vector space of dimension ≤ 3, concentrated in degree n. Let E be a subspace of $\mathcal{G}^2 V$. If $\mathrm{Ann}\, E \subseteq \mathrm{Grad}^2 V^\vee$ contains a non-degenerate element q, then $K[P^2 E] = 0$.*

Proof. It is convenient to consider the dual picture. By the duality of the sequences (6), $K[\mathcal{G}^2 E] = 0$ if and only if the restriction of $\psi : \mathrm{Sym}^2\mathrm{Grad}^2 V^\vee \to K\!\!\!/[\mathrm{Sym}^2\mathrm{Anti}^2 V^\vee]$ to $\mathrm{Ann}\, P^2 E \subseteq \mathrm{Sym}^2\mathrm{Grad}^2 V^\vee$ is surjective. In terms of the Kulkarni-Nomizu product $\owedge$ described in Remark 2.4, the image $\psi(\mathrm{Ann}\, P^2 E)$ is $\mathrm{Ann}\, E \owedge \mathrm{Grad}^2 V^\vee$.

The case when n is odd is essentially trivial, because then q is a symplectic form on V and so $\dim V = 0$ or 2. If $\dim V = 2$ then $K\!\!\!/[\mathrm{Sym}^2\mathrm{Sym}^2 V^\vee]$ is one-dimensional, and it is easy to see that $q \owedge q$ is non-zero. In particular $q \owedge \mathrm{Alt}^2 V^\vee = K\!\!\!/[\mathrm{Sym}^2\mathrm{Sym}^2 V^\vee]$, so $K[P^2 E] = 0$.

For the case when n is even, Besse [6, 1.119] explains that $q \owedge \mathrm{Sym}^2 V^\vee$ is all of $K\!\!\!/[\mathrm{Sym}^2\mathrm{Alt}^2 V^\vee]$ for $\dim V \leq 3$; this is the same algebraic result that leads to the well-known fact from Riemannian geometry that the Riemann curvature is determined by the Ricci curvature in dimension ≤ 3. □

5.2. **Bianchi-Massey tensors without Massey products.** Let us now consider the question of when the Bianchi-Massey tensor can be non-trivial even though all Massey triple products vanish. According to Lemma 2.9, Massey triple products on a closed oriented m-manifold M correspond to evaluating the Bianchi-Massey tensor on elements of $\mathcal{B}^{m+1}(M)$ that are ordinary in the sense of Definition 2.10. Therefore the algebraic version of the question is whether there exist m-dimensional Poincaré duality algebras H^* where $\mathcal{B}^{m+1}(H^*)$ is not generated by ordinary elements.

As above, we restrict to the case when M is $(n-1)$-connected of dimension $4n-1$. Then $\mathcal{B}^{4n}(M) = K[P^2E^{2n}]$, so the question boils down to whether for a vector space V there is a subspace E of P^2V or Λ^2V (according to whether n is even or odd) such that $K[P^2E]$ is not generated by ordinary elements.

Let us begin with the case when n is even. Example 5.3 gives an example where $r := \dim V = 5$, and $K[P^2E]$ is non-trivial even though it contains *no* ordinary elements at all (giving rise to $(n-1)$-connected $(4n-1)$-manifolds where any Massey triple products that are defined are forced to vanish due to the symmetries—*e.g.* ones of the form $\langle x, y, x\rangle$—but can still have a non-trivial Bianchi-Massey tensor). We find it helpful to first present a dimension-counting argument to show that this is not an uncommon phenomenon when $r > 5$.

Lemma 5.2. *For any $r := \dim V \geq 6$ there exist $E \subseteq P^2V$ such that $K[P^2E]$ is non-trivial but contains no ordinary elements.*

Proof. Note that the condition that $K[P^2E]$ contain an ordinary element means that there are some 2-planes $A, B \subseteq V$ such that $AB := \{xy \mid x \in A, y \in B\} \subseteq P^2V$ is contained in E. Let us first consider the case when $A \neq B$, so that AB has dimension 4 rather than 3. $Gr_2(V) \times Gr_2(V)$ has dimension $4r-8$, and the space of k-planes $E \subseteq P^2V$ containing a fixed AB has dimension $(k-4)\left(\binom{r+1}{2} - k\right)$. So the space of k-planes containing some AB with $A \neq B$ has positive codimension in $Gr_k(P^2V)$ if

$$(k-4)\left(\binom{r+1}{2} - k\right) + 4r - 8 \ < \ k\left(\binom{r+1}{2} - k\right)$$

which reduces to

$$r - 2 \ < \ \binom{r+1}{2} - k.$$

Similarly, for the the space of k-planes containing some P^2A to have positive codimension in $Gr_k(P^2V)$ reduces to

$$2r - 4 \ < \ 3\left(\binom{r+1}{2} - k\right),$$

which is weaker than the above condition. Thus if $k \leq \binom{r}{2} + 1$ and $E \in Gr_k(P^2V)$ is generic then $K[P^2E]$ contains no ordinary elements. Now for $k := \binom{r}{2} + 1$ and $r \geq 6$

$$\dim P^2E \ = \ \binom{\binom{r}{2}+2}{2} \ \geq \ \binom{k+3}{4} \ = \ \dim P^4V.$$

Hence in this case any $E \in Gr_k(P^2V)$ has $K[P^2E]$ non-trivial. □

When $k = \binom{r}{2} + 2$, *i.e.* when E has codimension $r-2$, the expected codimension of the space of k-planes $E \subset P^2V$ containing a fixed AB equals the dimension of $Gr_2(V) \times Gr_2(V)$, and we would expect each E to contain a finite number of AB. We can turn this round as follows.

For each $(A, B) \in Gr_2(V) \times Gr_2(V)$ and $q \in \operatorname{Ann} E \subset P^2V^*$, the condition that $AB \subseteq E$ implies that A and B are orthogonal with respect to the bilinear form q, which imposes 4 constraints on (A, B). Now recall that mapping an oriented 2-plane with orthonormal basis x, y (w.r.t. some standard inner product) to $\langle x + iy\rangle \in \mathbb{P}(V \otimes \mathbb{C})$ embeds $\widetilde{Gr}_2(V) \hookrightarrow \mathbb{C}P^{r-1}$ with image the quadric $Q := \{z : \sum z_i^2 = 0\}$. That A and B are q-orthogonal translates to the condition that the image of (A, B) in $\mathbb{C}P^{r-1} \times \mathbb{C}P^{r-1}$ lies in the subset of $Q \times Q$ cut out by $q(z, z) = q(z, \bar{z}) = 0$. These correspond to sections of the line bundles $\mathcal{O}(1, 1)$ and $\mathcal{O}(1, -1)$ respectively (but only the first is holomorphic).

Writing c_1 and c_2 for the generators of the H^2 of the two $\mathbb{C}P^{r-1}$ factors, we see that the topological intersection number of $Q \times Q$ with $r-2$ such subsets is the $(c_1c_2)^{r-1}$ coefficient of $(2c_1)(2c_2)(c_1{+}c_2)^{r-2}(c_1{-}c_2)^{r-2}$, which equals $\pm 4\binom{r-2}{\frac{r}{2}-1}$ when r is even, and vanishes when r is odd. This counts each ordered pair of unoriented 2-planes (A, B) with $AB \subseteq E$ 4 times, and possibly with some cancelling signs.

- For $r = 3$ we expect that a generic codimension 1 subspace $E \subset P^2V$ should contain no AB. Indeed, if the generator $q \in \operatorname{Ann} E$ is non-degenerate then there can obviously be no q-orthogonal 2-planes (and moreover we already argued in Proposition 5.1 that $K[P^2E] = 0$ in that case).
- For $r = 4$ we expect that for any $E \subset P^2V$ of codimension 2, there should be at least two ordered pairs (A, B) of unoriented 2-planes such that AB is contained in E (and possibly only one unordered pair). If $\operatorname{Ann} E$ is spanned by two non-degenerate elements, then we could also see this by applying the Lefschetz fixed point theorem to the composition of the maps $\perp\colon Gr_2(V) \to Gr_2(V)$ that they define (since $\chi(Gr_2(V)) = 2$).

  If $\operatorname{Ann} E$ is spanned by $x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $\lambda_1x_1^2 + \lambda_2x_2^2 + \lambda_3x_3^2 + \lambda_4x_4^2$ with λ_i distinct, then the coordinate planes corresponding to each partition of $\{1, 2, 3, 4\}$ into two halves gives 6 ordered pairs of simultaneously orthogonal planes.
- For $r = 5$ the calculation suggests that there may be some choices of $q_1, q_2, q_3 \in \operatorname{Sym}^2V^*$ such that $E := \operatorname{Ann}\langle q_1, q_2, q_3\rangle \subseteq P^2V$ does not contain any AB.

Example 5.3. Let $E := \operatorname{Ann}\langle q_1, q_2, q_3\rangle \subseteq P^2\mathbb{Q}^5$ for $q_i \in \operatorname{Sym}^2\mathbb{Q}^5$ defined by

$$\begin{aligned} q_1 &= x_1x_4 + x_3x_5, \\ q_2 &= x_2x_5 + x_3x_4, \\ q_3 &= x_1^2 + x_1x_2 + x_2^2 + x_3^2 + x_4^2 + x_5^2. \end{aligned}$$

Suppose that A, B are orthogonal with respect to q_1 and q_2. Let $\pi \subset \mathbb{Q}^5$ be the 3-dimensional subspace $\{x_4 = x_5 = 0\}$. One can check that

(i) If A is contained in π then so is B, and vice versa.
(ii) In fact, if A intersects π non-trivially then B is contained in π, except if $A \cap \pi$ is spanned by an element of the form $(a^2, b^2, \pm ab, 0, 0)$.
(iii) If A and B are both transverse to π then they are equal.

Now if A and B are both contained in π then they are definitely not orthogonal with respect to q_3, because its restriction to π is non-degenerate. If A and B are equal they also cannot be q_3-orthogonal because q_3 is positive-definite. Finally,

$$\begin{aligned} q_3\big((a^2, b^2, \pm ab, 0, 0), (c^2, d^2, \pm cd, 0, 0)\big) &= a^2c^2 + \tfrac{1}{2}(a^2d^2 + b^2c^2) + b^2d^2 \pm abcd \\ &= \tfrac{1}{2}(a^2 + b^2)(c^2 + d^2) + \tfrac{1}{2}(ac \pm bd)^2 > 0 \end{aligned}$$

for any non-zero (a, b) and (c, d). Hence E contains no AB, but $K[P^2E]$ has dimension at least $\binom{13}{2} - \binom{8}{4} = 78 - 70 = 8$.

We can also show that it happens for $r \geq 11$ that $K[P^2E]$ is non-trivial even though E does not even contain any monomials xy. The image of $\mathbb{P}(V) \times \mathbb{P}(V) \to \mathbb{P}(P^2V)$, $(\langle x\rangle, \langle y\rangle) \mapsto \langle xy\rangle$ has dimension $2r - 2$, so is disjoint from a generic $E \subseteq P^2V$ of dimension $\binom{r+1}{2} - 2r + 1$. For $r \geq 11$, this makes $\dim P^2E > \dim P^4V$, so E automatically has $K[P^2E]$ non-trivial. This gives rise to $(2k{-}1)$-connected $(8k{-}1)$-manifolds which can have a non-trivial Bianchi-Massey tensor, even though there are no defined Massey triple products at all.

Now let us consider the case when n is odd. In this case, the dimension count argument is not especially sharp. For $r = 4$, the fact that $Gr_2(V)$ is 4-dimensional leads one to "expect" a generic 2-dimensional $E \subseteq \Lambda^2V$ to contain (the one-dimensional) Λ^2A for some $A \in Gr_2(V)$. Nevertheless we have the following simple example.

Example 5.4. Let $V = \mathbb{Q}^4$ and

$$E := \langle v_1 \wedge v_2 + v_3 \wedge v_4,\ v_1 \wedge v_3 - v_2 \wedge v_4,\ v_1 \wedge v_4 + v_2 \wedge v_3\rangle \subset \Lambda^2V,$$

for a basis $v_1, \dots, v_4 \in V$. Then $K[P^2E]$ does not contain any ordinary elements: indeed E contains no decomposable elements at all, so for any $x, y \in V$, $x \wedge y \in E$ implies that x and y are linearly dependent. However, $\dim P^2E = 6$ while $\dim \Lambda^4 V = 1$, so $\dim K[P^2E] = 5$ (it is clear that P^2E maps onto $\Lambda^4 V$).

Remark 5.5. We emphasise that by Theorem 1.5, every triple $(V, E, \mathcal{F} \in K[P^2E]^\vee)$ covered by Lemma 5.2 and Examples 5.3 and 5.4 is realised as $\big(H^n(M), \ker(\mathcal{G}^2 H^n(M) \to H^{2n}(M)), \mathcal{F}_M\big)$ for some closed $(n-1)$-connected $(4n-1)$-manifold M. A corresponding integral statement follows from Theorem 1.8.

5.3. **Some remarks on the classification of simply-connected spin 7-manifolds.** We begin this subsection by determining which linking models and Pontrjagin classes are realised by simply-connected spin 7-manifolds M. Let $p_M \in H^4(M;\mathbb{Z})$ be the spin characteristic class, related to the first Pontrjagin class by $2p_M = p_1(M)$. By [11, Lemma 2.2(i)], p_M is always even.

Proposition 5.6. *Given a linking model $(H^{2*}, b, \overline{\mathcal{F}})$ and $p \in 2H^4$, there is a 1-connected spin 7-manifold M with*

$$(H^{2*}(M;\mathbb{Z}), b_M, \overline{\mathcal{F}}_M, p_M) = (H^{2*}, b, \overline{\mathcal{F}}, p)$$

if and only if b and $\overline{\mathcal{F}}$ are compatible. Moreover, we may always assume that $TH^3(M;\mathbb{Z}) = 0$.

Proof. Given M, we first show that the linking form b_M and Bianchi-Massey tensor $\mathcal{F}_M$ are compatible. Let $K(H^2(M;\mathbb{Z}), 2)$ be the indicated Eilenberg-MacLane space. By [7, Proposition 4.2] the spin bordism group $\Omega_7^{spin}\big(K(H^2(M;\mathbb{Z}), 2)\big)$ vanishes. Hence M bounds over $H^2(M;\mathbb{Z})$ and by Corollary 4.2 b_M and $\overline{\mathcal{F}}_M$ are compatible.

The existence statement for any linking model $(H^{2*}, b, \overline{\mathcal{F}})$ with b and $\overline{\mathcal{F}}$ compatible follows immediately from Theorem 4.4. Hence it remains to determine the possible values of p_M. By [27, §3] the manifolds M of Theorem 4.4 realising a given $(H^{2*}, \overline{\mathcal{F}})$ can be assumed spin with p_M any element of $2H^4(M;\mathbb{Z}) \cong 2H^4$. □

Corollary 1.10 implies in particular that the invariants in Proposition 5.6 determine the diffeomorphism type of M up to a finite number of possibilities. We conclude with a discussion of how to pin down the remaining finite ambiguity.

The further invariants needed include the quadratic linking family and generalised Eells-Kuiper invariant from the 2-connected classification [11]. When M bounds over its normal 2-type in the sense of Kreck [22] (in particular, whenever $\pi_2(M)$ is torsion-free), one can adapt the coboundary description of the Bianchi-Massey tensor from Lemma 4.1 to define mod m extensions of $\overline{\mathcal{F}}$ for any integer m. One should further expect to be able to use such coboundaries to define some further generalised version of the generalised Kreck-Stolz invariants of Hepworth [19], which are based on [22, Theorem 6]. Assuming that $\pi_2(M)$ is torsion-free, so that $H^2(M;\mathbb{Z}/m) \cong H^2(M;\mathbb{Z}) \otimes \mathbb{Z}/m$, may help evade some subtleties. (Note also that Proposition 5.6 lets us realise every possible integral cohomology ring compatible with $\pi_2 M$ being torsion-free.)

Conjecture 5.7 (*cf.* [10, §2.c]). *Simply-connected spin 7-manifolds M with torsion-free $\pi_2(M)$ are classified up to spin diffeomorphism by the cohomology ring $H^*(M;\mathbb{Z})$, the spin characteristic class p_M, the torsion linking form b_M on $TH^4(M;\mathbb{Z})$ and its family of quadratic refinements, the generalised Eells-Kuiper invariant from* [11]*, the Bianchi-Massey tensor $\overline{\mathcal{F}}_M$ and its mod m extensions, and some variation of the generalised Kreck-Stolz invariants from* [19].

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School of Mathematics and Statistics, University of Melbourne, Parkville, VIC, 3010, Australia
E-mail address: dcrowley@unimelb.edu.au

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK
E-mail address: j.nordstrom@bath.ac.uk