Besson, Olivier An alternative to diagonal loading for implementation of a white noise array gain constrained robust beamformer. (2018) Signal Processing, 152. 79-82. ISSN 0165-1684

Short communication

An alternative to diagonal loading for implementation of a white noise array gain constrained robust beamformer

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A B S T R A C T

Diagonal loading is one of the most popular methods of robust adaptive beamforming, and the solution to many different problems aimed at producing beamformers which are robust to finite samples effects or/and steering vector errors. Among the latter, constraining the white noise array gain (WNAG) is a meaningful approach. However, relating the loading level to the desired WNAG is not straightforward. In this communication, using a generalized sidelobe canceler structure of the beamformer, we prove that the WNAG constraint can be encoded directly in the beamformer, and the latter can be obtained in a rather simple way from a specific eigenvector and without going through the diagonal loading step.

Keywords:
Robust adaptive beamforming
White noise array gain
Generalized sidelobe canceler

1. Problem statement

For about forty years, driven by the practical need to cope with uncertainties that unavoidably arise in any radar, sonar or communication system, an uninterrupted thread of research about robust adaptive beamforming has given rise to a vast literature and a myriad of techniques, with many different approaches proposed [1–4]. Yet, one of the earliest proposed methods, namely diagonal loading (DL) [5–8], still stands as a reference to which any newly proposed method is systematically compared. The main reason for such a preeminence is that 1)it performs very well and 2)diagonal loading emerges naturally as the solution to various and different optimization problems, all aimed at producing beamformers robust to either finite samples effects or steering vector errors, or both. Indeed, let us start with the minimum power distortionless response (MPDR) beamformer which solves [1]

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \quad \text{subject to } \boldsymbol{w}^H \boldsymbol{a}_0 = 1 \tag{1}$$

where $\hat{\boldsymbol{R}} = K^{-1} \sum_{k=1}^{K} \boldsymbol{x}_k \boldsymbol{x}_k^H$ stands for the sample covariance matrix computed from K independent snapshots $\boldsymbol{x}_k \in \mathbb{C}^N$ and $\boldsymbol{a}_0$ is the assumed signal of interest (SOI) steering vector. Assuming that $K \geq N$, the solution to (2) is given by

$$\boldsymbol{w}_{\text{MPDR}} = \frac{\hat{\boldsymbol{R}}^{-1} \boldsymbol{a}_0}{\boldsymbol{a}_0^H \hat{\boldsymbol{R}}^{-1} \boldsymbol{a}_0} \tag{2}$$

while a diagonally loaded beamformer writes

$$\boldsymbol{w}_{\text{DL}} = \alpha \left(\hat{\boldsymbol{R}} + \mu \boldsymbol{I} \right)^{-1} \boldsymbol{a}_0 \tag{3}$$

where μ stands for the loading level and α is some normalizing factor. The weight vector in (3) is in fact the solution to many problems. Indeed, with $\alpha = [\boldsymbol{a}_0^H (\hat{\boldsymbol{R}} + \mu \boldsymbol{I})^{-1} \boldsymbol{a}_0]^{-1}$, it solves (under the constraint that $\hat{\boldsymbol{R}} + \mu \boldsymbol{I} > \boldsymbol{0}$)

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} + \mu \|\boldsymbol{w}\|^2 \text{ subject to } \boldsymbol{w}^H \boldsymbol{a}_0 = 1 \tag{4}$$

which can be interpreted as a regularization of (1). Similarly, in the landmark paper by Cox et al. [9], DL is shown to be the solution when one wants to constrain the *white noise array gain* (WNAG), i.e., it is the solution to

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \text{ subject to } \boldsymbol{w}^H \boldsymbol{a}_0 = 1 \text{ and } \|\boldsymbol{w}\|^2 \leq A_{\text{WN}}^{-1} \tag{5}$$

where $A_{\text{WN}} \leq N$ is the desired WNAG. In the same vein, the minimization problems (6)–(10) which are stated and solved respectively in [10–14]

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \text{ subject to } \min_{\|\boldsymbol{a}-\boldsymbol{a}_0\| \leq \epsilon} \boldsymbol{w}^H \boldsymbol{a} \geq 1 \tag{6}$$

$$\min_{\boldsymbol{w}} \boldsymbol{w}^H \hat{\boldsymbol{R}} \boldsymbol{w} \text{ subject to } \min_{\|\boldsymbol{a}-\boldsymbol{a}_0\| \leq \epsilon} \text{Re}\left(\boldsymbol{w}^H \boldsymbol{a}\right) \geq 1 \tag{7}$$

$$\max_{P, \boldsymbol{a}} P \text{ subject to } \hat{\boldsymbol{R}} - P \boldsymbol{a} \boldsymbol{a}^H \geq \boldsymbol{0} \text{ for } \|\boldsymbol{a} - \boldsymbol{a}_0\|^2 \leq \epsilon^2 \tag{8}$$

$$\max_{P, \boldsymbol{a}} P \text{ subject to } \hat{\boldsymbol{R}} - P \boldsymbol{a} \boldsymbol{a}^H \geq \boldsymbol{0} \text{ for } \|\boldsymbol{a} - \boldsymbol{a}_0\|^2 \leq \epsilon^2 \text{ and } \|\boldsymbol{a}\|^2 = N \tag{9}$$

$$\min_{\boldsymbol{w}} \max_{\|\boldsymbol{\Delta}_1\| \leq \gamma} \boldsymbol{w}^H \left(\hat{\boldsymbol{R}} + \boldsymbol{\Delta}_1 \right) \boldsymbol{w} \text{ subject to } \min_{\|\boldsymbol{\Delta}_2\| \leq \epsilon} \boldsymbol{w}^H \left(\boldsymbol{a}_0 \boldsymbol{a}_0^H + \boldsymbol{\Delta}_2 \right) \boldsymbol{w} \geq 1 \tag{10}$$

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https://doi.org/10.1016/j.sigpro.2018.05.019

are all meant to produce beamformers which are robust to steering vector errors and/or to finite samples errors in $\hat{\boldsymbol{R}}$. It turns out that their solution can always be equivalently written as in (3) even if the solution is not computed explicitly in this form. For instance, in [10], the problem in (6) is recognized and formulated as a second order cone program and the solution is computed accordingly without resorting to computation of a loading level and a vector as in (3). Similarly, the approach we propose below computes the solution to (5) as (25), which is different from (3) but equivalent. In contrast, the problems in (7)–(10) are solved using Lagrange multipliers and the solution is indeed computed as in (3) where α and μ are calculated from the parameters of the problem, i.e., ϵ, γ. Therefore, while different implementations could be used to obtain the solutions to these optimization problems, the latter can always be equivalently written as a diagonally loaded beamformer.

That being said, diagonal loading is not the unique possible way to achieve robustness and many different approaches have been proposed in the literature. One of them is based on producing a better estimate of the SOI steering vector or/and a better estimate of the interference plus noise covariance matrix, see [15–19] for examples. Imposing additional constraints [20–25] or adopting a Bayesian perspective [26,27] to take into account steering vector errors also produces effective methods. This is usually achieved at the price of more complicated optimization problems. Another drawback of diagonal loading is the need to fix the loading level, or equivalently to fix A_{WN}, ϵ or γ in the beamformers (5)–(10). Therefore, parameter-free beamformers are of interest [28] and a number of papers have focused on finding automatically the optimal loading level, see e.g., [29,30]. In [29], ridge regression, also referred to as Hoerl–Kannard–Baldwin (HKB) method [31], is advocated. The idea is to adopt a generalized sidelobe canceler (GSC) structure so that the unit-gain constraint on the SOI is automatically fulfilled and one needs to solve a simple unconstrained least-squares (LS) problem with respect to the weight vector of the auxiliary channels. This LS problem is regularized and various choices of the regularizing parameter are proposed. Reference [30] considers a generalized linear combination $\tilde{\boldsymbol{R}} = \alpha \boldsymbol{I} + \beta \hat{\boldsymbol{R}}$ and proposes to estimate α and β from the data so as to minimize the mean-square error of $\tilde{\boldsymbol{R}}$.

In this communication, we still consider diagonal loading, or at least a problem whose solution is diagonal loading. Now, while the idea of automatic computation of the loading level is seducing, many engineers and practitioners may wish to have available a parameter that they can tune according to the application they consider, and their experience and knowledge about it. Considering the problems of (5)–(10), it seems to us that the WNAG constraint is the most physically appealing [9]. Indeed, WNAG can be interpreted as the area under the array beampattern. Moreover, we have an upper limit (N) for it and it makes sense to fix the WNAG with respect to this upper limit, depending on the tradeoff between mainbeam control and good sidelobe adaptation [32]. In contrast, it may not be easy to have a good idea of ϵ or γ, that is of the norm of the errors on either the steering vector or the sample covariance matrix. Therefore, in the sequel, we consider the problem in (5). Unless $\|\boldsymbol{w}_{\mathrm{MPDR}}\|^2 \leq A_{\mathrm{WN}}^{-1}$ in which case $\boldsymbol{w}_{\mathrm{MPDR}}$ is the solution, the latter is of the form (3) with $\alpha = [\boldsymbol{a}_0^H(\hat{\boldsymbol{R}} + \mu \boldsymbol{I})^{-1}\boldsymbol{a}_0]^{-1}$ and where the WNAG is equal to A_{WN} [13]. However, solving

$$\|\boldsymbol{w}_{\mathrm{DL}}\|^2 = A_{\mathrm{WN}}^{-1} \Leftrightarrow \frac{\boldsymbol{a}_0^H\left(\hat{\boldsymbol{R}} + \mu \boldsymbol{I}\right)^{-2}\boldsymbol{a}_0}{\left[\boldsymbol{a}_0^H\left(\hat{\boldsymbol{R}} + \mu \boldsymbol{I}\right)^{-1}\boldsymbol{a}_0\right]^2} = A_{\mathrm{WN}}^{-1} \tag{11}$$

is not straightforward [12,13]. Indeed, this requires eigenvalue decomposition of $\hat{\boldsymbol{R}}$ followed by the search of the (hopefully) unique solution to a scalar equation [12,13]. Our objective here is to provide an *alternative approach that can produce the solution to* (5) *without resorting to the diagonally loaded form, see Eq.* (25) *for an expression of this solution as a function of WNAG.*

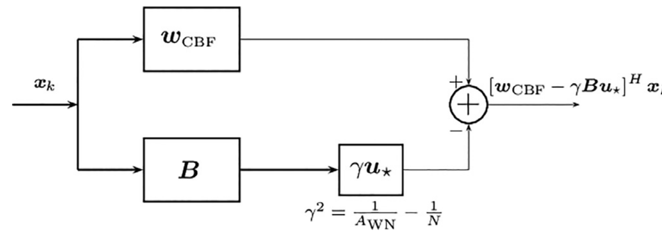

Fig. 1. Generalized sidelobe canceler structure of the solution. The white noise array gain is related to γ through $A_{\mathrm{WN}} = N/(1 + N\gamma^2)$.

2. Deriving the white noise array gain constrained beamformer

The basic idea behind our approach is to use a GSC structure, as illustrated in Fig. 1. Resorting to such a structure in order to derive a quadratically constrained beamformer has already been advocated, see e.g., [33]. The solution to (5) can be written as

$$\boldsymbol{w}_\star = \boldsymbol{w}_{\mathrm{CBF}} - \boldsymbol{B}\boldsymbol{w}_a \tag{12}$$

where $\boldsymbol{B} \in \mathbb{C}^{N \times (N-1)}$ is a semi-unitary blocking matrix, i.e., $\boldsymbol{B}^H\boldsymbol{B} = \boldsymbol{I}$, $\boldsymbol{B}^H\boldsymbol{a}_0 = \boldsymbol{0}$, $\boldsymbol{w}_{\mathrm{CBF}} = \boldsymbol{a}_0/(\boldsymbol{a}_0^H\boldsymbol{a}_0)$ and the output power can be minimized in an unconstrained manner with respect to $\boldsymbol{w}_a$. Noting that $\|\boldsymbol{w}_{\mathrm{CBF}} - \boldsymbol{B}\boldsymbol{w}_a\|^2 = \|\boldsymbol{w}_{\mathrm{CBF}}\|^2 + \|\boldsymbol{w}_a\|^2$, (5) is equivalent to

$$\min_{\|\boldsymbol{w}_a\| \leq \gamma} [\boldsymbol{w}_{\mathrm{CBF}} - \boldsymbol{B}\boldsymbol{w}_a]^H \hat{\boldsymbol{R}} [\boldsymbol{w}_{\mathrm{CBF}} - \boldsymbol{B}\boldsymbol{w}_a] \tag{13}$$

with $\gamma^2 = A_{\mathrm{WN}}^{-1} - N^{-1}$. Let us rewrite $\boldsymbol{w}_a = \gamma \boldsymbol{u}$ where $\boldsymbol{u}$ belongs to the complex sphere, i.e., $\|\boldsymbol{u}\| \leq 1$. The interest of this approach is that the *weight vector* $\boldsymbol{w}_{\mathrm{CBF}} - \gamma \boldsymbol{B}\boldsymbol{u}$ *is written as a function of* γ*, which is directly related to the desired WNAG*. Now, one can write

$$\begin{aligned} J(\boldsymbol{u}) &= [\boldsymbol{w}_{\mathrm{CBF}} - \gamma \boldsymbol{B}\boldsymbol{u}]^H \hat{\boldsymbol{R}} [\boldsymbol{w}_{\mathrm{CBF}} - \gamma \boldsymbol{B}\boldsymbol{u}] \\ &= \gamma^2 \boldsymbol{u}^H \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{B}\boldsymbol{u} - \gamma \boldsymbol{u}^H \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}} - \gamma \boldsymbol{w}_{\mathrm{CBF}}^H \hat{\boldsymbol{R}} \boldsymbol{B}\boldsymbol{u} + \boldsymbol{w}_{\mathrm{CBF}}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}} \\ &= \boldsymbol{u}^H \boldsymbol{\Omega} \boldsymbol{u} - \boldsymbol{u}^H \boldsymbol{\eta} - \boldsymbol{\eta}^H \boldsymbol{u} + \boldsymbol{w}_{\mathrm{CBF}}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}} \end{aligned} \tag{14}$$

with $\boldsymbol{\Omega} = \gamma^2 \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{B}$ and $\boldsymbol{\eta} = \gamma \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}}$. It thus remains to solve

$$\min_{\|\boldsymbol{u}\| \leq 1} \boldsymbol{u}^H \boldsymbol{\Omega} \boldsymbol{u} - \boldsymbol{u}^H \boldsymbol{\eta} - \boldsymbol{\eta}^H \boldsymbol{u}. \tag{15}$$

We now examine the various options to solve the problem in (15). The latter entails optimization over the complex unit sphere, a type of problem for which theory is now well grounded [34]. In fact, solving (15) can now be done conveniently with the help of existing off the shelf toolboxes such as Manopt [35]. We used the latter and checked that it provides the same solution as the diagonally loaded beamformer which satisfies the WNAG constraint (11). However, a somewhat simpler route can be taken to solve (15).

First, note that if $\boldsymbol{\Omega}$ is full-rank

$$J(\boldsymbol{u}) = \left[\boldsymbol{u} - \boldsymbol{\Omega}^{-1}\boldsymbol{\eta}\right]^H \boldsymbol{\Omega} \left[\boldsymbol{u} - \boldsymbol{\Omega}^{-1}\boldsymbol{\eta}\right] - \boldsymbol{\eta}^H \boldsymbol{\Omega}^{-1} \boldsymbol{\eta}. \tag{16}$$

Therefore, if $\boldsymbol{\Omega}$ is full-rank and $\left\|\boldsymbol{\Omega}^{-1}\boldsymbol{\eta}\right\| \leq 1$, then the solution is $\boldsymbol{u}_\star = \boldsymbol{\Omega}^{-1}\boldsymbol{\eta}$. Note that $\left\|\boldsymbol{\Omega}^{-1}\boldsymbol{\eta}\right\|^2 \leq 1 \Leftrightarrow \|\boldsymbol{w}_{\mathrm{MPDR}}\|^2 \leq A_{\mathrm{WN}}^{-1}$: in this case the MPDR beamformer satisfies the WNAG constraint without diagonal loading.

Next, let us consider the alternative cases, namely when $\boldsymbol{\Omega}$ is full-rank and $\left\|\boldsymbol{\Omega}^{-1}\boldsymbol{\eta}\right\| > 1$ or when $\boldsymbol{\Omega}$ is rank-deficient. It is known [36] that a solution $\boldsymbol{u}_\star$ ($\|\boldsymbol{u}_\star\| \leq 1$) to the problem exists if there exists $\lambda_\star \geq 0$ such that $\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I} > \boldsymbol{0}$ and $(\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I})\boldsymbol{u}_\star = \boldsymbol{\eta}$. For the two cases mentioned above, one necessarily has $\lambda_\star > 0$. Indeed, with a full-rank $\boldsymbol{\Omega}$, if $\lambda_\star = 0$ then $\boldsymbol{\Omega}\boldsymbol{u}_\star = \boldsymbol{\eta}$ implies that $\|\boldsymbol{u}_\star\| = \|\boldsymbol{\Omega}^{-1}\boldsymbol{\eta}\| > 1$ which violates the inequality constraint. If $\boldsymbol{\Omega}$ is rank-deficient, then $\lambda_\star = 0 \Rightarrow \boldsymbol{\Omega}\boldsymbol{u}_\star = \boldsymbol{\eta} \Rightarrow \boldsymbol{\eta} \in \mathcal{R}(\boldsymbol{\Omega})$. The latter condition would imply that $\boldsymbol{a}_0 \in \mathcal{R}(\hat{\boldsymbol{R}})$ which happens with probability 0. Note also that with $\boldsymbol{\Omega}$ rank-deficient and $\lambda_\star = 0$ the matrix $\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I}$ is no longer positive-definite. Therefore, in these two latter cases, $\lambda_\star > 0$. As already known, if this solution exists, then the corresponding weight vector in (12) can be written in the form of diagonal loading, as in (3). This can be easily checked:

$$\begin{aligned}\boldsymbol{w}_\star &= \boldsymbol{w}_{\mathrm{CBF}} - \gamma \boldsymbol{B}(\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I})^{-1}\boldsymbol{\eta} \\ &= \boldsymbol{w}_{\mathrm{CBF}} - \gamma^2 \boldsymbol{B}\left(\gamma^2 \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{B} + \lambda_\star \boldsymbol{I}\right)^{-1} \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}} \\ &= \boldsymbol{w}_{\mathrm{CBF}} - \boldsymbol{B}\left(\boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{B} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)^{-1} \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}}. \end{aligned} \tag{17}$$

Pre-mutiplying the previous equation by $\boldsymbol{B}^H\left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)$ and noting that $\boldsymbol{B}^H\boldsymbol{B} = \boldsymbol{I}$, it ensues that

$$\boldsymbol{B}^H\left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)\boldsymbol{w}_\star = \boldsymbol{B}^H\left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)\boldsymbol{w}_{\mathrm{CBF}} - \boldsymbol{B}^H \hat{\boldsymbol{R}} \boldsymbol{w}_{\mathrm{CBF}} = \boldsymbol{0}. \tag{18}$$

Therefore, since the orthogonal complement of $\boldsymbol{B}$ is spanned by vector $\boldsymbol{a}_0$, we have

$$\begin{aligned}&\left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)\boldsymbol{w}_\star \propto \boldsymbol{a}_0 \\ &\quad \Rightarrow \boldsymbol{w}_\star \propto \left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)^{-1}\boldsymbol{a}_0 \\ &\quad \Rightarrow \boldsymbol{w}_\star = \frac{\left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)^{-1}\boldsymbol{a}_0}{\boldsymbol{a}_0^H\left(\hat{\boldsymbol{R}} + \gamma^{-2}\lambda_\star \boldsymbol{I}\right)^{-1}\boldsymbol{a}_0}\end{aligned} \tag{19}$$

which proves that $\boldsymbol{w}_\star = \boldsymbol{w}_{\mathrm{CBF}} - \gamma \boldsymbol{B}\boldsymbol{u}_\star$ is indeed a diagonally loaded beamformer with $\gamma^{-2}\lambda_\star$ as the loading level. Simply, it is written in a different form.

Let us go back to solving (15) and let us write

$$\begin{aligned}\|\boldsymbol{u}_\star\|^2 &= \boldsymbol{\eta}^H(\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I})^{-2}\boldsymbol{\eta} \\ &= \boldsymbol{\eta}^H \boldsymbol{U}(\boldsymbol{\Lambda} + \lambda_\star \boldsymbol{I})^{-2}\boldsymbol{U}^H\boldsymbol{\eta} \quad \left[\boldsymbol{\Omega} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{U}^H\right] \\ &= \boldsymbol{z}^H(\boldsymbol{\Lambda} + \lambda_\star \boldsymbol{I})^{-2}\boldsymbol{z} \quad \left[\boldsymbol{z} = \boldsymbol{U}^H\boldsymbol{\eta}\right] \\ &= \sum_{n=1}^{N-1} \frac{|z_n|^2}{(\lambda_n + \lambda_\star)^2} = f(\lambda_\star).\end{aligned} \tag{20}$$

Ordering the eigenvalues in decreasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{N-1}$, $\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I} > \boldsymbol{0}$ implies that $\lambda_\star > -\lambda_{N-1}$. Moreover, it is easy to see that $f(\lambda)$ is monotonically decreasing over $]-\lambda_{N-1}, +\infty[$ and that $\lim_{\lambda \to -\lambda_{N-1}^+} f(\lambda) = +\infty$ and $\lim_{\lambda \to +\infty} f(\lambda) = 0$ so that there is a *unique* solution $\lambda_\star \in]-\lambda_{N-1}, +\infty[$ to $f(\lambda) = 1$. One could solve numerically Eq. (20) and then compute $\boldsymbol{u}_\star = (\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I})^{-1}\boldsymbol{\eta}$. This is essentially what was proposed in [33] and it much resembles the approach of [12,13]. However, we now prove that $\boldsymbol{u}_\star$ *can be obtained directly, as a specific eigenvector of a certain matrix.*

Let us rewrite $1 - f(\lambda)$ as

$$\begin{aligned}g(\lambda) &= 1 - \boldsymbol{z}^H(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-2}\boldsymbol{z} \\ &= \det\left(\boldsymbol{I} - \boldsymbol{z}\boldsymbol{z}^H(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-2}\right) \\ &= \det(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-1}\det\left((\boldsymbol{\Lambda} + \lambda \boldsymbol{I}) - \boldsymbol{z}\boldsymbol{z}^H(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-1}\right) \\ &= \det(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-2}\det\left(\begin{bmatrix}\boldsymbol{\Lambda} + \lambda \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{z}\boldsymbol{z}^H & \boldsymbol{\Lambda} + \lambda \boldsymbol{I}\end{bmatrix}\right) \\ &= \det(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-2}\det\left(\begin{bmatrix}\boldsymbol{\Omega} + \lambda \boldsymbol{I} & \boldsymbol{I} \\ \boldsymbol{\eta}\boldsymbol{\eta}^H & \boldsymbol{\Omega} + \lambda \boldsymbol{I}\end{bmatrix}\right) \\ &= \det(\boldsymbol{\Lambda} + \lambda \boldsymbol{I})^{-2}\det\left(\begin{bmatrix}\boldsymbol{\Omega} & \boldsymbol{I} \\ \boldsymbol{\eta}\boldsymbol{\eta}^H & \boldsymbol{\Omega}\end{bmatrix} + \lambda \boldsymbol{I}\right).\end{aligned} \tag{21}$$

Therefore $-\lambda_\star$ is an eigenvalue of $\boldsymbol{A} = \begin{bmatrix}\boldsymbol{\Omega} & \boldsymbol{I} \\ \boldsymbol{\eta}\boldsymbol{\eta}^H & \boldsymbol{\Omega}\end{bmatrix}$. However, we know that $\lambda_\star$ is the unique solution of $f(\lambda) = 1$ in $]-\lambda_{N-1}, +\infty[$. Therefore $\lambda_\star$ corresponds to the largest eigenvalue of $-\boldsymbol{A}$ or to the opposite of the smallest eigenvalue of $\boldsymbol{A}$. Let $\boldsymbol{v}_\star = \begin{bmatrix}\boldsymbol{v}_{1\star} \\ \boldsymbol{v}_{2\star}\end{bmatrix}$ denote the corresponding eigenvector. Then, $\boldsymbol{A}\boldsymbol{v}_\star = -\lambda_\star \boldsymbol{v}_\star$ implies that

$$\boldsymbol{\Omega}\boldsymbol{v}_{1\star} + \boldsymbol{v}_{2\star} = -\lambda_\star \boldsymbol{v}_{1\star} \tag{22a}$$

$$\boldsymbol{\eta}\boldsymbol{\eta}^H\boldsymbol{v}_{1\star} + \boldsymbol{\Omega}\boldsymbol{v}_{2\star} = -\lambda_\star \boldsymbol{v}_{2\star}. \tag{22b}$$

It ensues that $(\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I})\boldsymbol{v}_{2\star} = -(\boldsymbol{\eta}^H\boldsymbol{v}_{1\star})\boldsymbol{\eta}$. Now, we cannot have $\boldsymbol{\eta}^H\boldsymbol{v}_{1\star} = 0$, otherwise $\boldsymbol{\Omega}\boldsymbol{v}_{2\star} = -\lambda_\star \boldsymbol{v}_{2\star}$ and $\boldsymbol{v}_{2\star}$ would be an eigenvector of $\boldsymbol{\Omega}$ associated with eigenvalue $-\lambda_\star < 0$. Since $\boldsymbol{\Omega}$ is positive semi-definite, this implies that $\boldsymbol{v}_{2\star} = \boldsymbol{0}$, which in turn implies that $\boldsymbol{\Omega}\boldsymbol{v}_{1\star} = -\lambda_\star \boldsymbol{v}_{1\star}$ and would also lead to $\boldsymbol{v}_{1\star} = \boldsymbol{0}$ which is in contradiction with the fact that $\boldsymbol{v}_\star$ is an eigenvector. Additionally, we know that $(\boldsymbol{\Omega} + \lambda_\star \boldsymbol{I})\boldsymbol{u}_\star = \boldsymbol{\eta}$ which necessarily implies that

$$\boldsymbol{u}_\star = -\frac{\boldsymbol{v}_{2\star}}{\boldsymbol{\eta}^H\boldsymbol{v}_{1\star}}. \tag{23}$$

Therefore, we end up with a *direct solution to* (15), *which is obtained from the eigenvector of* $-\boldsymbol{A}$ *associated with its largest eigenvalue*. It turns out that $\boldsymbol{u}_\star$ is unit-norm (as expected) since premultiplying the second line of (22) by $\boldsymbol{v}_{1\star}^H$ gives

$$\begin{aligned}\boldsymbol{v}_{1\star}^H\boldsymbol{\eta}\boldsymbol{\eta}^H\boldsymbol{v}_{1\star} &= -\boldsymbol{v}_{1\star}^H\boldsymbol{\Omega}\boldsymbol{v}_{2\star} - \lambda_\star \boldsymbol{v}_{1\star}^H\boldsymbol{v}_{2\star} \\ &= -\left[\boldsymbol{\Omega}\boldsymbol{v}_{1\star} + \lambda_\star \boldsymbol{v}_{1\star}\right]^H\boldsymbol{v}_{2\star} \\ &= \boldsymbol{v}_{2\star}^H\boldsymbol{v}_{2\star}\end{aligned} \tag{24}$$

which implies that $\|\boldsymbol{u}_\star\| = \frac{\|\boldsymbol{v}_{2\star}\|}{(\boldsymbol{v}_{1\star}^H\boldsymbol{\eta}\boldsymbol{\eta}^H\boldsymbol{v}_{1\star})^{1/2}} = 1$.

To conclude, our WNAG-constrained GSC beamformer is thus

$$\boldsymbol{w}_\star = \boldsymbol{w}_{\mathrm{CBF}} - \left(\frac{N - A_{\mathrm{WN}}}{NA_{\mathrm{WN}}}\right)^{1/2}\boldsymbol{B}\boldsymbol{u}_\star \tag{25}$$

where $\boldsymbol{u}_\star$ is obtained from the eigenvector of $\boldsymbol{A}$ associated with its smallest eigenvalue. This beamformer guarantees a white noise array gain equal to A_{WN} while minimizing output power and enforcing a unit-gain in the SOI direction.

3. Conclusions

In this communication, we considered the problem of finding a robust beamformer with given white noise array gain. We proposed a solution different from diagonal loading. Using a generalized sidelobe canceler structure, we show that the WNAG constraint can be easily enforced and one is left with solving an optimization problem on the complex sphere. We showed that the solution to this problem is obtained as the eigenvector of a given matrix. Although the new beamformer can be written in an equivalent diagonal loading form, its implementation is significantly different and, moreover, the WNAG is directly encoded in the weight vector without the need to go through the step of diagonal loading.

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