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ON IDEMPOTENT N-ARY SEMIGROUPS

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Abstract

This thesis, which consists of two parts, focuses on characterizations and descriptions of classes of idempotent n-ary semigroups where $n \geq 2$ is an integer. Part I is devoted to the study of various classes of idempotent semigroups and their link with certain concepts stemming from social choice theory. In Part II, we provide constructive descriptions of various classes of idempotent n-ary semigroups.

More precisely, after recalling and studying the concepts of single-peakedness and rectangular semigroups in Chapters 1 and 2, respectively, in Chapter 3 we provide characterizations of the classes of idempotent semigroups and totally ordered idempotent semigroups, in which the latter two concepts play a central role. Then in Chapter 4 we particularize the latter characterizations to the classes of quasitrivial semigroups and totally ordered quasitrivial semigroups. We then generalize these results to the class of quasitrivial n-ary semigroups in Chapter 5. Chapter 6 is devoted to characterizations of several classes of idempotent n-ary semigroups satisfying quasitriviality on certain subsets of the domain. Finally, Chapter 7 focuses on characterizations of the class of symmetric idempotent n-ary semigroups.

Throughout this thesis, we also provide several enumeration results which led to new integer sequences that are now recorded in The On-Line Encyclopedia of Integer Sequences (OEIS). For instance, one of these enumeration results led to a new definition of the Catalan numbers.

Keywords. Semigroup, n-ary semigroup, band, quasitrivial semigroup, semilattice, Abelian group, reducibility, enumeration, Catalan numbers.

2020 *Mathematics Subject Classification.* Primary 06A12, 06F05, 20K25, 20M10, 20M14, 20N15 ; Secondary 05A15, 16B99.

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Introduction

Throughout the first part of this manuscript, X is a nonempty set and $n \geq 1$ is an integer. We often use the symbol X_n if X contains $n \geq 1$ elements, in which case we assume without loss of generality that $X_n = \{1, \ldots, n\}$. When considering enumeration problems, we often denote the empty set by X_0. Also, we denote the size of any set S by $|S|$. Finally, as we will only consider totally ordered algebraic structures we will often say that these algebraic structures are ordered.

A semigroup (X, F) is a nonempty set X together with an associative binary operation F. Natural examples of semigroups are the set of integers together with the addition and the set of functions from a set to itself together with the composition of functions. Since the concept of semigroup is very natural and fundamental to define the concepts of groups and rings, it can be said that the concept of semigroup has been omnipresent in mathematics since its earliest origins. However, the algebraic theory of semigroups is a much more recent development. Indeed, its origin stems back to the 20th century with most of the developments taking place after the Second World War. We refer to [55] for further historical background on the development of the algebraic theory of semigroups.

Among the pioneers of the algebraic theory of semigroups, Clifford [15] introduced and studied in 1954 the class of bands (i.e., idempotent semigroups), that is the class of semigroups whose associated binary operation F satisfies $F(x, x) = x$ for all $x \in X$. In particular, he showed that any band is a semilattice of rectangular semigroups, where a rectangular semigroup is nothing other than a direct sum of a left zero semigroup and a right zero semigroup up to isomorphism (as shown later in 1958 by Kimura [64]). At the same time, Clifford [16–20] also studied from a topological point of view the class of totally ordered semigroups, that is the class of semigroups whose associated binary operation F preserves a given total order on X. Then in 1962, Saîto [91] provided a technical characterization of the class of totally ordered bands based on properties of tree semilattices. In 1973, he also provided a characterization of the class of orderable bands [92], that is the class of bands for which there exists a total order on X that is preserved by its associated binary operation. Also, in 1971, Lyapin [72] provided a characterization of the class of quasitrivial semigroups, that is the class of semigroups whose associated binary operation F always outputs one of its input values, i.e. $F(x, y) \in \{x, y\}$ for all $x, y \in X$. In particular, he showed that a semigroup is quasitrivial if and only if it is an ordinal sum of projections (see also [73]). Prior to this, in 1928, Dörnte [44] proposed a generalization of the concept of semigroup to n-ary operations, where $n \geq 2$ is an integer. This generalization has been further developed by Post [82] in 1940 in the framework of n-ary groups. Since then, many authors have investigated the classes of n-ary semigroups and n-ary groups (see, e.g., [23, 37, 38, 40–43, 51, 66, 68–70, 75]). In this context, several authors have been interested in the quest for conditions under which an n-ary associative operation can be expressed as a composition of a single associative operation (see, e.g., [1, 23, 39, 43, 66, 70, 74, 75]). In that case, the n-ary associative operation is said to be reducible to that associative operation. For instance, Dudek and Mukhin [43] showed in 2006 that an n-ary semigroup is reducible to a semigroup if and only if one can adjoin a neutral element to the n-ary semigroup. However, this result does not entirely solve the reducibility problem as it is in general not easier to check whether one can adjoin a neutral element to an n-ary semigroup than to check whether it has a binary reduction. Moreover, it seems that the class of idempotent n-ary semigroups has been barely investigated in the literature thus far. For instance, Ackerman [1] provided only in 2011 a characterization of the class of quasitrivial n-ary semigroups in terms of binary reductions. More precisely, he showed that almost every quasitrivial n-ary semigroup is reducible to a semigroup.

In this thesis, we essentially provide and investigate characterizations and descriptions of classes of idempotent semigroups as well as some relevant subclasses of idempotent n-ary semigroups.

The first part is devoted to the study of several classes of ordered bands and their surprising link to some concepts arising in social choice theory such as single-peakedness. More precisely, in Chapter 1 we introduce and extend several concepts related to single-peakedness for weak orders. In particular, we provide characterizations of these concepts that enable us to visualize them through the Hasse diagram of the considered weak order. Then in Chapter 2 we investigate the class of rectangular semigroups that will be useful in the subsequent chapters. Chapter 3 is devoted to the study of the class of bands. Specifically, we recall characterizations of several subclasses of bands, including the class of ordered bands. Then we particularize the latter characterization to the class of commutative ordered bands by introducing a generalization of the concept of single-peakedness to semilattice orders. Surprisingly, the enumeration of the class of commutative ordered bands provides a new definition of the Catalan numbers. In Chapter 4, we investigate the class of quasitrivial semigroups. In particular, we recall a characterization of this class and particularize it to the subclass of ordered quasitrivial semigroups, in which the concept of single-peakedness plays a central role. Then we further investigate the class of quasitrivial semigroups by classifying its elements into subclasses defined by relevant equivalence relations. We also provide several enumeration results, which lead to new integer sequences that are now available in The On-Line Encyclopedia of Integer Sequences (OEIS) [94].

The second part of this thesis is devoted to the study of classes of idempotent n-ary semigroups, which have been barely investigated in the literature thus far. Chapter 5 focuses on the class of quasitrivial n-ary semigroups. After recalling the main results obtained by Ackerman [1], we show that every quasitrivial n-ary semigroup is reducible to a semigroup. Then we provide necessary and sufficient conditions for which this binary reduction is quasitrivial and unique. In particular, we provide characterizations of the class of quasitrivial n-ary semigroups based on binary reductions which are essential to construct those n-ary semigroups. In Chapter 6, we investigate the class of idempotent n-ary semigroups that satisfy quasitriviality on certain subsets of the domain. This investigation leads to nested subclasses of idempotent n-ary semigroups for which we provide various characterizations. In particular, we show that every n-ary semigroup in these classes is reducible to a semigroup. This result provides an easy way to construct those n-ary semigroups. In Chapter 7, we provide a characterization of the class of symmetric idempotent n-ary semigroups in terms of strong semilattices of right zero semigroups and n-ary extensions of Abelian groups whose exponents divide $n-1$. This characterization enables us to easily construct symmetric idempotent n-ary semigroups. Moreover, in contrast to the previous n-ary semigroups, we observe that symmetric idempotent n-ary semigroups are in general not reducible to semigroups. Therefore, we also provide necessary and sufficient conditions that ensure the existence of a binary reduction for a symmetric idempotent n-ary semigroup.

The main results of this thesis constitute new contributions to the problems of characterizing and enumerating classes of idempotent n-ary semigroups. Most of these contributions are reported in the following articles:

- [22] M. Couceiro and J. Devillet. Every quasitrivial n-ary semigroup is reducible to a semigroup. *Algebra Universalis*, 80(4), 2019.
- [24] M. Couceiro, J. Devillet, and J.-L. Marichal. Characterizations of idempotent discrete uninorms. *Fuzzy Sets and Syst.*, 334:60–72, 2018.
- [25] M. Couceiro, J. Devillet, and J.-L. Marichal. Quasitrivial semigroups: characterizations and enumerations. *Semigroup Forum*, 98(3):472–498, 2019.
- [26] M. Couceiro, J. Devillet, J.-L. Marichal, and P. Mathonet. Reducibility of n-ary semigroups: from quasitriviality towards idempotency. *Contributions to Algebra and Geometry*, submitted for revision. arXiv:1909.10412.
- [31] J. Devillet. Bisymmetric and quasitrivial operations: characterizations and enumerations. *Aequat. Math.*, 93(3):501–526, 2019.
- [32] J. Devillet and G. Kiss. Characterizations of biselective operations. *Acta Math. Hungar.*, 157(2):387–407, 2019.
- [33] J. Devillet, G. Kiss, and J.-L. Marichal. Characterizations of quasitrivial symmetric nondecreasing associative operations. *Semigroup Forum*, 98(1):154–171, 2019.
- [34] J. Devillet, J.-L. Marichal, and B. Teheux. Classifications of quasitrivial semigroups. *Semigroup Forum*, in press. https://doi.org/10.1007/s00233-020-10087-5.
- [35] J. Devillet and P. Mathonet. On the structure of symmetric n-ary bands. arXiv:2004.12423.
- [36] J. Devillet and B. Teheux. Associative, idempotent, symmetric, and order-preserving operations on chains. *Order*, in press. https://doi.org/10.1007/s11083-019-09490-7.

We end this introduction by giving a list of the top five of the new contributions presented in this thesis:

1. We provide characterizations of single-peakedness and related properties (see Propositions 1.5, 1.9, and 1.12) to establish descriptions of classes of ordered quasitrivial semigroups (see Proposition 4.21 and Corollary 4.31). These results reveal surprising links between semigroup theory and social choice theory.

2. By solving several enumeration issues, we show that the number of ordered commutative bands on X_n is precisely the nth Catalan number (see Proposition 3.36).

3. We prove that every quasitrivial n-ary semigroup is reducible to a semigroup (see Corollary 5.6). We also provide necessary and sufficient conditions for this binary reduction to be unique and quasitrivial (see Theorem 5.26).

4. We provide a characterization of the class of quasitrivial n-ary semigroups in terms of binary reductions which enables us to easily construct those n-ary semigroups (see Corollaries 5.25 and 5.28).

5. We characterize and describe the class of symmetric idempotent n-ary semigroups (see Theorem 7.22). We also provide necessary and sufficient conditions that ensure the reducibility of any symmetric idempotent n-ary semigroup to a semigroup (see Theorem 7.26).

Part I

Idempotent semigroups and single-peakedness

Chapter 1

Single-peakedness and related properties

Single-peakedness is a property of total orders that arose in social choice theory,[1] where it provides a way to overcome the Condorcet paradox [10]. In this chapter, we recall the definition of single-peakedness and extend it to the general case where the domain X may be infinite. To this extent, we first recall some concepts related to ordered and preordered sets (Section 1.1). Then we introduce and study other properties stemming from social choice theory that extend the single-peakedness property to weak orders [9–11, 71]. In particular, we provide characterization results based on these properties in terms of properties of the Hasse graph of the corresponding orders (Section 1.2). These properties will be used in Chapter 4 in order to characterize the classes of ordered and orderable quasitrivial semigroups, respectively. When X is finite, we compute the sizes of various classes of orders satisfying those properties (Section 1.3). Most of the contributions presented in this chapter stem from [25, 31, 33, 34].

1.1 Ordered and preordered sets

In this section, we introduce some relevant concepts from order theory and set up the terminology that will be used throughout this chapter. We refer to [29, 50] for an introduction to order theory.

Recall that a binary relation R on X is said to be

- *total* if $\forall x, y$: xRy or yRx;
- *reflexive* if $\forall x$: xRx;
- *transitive* if $\forall x, y, z$: xRy and yRz implies xRz;
- *antisymmetric* if $\forall x, y$: xRy and yRx implies $x = y$;
- *asymmetric* if $\forall x, y$: xRy implies $\neg(yRx)$;
- *symmetric* if $\forall x, y$: xRy if and only if yRx;

[1] Social choice theory is a multidisciplinary research field that lies at the intersection of welfare economics, decision theory, and voting theory. In particular, it focuses on the aggregation of individual inputs (such as votes or preferences) into collective outputs (such as collective decisions or preferences). We refer to [11] and [71] for a historical background on social choice theory.

It is easy to see that any total binary relation on X is reflexive.

An *equivalence relation on* X is a binary relation on X that is reflexive, transitive, and symmetric. In what follows, unless stated otherwise, we use the notation $\sim$ for any equivalence relation.

A *partial order on* X is a binary relation $\preceq$ on X that is reflexive, transitive, and antisymmetric. In that case, the ordered pair $(X, \preceq)$ is called a *partially ordered set* (or a *poset* for short). We denote the asymmetric part of $\preceq$ by $\prec$. Also, if a partial order $\preceq$ on X is total, then it is called a *total order* and $(X, \preceq)$ is called a *totally ordered set* (or a *chain*). In what follows, unless stated otherwise, we use the notation $\leq$ for any total order. For any integer $n \geq 1$, the pair $(X_n, \leq_n)$ represents the set $X_n = \{1, \ldots, n\}$ endowed with the total order $\leq_n$ defined by $1 <_n \cdots <_n n$.

More generally, a *preorder on* X is a binary relation $\precsim$ on X that is reflexive and transitive. The ordered pair $(X, \precsim)$ is then called a *preordered set* . We denote the symmetric and asymmetric parts of $\precsim$ by $\sim$ and $\prec$, respectively. Thus, we have $x \sim y$ if and only if $x \precsim y$ and $y \precsim x$. Also, we have $x \prec y$ if and only if $x \precsim y$ and $\neg(y \precsim x)$. Recall also that $\sim$ is an equivalence relation on X and that $\prec$ induces a partial order on the quotient set $X/{\sim}$. Thus, defining a preorder on X amounts to defining a partially ordered partition of X. For any $a \in X$, we use the notation $[a]_\sim$ to denote the equivalence class of a for $\sim$, i.e., $[a]_\sim = \{x \in X : x \sim a\}$. Also, we say that two elements $x, y \in X$ are *incomparable*, and we write $x \,||\, y$, if $x \not\precsim y$ and $y \not\precsim x$. For any $Y \subseteq X$, we denote by $\precsim\!|_Y$ the *restriction of* $\precsim$ *to* Y. For simplicity, we often write $(Y, \precsim)$ for $(Y, \precsim\!|_Y)$. Moreover, for any nonempty subsets Y, Z of X we write $Y \precsim Z$ if $y \precsim z$ for any $y \in Y$ and any $z \in Z$. Finally, if a preorder $\precsim$ on X is total, then it is called a *weak order* and $(X, \precsim)$ is called a *weakly ordered set*.

If $(X, \precsim)$ is a preordered set, then an element $a \in X$ is said to be *maximal* (resp. *minimal*) *for* $\precsim$ if for any $x \in X$ such that $a \precsim x$ (resp. $x \precsim a$) we have $a \sim x$. We denote the set of maximal (resp. minimal) elements of X for $\precsim$ by $\max_{\precsim} X$ (resp. $\min_{\precsim} X$). Note that this set need not be nonempty (consider, e.g., the set $\mathbb{Z}$ of integers endowed with the usual total order $\leq$).

Recall that for a given preorder $\precsim$ on X the *dual preorder* of $\precsim$ is the preorder $\precsim^d$ on X defined by $x \precsim^d y$ if and only if $y \precsim x$.

A nonempty subset I of a preordered set $(X, \precsim)$ is an *ideal* if it is a directed lower set, that is, if $x \in X$ and $y, z \in I$ are such that $x \precsim y$, then $x \in I$ and there exists $u \in I$ such that $y \precsim u$ and $z \precsim u$. A nonempty subset H of a preordered set $(X, \precsim)$ is a *filter* if it is an ideal in $(X, \precsim^d)$. For every element x of a preordered set $(X, \precsim)$, the sets $(x]_{\precsim} = \{y \in X : y \precsim x\}$ and $[x)_{\precsim} = \{y \in X : x \precsim y\}$ are the *ideal* and the *filter* generated by x, respectively.

An ideal I is said to be *principal* if there is $x \in X$ such that $I = (x]_{\precsim}$. *Principal filters* are defined dually. In particular, in a finite preordered set, all ideals are principal. In a partially ordered set $(X, \preceq)$, we set $[a, b]_{\preceq} = \{x \in X : a \preceq x \preceq b\}$ and $]a, b[_{\preceq} = \{x \in X : a \prec x \prec b\}$ for every $a \preceq b$ in X. When there is no risk of confusion, we might write $[a, b]$ (resp., $]a, b[$) instead of $[a, b]_{\preceq}$ (resp., $]a, b[_{\preceq}$). A subset C of a partially ordered set $(X, \preceq)$ is said to be *convex (for* $\preceq$*)* if it contains $[a, b]$ for any $a, b \in C$ with $a \preceq b$.

If $(X, \precsim)$ is a preordered set, then an element $x \in X$ is said to *cover* an element $y \in X$, and we write $y \prec\!\cdot\, x$, if $y \prec x$ and there exist no $z \in X$ such that $y \prec z \prec x$.

Now, let $\precsim$ be a preorder on X. The *Hasse graph* of $(X, \precsim)$ is the undirected graph $(X/{\sim}, E)$, where

$$E = \{\{[x]_\sim, [y]_\sim\} : [x]_\sim \prec\!\cdot\, [y]_\sim \text{ or } [y]_\sim \prec\!\cdot\, [x]_\sim\}.$$

We can always represent the Hasse graph of $(X_n, \precsim)$ in the following way. To any element in $X_n/{\sim}$ we assign exactly one point in the plane $\mathbb{R}^2$. Also, an edge joins two points $[x]_\sim, [y]_\sim \in$

$X_n/{\sim}$ if $\{[x]_\sim, [y]_\sim\} \in E$. Finally, we draw a point $[x]_\sim \in X_n/{\sim}$ below another point $[y]_\sim \in X_n/{\sim}$ if $[y]_\sim$ covers $[x]_\sim$. This representation of the Hasse graph of $(X_n, \precsim)$ is also called the *Hasse diagram* of $(X_n, \precsim)$. For instance, the Hasse diagram of the preordered set $(X_5, \precsim)$, where $\precsim$ is defined by $1 \parallel 2$, $1 \prec 3$, $2 \prec 3$, and $3 \prec 4 \sim 5$, is depicted in Figure 1.1.

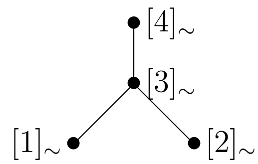

Figure 1.1: Hasse diagram of $(X_5, \precsim)$

Let $(Y, \precsim)$ and $(Z, \precsim')$ be two preordered sets. Recall that the preorders $\precsim$ and $\precsim'$ are said to be *isomorphic*, and we write $\precsim \simeq \precsim'$, if there exists a bijection $\phi\colon Z \to Y$ such that

$$x \precsim' y \quad \Leftrightarrow \quad \phi(x) \precsim \phi(y), \qquad x, y \in Z.$$

The bijection ϕ is then said to be an *isomorphism* from $(Z, \precsim')$ to $(Y, \precsim)$. It is said to be an *automorphism* of $(Y, \precsim)$ if $(Y, \precsim) = (Z, \precsim')$. The latter concepts are used in Chapter 4 in order to classify and enumerate the class of quasitrivial semigroups.

1.2 Characterizations of single-peakedness

The following example [89] provides a motivation for the use of single-peakedness in social choice.

Example 1.1. Suppose you are asked to order the following six objects in decreasing preference:

1 :	0 sandwich
2 :	1 sandwich
3 :	2 sandwiches
4 :	3 sandwiches
5 :	4 sandwiches
6 :	more than 4 sandwiches

We write $i <' j$ if i is preferred to j. We list below four possible rankings.

- After a good lunch: $1 <'_\alpha 2 <'_\alpha 3 <'_\alpha 4 <'_\alpha 5 <'_\alpha 6$.
- If you are starving: $6 <'_\beta 5 <'_\beta 4 <'_\beta 3 <'_\beta 2 <'_\beta 1$.
- A possible intermediate situation: $3 <'_\gamma 4 <'_\gamma 5 <'_\gamma 6 <'_\gamma 2 <'_\gamma 1$.
- A quite unlikely preference: $6 <'_\delta 3 <'_\delta 5 <'_\delta 2 <'_\delta 4 <'_\delta 1$.

We observe that, for any $\ell \in \{\alpha, \beta, \gamma\}$ and any $i, j, k \in X_6$ such that $i <_6 j <_6 k$, we have $j <'_\ell i$ or $j <'_\ell k$. In social choice theory, the total order $\leq'_\ell$ is then said to be single-peaked for $\leq_6$. We now see that $\leq'_\delta$ is not single-peaked for $\leq_6$ since $3 <_6 5 <_6 6$ and $6 <'_\delta 3 <'_\delta 5$.

The concept of single-peakedness was first introduced for finite chains by Black [9, 10] (see [8, 49] for more recent references). We can easily generalize this concept to arbitrary chains as follows.

Definition 1.2 (see [33])**.** Let $\leq$ be a total order on X. A total order $\leq'$ on X is said to be *single-peaked for* $\leq$ if for any $a, b, c \in X$,

$$a < b < c \quad \Rightarrow \quad b <' a \quad \text{or} \quad b <' c. \tag{1.1}$$

Note that the single-peakedness condition is self-dual with respect to the total order $\leq$, that is, a total order $\leq'$ on X is single-peaked for $\leq$ if and only if it is single-peaked for $\leq^d$.

Example 1.3. There are exactly four total orders $\leq'$ on X_3 that are single-peaked for $\leq_3$, namely $1 <' 2 <' 3$, $2 <' 1 <' 3$, $2 <' 3 <' 1$, and $3 <' 2 <' 1$.

Let $\leq$ be a fixed (reference) total order on X_n for some integer $n \geq 1$. For any weak order $\precsim$ on X_n, let also $\mathcal{G}_{\precsim}$ be the graph of the identity function $i_{X_n} \colon X_n \to X_n$ represented in the Cartesian coordinate system obtained by considering the reference totally ordered set $(X_n, \leq)$ on the horizontal axis and the dual version of the weakly ordered set $(X_n, \precsim)$ on the vertical axis.

Example 1.4. Let us consider the reference totally ordered set $(X_6, \leq_6)$ and the weak order $\precsim$ on X_6 defined by $2 \sim 3 \sim 4 \prec 1 \prec 5 \sim 6$. The associated graph $\mathcal{G}_{\precsim}$ is represented in Figure 1.2.

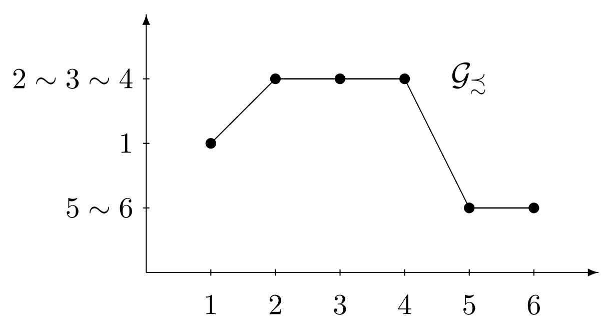

Figure 1.2: Example 1.4

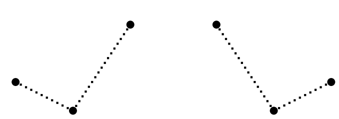

Figure 1.3: The two patterns excluded by condition (1.1)

Black [10] observed that the single-peakedness property of a total order $\leq'$ on X_n for $\leq_n$ can be easily checked by analyzing its corresponding graph $\mathcal{G}_{\leq'}$. In fact, the total order $\leq'$ is single-peaked for $\leq_n$ if and only if $\mathcal{G}_{\leq'}$ is "V-free" in the sense that we cannot find three points (i, i), (j, j), (k, k) in V-shape in $\mathcal{G}_{\leq'}$, which means that the patterns shown in Figure 1.3 are forbidden. Equivalently, the function whose graph is represented by $\mathcal{G}_{\leq'}$ has only one local maximum.

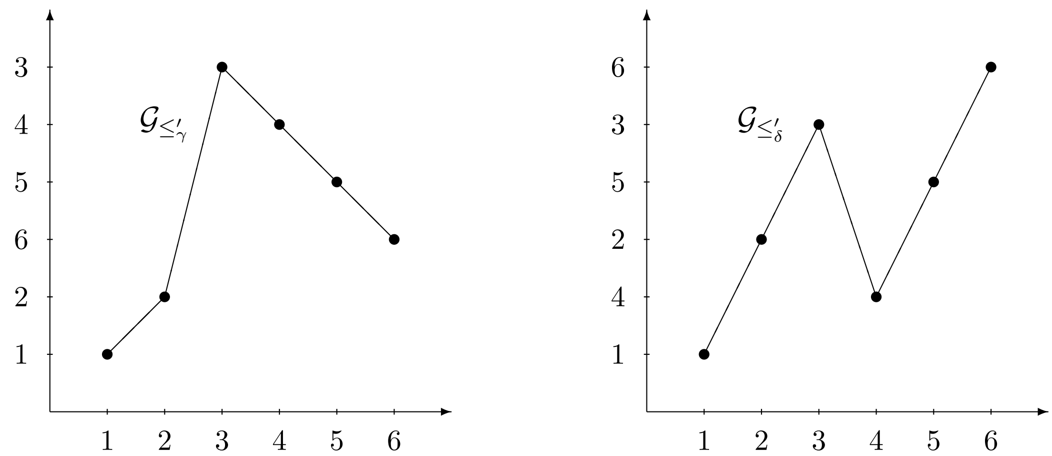

Figure 1.4: $\leq'_\gamma$ is single-peaked (left) while $\leq'_\delta$ is not (right)

For instance, Figure 1.4 gives the graphs $\mathcal{G}_{\leq'_\gamma}$ and $\mathcal{G}_{\leq'_\delta}$ defined in Example 1.1 for the reference total order $\leq_6$. We see that the function whose graph is given by $\mathcal{G}_{\leq'_\gamma}$ has only one local maximum while it has two local maxima in the case of $\mathcal{G}_{\leq'_\delta}$.

The following result provides characterizations of single-peakedness. The equivalence among (i), (iv), and (v) was shown in [33].

Proposition 1.5. *Let $\leq$ and $\leq'$ be two total orders on X. The following assertions are equivalent.*

(i) *$\leq'$ is single-peaked for $\leq$.*

(ii) *Every ideal of $(X, \leq')$ is a convex subset of $(X, \leq)$.*

(iii) *Every principal ideal of $(X, \leq')$ is a convex subset of $(X, \leq)$.*

(iv) *For any $x_0, x_1, x_2 \in X$ such that $x_0 <' x_1$ and $x_0 <' x_2$ we have*

$$x_0 < x_1 < x_2 \quad \textit{or} \quad x_2 < x_1 < x_0 \quad \Rightarrow \quad x_1 <' x_2. \tag{1.2}$$

If X has a minimal element x_0 for $\leq'$, then any of the assertions $(i) - (iv)$ is equivalent to the following one.

(v) *(1.2) holds for any $x_1, x_2 \in X$.*

Proof. We first prove the implication $(i) \Rightarrow (ii)$. So, suppose that $\leq'$ is single-peaked for $\leq$ and suppose to the contrary that there exists an ideal I of $(X, \leq')$ that is not a convex subset of $(X, \leq)$. Then there exist $a, b, c \in X$ satisfying $a < b < c$ such that $a, c \in I$ and $b \notin I$. This means that $a <' b$ and $c <' b$, which contradicts our assumption. The implication $(ii) \Rightarrow (iii)$ is obvious. Now, let us show that $(iii) \Rightarrow (iv)$. So, suppose that every principal ideal of $(X, \leq')$ is a convex subset of $(X, \leq)$ and suppose to the contrary that there exist $x_0, x_1, x_2 \in X$ satisfying $x_0 <' x_1$ and $x_0 <' x_2$ for which (1.2) fails to hold, i.e., either ($x_0 < x_1 < x_2$ and $x_0 <' x_2 <' x_1$) or ($x_2 < x_1 < x_0$ and $x_0 <' x_2 <' x_1$). But then the principal ideal $(x_2]_{\leq'}$ is not a convex subset of $(X, \leq)$, which contradicts our assumption. The implication $(iv) \Rightarrow (v)$ is obvious. Finally, let

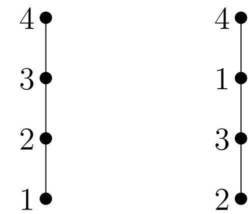

Figure 1.5: Hasse diagrams of $(X_4, \leq_4)$ and $(X_4, \leq')$.

us show that $(v) \Rightarrow (i)$. We proceed again by contradiction. Suppose that there exist $a, b, c \in X$ satisfying $a < b < c$ such that $a <' b$ and $c <' b$. Since x_0 is the minimal element of $(X, \leq')$, we must have $x_0 \neq b$. If $x_0 < b$, then setting $x_1 = b$ and $x_2 = c$, we obtain $x_0 < x_1 < x_2$ and $x_2 <' x_1$, which contradicts (1.2). We arrive at a similar contradiction if $b < x_0$. $\square$

Proposition 1.5 is of particular interest as it enables us to check whether a total order $\leq'$ on X is single-peaked for a reference total order $\leq$ on X simply by looking at the Hasse diagram of $(X, \leq')$. For instance, the total order $\leq'$ on X_4 defined by $2 <' 3 <' 1 <' 4$ is single-peaked for $\leq_4$. Indeed, it is not difficult to see that every principal ideal of $(X_4, \leq')$ is a convex subset of $(X_4, \leq_4)$; see Figure 1.5.

It is natural to use weak orders in election systems, where two or more candidates can be considered as equivalent by the voters. In this context, the property of single-peakedness for finite chains was generalized to weak orders by Black [9, 10] and this generalization was reformulated mathematically by Fitzsimmons [46]. We now extend this concept to arbitrary chains, possibly infinite.

Definition 1.6. Let $\leq$ be a total order on X. A weak order $\precsim$ on X is said to be *single-plateaued for* $\leq$ if the following two conditions hold.

(a) For any $a, b, c \in X$ such that $a < b < c$, we have $b \precsim a$ or $b \precsim c$.

(b) For any $a, b, c \in X$ such that $a \neq c$ and $b \prec a \sim c$, we have $a < b < c$ or $c < b < a$.

Remark 1.7. Let $(X, \leq)$ be a chain, let $\precsim$ be a weak order on X, and let $P \subseteq X$ be such that the restriction $\precsim|_P$ of $\precsim$ to P is a total order. Then $\precsim|_P$ satisfies condition (a) of Definition 1.6 for $\leq|_P$ if and only if it is single-peaked for $\leq|_P$.

Black [10] observed that the single-plateauedness property of a weak order $\precsim$ on X_n for $\leq_n$ can be easily checked by analyzing the graph $\mathcal{G}_{\precsim}$. Actually, condition (a) of Definition 1.6 says that the graph $\mathcal{G}_{\precsim}$ is V-free, i.e., we cannot find three points (i, i), (j, j), (k, k) in V-shape in $\mathcal{G}_{\precsim}$. Condition (b) of Definition 1.6 says that the graph $\mathcal{G}_{\precsim}$ is both reversed L-free and L-free, which means that the two patterns shown in Figure 1.6 (reversed L-shape and L-shape), are forbidden.

Example 1.8. The weak order $\precsim'$ on X_4 defined by $2 \sim' 3 \prec' 4 \prec' 1$ is single-plateaued for $\leq_4$. Indeed, the graph $\mathcal{G}_{\precsim'}$ is V-free, reversed L-free, and L-free; see Figure 1.7 (left). However, the weak order $\precsim$ on X_4 defined by $1 \prec 2 \sim 3 \prec 4$ is not single-plateaued for $\leq_4$. Indeed, the graph $\mathcal{G}_{\precsim}$ is not L-free as the points $(1, 1)$, $(2, 2)$, and $(3, 3)$ are in L-shape; see Figure 1.7 (right).

The following proposition provides a characterization of condition (a) of Definition 1.6 in terms of ideals of $(X, \precsim)$.

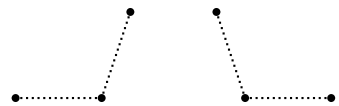

Figure 1.6: The two patterns excluded by condition (b) of Definition 1.6

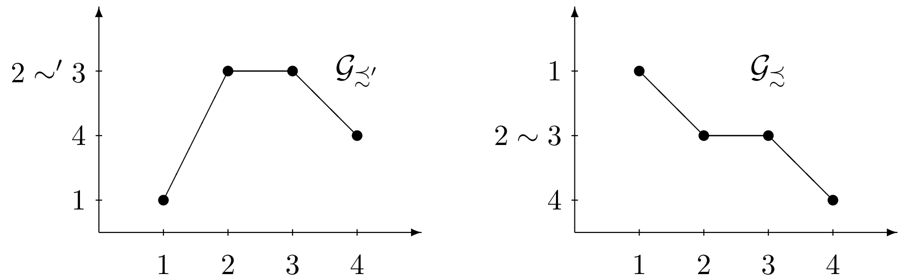

Figure 1.7: Example 1.8

Proposition 1.9. *Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X. The following assertions are equivalent.*

(i) Condition (a) of Definition 1.6 holds.

(ii) Every ideal of $(X, \precsim)$ is a convex subset of $(X, \leq)$.

(iii) Every principal ideal of $(X, \precsim)$ is a convex subset of $(X, \leq)$.

Proof. The implication $(i) \Rightarrow (ii)$ is straightforward. Also, the implication $(ii) \Rightarrow (iii)$ is obvious. Let us show that $(iii) \Rightarrow (i)$. For the sake of a contradiction, suppose that there exist $a, b, c \in X$ such that $a < b < c$, $a \prec b$, and $c \prec b$. Set $t_0 = c$ if $a \prec c$, and $t_0 = a$, otherwise. We then have $a, c \in (t_0]_{\precsim}$. By convexity for $\leq$ we also have $b \in (t_0]_{\precsim}$. Therefore we have $a \prec b \precsim t_0$ and $c \prec b \precsim t_0$, which contradicts the definition of t_0. $\square$

Proposition 1.9 is of particular interest as it allows us to check whether a weak order $\precsim$ on X satisfies condition (a) of Definition 1.6 for a reference total order $\leq$ on X by analyzing the Hasse diagram of $(X, \precsim)$. For instance, the weak order $\precsim$ on X_4 defined by $2 \prec 1 \sim 3 \prec 4$ satisfies condition (a) of Definition 1.6 for $\leq_4$. Indeed, it is not difficult to see that every principal ideal of $(X_4, \precsim)$ is a convex subset of $(X_4, \leq_4)$; see Figure 1.8.

In what follows, we provide several characterizations of single-plateauedness. To this extent, we first introduce the notion of plateau.

Definition 1.10 (see [25])**.** Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X. We say that a subset P of X of size $|P| \geq 2$ is *a plateau for* $(\leq, \precsim)$ if P is convex for $\leq$ and if there exists $x \in X$ such that $P \subseteq [x]_{\sim}$.

Remark 1.11. Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X. A set $C \in X/{\sim}$ with $|C| \geq 2$ is a plateau for $(\leq, \precsim)$ if and only if it is convex for $\leq$.

The following proposition provides characterizations of single-plateauedness.

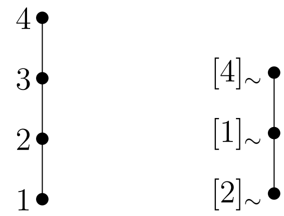

Figure 1.8: Hasse diagrams of $(X_4, \leq_4)$ and $(X_4, \precsim)$.

Proposition 1.12 (see [25]). *Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X. The following assertions are equivalent.*

(i) *$\precsim$ is single-plateaued for $\leq$.*

(ii) *For any $a, b, c \in X$ such that $a < b < c$, we have $b \prec a$ or $b \prec c$ or $a \sim b \sim c$.*

(iii) *For any $a, b, c \in X$ such that $a < b < c$, we have*

$$\begin{cases} a \prec b \ \Rightarrow \ b \prec c, \\ c \prec b \ \Rightarrow \ b \prec a. \end{cases}$$

(iv) *Condition (a) of Definition 1.6 as well as the following one hold.*

(b') *If $P \subseteq X$, $|P| \geq 2$, is a plateau for $(\leq, \precsim)$, then it is $\precsim$-minimal in the sense that for every $a \in X$ satisfying $a \precsim P$ there exists $z \in P$ such that $z \sim a$.*

Proof. Let us first show that $(i) \Rightarrow (ii)$. Suppose to the contrary that there exist $a, b, c \in X$ with $a < b < c$ such that $a \precsim b$ and $c \precsim b$ and $\neg(a \sim b \sim c)$. By condition (a) of Definition 1.6 we have $a \prec b \sim c$ or $c \prec a \sim b$, which contradicts condition (b) of Definition 1.6. Now, let us show that $(ii) \Rightarrow (iii)$. Suppose to the contrary that there exist $a, b, c \in X$ with $a < b < c$ such that any of the following conditions hold.

- $a \prec b$ and $c \precsim b$.
- $c \prec b$ and $a \precsim b$.

This clearly contradicts condition (ii). Now, let us show that $(iii) \Rightarrow (iv)$. It is easy to see that condition (a) of Definition 1.6 holds. So, suppose to the contrary that there exists a plateau P for $(\leq, \precsim)$ that is not $\precsim$-minimal, that is, there exist $a, b, c \in X$ with $a < c$ such that $a, c \in P$, $b \prec a \sim c$, and $b \notin [a, c]$. This clearly contradicts condition (iii). Finally, let us show that $(iv) \Rightarrow (i)$. Let $a, b, c \in X$ such that $a < c$ and $b \prec a \sim c$ and suppose that $b \notin [a, c]$. Assume without loss of generality that $b < a$. If $[a, c]$ is a plateau for $(\leq, \precsim)$, then it cannot be $\precsim$-minimal, which contradicts (b'). Hence $[a, c]$ is not a plateau for $(\leq, \precsim)$, which means that there exists $z \in [a, c]$ such that $\neg(z \sim a)$. By condition (a) of Definition 1.6 we then have $z \prec a$. But then the triplet (b, a, z) violates condition (a) of Definition 1.6 since $z \prec a$ and $b \prec a$. $\square$

Remark 1.13. Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X. It is easy to see that $\precsim$ satisfies condition (b') of Proposition 1.12(iv) whenever it satisfies condition (b) of Definition 1.6. The converse is not true in general. For instance, consider the chain $(X_4, \leq_4)$ and the weak order $\precsim$ on X_4 defined by $1 \prec 2 \sim 4 \prec 3$. Then $\precsim$ satisfies condition (b') of Proposition 1.12(iv) but it does not satisfy condition (b) of Definition 1.6. Indeed, we have $1 \prec 2 \sim 4$ and $1 <_4 2 <_4 4$.

Given a weak order $\precsim$ on X, we now provide a necessary and sufficient condition on $\precsim$ that ensures the existence of a total order on X for which $\precsim$ is single-plateaued.

Definition 1.14 (see [34])**.** We say that a weak order $\precsim$ on X is 2-*quasilinear* if there are no pairwise distinct $a, b, c, d \in X$ such that $a \prec b \sim c \sim d$.

Thus, a weak order $\precsim$ on X is 2-quasilinear if and only if every set $C \in X/{\sim}$ that is not minimal for $\precsim$ contains at most two elements of X.

The following proposition provides a quite surprising characterization of 2-quasilinearity.

Proposition 1.15 (see [34])**.** *A weak order on X is 2-quasilinear if and only if it is single-plateaued for some total order on X.*

Proof. (Necessity) Let $\precsim$ be a 2-quasilinear weak order on X. For any $x \in X$, let $S_{[x]_\sim}$ be a total order on $[x]_\sim$.

Consider the binary relation $\leq$ on X whose symmetric part is the identity relation on X and the asymmetric part $<$ is defined as follows. Let $x, y \in X$ be such that $x \neq y$ and $x \precsim y$.

$$- \text{ If } x \sim y \quad \text{and} \quad \begin{cases} x\, S_{[x]_\sim}\, y\,, & \text{then we set } x < y. \\ y\, S_{[x]_\sim}\, x\,, & \text{then we set } y < x. \end{cases}$$

$$- \text{ If } x \prec y \quad \text{and} \quad \begin{cases} y = \min_{S_{[y]_\sim}} [y]_\sim\,, & \text{then we set } y < x. \\ y = \max_{S_{[y]_\sim}} [y]_\sim \text{ and } |[y]_\sim| = 2\,, & \text{then we set } x < y. \end{cases}$$

Let us show that, thus defined, the relation $\leq$ is a total order on X. It is clearly total by definition. It is also antisymmetric. Indeed, it is clear that there are no $x, y \in X$ such that $x < y$ and $y < x$. Finally, let us prove by contradiction that it is transitive. Suppose that there are $x, y, z \in X$ such that $x < y$, $y < z$, and $z < x$. Also, suppose for instance that $x \sim y \prec z$ (the other 12 cases can be verified similarly). Since $y \prec z$, we must have $z = \max_{S_{[z]_\sim}} [z]_\sim$ and $|[z]_\sim| = 2$. Also, since $x \prec z$, we must have $z = \min_{S_{[z]_\sim}} [z]_\sim$, a contradiction.

Let us now show that $\precsim$ is single-plateaued for $\leq$. To this extent, we only need to show that condition (ii) of Proposition 1.12 holds. So, let $a, b, c \in X$ such that $a < b < c$, $\neg(a \sim b \sim c)$, and $c \precsim b$. We only need to show that $b \prec a$. We have two exclusive cases to consider.

- If $c \sim b$, then we clearly have $b\, S_{[b]_\sim}\, c$. It follows that we cannot have $a \prec b$ and $b = \max_{S_{[b]_\sim}} [b]_\sim$ and $|[b]_\sim| = 2$. We cannot have $a \sim b$ either for we have $\neg(a \sim b \sim c)$. Therefore, we must have $b \prec a$.

- If $c \prec b$, then we have $b = \min_{S_{[b]_\sim}} [b]_\sim$. It follows that we cannot have $a \prec b$ and $b = \max_{S_{[b]_\sim}} [b]_\sim$ and $|[b]_\sim| = 2$. Clearly, we cannot have $a \sim b$ and $a\, S_{[b]_\sim}\, b$ either. Therefore, we must have $b \prec a$.

(Sufficiency) We proceed by contradiction. Suppose that there exist pairwise distinct $a, b, c, d \in X$ such that $a \prec b \sim c \sim d$. Assume without loss of generality that $b < c < d$. If $a < b < c$, then the triplet (a, b, c) violates single-plateauedness of $\precsim$. If $b < a < c$, then the triplet (a, c, d) violates single-plateauedness of $\precsim$. In the two other cases the triplet (b, c, a) violates single-plateauedness of $\precsim$. $\square$

In the finite case, i.e. when $X = X_n$ for some $n \geq 1$, an algorithm can be easily derived from Proposition 1.15 to construct a total order on X_n for which a given 2-quasilinear weak order is single-plateaued. For any weak order $\precsim$ on X_n, let $k = |X_n/{\sim}|$ and let $C_1, \ldots, C_k$ denote the elements of $X_n/{\sim}$ ordered by the relation induced by $\precsim$, that is, $C_1 \prec \cdots \prec C_k$ (where $C_i \prec C_j$ means that we have $x \prec y$ for all $x \in C_i$ and all $y \in C_j$). For a 2-quasilinear weak order $\precsim$ on X_n, the total order $\leq$ on X_n mentioned in Proposition 1.15 can be very easily constructed as follows. First, choose a total order S_i on each set C_i ($i = 1, \ldots, k$). Then execute the following four-step algorithm.

1. Let L be the empty list.
2. For $i = k, \ldots, 2$, append the element $\min_{S_i} C_i$ to L.
3. Append to L the elements of C_1 in the order given by S_1.
4. For $i = 2, \ldots, k$ such that $|C_i| = 2$, append the element $\max_{S_i} C_i$ to L.

The order given by L defines a suitable total order $\leq$ on X_n. For instance, suppose that we have the following 2-quasilinear weak order $\precsim$ on X_8:

$$1 \sim 2 \sim 3 \prec 4 \sim 5 \prec 6 \prec 7 \sim 8.$$

In each set C_i ($i = 1, 2, 3, 4$), we let S_i be the restriction of $\leq_n$ to C_i. The list constructed by the algorithm above is then given by $L = (7, 6, 4, 1, 2, 3, 5, 8)$ and provides the following total order:

$$7 < 6 < 4 < 1 < 2 < 3 < 5 < 8.$$

Finally, we can readily verify that $\precsim$ is single-plateaued for $\leq$ (see Figure 1.9).

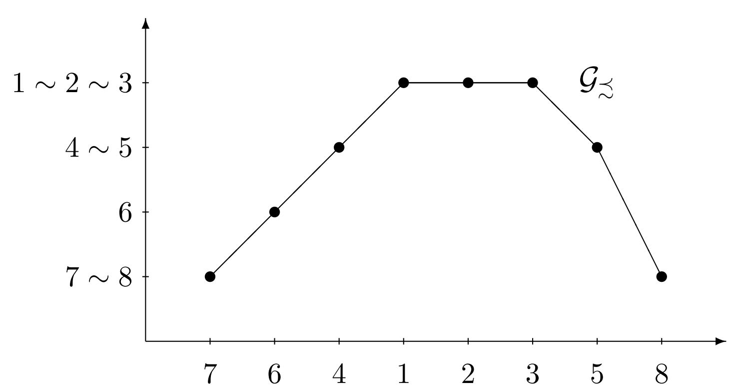

Figure 1.9: $\precsim$ is single-plateaued for $\leq$

Remark 1.16. In order to aggregate votes in a consistent way, social choice theorists usually face the problem of finding a reference total order, if any, for which a number of given weak orders are single-plateaued. To formalize this concept, let $k \geq 1$ be an integer and let $\{\precsim_1, \ldots, \precsim_k\}$ be a set of weak orders on X_n. The set $\{\precsim_1, \ldots, \precsim_k\}$ is said to be *single-plateaued consistent* [46] if there exists a total order $\leq$ on X_n such that $\precsim_i$ is single-plateaued for $\leq$ for any $i \in \{1, \ldots, k\}$. Single-plateaued consistency was characterized in [46] by means of properties of matrices. We observe that if a set of weak orders on X_n is single-plateaued consistent, then each weak order must be 2-quasilinear by Proposition 1.15. However, the converse is not true in general. For instance, it is not difficult to see that the weak orders $\precsim_1$ and $\precsim_2$ on X_3 defined by $1 \sim_1 2 \prec_1 3$ and $3 \prec_2 1 \sim_2 2$, respectively, are 2-quasilinear but the set $\{\precsim_1, \precsim_2\}$ is not single-plateaued consistent.

For any total order $\leq$ on X and any weak order $\precsim$ on X, we say that $\leq$ *extends* (or *is subordinated to*) $\precsim$ if, for any $x, y \in X$, we have that $x \prec y$ implies $x < y$.

Definition 1.17 (see [31]). We say that a weak order $\precsim$ on X is *quasilinear* if there are no pairwise distinct $a, b, c \in X$ such that $a \prec b \sim c$.

Thus, a weak order $\precsim$ on X is quasilinear if and only if every set $C \in X/{\sim}$ that is not minimal for $\precsim$ contains exactly one element of X. Clearly, such a weak order is also 2-quasilinear. We also have the following proposition, which is the counterpart of Proposition 1.15 for quasilinear weak orders.

Proposition 1.18 (see [34]). *A weak order on X is quasilinear if and only if it is single-plateaued for any total order on X that extends it.*

Proof. (Necessity) Let $\precsim$ be a quasilinear weak order on X. Suppose that there exists a total order $\leq$ on X that extends $\precsim$ and such that $\precsim$ is not single-plateaued for $\leq$. That is, there exist $a, b, c \in X$ such that $a < b < c$, $c \precsim b$, $a \precsim b$, and $\neg(a \sim b \sim c)$. Then we must have $a \prec b \sim c$, which contradicts quasilinearity.

(Sufficiency) Let $\precsim$ be a weak order on X that is single-plateaued for any total order on X that extends it. Suppose that $\precsim$ is not quasilinear, that is, there exist pairwise distinct $a, b, c \in X$ such that $a \prec b \sim c$. It is then clear that $\precsim$ is not single-plateaued for any total order $\leq$ on X that extends $\precsim$. Thus, we reach a contradiction. $\square$

Remark 1.19. For any integer $n \geq 1$, the weak orders on X_n that are quasilinear are known in social choice theory as *top orders* (see, e.g., [46]). Several election systems such as the Borda count can be extended for top orders (see, e.g., [46]).

Now, we provide a characterization of quasilinearity under single-plateauedness.

Proposition 1.20 (see [31]). *Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X that is single-plateaued for $\leq$. The following conditions are equivalent.*

(i) $\precsim$ is quasilinear.

(ii) If there exist $a, b \in X$, with $a < b$, such that $a \sim b$, then $[a, b]$ is a plateau for $(\leq, \precsim)$ and it is $\precsim$-minimal in the sense that for every $a \in X$ satisfying $a \precsim P$ there exists $z \in P$ such that $z \sim a$.

Proof. Let us first show that $(i) \Rightarrow (ii)$. We proceed by contradiction. Let $a, b \in X$ with $a < b$, such that $a \sim b$ and suppose that $[a, b]$ is not a plateau for $(\leq, \precsim)$. But then there exists $u \in [a, b]$ such that either $u \prec a \sim b$ which contradicts quasilinearity, or $a \sim b \prec u$ which contradicts single-plateauedness. Thus, $[a, b]$ is a plateau for $(\leq, \precsim)$ and it is $\precsim$-minimal by quasilinearity. Now, let us show that $(ii) \Rightarrow (i)$. We proceed again by contradiction. Suppose that there exist pairwise distinct $a, b, c \in X$ such that $a \prec b \sim c$. We can suppose without loss of generality that $b < c$. But then $[b, c]$ is a plateau for $(\leq, \precsim)$ which is not $\precsim$-minimal, a contradiction to (ii). □

When $X = X_n$ for some $n \geq 1$, Proposition 1.20 is of particular interest as it enables us to easily check whether a weak order $\precsim$ on X_n is quasilinear and single-plateaued for $\leq_n$. Indeed, as discussed, single-plateauedness says that the graph $\mathcal{G}_{\precsim}$ is V-free, L-free, and reversed L-free. Also, by Proposition 1.20, quasilinearity says that any two elements which have the same position on the vertical axis form a plateau. For instance, the weak order $\precsim$ on X_8 whose graph $\mathcal{G}_{\precsim}$ is depicted in Figure 1.9 is not quasilinear since the elements 7 and 8 have the same position on the vertical axis but do not form a plateau.

We conclude this section by studying another extension of single-peakedness for weak orders [46, 67].

Definition 1.21. Let $\leq$ be a total order on X. A weak order $\precsim$ on X is said to be *existentially single-peaked for* $\leq$ if it can be extended to a total order $\leq'$ that is single-peaked for $\leq$.

Remark 1.22. The concept of existential single-peakedness was first introduced for finite chains in social choice theory [46, 67]. In Definition 1.21 we extended this concept to arbitrary chains.

Example 1.23. The weak order $\precsim$ on X_3 defined by $3 \prec 1 \sim 2$ is existentially single-peaked for $\leq_3$. Indeed, the total order $\leq'$ on X_3 defined by $3 <' 2 <' 1$ extends $\precsim$ and is single-peaked for $\leq_3$. Also, it is not difficult to see that the weak order $\precsim'$ on X_3 defined by $1 \sim' 3 \prec' 2$ is not existentially single-peaked for $\leq_3$.

The following proposition provides a characterization of existential single-peakedness.

Proposition 1.24. *Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X. The following assertions are equivalent.*

(i) $\precsim$ is existentially single-peaked for $\leq$.

(ii) Condition (a) of Definition 1.6 holds.

Proof. The implication $(i) \Rightarrow (ii)$ is straightforward. Let us show that $(ii) \Rightarrow (i)$. Consider the binary relation $\leq'$ on X whose symmetric part is the identity relation on X and the asymmetric part $<'$ is defined as follows. Let $x, y \in X$ be such that $x \neq y$ and $x \precsim y$.

– If $x \prec y$, then we set $x <' y$.

– If $x \sim y$, $x < y$, and $\begin{cases} \exists\, z \in X \text{ such that } z \prec y \text{ and } y < z\,, & \text{then we set } y <' x. \\ \nexists\, z \in X \text{ such that } z \prec y \text{ and } y < z\,, & \text{then we set } x <' y. \end{cases}$

Let us show that, thus defined, the relation $\leq'$ is a total order on X. It is clearly total by definition. It is also antisymmetric. Indeed, it is clear that there are no $x, y \in X$ such that $x <' y$ and $y <' x$. Finally, let us prove by contradiction that it is transitive. Suppose that there are $x, y, z \in X$ such that $x <' y$, $y <' z$, and $z <' x$. By definition of $\leq'$ we must have $x \sim y \sim z$. Also, suppose for instance that $x < y < z$ (the other 5 cases can be verified similarly). Since $z <' x$ and $x < z$, there exists $u \in X$ such that $u \prec z$ and $z < u$, which contradicts the fact that $y <' z$.

Let us now show that $\leq'$ is single-peaked for $\leq$. So, let $a, b, c \in X$ such that $a < b < c$ and $c <' b$. We only need to show that $b <' a$. We have two exclusive cases to consider.

- If $c \prec b$, then by condition (ii) we have $b \precsim a$. By definition of $\leq'$ we then have $b <' a$.
- If $c \sim b$, then by definition of $\leq'$, there exists $u \in X$ such that $u \prec c$ and $c < u$. We first observe that we cannot have $a \prec b$. Otherwise, we would have $a < c < u$, $a \prec c$, and $u \prec c$, which would contradict condition (ii). Therefore, we have two exclusive subcases to consider.
  - If $b \prec a$, then by definition of $\leq'$ we have $b <' a$.
  - If $a \sim b$, then since $u \prec b$ and $b < u$, we must have $b <' a$. □

Remark 1.25. The equivalence between conditions (i) and (ii) of Proposition 1.24 was stated without proof for finite chains in [46]. It was then recently proven for finite chains in [47]. Here we provide a constructive proof of this equivalence for arbitrary chains.

When $X = X_n$ for some $n \geq 1$, Proposition 1.24 is of particular interest as it enables us to easily check whether a weak order is existentially single-peaked for $\leq_n$. In fact, by Proposition 1.24, a weak order $\precsim$ on X_n is existentially single-peaked for $\leq_n$ if and only if the graph $\mathcal{G}_{\precsim}$ is V-free.

Example 1.26. The weak order $\precsim$ on X_5 defined by $4 \sim 5 \prec 3 \prec 1 \sim 2$ is existentially single-peaked for $\leq_5$. Indeed, the graph $\mathcal{G}_{\precsim}$ is V-free; see Figure 1.10 (left). For instance, the total order $\leq'$ on X_5 defined by $4 <' 5 <' 3 <' 2 <' 1$ extends $\precsim$ and is single-peaked for $\leq_5$; see Figure 1.10 (right).

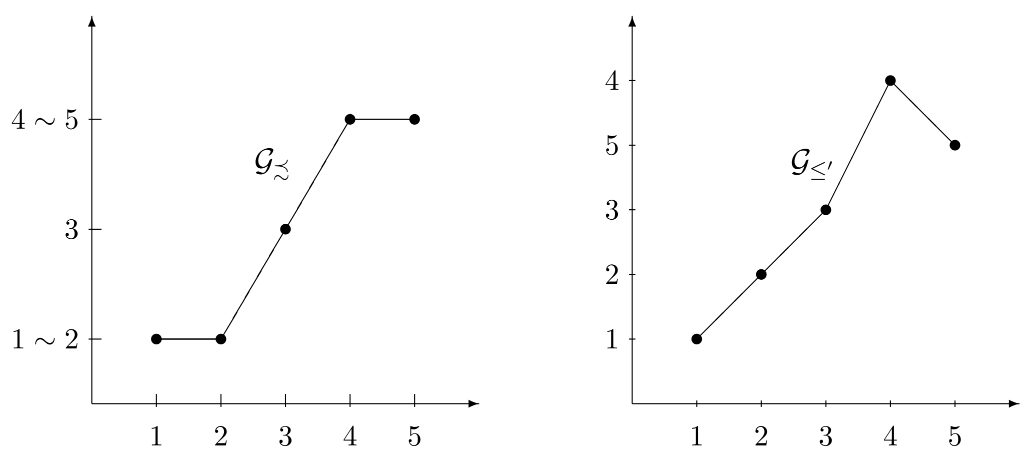

Figure 1.10: Example 1.26

Let $\leq$ be a total order on X. By Proposition 1.24, we observe that if a weak order on X is single-plateaued for $\leq$, then it is existentially single-peaked for $\leq$. However, the converse of the latter statement is not true in general. For instance, the weak order $\precsim$ on X_3 defined by $1 \prec 2 \sim 3$ is existentially single-peaked for $\leq_3$. However, it is not single-plateaued for $\leq_3$. The following result shows that the latter two properties are equivalent for quasilinear weak orders.

Proposition 1.27 (see [31]). *Let $\leq$ be a total order on X and let $\precsim$ be a quasilinear weak order on X. Then $\precsim$ is single-plateaued for $\leq$ if and only if it is existentially single-peaked for $\leq$.*

Proof. (Necessity) This follows from Proposition 1.24.

(Sufficiency) Suppose that $\precsim$ is existentially single-peaked for $\leq$. Then $\precsim$ satisfies condition (a) of Definition 1.6 by Proposition 1.24. Also, $\precsim$ satisfies condition (b) of Definition 1.6 by quasilinearity. Thus, $\precsim$ is single-plateaued for $\leq$. $\square$

1.3 Enumeration results

In this section we consider the problem of enumerating the special weak orders introduced in Section 1.2. For instance, for any integer $n \geq 1$, we provide in Proposition 1.28 the exact number of 2-quasilinear weak orders on X_n. We posted the corresponding sequence in Sloane's On-Line Encyclopedia of Integer Sequences (OEIS, see [94]) as sequence A307005.

The results reported in this section will not be used in the subsequent chapters. The reader who is less interested in enumeration issues may want to skip this section.

We often consider either the (ordinary) generating function (GF) or the exponential generating function (EGF) of a given integer sequence $(s_n)_{n\geq 0}$. Recall [53] that, when these functions exist, they are respectively defined by the formal power series

$$S(z) ~=~ \sum_{n\geq 0} s_n\, z^n \quad\text{and}\quad \widehat{S}(z) ~=~ \sum_{n\geq 0} s_n\, \frac{z^n}{n!}\,.$$

Recall [53] also that for any integers $0 \leq k \leq n$, the *Stirling number of the second kind* $\left\{ n \atop k \right\}$ is defined by

$$\left\{ n \atop k \right\} ~=~ \frac{1}{k!}\,\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}\, i^n.$$

For any integer $n \geq 0$, let $p(n)$ denote the number of weak orders on X_n, or equivalently, the number of totally ordered partitions of X_n. Setting $p(0) = 1$, the number $p(n)$ is explicitly given by

$$p(n) ~=~ \sum_{k=0}^{n}\left\{ n \atop k \right\} k!\,, \qquad n \geq 0.$$

Actually, the corresponding sequence $(p(n))_{n\geq 0}$ consists of the *ordered Bell numbers* (Sloane's A000670) and satisfies the following recurrence equation

$$p(n+1) ~=~ \sum_{k=0}^{n}\binom{n+1}{k}\, p(k)\,, \qquad n \geq 0,$$

with $p(0) = 1$. Moreover, its EGF is given by $\widehat{P}(z) = 1/(2 - e^z)$.

Now, for any integer $n \geq 0$, let $q(n)$ be the number of 2-quasilinear weak orders on X_n and let $r(n)$ be the number of quasilinear weak orders on X_n. By convention, we set $q(0) = r(0) = 1$.

The next two propositions [31, 34] provide explicit expressions for these sequences. Also, the first few values are given in Table 1.1.[2]

Proposition 1.28. *The sequence $(q(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$q(n+2) \;=\; 1+(n+2)\,q(n+1)+\frac{1}{2}(n+2)(n+1)\,q(n), \qquad n\geq 1,$$

with $q(1)=1$ and $q(2)=3$, and we have

$$q(n) \;=\; \sum_{k=0}^{n}\frac{n!}{(n+1-k)!}\,G_k\,, \qquad n\geq 1,$$

where $G_n=\frac{\sqrt{3}}{3}\,(\frac{1+\sqrt{3}}{2})^n-\frac{\sqrt{3}}{3}\,(\frac{1-\sqrt{3}}{2})^n$. Moreover, its EGF is given by

$$\widehat{Q}(z)=(2e^z-2z-z^2)/(2-2z-z^2).$$

Proof. We clearly have $q(1)=1$ and $q(2)=3$. Now, let $n\geq 3$, let $\precsim$ be a 2-quasilinear weak order on X_n, and let m be the number of maximal elements of X_n for $\precsim$. By definition of 2-quasilinearity, we necessarily have $m\in\{1,2,n\}$. Moreover, the restriction of $\precsim$ to the $(n-m)$-element set obtained from X_n by removing its maximal elements for $\precsim$ is 2-quasilinear. It follows that the sequence $q(n)$ satisfies the second order linear recurrence equation

$$q(n) \;=\; 1+n\,q(n-1)+\frac{1}{2}\,n(n-1)\,q(n-2), \qquad n\geq 3,$$

as claimed. Thus, the sequence $(a(n))_{n\geq 0}$ defined by $a(n)=q(n)/n!$ for every $n\geq 0$ satisfies the second order linear recurrence equation (with constant coefficients)

$$a(n) \;=\; \frac{1}{n!}+a(n-1)+\frac{1}{2}\,a(n-2), \qquad n\geq 3.$$

The expression for the EGF of $(q(n))_{n\geq 0}$ (which is exactly the GF of $(a(n))_{n\geq 0}$) follows straightforwardly. The claimed closed form for $q(n)$ is then obtained by solving the latter recurrence equation (using the method of variation of parameters). $\square$

Proposition 1.29. *The sequence $(r(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$r(n+1)-(n+1)r(n) \;=\; 1, \qquad n\geq 1,$$

with $r(1)=1$, and we have the explicit expression

$$r(n) \;=\; n!\sum_{i=1}^{n}\frac{1}{i!}, \qquad n\geq 1.$$

We also have the closed-form expression $r(n)=\lfloor n!(e-1)\rfloor$ for every $n\geq 1$. Moreover, its EGF is given by $\widehat{R}(z)=(e^z-z)/(1-z)$.

[2] Actually, we have $r(n)=\mathrm{A002627}(n)$ for every integer $n\geq 1$.

Proof. We clearly have

$$r(n) \;=\; \sum_{i=1}^{n} \binom{n}{i,1,\ldots,1}, \quad n \geq 1,$$

where the multinomial coefficient $\binom{n}{i,1,\ldots,1}$ provides the number of ways to put the elements $1,\ldots,n$ into $(n-i+1)$ classes of sizes $i,1,\ldots,1$. The claimed linear recurrence equation as well as the EGF of $(r(n))_{n\geq 0}$ follow straightforwardly. We then have $r(n) = A002627(n)$ for $n \geq 1$ and the closed-form expression of $(r(n))_{n\geq 0}$ follows immediately [94]. $\square$

n	$p(n)$	$q(n)$	$r(n)$
1	1	1	1
2	3	3	3
3	13	13	10
4	75	71	41
5	541	486	206
6	4 683	3 982	1 237
OEIS	A000670	A307005	A002627

Table 1.1: First few values of $p(n)$, $q(n)$, and $r(n)$

Now, we focus on the enumeration of both single-plateaued and existentially single-peaked weak orders.

Assume that X_n is endowed with $\leq_n$. For any integer $n \geq 0$, we denote by

- $u(n)$ the number of total orders $\leq'$ on X_n that are single-peaked for $\leq_n$,
- $v(n)$ the number of weak orders $\precsim$ on X_n that are single-plateaued for $\leq_n$,
- $w(n)$ the number of weak orders $\precsim$ on X_n that are existentially single-peaked for $\leq_n$,
- $s(n)$ the number of quasilinear weak orders $\precsim$ on X_n that are single-plateaued for $\leq_n$.

As a convention, we set $u(0) = v(0) = w(0) = s(0) = 1$. It is known (see, e.g., [8]) that there are exactly $u(n) = 2^{n-1}$ single-peaked total orders on X_n for $\leq_n$. However, to our knowledge, the number of single-plateaued weak orders on X_n for $\leq_n$ was unknown prior to the contributions recorded in [25]. Propositions 1.31 and 1.33 below provide explicit formulas for the sequences $(v(n))_{n\geq 0}$ and $(w(n))_{n\geq 0}$. The first few values of these sequences are shown in Table 1.2.[3]

Lemma 1.30 (see [25]). *Let $\leq$ be a total order on X and let $\precsim$ be a weak order on X that is single-plateaued for $\leq$. Assume that both $\min_{\leq} X$ and $\max_{\leq} X$ are nonempty and let $a = \min_{\leq} X$ and $b = \max_{\leq} X$. If $\max_{\precsim} X \neq X$, then $\max_{\precsim} X \subseteq \{a, b\}$.*

Proof. By Proposition 1.15 the set $\max_{\precsim} X$ contains at most two elements. Now suppose that there exists $x \in (\max_{\precsim} X) \setminus \{a, b\}$. Then the triplet (a, x, b) violates single-plateauedness of $\precsim$. $\square$

[3]Note that the sequences A048739 and A007052 are shifted versions of $(v(n))_{n\geq 0}$ and $(w(n))_{n\geq 0}$, respectively. More precisely, we have $v(n) = \text{A048739}(n-1)$ and $w(n) = \text{A007052}(n-1)$ for every integer $n \geq 1$.

Proposition 1.31 (see [25]). *The sequence $(v(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$v(n+2)-2\,v(n+1)-v(n)\;=\;1\,,\qquad n\geq 1,$$

with $v(0)=v(1)=1$ and $v(2)=3$, and we have

$$\begin{aligned} 2\,v(n)+1 &= \tfrac{1}{2}(1+\sqrt{2})^{n+1}+\tfrac{1}{2}(1-\sqrt{2})^{n+1}\\ &= \textstyle\sum_{k\geq 0}\binom{n+1}{2k}\,2^k\,,\qquad n\geq 1. \end{aligned}$$

Moreover, its GF is given by $V(z)=(z^3+z^2-2z+1)/(z^3+z^2-3z+1)$.

Proof. We clearly have $v(0)=1$ and $v(1)=1$. So let us assume that $n\geq 2$. If $\precsim$ is a weak order on X_n that is single-plateaued for $\leq_n$, then by Lemma 1.30 either $\max_{\precsim} X_n = X_n$, or $\max_{\precsim} X_n = \{1\}$, or $\max_{\precsim} X_n = \{n\}$, or $\max_{\precsim} X_n = \{1,n\}$. In the three latter cases it is clear that the restriction of $\precsim$ to $X_n\setminus\max_{\precsim} X_n$ is single-plateaued for the restriction of $\leq_n$ to $X_n\setminus\max_{\precsim} X_n$. It follows that the number $v(n)$ of single-plateaued weak orders on X_n for $\leq_n$ satisfies the following second order linear equation

$$v(n)\;=\;1+v(n-1)+v(n-1)+v(n-2),\qquad n\geq 2.$$

The claimed expressions of $v(n)$ and the GF of $(v(n))_{n\geq 0}$ follow straightforwardly. □

Lemma 1.32. *Let $\precsim$ be a weak order on X_n that is existentially single-peaked for $\leq_n$ and let $k\in\{1,\ldots,n\}$. Then $|\max_{\precsim} X_n|=k$ if and only if $\max_{\precsim} X_n=[1,i-1]\cup[n-k+i,n]$ for some $i\in\{1,\ldots,k+1\}$.*

Proof. (Necessity) By Proposition 1.24(ii) we have that $1\in\max_{\precsim} X_n$ or $n\in\max_{\precsim} X_n$. Thus, if $k\in\{1,n-1,n\}$, then the result follows immediately. Now, suppose that $k\in\{2,\ldots,n-2\}$ and suppose to the contrary that

$$\max\nolimits_{\precsim} X_n\;\neq\;[1,i-1]\cup[n-k+i,n],\qquad i\in\{1,\ldots,k+1\}. \tag{1.3}$$

We have three cases to consider.

- If $1\in\max_{\precsim} X_n$ and $n\notin\max_{\precsim} X_n$, then there exist $a,b\in X$ such that $1<_n a<_n b<_n n$ and $a\prec 1\sim b$ by (1.3). But then the triplet (a,b,n) violates existential single-peakedness by Proposition 1.24(ii).

- If $1\notin\max_{\precsim} X_n$ and $n\in\max_{\precsim} X_n$, then there exist $a,b\in X$ such that $1<_n a<_n b<_n n$ and $b\prec a\sim n$ by (1.3). But then the triplet $(1,a,b)$ violates existential single-peakedness by Proposition 1.24(ii).

- If $1,n\in\max_{\precsim} X_n$, then $k\geq 3$ by (1.3). Also, by (1.3), there exist $a,b,c\in X$ such that $1<_n a<_n b<_n c<_n n$, $a\prec 1\sim b\sim n$, and $c\prec 1\sim b\sim n$. But then the triplet (a,b,c) violates existential single-peakedness by Proposition 1.24(ii).

(Sufficiency) Obvious. □

Proposition 1.33. *The sequence $(w(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$w(n+2)-4w(n+1)+2w(n) \;=\; 0\,, \qquad n\geq 1,$$

with $w(0)=w(1)=1$ and $w(2)=3$, and we have

$$\begin{aligned} w(n) &= \tfrac{1}{4}(2+\sqrt{2})^n+\tfrac{1}{4}(2-\sqrt{2})^n \\ &= \textstyle\sum_{k\geq 0}\binom{n}{2k}\,2^{n-k-1}\,, \qquad n\geq 1. \end{aligned}$$

Moreover, its GF is given by $W(z)=(1-3z+z^2)/(1-4z+2z^2)$.

Proof. We clearly have $w(0)=w(1)=1$ and $w(2)=3$. Now, let $n\geq 3$, let $\precsim$ be an existentially single-peaked weak order on X_n, and let k be the number of maximal elements of X_n for $\precsim$. By Lemma 1.32, we have $\max_{\precsim} X_n=[1,i-1]\cup[n-k+i,n]$ for some $i\in\{1,\ldots,k+1\}$. Moreover, the restriction of $\precsim$ to $X_n\setminus\max_{\precsim} X_n$ is existentially single-peaked for the restriction of $\leq_n$ to $X_n\setminus\max_{\precsim} X_n$. Thus, it follows that the sequence $w(n)$ satisfies the linear recurrence equation

$$w(n) \;=\; 1+\sum_{k=1}^{n-1}(k+1)w(n-k), \qquad n\geq 3.$$

We then conclude that the sequence $w(n)$ satisfies the second order linear recurrence equation $w(n)-4w(n-1)+2w(n-2)=0$ for every $n\geq 3$. The expression for the GF of $(w(n))_{n\geq 0}$ follows straightforwardly. The claimed closed form for $w(n)$ is then obtained by solving the latter recurrence equation. □

Lemma 1.34 (see [31])**.** *Let $\precsim$ be a quasilinear weak order on X_n that is single-plateaued for $\leq_n$. If $\max_{\precsim} X_n\neq X_n$, then $\max_{\precsim} X_n\subseteq\{1,n\}$ and $|\max_{\precsim} X_n|=1$.*

Proof. We proceed by contradiction. Since $\precsim$ is quasilinear, the set $\max_{\precsim} X_n$ contains exactly one element. Suppose that $\max_{\precsim} X_n=\{x\}$, where $x\in X_n\setminus\{1,n\}$. Then the triplet $(1,x,n)$ violates the single-plateauedness of $\precsim$. □

Proposition 1.35 (see [31])**.** *The sequence $(s(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$s(n+1) \;=\; 2s(n)+1, \qquad n\geq 1,$$

with $s(0)=s(1)=1$, and we have the closed-form expression

$$s(n) \;=\; 2^n-1, \qquad n\geq 1.$$

Moreover, its GF is given by $S(z)=(2z^2-2z+1)/(2z^2-3z+1)$.

Proof. We clearly have $s(0)=s(1)=1$. So let us assume that $n\geq 2$. If $\precsim$ is a quasilinear weak order on X_n that is single-plateaued for $\leq_n$, then by Lemma 1.34, either $\max_{\precsim} X_n=X_n$ or $\max_{\precsim} X_n=\{1\}$ or $\max_{\precsim} X_n=\{n\}$. In the two latter cases, it is clear that the restriction of $\precsim$ to $X_n\setminus\max_{\precsim} X_n$ is quasilinear and single-plateaued for the restriction of $\leq_n$ to $X_n\setminus\max_{\precsim} X_n$. It follows that the number $s(n)$ of quasilinear weak orders on X_n that are single-plateaued for $\leq_n$ satisfies the first order linear equation

$$s(n) \;=\; 1+s(n-1)+s(n-1), \qquad n\geq 2.$$

The stated closed-form expression of $s(n)$ and the GF of $(s(n))_{n\geq 0}$ follow straightforwardly. □

n	$u(n)$	$v(n)$	$w(n)$	$s(n)$
0	1	1	1	1
1	1	1	1	1
2	2	3	3	3
3	4	8	10	7
4	8	20	34	15
5	16	49	116	31
6	32	119	396	63
OEIS	A011782	A048739	A007052	A000225

Table 1.2: First few values of $u(n)$, $v(n)$, $w(n)$, and $s(n)$

Example 1.36. The $w(3) = 10$ weak orders on X_3 that are existentially single-peaked for $\leq_3$ are: $1 \prec 2 \prec 3$, $2 \prec 1 \prec 3$, $2 \prec 3 \prec 1$, $3 \prec 2 \prec 1$, $1 \prec 2 \sim 3$, $2 \prec 1 \sim 3$, $3 \prec 1 \sim 2$, $1 \sim 2 \prec 3$, $2 \sim 3 \prec 1$, and $1 \sim 2 \sim 3$. $v(3) = 8$ of those are single-plateaued for $\leq_3$, $s(3) = 7$ of those are quasilinear and single-plateaued for $\leq_3$, and $u(3) = 4$ of those are total orders that are single-peaked for $\leq_3$.

For any sequence

$$(t(n))_{n\geq 0} \in \{(p(n))_{n\geq 0}, (q(n))_{n\geq 0}, (r(n))_{n\geq 0}, (u(n))_{n\geq 0}, (v(n))_{n\geq 0}, (w(n))_{n\geq 0}, (s(n))_{n\geq 0}\},$$

we denote by

- $(t_{\text{iso}}(n))_{n\geq 0}$ the number of those weak orders on X_n that are defined up to an isomorphism. As a convention, we set $t_{\text{iso}}(0) = 1$.
- $(t_e(n))_{n\geq 0}$ the number of those weak orders $\precsim$ on X_n for which X_n has exactly one minimal element for $\precsim$. As a convention, we set $t_e(0) = 0$.
- $(t_a(n))_{n\geq 0}$ the number of those weak orders $\precsim$ on X_n for which X_n has exactly one maximal element for $\precsim$. As a convention, we set $t_a(0) = 0$.
- $(t_{ae}(n))_{n\geq 0}$ the number of those weak orders $\precsim$ on X_n for which X_n has exactly one minimal element and exactly one maximal element for $\precsim$, the two elements being distinct. As a convention, we set $t_{ae}(0) = 0$. Also, by definition we have $t_{ae}(1) = 0$.

It is clear that the number of weak orders on X_n that are defined up to an isomorphism is precisely the number of totally ordered partitions of a set of n unlabeled items (Sloane's A011782), that is, $p_{\text{iso}}(n) = 2^{n-1}$ for all $n \geq 1$. Also, for a weak order $\precsim$ on X_n that is existentially single-peaked for $\leq_n$ we have $|\max_{\precsim} X_n| \leq n$. Moreover, existential single-peakedness is a property that depends on $\leq_n$. Thus, we conclude that $w_{\text{iso}}(n) = p_{\text{iso}}(n) = 2^{n-1}$ for all $n \geq 1$. Finally, since single-peakedness is also a property that depends on $\leq_n$, it is not difficult to see that the number of single-peaked total orders on X_n for $\leq_n$ that are defined up to an isomorphism is precisely the number of total orders on X_n that are defined up to an isomorphism, that is, $u_{\text{iso}}(n) = 1$ for all $n \geq 1$. The next proposition, provides explicit formulas for the sequences $(q_{\text{iso}}(n))_{n\geq 0}$, $(r_{\text{iso}}(n))_{n\geq 0}$, and $(v_{\text{iso}}(n))_{n\geq 0}$.

Proposition 1.37 (see [34]). *We have $q_{iso}(n) = v_{iso}(n) = F_{n+2} - 1$ for every $n \geq 1$, where F_n is the nth Fibonacci number. Moreover, we have $r_{iso}(n) = s_{iso}(n) = n$ for every $n \geq 1$.*

Proof. Let us consider the sequence $q_{\text{iso}}(n)$. We clearly have $q_{\text{iso}}(1) = 1$ and $q_{\text{iso}}(2) = 2$. Let $n \geq 3$. Proceeding as in the proof of Proposition 1.28, we see that the sequence $q_{\text{iso}}(n)$ satisfies the second order linear recurrence equation

$$q_{\text{iso}}(n) \;=\; 1 + q_{\text{iso}}(n-1) + q_{\text{iso}}(n-2), \qquad n \geq 3.$$

The explicit expression for $q_{\text{iso}}(n)$ then follows immediately. Also, for a weak order $\precsim$ on X_n that is single-plateaued for $\leq_n$ we have $|\max_{\precsim} X_n| \leq 2$ by Lemma 1.30. Moreover, since single-plateauedness is a property that depends on $\leq_n$, we conclude that $v_{\text{iso}}(n) = q_{\text{iso}}(n)$ for every $n \geq 1$.

Let us now consider the sequence $r_{\text{iso}}(n)$. We clearly have $r_{\text{iso}}(1) = 1$. Let $n \geq 2$. Proceeding as in the proof of Proposition 1.28, we see that the sequence $r_{\text{iso}}(n)$ satisfies the first order linear recurrence equation

$$r_{\text{iso}}(n) \;=\; 1 + r_{\text{iso}}(n-1), \qquad n \geq 2.$$

The explicit expression for $r_{\text{iso}}(n)$ then follows immediately. Moreover, since single-plateauedness is again a property that depends on $\leq_n$, we conclude that $s_{\text{iso}}(n) = r_{\text{iso}}(n)$ for every $n \geq 1$. $\square$

It is clear that the number of weak orders $\precsim$ on X_n that have exactly one minimal element (resp. maximal element) is precisely the number of totally ordered partitions of X_n such that the minimal (resp. maximal) set $C \in X/{\sim}$ is a singleton (Sloane's A052882), that is, $p_e(n) = p_a(n) = np(n-1)$ for all $n \geq 1$. Also, we observe that $p_{ae}(n) = np_e(n-1)$ for all $n \geq 1$. Similarly, for all $n \geq 1$, we have $q_a(n) = nq(n-1)$ and $q_{ae}(n) = nq_e(n-1)$. Also, a straightforward adaptation of the proof of Proposition 1.28 shows that $q_e(n) = n!G_n$ for all $n \geq 0$, where $G_n = \frac{\sqrt{3}}{3}\,(\frac{1+\sqrt{3}}{2})^n - \frac{\sqrt{3}}{3}\,(\frac{1-\sqrt{3}}{2})^n$. Moreover, it is clear that $u_e(n) = u_a(n) = u_{ae}(n) = u(n)$ for all $n \geq 1$. Furthermore, for all $n \geq 2$, it is not difficult to see that $r_e(n) = r_{ae}(n) = n!$ and $r_a(n) = r(n) - 1$. Finally, for all $n \geq 2$, it is easy to see that $s_e(n) = s_{ae}(n) = u_e(n)$ and $s_a(n) = s(n) - 1$.

Propositions 1.38 and 1.40 below provide explicit formulas for the remaining sequences. The first few values of these sequences are shown in Tables 1.3 and 1.4.[4] It turns out that the sequence $(v_e(n))_{n\geq 0}$ consists of the so-called *Pell numbers* (Sloane's A000129).

Proposition 1.38 (see [25]). *The sequence $(v_e(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$v_e(n+2) - 2\,v_e(n+1) - v_e(n) \;=\; 0\,, \qquad n \geq 0,$$

with $v_e(0) = 0$ and $v_e(1) = 1$, and we have

$$\begin{aligned} v_e(n) &= \tfrac{\sqrt{2}}{4}(1+\sqrt{2})^n - \tfrac{\sqrt{2}}{4}(1-\sqrt{2})^n \\ &= \textstyle\sum_{k\geq 0} \binom{n}{2k+1} 2^k\,, \qquad n \geq 0. \end{aligned}$$

Moreover, its GF is given by $V_e(z) = -z/(z^2 + 2z - 1)$. Furthermore, for any integer $n \geq 1$, we have $v_a(n) = 2v(n-1)$ and $v_{ae}(n) = 2v_e(n-1)$.

[4] Note that the sequences A163271, A003480, and A006012 are shifted versions of $(v_{ae}(n))_{n\geq 0}$, $(w_e(n))_{n\geq 0}$, and $(w_a(n))_{n\geq 0}$, respectively. More precisely, we have $v_{ae}(n) = \text{A163271}(n-1)$, $w_e(n) = \text{A003480}(n-1)$, and $w_a(n) = \text{A006012}(n-1)$ for every integer $n \geq 1$.

Proof. The formula describing the sequence $(v_e(n))_{n\geq 0}$ is obtained by following the same steps as in the proof of Proposition 1.31, except that in this case we always have $\max_{\precsim} X_n \neq X_n$. As for the sequence $(v_a(n))_{n\geq 0}$ we note that $\max_{\precsim} X_n$ must be either $\{1\}$ or $\{n\}$ and that the restriction of $\precsim$ to $X_n \setminus \max_{\precsim} X_n$ is single-plateaued for the restriction of $\leq_n$ to $X_n \setminus \max_{\precsim} X_n$. We proceed similarly for the sequence $(v_{ae}(n))_{n\geq 0}$. $\square$

n	$v_e(n)$	$v_a(n)$	$v_{ae}(n)$
0	0	0	0
1	1	1	0
2	2	2	2
3	5	6	4
4	12	16	10
5	29	40	24
6	70	98	58
OEIS	A000129	A293004	A163271

Table 1.3: First few values of $v_e(n)$, $v_a(n)$, and $v_{ae}(n)$

Example 1.39. The $v(3) = 8$ weak orders on X_3 that are single-plateaued for $\leq_3$ are: $1 \prec 2 \prec 3$, $2 \prec 1 \prec 3$, $2 \prec 3 \prec 1$, $3 \prec 2 \prec 1$, $2 \prec 1 \sim 3$, $1 \sim 2 \prec 3$, $2 \sim 3 \prec 1$, and $1 \sim 2 \sim 3$. $v_e(3) = 5$ of those have exactly one minimal element and $v_a(3) = 6$ of those have exactly one maximal element. $v_{ae}(3) = 4$ of those have exactly one minimal element and exactly one maximal element. These four weak orders correspond to the $u(4) = 2^{3-1} = 4$ total orders on X_3 that are single-peaked for $\leq_3$.

n	$w_e(n)$	$w_a(n)$	$w_{ae}(n)$
0	0	0	0
1	1	1	0
2	2	2	2
3	7	6	4
4	24	20	14
5	82	68	48
6	280	232	164
OEIS	A003480	A006012	

Table 1.4: First few values of $w_e(n)$, $w_a(n)$, and $w_{ae}(n)$

Proposition 1.40. *The sequence $(w_e(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$w_e(n+2) - 4\,w_e(n+1) + 2\,w_e(n) \;=\; 0\,, \qquad n \geq 2,$$

with $w_e(0) = 0$, $w_e(1) = 1$, $w_e(2) = 2$, and $w_e(3) = 7$, and we have

$$\begin{aligned} w_e(n) \;&=\; \tfrac{\sqrt{2}}{8}(2+\sqrt{2})^n - \tfrac{\sqrt{2}}{8}(2-\sqrt{2})^n \\ &=\; \textstyle\sum_{k\geq 0} \binom{n}{2k+1}\, 2^{n-k-2}\,, \qquad n \geq 2. \end{aligned}$$

Moreover, its GF is given by $W_e(z) = z(1-z)^2/(1-4z+2z^2)$. *Furthermore, for any integer* $n \geq 1$, *we have* $w_a(n) = 2w(n-1)$ *and* $w_{ae}(n) = 2w_e(n-1)$.

Proof. The formula describing the sequence $(w_e(n))_{n\geq 0}$ is obtained by following the same steps as in the proof of Proposition 1.33, except that in this case we always have $\max_{\precsim} X_n \neq X_n$. As for the sequence $(w_a(n))_{n\geq 0}$ we note that $\max_{\precsim} X_n$ must be either $\{1\}$ or $\{n\}$ and that the restriction of $\precsim$ to $X_n \setminus \max_{\precsim} X_n$ is existentially single-peaked for the restriction of $\leq_n$ to $X_n \setminus \max_{\precsim} X_n$. We proceed similarly for the sequence $(w_{ae}(n))_{n\geq 0}$. $\square$

Example 1.41. The $w(3) = 10$ weak orders on X_3 that are existentially single-peaked for $\leq_3$ are: $1 \prec 2 \prec 3$, $2 \prec 1 \prec 3$, $2 \prec 3 \prec 1$, $3 \prec 2 \prec 1$, $1 \prec 2 \sim 3$, $2 \prec 1 \sim 3$, $3 \prec 1 \sim 2$, $1 \sim 2 \prec 3$, $2 \sim 3 \prec 1$, and $1 \sim 2 \sim 3$. $w_e(3) = 7$ of those have exactly one minimal element and $w_a(3) = 6$ of those have exactly one maximal element. $w_{ae}(3) = 4$ of those have exactly one minimal element and exactly one maximal element. These four weak orders correspond to the $u(4) = 2^{3-1} = 4$ total orders on X_3 that are single-peaked for $\leq_3$.

Chapter 2

Semigroups

Some sources (see, e.g., [21]) attribute the first use of the term semigroup in mathematical literature to J.-A. de Séguier in *Éléments de la Théorie des Groupes Abstraits* (Paris, 1904). Since then the theory of semigroups has evolved with the study of its applications in other fields such as automata theory and functional analysis. We refer to [21, 55] for a historical background on semigroup theory.

In this chapter, we first introduce the basic semigroup theory needed in this thesis (Section 2.1). Then, we study the class of rectangular semigroups which is the key object towards the characterization of the class of idempotent semigroups (Section 2.2). Most of the contributions presented in this chapter stem from [32].

2.1 Preliminaries

The concepts introduced in this section are stemming from standard books on introductory semigroup theory (see, e.g., [21, 56, 80, 81]).

An operation $F\colon X^2 \to X$ is said to be

- *associative* if $F(F(x,y),z) = F(x,F(y,z))$ for all $x,y,z \in X$;
- *idempotent* if $F(x,x) = x$ for all $x \in X$;
- *quasitrivial* (or *conservative*) if $F(x,y) \in \{x,y\}$ for all $x,y \in X$;
- *commutative* (or *symmetric*) if $F(x,y) = F(y,x)$ for all $x,y \in X$;
- *anticommutative* if $\forall x,y \in X\colon F(x,y) = F(y,x) \Rightarrow x = y$.

Recall that if $F\colon X^2 \to X$ is an operation, then the pair (X,F) is said to be a *groupoid*. Moreover, if $F\colon X^2 \to X$ is an associative operation, then (X,F) is said to be a *semigroup*. An idempotent semigroup is also said to be a *band*.

A groupoid (X,F) is said to be *trivial* if $|X| = 1$. Also, we say that X is *trivial* if $|X| = 1$.

Recall that an element $e \in X$ is a *neutral element* of an operation $F\colon X^2 \to X$ if $F(x,e) = F(e,x) = x$ for every $x \in X$. A semigroup (X,F) for which F has a neutral element is called a *monoid*. If $F\colon X^2 \to X$ is an associative operation that has no neutral element, then we can *adjoin to X a neutral element e for F*; that is, there is a binary associative operation $F^*\colon (X \cup \{e\})^2 \to X \cup \{e\}$ such that e is a neutral element for F^* and $F^*|_{X^2} = F$.

Remark 2.1. Suppose that (X,F) is a monoid (otherwise, we can adjoin to X a neutral element e for F and consider the monoid $(X\cup\{e\},F^*)$). We can define five equivalence relations L,R,J,H,D on X by

- $xLy \quad\Leftrightarrow\quad \{F(z,x)\colon z\in X\}=\{F(z,y)\colon z\in X\}, \qquad x,y\in X.$
- $xRy \quad\Leftrightarrow\quad \{F(x,z)\colon z\in X\}=\{F(y,z)\colon z\in X\}, \qquad x,y\in X.$
- $xJy \quad\Leftrightarrow\quad \{F(F(u,x),v)\colon u,v\in X\}=\{F(F(u,y),v)\colon u,v\in X\}, \qquad x,y\in X.$
- $xHy \quad\Leftrightarrow\quad xLy$ and $xRy, \qquad x,y\in X.$
- $xDy \quad\Leftrightarrow\quad \exists z\in X$ such that xLz and $zRy, \qquad x,y\in X.$

These equivalence relations are called *Green's relations* [54]. Green's relations are fundamental tools in the study of semigroups. They were used in order to understand the structure of several classes of semigroups. In this thesis, we will not make use of Green's relations but we will use other tools that are convenient for the study of bands (see Chapter 3).

Recall that an equivalence relation $\sim$ on X is said to be a *congruence* for $F\colon X^2\to X$ if it is compatible with F, that is, for any $x,y,z\in X$,

$$x\sim y \quad\Rightarrow\quad F(x,z)\sim F(y,z) \text{ and } F(z,x)\sim F(z,y).$$

In that case, $\sim$ is also said to be a congruence on the groupoid (X,F). Also, we denote by $\tilde{F}$ the operation induced by F on $X/\sim$, that is,

$$\tilde{F}([x]_\sim,[y]_\sim) \;=\; [F(x,y)]_\sim, \qquad x,y\in X.$$

We denote the *range* of an operation $F\colon X^2\to X$ by $\mathrm{ran}(F)$. Also, the *diagonal section* $\delta_F\colon X\to X$ of an operation $F\colon X^2\to X$ is defined by $\delta_F(x)=F(x,x)$ for any $x\in X$.

Two groupoids (X,F) and (Y,G) are said to be *isomorphic* if there is a bijection $\phi\colon X\to Y$ such that $\phi(F(x,y))=G(\phi(x),\phi(y))$ for every $x,y\in X$. In that case, the operations F and G are said to be *conjugate to each other*. Also, two sets X and Y are said to be *equipollent* if there exists a bijection $\phi\colon X\to Y$.

Let Y be a nonempty set. For any operations $F\colon X^2\to X$ and $G\colon Y^2\to Y$ we define the operation $F\times G\colon (X\times Y)^2\to X\times Y$ by $(F\times G)((x,y),(u,v))=(F(x,u),G(y,v))$ for all $(x,y),(u,v)\in X\times Y$. In that case, the groupoid $(X\times Y,F\times G)$ is said to be the *direct sum of* (X,F) *and* (Y,G). It is easy to see that $(X\times Y,F\times G)$ is isomorphic to (Y,G) (resp. (X,F)) whenever X is trivial (resp. Y is trivial). Of course, if (X,F) and (Y,G) are semigroups, then their direct sum $(X\times Y,F\times G)$ is again a semigroup.

Recall that the *projection operations* $\pi_1\colon X^2\to X$ and $\pi_2\colon X^2\to X$ are respectively defined by $\pi_1(x,y)=x$ and $\pi_2(x,y)=y$ for all $x,y\in X$. In that case, the pairs (X,π_1) and (X,π_2) are called *left zero semigroup* and *right zero semigroup* , respectively.

The following fact will be useful as we continue.

Fact 2.2. *Let $F\colon X^2\to X$ be an operation, let Y be a set that is equipollent to X, and let $i\in\{1,2\}$. Then $F=\pi_i$ if and only if (X,F) is isomorphic to (Y,π_i).*

The preimage of an element $x \in X$ under an operation $F\colon X^2 \to X$ is denoted by $F^{-1}[x]$. When $X = X_n$ for some integer $n \geq 1$, we also define the *preimage sequence of* F as the nondecreasing n-element sequence of the numbers $|F^{-1}[x]|$, $x \in X_n$. We denote this sequence by $|F^{-1}|$.

Example 2.3. Let $i \in \{1,2\}$ and let us consider the operation $\pi_i\colon X_n^2 \to X_n$. Then we have $|\pi_i^{-1}| = (n,\ldots,n)$. Also, if $\max_{\leq_n}\colon X_n^2 \to X_n$ denotes the maximum operation on X_n for $\leq_n$, then $|\max_{\leq_n}^{-1}| = (1,3,\ldots,2n-1)$.

Fact 2.4. *Let Y, Z be two non-empty sets and let $F\colon Y^2 \to Y$ and $G\colon Z^2 \to Z$ be two operations. For any $(u,v) \in Y \times Z$ we have that $(F \times G)^{-1}[(u,v)]$ is equipollent to $F^{-1}[u] \times G^{-1}[v]$.*

2.2 Rectangular semigroups

In this section we introduce and study the class of rectangular semigroups. In particular, we provide characterizations of this class as well as enumeration results [32].

An associative operation $F\colon X^2 \to X$ is said to be *rectangular* [64] if $F(x, F(y,x)) = x$ for all $x, y \in X$. It turns out that any such operation is idempotent.

Fact 2.5 (see [64]). *If an associative operation $F\colon X^2 \to X$ is rectangular, then it is idempotent.*

The converse of Fact 2.5 is not true in general. For instance, consider the chain $(X, \leq)$ together with the maximum operation $\max_{\leq}\colon X^2 \to X$ defined by $\max_{\leq}(x,y) = y$ if $x \leq y$. Then $\max_{\leq}$ is associative and idempotent but not rectangular since $\max_{\leq}(x, \max_{\leq}(y,x)) = y$ whenever $x \leq y$.

A typical example of rectangular semigroup is given by the semigroup $(X \times Y, \pi_1 \times \pi_2)$. The next theorem shows that any rectangular semigroup is isomorphic to the direct sum of a left zero semigroup and a right zero semigroup.

Theorem 2.6 (see [64]). *An operation $F\colon X^2 \to X$ is associative and rectangular if and only if there exist two non-empty sets Y, Z such that (X, F) is isomorphic to $(Y \times Z, \pi_1 \times \pi_2)$.*

Example 2.7. Let us construct an associative and rectangular operation on X_4. To this extent, we consider the bijection $\phi\colon X_4 \to \{1,2\} \times \{3,4\}$ defined by $\phi(1) = (1,3)$, $\phi(2) = (1,4)$, $\phi(3) = (2,3)$, and $\phi(4) = (2,4)$. The operation $F\colon X_4^2 \to X_4$ defined by

$$F(x,y) \;=\; \phi^{-1}((\pi_1 \times \pi_2)(\phi(x), \phi(y))), \qquad x, y \in X_4,$$

is then associative and rectangular by Theorem 2.6.

The following corollary, which provides a characterization of quasitrivial rectangular semigroups, follows from Fact 2.2 and Theorem 2.6.

Corollary 2.8. *Let $F\colon X^2 \to X$ be an associative and rectangular operation. Then F is quasitrivial if and only if $F = \pi_i$ for some $i \in \{1,2\}$.*

The following corollary follows from Fact 2.4 and Theorem 2.6.

Corollary 2.9. *If $F\colon X_n^2 \to X_n$ is associative and rectangular, then $|F^{-1}| = (n,\ldots,n)$.*

Remark 2.10. The converse of Corollary 2.9 is not true in general. For instance, let us consider the operation $F\colon X_3^2 \to X_3$ defined by the following conditions:

- $F(x,x) = x$ for all $x \in X_3$.
- $F(1,1) = F(1,2) = F(2,3)$.
- $F(2,2) = F(1,3) = F(3,2)$.
- $F(3,3) = F(2,1) = F(3,1)$.

Then $|F^{-1}| = (3,3,3)$ but F is neither associative nor rectangular. Indeed, we have

$$F(1,F(2,1)) \; = \; F(1,3) \; = \; 2 \quad \text{and} \quad F(F(1,2),1) \; = \; F(1,1) \; = \; 1.$$

For all integer $n \geq 1$, let $\alpha(n)$ (resp. $\beta(n)$) denote the number of associative and rectangular operations on X_n (resp. the number of associative and rectangular operations on X_n that are defined up to an isomorphism). In the following propositions we show that $\alpha(n) = A121860(n)$ and $\beta(n) = d(n) = A000005(n)$ (see [94]), where $d(n)$ denotes the number of positive integer divisors of n.

Proposition 2.11 (see [32]). *For all integer $n \geq 1$, we have*

$$\alpha(n) \; = \; \sum_{d|n} \frac{n!}{d!\,\left(\frac{n}{d}\right)!}.$$

Proof. By Theorem 2.6, we clearly have $\alpha(1) = 1$, $\alpha(2) = 2$, and $\alpha(3) = 2$. So, let $n \geq 4$ and let $F\colon X_n^2 \to X_n$ be an associative and rectangular operation. By Theorem 2.6, there exist two sets Y, Z with $|Y|, |Z| \geq 1$ such that (X_n, F) is isomorphic to $(Y \times Z, \pi_1 \times \pi_2)$. Thus, due to the usual representation of a rectangular semigroup as a direct sum of a left zero semigroup and a right zero semigroup (see Figure 2.1 and Corollary 2.27), counting the number of associative and rectangular operations on X_n is equivalent to counting the number of ways to partition X_n into k equivalence classes of sizes $l, \ldots, l$ and the number of bijections between two consecutive equivalence classes. Thus, we have

$$\alpha(n) \; = \; \sum_{\substack{k,l \\ kl=n}} \frac{\binom{n}{l,\ldots,l}}{k!} (l!)^{k-1} \; = \; \sum_{\substack{k,l \\ kl=n}} \frac{n!}{k!l!},$$

where the multinomial coefficient $\binom{n}{l,\ldots,l}$ provides the number of ways to put the elements $1, \ldots, n$ into k classes of sizes $l, \ldots, l$ and $l!$ is the number of bijections between two such classes. $\square$

Proposition 2.12 (see [32]). *For all integer $n \geq 1$, we have $\beta(n) = d(n)$.*

Proof. By Theorem 2.6, counting the number of associative and rectangular operations on X_n that are defined up to an isomorphism is equivalent to counting the number of finite sets Y, Z such that $|Y \times Z| = |X_n| = n$ up to a bijection. Thus, $\beta(n)$ provides the number of ways to write n into a product of two elements $k, l \in \{1, \ldots, n\}$. This is in turn the number of divisors of n. $\square$

The following corollaries are immediate consequences of Theorem 2.6 and Propositions 2.11 and 2.12.

Corollary 2.13 (see [32]). *$\alpha(n) = 2$ (resp. $\beta(n) = 2$) if and only if n is prime.*

Corollary 2.14 (see [32]). *Let $F\colon X_n^2 \to X_n$ be an associative and rectangular operation. If n is prime, then $F = \pi_1$ or $F = \pi_2$.*

The following lemma provides a characterization of rectangular semigroups by means of alternative properties.

Lemma 2.15 (see [56, 64, 78]). *Let $F\colon X^2 \to X$ be an associative operation. The following assertions are equivalent.*

(i) F is rectangular.

(ii) F is surjective and $F(F(x,y),z) = F(x,z)$ for all $x,y,z \in X$.

(iii) F is idempotent and $F(F(x,y),z) = F(x,z)$ for all $x,y,z \in X$.

(iv) F is anticommutative.

In view of Lemma 2.15, we now study the class of operations $F\colon X^2 \to X$ satisfying

$$F(F(x,y),z) \;=\; F(x,F(y,z)) \;=\; F(x,z), \qquad x,y,z \in X. \tag{2.1}$$

Any rectangular operation $F\colon X^2 \to X$ satisfies (2.1) by Lemma 2.15. However, the converse of the latter statement is not true in general. For instance, the operation $F\colon X_2^2 \to X_2$ defined by $F(x,y) = 1$ for all $x,y \in X_2$ satisfies (2.1) but it is not rectangular.

The following lemma provides a partial description of the latter class of semigroups.

Lemma 2.16 (see [64]). *If $F\colon X^2 \to X$ satisfies (2.1), then $(\mathrm{ran}(\delta_F), F|_{\mathrm{ran}(\delta_F)^2})$ is a rectangular semigroup and $F(x,y) = F(F(x,x),F(y,y))$ for all $x,y \in X$.*

We now introduce a simple functional equation that turns out to be equivalent to (2.1).

Definition 2.17 (see [32]). An operation $F\colon X^2 \to X$ is said to be $(1,4)$-*selective* if

$$F(F(x,y),F(u,v)) \;=\; F(x,v), \qquad x,y,u,v \in X.$$

If $F\colon X^2 \to X$ is $(1,4)$-selective, then the groupoid (X,F) is said to be *generalized diagonal* [86]. Moreover, if F is also idempotent, then the groupoid (X,F) is said to be *diagonal* [85].

Remark 2.18. Let $i,j \in \{1,2,3,4\}$. An operation $F\colon X^2 \to X$ is said to be (i,j)-*selective* [32] if $F(F(x_1,x_2),F(x_3,x_4)) = F(x_i,x_j)$ for all $x_1,x_2,x_3,x_4 \in X$. This class of operations was completely characterized in [32].

The following proposition provides a characterization of the class of operations $F\colon X^2 \to X$ satisfying (2.1).

Proposition 2.19 (see [86, 99]). *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) F is $(1,4)$-selective.

(ii) $F|_{\mathrm{ran}(F)^2}$ *is* $(1,4)$*-selective and* $F(x,y) = F(F(x,x), F(y,y))$ *for all* $x, y \in X$.

(iii) F *satisfies* (2.1).

The following corollary is an immediate consequence of Lemma 2.15 and Proposition 2.19.

Corollary 2.20. *An operation* $F\colon X^2 \to X$ *satisfies* (2.1) *if and only if* $(\mathrm{ran}(F), F|_{\mathrm{ran}(F)^2})$ *is a rectangular semigroup and* $F(x,y) = F(F(x,x), F(y,y))$ *for all* $x, y \in X$.

Corollary 2.20 is of particular interest as it enables us to easily enumerate the class of operations $F\colon X_n^2 \to X_n$ satisfying (2.1). For all integer $n \geq 1$, let $\rho(n)$ denote the number of operations on X_n satisfying (2.1). The following proposition provides an explicit formula for $\rho(n)$. Also, the first few values are given in Table 2.1.

Proposition 2.21. *For all integer* $n \geq 1$*, we have*

$$\rho(n) \ = \ \sum_{k=1}^{n} \binom{n}{k} k^{n-k} \alpha(k).$$

Proof. We clearly have $\rho(1) = 1$. So let $n \geq 2$, let $F\colon X_n^2 \to X_n$ be an operation satisfying (2.1), and let k be the number of elements of $\mathrm{ran}(F)$. We have that $(\mathrm{ran}(F), F|_{\mathrm{ran}(F)^2})$ is a rectangular semigroup by Corollary 2.20. Also, we have that $F(x,y) = F(F(x,x), F(y,y))$ for all $x, y \in X$ by Corollary 2.20. That is, for any $x, y \in X$, the value of $F(x,y)$ is determined by the values of $F(x,x)$ and $F(y,y)$. Thus, F is completely determined by $F|_{\mathrm{ran}(F)^2}$ and δ_F. Therefore, we obtain the claimed formula where k^{n-k} provides the number of functions from $X \setminus \mathrm{ran}(F)$ to $\mathrm{ran}(F)$. □

n	$\alpha(n)$	$\beta(n)$	$\rho(n)$
1	1	1	1
2	2	2	4
3	2	2	17
4	8	3	79
5	2	2	407
6	122	4	2 350
OEIS	A121860	A000005	

Table 2.1: First few values of $\alpha(n)$, $\beta(n)$, and $\rho(n)$

For any operation $F\colon X^2 \to X$ we define two binary relations $\sim_1$ and $\sim_2$ on X by

$$x \sim_1 y \quad \Leftrightarrow \quad F(x,y) = x \qquad x, y \in X,$$

and

$$x \sim_2 y \quad \Leftrightarrow \quad F(x,y) = y \qquad x, y \in X.$$

It is easy to see that $\sim_1$ is reflexive if and only if $\sim_2$ is reflexive.

Remark 2.22. We observe that for all $F\colon X^2 \to X$ the binary relation $\sim_2$ on X was already introduced in [13] and was called the *trace* of F.

The following observation will be useful as we continue.

Fact 2.23 (see [32]). *If F is an associative operation, then $\sim_1$ and $\sim_2$ are transitive.*

Proposition 2.24 (see [32]). *Let $F\colon X^2 \to X$ be $(1,4)$-selective. The following assertions are equivalent.*

(i) *F is anticommutative.*

(ii) *F is surjective.*

(iii) *F is idempotent.*

(iv) *$\sim_1$ is an equivalence relation on X.*

(v) *$\sim_2$ is an equivalence relation on X.*

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follow from Lemma 2.15 and Proposition 2.19.

Now, let us show that $(iii) \Rightarrow ((iv)$ and $(v))$. The binary relations $\sim_1$ and $\sim_2$ are clearly reflexive since F is idempotent. Also, by Proposition 2.19 and Fact 2.23 we have that $\sim_1$ and $\sim_2$ are transitive. Now, let us show that $\sim_1$ and $\sim_2$ are symmetric. Let $x, y, u, v \in X$ such that $x \sim_1 y$ and $u \sim_2 v$, that is, $F(x, y) = x$ and $F(u, v) = v$. Then, using idempotency and $(1,4)$-selectiveness, we get

$$F(y, x) \ = \ F(F(y, y), F(x, y)) \ = \ F(y, y) \ = \ y,$$

and

$$F(v, u) \ = \ F(F(u, v), F(u, u)) \ = \ F(u, u) \ = \ u,$$

that is, $y \sim_1 x$ and $v \sim_2 u$.

Finally, the implications $(iv) \Rightarrow (iii)$ and $(v) \Rightarrow (iii)$ are obvious. $\square$

Proposition 2.25 (see [32]). *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) *F is $(1,4)$-selective and satisfies any of the conditions $(i) - (v)$ of Proposition 2.24.*

(ii) *$\sim_1$ and $\sim_2$ are equivalence relations on X such that $[x]_{\sim_2} \cap [y]_{\sim_1} = \{F(x, y)\}$ for all $x, y \in X$.*

(iii) *The following conditions hold.*

- *(a)* *$\sim_1$ and $\sim_2$ are equivalence relations on X.*
- *(b)* *For all $y, z \in X$ and all $x \in [y]_{\sim_1}$ there exists a unique $u \in [z]_{\sim_1}$ such that $x \sim_2 u$.*
- *(c)* *For all $x, y, z \in X$ such that $y \sim_1 z$ we have $F(x, y) = F(x, z)$.*

Moreover, if any of the assertions $(i) - (iii)$ are satisfied, then $\sim_1$ and $\sim_2$ are congruences for F such that $\overset{\sim_1}{F} = \pi_2$ and $\overset{\sim_2}{F} = \pi_1$.

Proof. Let us first show that $(i) \Rightarrow (ii)$. By Proposition 2.24 we have that $\sim_1$ and $\sim_2$ are equivalence relations on X. Let $x, y \in X$ and let us show that $[x]_{\sim_2} \cap [y]_{\sim_1} = \{F(x,y)\}$. By Lemma 2.15 and Proposition 2.19 we have $F(x,y) \in [x]_{\sim_2} \cap [y]_{\sim_1}$. Also, if $z \in [x]_{\sim_2} \cap [y]_{\sim_1}$, then $z \sim_1 F(x,y)$ and $z \sim_2 F(x,y)$ which by definition implies that $z = F(x,y)$.

Now, let us show that $(ii) \Rightarrow (iii)$. Condition (a) is clearly satisfied.

So, let $x, y, z \in X$ such that $x \in [y]_{\sim_1}$. By (ii), we have $[x]_{\sim_2} \cap [z]_{\sim_1} = \{F(x,z)\}$, which proves condition (b).

Now, let $x, y, z \in X$ such that $y \sim_1 z$, that is, $[y]_{\sim_1} = [z]_{\sim_1}$. By (ii) and the previous assumption, we get

$$\{F(x,y)\} \;=\; [x]_{\sim_2} \cap [y]_{\sim_1} \;=\; [x]_{\sim_2} \cap [z]_{\sim_1} \;=\; \{F(x,z)\},$$

which proves condition (c).

Finally, let us show that $(iii) \Rightarrow (i)$. Since $\sim_1$ and $\sim_2$ are equivalence relations on X, it follows that F is idempotent. Let $x, y, u, v \in X$ and let us show that $F(F(x,y), F(u,v)) = F(x,v)$. We clearly have that $t \in [t]_{\sim_1}$ for all $t \in X$. By conditions (b) and (c), we have that there exists a unique $s \in [y]_{\sim_1}$ such that $F(x,y) = F(x,s) = s$, that is, $x \sim_2 s$. Also, by conditions (b) and (c), we have that there exists a unique $t \in [v]_{\sim_1}$ such that $F(u,v) = F(u,t) = t$, that is, $u \sim_2 t$. Moreover, by conditions (b) and (c), we have that there exists a unique $z \in [v]_{\sim_1}$ such that $F(s,t) = F(s,z) = z$, that is, $s \sim_2 z$. By transitivity of $\sim_2$ we have that $x \sim_2 z$ and by condition (b) we have that z is unique. Thus, we obtain $F(F(x,y), F(u,v)) = F(s,t) = z = F(x,z) = F(x,v)$ which concludes the proof.

Now, assuming any of the conditions $(i) - (iii)$, it is not difficult to see that $\sim_1$ and $\sim_2$ are congruences for F such that $\overset{\sim_1}{F} = \pi_2$ and $\overset{\sim_2}{F} = \pi_1$. □

Remark 2.26. In Proposition 2.25(iii), conditions (b) and (c) can be replaced by the following two conditions.

(b') For all $y, z \in X$ and all $x \in [y]_{\sim_2}$ there exists a unique $u \in [z]_{\sim_2}$ such that $x \sim_1 u$.

(c') For all $x, y, z \in X$ such that $y \sim_2 z$ we have $F(y,x) = F(z,x)$.

The following corollary is an equivalent form of Proposition 2.25.

Corollary 2.27 (see [32]). *An operation $F\colon X^2 \to X$ is $(1,4)$-selective and satisfies any of the conditions $(i) - (v)$ of Proposition 2.24 if and only if the following conditions hold.*

(i) $\sim_1$ and $\sim_2$ are equivalence relations on X and for all $x, y \in X$ there exists a bijection $f\colon [x]_{\sim_1} \to [y]_{\sim_1}$ defined by

$$f(u) \;=\; v \quad\Leftrightarrow\quad u \;\sim_2\; v, \qquad u \in [x]_{\sim_1}, v \in [y]_{\sim_1}.$$

(ii) For all $x, y, z \in X$ such that $y \sim_1 z$ we have $F(x,y) = F(x,z)$.

According to Corollary 2.27, any $(1,4)$-selective and idempotent operation $F\colon X^2 \to X$ gives rise to two partitions of X that group its elements in a grid form as follows. Two elements $x, y \in X$ belong to the same column (resp. row) if and only if $x \sim_1 y$ (resp. $x \sim_2 y$). Conversely, any operation $F\colon X^2 \to X$ such that $\sim_1$ and $\sim_2$ are equivalence relations that group the elements of X in such a grid form with the convention that $F(x,y) = F(x,z)$ for all $x, y, z \in X$ such that $y \sim_1 z$, is $(1,4)$-selective and idempotent (see Figure 2.1).

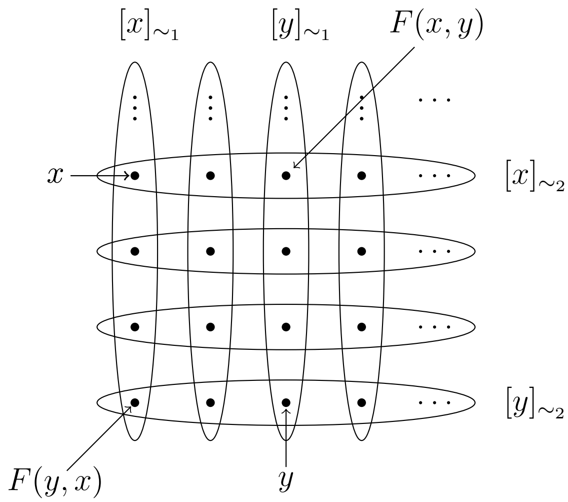

Figure 2.1: Partition of X for $\sim_1$ and $\sim_2$

Example 2.28. Let us construct an associative and rectangular operation on X_4. To this extent, we first arrange the elements of X_4 in the grid form depicted in Figure 2.2. Let us now consider the operation $F\colon X_4^2 \to X_4$ defined by the following conditions:

- $F(x,x) = x$ for all $x \in X_4$,
- $F|_{\{1,2\}^2} = \pi_1|_{\{1,2\}^2}$ and $F|_{\{3,4\}^2} = \pi_1|_{\{3,4\}^2}$,
- $F|_{\{1,4\}^2} = \pi_2|_{\{1,4\}^2}$ and $F|_{\{2,3\}^2} = \pi_2|_{\{2,3\}^2}$,
- $F(1,3) = 4$, $F(3,1) = 2$, $F(2,4) = 3$, and $F(4,2) = 1$.

Then F is associative and rectangular by Proposition 2.25. Now, let us consider the operation $G\colon X_5^2 \to X_5$ defined by the following conditions:

- $G|_{X_4^2} = F$ and $G(5,5) = 4$,
- $G(x,5) = G(x,4)$ and $G(5,x) = G(4,x)$ for all $x \in X_4$.

Then G satisfies (2.1) by Corollary 2.20.

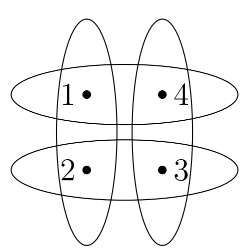

Figure 2.2: Example 2.28

Chapter 3

Idempotent semigroups

Semigroups are ubiquitous in mathematics, as many algebraic structures are defined with associative operations (groups, rings, Lie groups, *etc*). Semigroups also appear in the algebraic treatment of classical and non-classical logics [14, 57]. They have also been studied by several authors in the theory of functional equations (see, e.g., [4, 6, 27] and the references therein). Among these classes of semigroups, the class of idempotent semigroups is the center of our investigations.

In this chapter, we first introduce the basic semilattice theory needed in the rest of this thesis (Section 3.1). Then we study the class of idempotent semigroups, which are one of the key objects of this manuscript (Section 3.2). Finally, we focus on the study of ordered commutative bands and provide a characterization of the latter class by means of a concept that extends the concept of single-peakedness to semilattice orders (Section 3.3). When X is finite, the enumeration of the class of ordered commutative bands leads to a definition of the Catalan numbers that was previously unknown. Most of the contributions presented in this chapter stem from [36].

3.1 Semilattices

In this section we recall the concept of semilattice [29] which will be useful in the subsequent sections.

An element z of a partially ordered set $(X, \preceq)$ is an *upper bound* of $Y \subseteq X$ if $y \preceq z$ for every $y \in Y$. An upper bound z of Y is a *supremum* of Y if $z \preceq z'$ for every upper bound z' of Y. *Lower bounds* and *infimum* are defined dually. Partial orders $\preceq$ on X for which every subset $\{x, y\} \subseteq X$ has a *supremum* $x \curlyvee y$ are called *join-semilattice orders*, and in this case $(X, \preceq)$ is called a *join-semilattice*. If $\preceq$ is a join-semilattice order, then it is known [21] that the *join operation* $\curlyvee \colon X^2 \to X$ defined by $\curlyvee(x, y) = x \curlyvee y$ is associative, symmetric, and idempotent, and the pair $(X, \curlyvee)$ is called the *semilattice associated with* $\preceq$. We denote by $\vee$ the join operation of a total order $\leq$. It is easily seen that such an operation $\vee$ is quasitrivial. Groupoids (X, F) where F is associative, idempotent, and symmetric are called *semilattices*, and F is a *semilattice operation*. It is well known [21] that every semilattice (X, F) is the join-semilattice associated with the partial order $\preceq_F$ defined by

$$x \preceq_F y \quad \text{if} \quad F(x, y) = y. \tag{3.1}$$

That is, the join operation of $\preceq_F$ is F. We say that $\preceq_F$ is the *(join-semilattice) order associated with* F, and we denote $\preceq_F$ by $\preceq$ if no confusion is possible. The semilattice operation F is quasitrivial if and only if $\preceq_F$ is a total order. The mappings $(X, \preceq) \mapsto (X, \curlyvee)$ and $(X, F) \mapsto (X, \preceq_F)$

are inverse to each other, and define a one-to-one correspondence between join-semilattices and semilattices. By this correspondence, we use either $(X, \preceq)$ or $(X, \curlyvee)$ to denote a join-semilattice, and every semilattice order will be a join-semilattice order. Due to this convention, we write *semilattice* for join-semilattice.

A semilattice $(X, \preceq)$ is said to be a *tree semilattice* [91] if for all $x, y, u, v \in X$ such that $u \preceq x$, $v \preceq y$, and $x \parallel y$, we have $u \parallel v$. For instance, any chain is a tree semilattice. The following fact provides a characterization of tree semilattices.

Fact 3.1 (see [91])**.** *A semilattice $(X, \preceq)$ is not a tree semilattice if and only if there exist $a, b, c \in X$ such that $a \preceq b$, $a \preceq c$, and $b \parallel c$.*

Also, we say that a tree semilattice $(X, \preceq)$ is a *binary semilattice* if every $x \in X$ covers at most two elements.

3.2 Characterizations of idempotent semigroups

In this section we recall characterizations of several classes of idempotent semigroups [15, 78, 81, 91, 92].

Recall that a congruence $\sim$ on a groupoid (X, F) is said to be a *semilattice congruence* if $(X/{\sim}, \tilde{F})$ is a semilattice. The partition of X induced by a semilattice congruence is called a *semilattice decomposition of* X. It is well known [84, 100] that every semigroup admits a smallest semilattice congruence. For instance [81], the smallest semilattice congruence $\sim$ on a band (X, F) is defined by

$$x \sim y \quad \Leftrightarrow \quad F(F(x,y),x) = x \quad \text{and} \quad F(F(y,x),y) = y, \qquad x, y \in X. \tag{3.2}$$

Let $(Y, \curlyvee)$ be a semilattice and let $\{(X_\alpha, F_\alpha) \colon \alpha \in Y\}$ be a set of semigroups such that $X_\alpha \cap X_\beta = \varnothing$ for any $\alpha \neq \beta$. A groupoid (X, F) is said to be a *semilattice* $(Y, \curlyvee)$ *of semigroups* (X_α, F_α) if $X = \bigcup_{\alpha \in Y} X_\alpha$, $F|_{X_\alpha^2} = F_\alpha$ for every $\alpha \in Y$, and

$$F(X_\alpha \times X_\beta) \subseteq X_{\alpha \curlyvee \beta}, \qquad \alpha, \beta \in Y. \tag{3.3}$$

In this case, we write $(X, F) = ((Y, \curlyvee), (X_\alpha, F_\alpha))$ and we simply say that (X, F) is a semilattice of semigroups.

It is well known [56] that a semigroup (X, F) is a semilattice of semigroups if and only if there exists a semilattice congruence on (X, F). As a consequence, we obtain the following result.

Corollary 3.2. *A groupoid (X, F) is a semilattice of semigroups if and only if there exists a semilattice congruence on (X, F) such that $([x]_\sim, F|_{[x]_\sim^2})$ is a semigroup for any $x \in X$.*

Also, a semilattice of semigroups is in general not a semigroup. For instance, consider the operation $F \colon X_5^2 \to X_5$ defined by the following conditions:

- $F(x, x) = x$ for all $x \in X_5$,
- $F|_{\{2,3\}^2} = \pi_2|_{\{2,3\}^2}$ and $F|_{\{4,5\}^2} = \pi_1|_{\{4,5\}^2}$,
- $F(x, y) = F(y, x) = y$ for all $x \in \{1, 2, 3\}$ and all $y \in \{4, 5\}$,

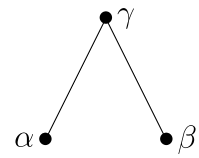

Figure 3.1: Hasse diagram of $(Y, \preceq)$

- $F(1,2) = F(2,1) = 4$ and $F(1,3) = F(3,1) = 5$.

Then (X_5, F) is a semilattice of semigroups. Indeed, consider the set $Y = \{\alpha, \beta, \gamma\}$ together with the semilattice order $\preceq$ defined by $\alpha \prec \gamma$, $\beta \prec \gamma$, and $\alpha \,||\, \beta$. Moreover, consider the semigroups $(X_\alpha, F_\alpha) = (\{1\}, F|_{\{1\}^2})$, $(X_\beta, F_\beta) = (\{2,3\}, F|_{\{2,3\}^2})$, and $(X_\gamma, F_\gamma) = (\{4,5\}, F|_{\{4,5\}^2})$. Then $(X, F) = ((Y, \curlyvee), (X_\alpha, F_\alpha))$. However, F is not associative since $F(1, F(2,3)) = F(1,3) = 5 \neq 4 = F(4,3) = F(F(1,2),3)$. The Hasse diagram of $(Y, \preceq)$ is depicted in Figure 3.1.

The next result provides a partial description of the class of bands.

Theorem 3.3 (see [15,78])**.** *If (X, F) is a band, then it is a semilattice of rectangular semigroups. Moreover, its semilattice congruence $\sim$ is defined by* (3.2)*.*

The following immediate corollary of Theorem 3.3 is of particular interest as it characterizes the idempotency property in the class of semigroups.

Corollary 3.4. *Let $F\colon X^2 \to X$ be an associative operation. Then F is idempotent if and only if there exists a semilattice congruence $\sim$ on (X, F) such that $([x]_\sim, F|_{[x]^2_\sim})$ is a rectangular semigroup for any $x \in X$. Moreover, the semilattice congruence $\sim$ on (X, F) is defined by* (3.2)*.*

The following result follows from Facts 2.4 and 2.5, Theorems 2.6 and 3.3 and Corollary 2.9.

Corollary 3.5. *Let $F\colon X_n^2 \to X_n$ be an associative operation. The following assertions are equivalent.*

(i) *F is rectangular.*

(ii) *F is idempotent and $|F^{-1}| = (n, \ldots, n)$.*

Now, we provide a characterization of the class of idempotent semigroups.

Theorem 3.6 (see [81])**.** *Let $(Y, \preceq)$ be a semilattice. For any $\gamma \in Y$, let L_γ and R_γ be nonempty sets such that $L_\gamma \cap L_\delta = R_\gamma \cap R_\delta = \varnothing$ for any $\delta \in Y \setminus \{\gamma\}$. Also, for any $\gamma \in Y$, let $S_\gamma = L_\gamma \times R_\gamma$. For any $\gamma \preceq \delta$, assume that there exist functions $f_{\gamma,\delta}\colon S_\gamma \times L_\delta \to L_\delta$ and $g_{\gamma,\delta}\colon S_\gamma \times R_\delta \to R_\delta$ satisfying the following conditions.*

(a) *For any $(i,j) \in S_\gamma$, any $x \in L_\gamma$, and any $y \in R_\gamma$, we have $f_{\gamma,\gamma}((i,j),x) = i$ and $g_{\gamma,\gamma}((i,j),y) = j$.*

(b) *For any $(i,j) \in S_\gamma$, any $(k,l) \in S_\delta$, any $x \in L_{\gamma\curlyvee\delta}$, and any $y \in R_{\gamma\curlyvee\delta}$, we have $f_{\gamma,\gamma\curlyvee\delta}((i,j), f_{\delta,\gamma\curlyvee\delta}((k,l),x)) = \mu$ and $g_{\delta,\gamma\curlyvee\delta}((k,l), g_{\gamma,\gamma\curlyvee\delta}((i,j),y)) = \nu$ for some $(\mu,\nu) \in S_{\gamma\curlyvee\delta}$.*

(c) For any $(i,j) \in S_\gamma$, any $(k,l) \in S_\delta$, any $\epsilon \succeq \gamma \curlyvee \delta$, any $x \in L_\epsilon$, and any $y \in R_\epsilon$, we have $f_{\gamma\curlyvee\delta,\epsilon}((\mu,\nu),x) = f_{\gamma,\epsilon}((i,j), f_{\delta,\epsilon}((k,l),x))$ and $g_{\gamma\curlyvee\delta,\epsilon}((\mu,\nu),y) = g_{\delta,\epsilon}((k,l), g_{\gamma,\epsilon}((i,j),y))$ for some $(\mu,\nu) \in S_{\gamma\curlyvee\delta}$.

Let $S = \bigcup_{\alpha\in Y} S_\alpha$ and define an operation $F\colon S^2 \to S$ by

$$F((i,j),(k,l)) \;=\; (\mu,\nu), \qquad (i,j) \in S_\gamma, (k,l) \in S_\delta,$$

where $\mu = f_{\gamma,\gamma\curlyvee\delta}((i,j), f_{\delta,\gamma\curlyvee\delta}((k,l),x))$ for any $x \in L_{\gamma\curlyvee\delta}$ and $\nu = g_{\delta,\gamma\curlyvee\delta}((k,l), g_{\gamma,\gamma\curlyvee\delta}((i,j),y))$ for any $y \in R_{\gamma\curlyvee\delta}$. Then (S,F) is a band, and conversely, any band can be so constructed.

It was observed in [81] that Theorem 3.6 illustrates how complicated the construction of arbitrary bands can be. Indeed, the parameters in this theorem and the conditions they satisfy are so technical that it is almost impossible to construct all bands from it. However, the sets and functions appearing in this theorem can be considered as useful tools to derive the structure of particular subclasses of bands as will be seen in Chapter 7.

Now, we consider characterizations of two subclasses of bands whose commutative counterparts will be further investigated in the next section.

Let $\leq$ be a total order on X. We say that an operation $F\colon X^2 \to X$ is *$\leq$-preserving* if $F(x,y) \leq F(x',y')$ whenever $x \leq x'$ and $y \leq y'$. In that case, the groupoid (X,F) is said to be *ordered for* $\leq$. Also, we say that an operation $F\colon X^2 \to X$ is *order-preservable* [34] if it is $\leq'$-preserving for some total order $\leq'$ on X. In that case, the groupoid (X,F) is said to be *orderable*. Finally, an operation $F\colon X^2 \to X$ is said to be *internal* [52] if $x \leq F(x,y) \leq y$ for all $x \leq y$ in X.

The following theorems provide a characterization of the class of orderable idempotent semigroups and the class of ordered idempotent semigroups.

Theorem 3.7 (see [92]). *Let $F\colon X^2 \to X$ be an associative operation. Then F is idempotent and order-preservable if and only if there exists a congruence $\sim$ on X for F such that the following conditions hold.*

(a) $(X/{\sim}, \tilde{F})$ is a tree semilattice.

(b) For any $x \in X$ we have $F|_{[x]^2_\sim} = \pi_i|_{[x]^2_\sim}$ for some $i \in \{1,2\}$.

(c) For any $C \in X/{\sim}$ and $x \in X$ such that $[x]_\sim \prec_{\tilde{F}} C$ we have $|\{F(x,y)\colon y \in C\}| \leq 2$ or $|\{F(y,x)\colon y \in C\}| \leq 2$.

(d) For any $C \in X/{\sim}$ such that $F|_{C^2} = \pi_1|_{C^2}$ (resp. $F|_{C^2} = \pi_2|_{C^2}$) and any $x,y,z \in X$ such that $[F(x,y)]_\sim \prec_{\tilde{F}} C$ and $z \in C$ we have $F(x,z) = F(y,z)$ (resp. $F(z,x) = F(z,y)$).

(e) For any $C \in X/{\sim}$ such that $F|_{C^2} = \pi_1|_{C^2}$ (resp. $F|_{C^2} = \pi_2|_{C^2}$) and any $x,y,z \in X$ such that $x,y \notin C$, $F(x,y) \in C$, $z \in C$, and $F(x,y) \neq F(x,z)$ (resp. $F(y,x) \neq F(z,x)$) we have $F(y,x) = F(y,z)$ (resp. $F(x,y) = F(z,y)$).

(f) For any $C \in X/{\sim}$ such that $F|_{C^2} = \pi_1|_{C^2}$ (resp. $F|_{C^2} = \pi_2|_{C^2}$) and any $x,y,z \in X$ such that $x,y,z \notin C$, $F(x,y) \in C$, $F(x,y) = F(x,z)$, and $F(y,z) = F(y,x)$ (resp. $F(y,x) = F(z,x)$ and $F(z,y) = F(x,y)$) we have $F(z,x) \neq F(z,y)$ (resp. $F(x,z) \neq F(y,z)$).

(g) *For any* $C \in X/{\sim}$ *such that* $F|_{C^2} = \pi_1|_{C^2}$ *(resp.* $F|_{C^2} = \pi_2|_{C^2}$*) and any* $x, y, z \in X$ *such that* $x, y \notin C$*,* $F(x, y) \in C$*,* $z \in C$*, and* $F(x, z) = F(y, z)$ *(resp.* $F(z, x) = F(z, y)$*) we have* $z = F(x, y)$ *(resp.* $z = F(y, x)$*).*

Moreover, $(X/{\sim}, \tilde{F})$ *is a binary semilattice and the congruence* $\sim$ *on* (X, F) *is defined by* (3.2).

Theorem 3.8 (see [91]). *Let* $F\colon X^2 \to X$ *be an associative operation and let* $\leq$ *be a total order on* X*. Then* F *is idempotent and* $\leq$*-preserving if and only if there exists a congruence* $\sim$ *on* X *for* F *such that the conditions* $(a) - (g)$ *of Theorem 3.7 as well as the following three conditions hold.*

(a') *Every ideal of* $(X/{\sim}, \tilde{F})$ *is convex for* $\leq$*.*

(b') F *is internal for* $\leq$*.*

(c') *For any* $x, y, z \in X$ *such that* $x \neq z$*,* $x \sim z$*, and* $[y]_\sim \prec_{\tilde{F}} [x]_\sim$*, we have* $x < y < z$ *or* $z < y < x$*.*

Moreover, $(X/{\sim}, \tilde{F})$ *is a binary semilattice and the congruence* $\sim$ *on* (X, F) *is defined by* (3.2).

We observe that Theorems 3.7 and 3.8 illustrate how technical the construction of arbitrary orderable bands and ordered bands can be. However, these theorems will be useful in the next section in order to characterize the subclass of commutative orderable bands as well as the subclass of commutative ordered bands. Indeed, as we will see, the particularization of Theorems 3.7 and 3.8 to commutative bands will enable us to easily construct commutative orderable bands and commutative ordered bands.

3.3 Characterizations of commutative idempotent semigroups

In this section, we provide an alternative characterization of the class of ordered commutative bands by means of a concept that extends the concept of single-peakedness to semilattice orders. In the case where the underlying set is finite, we enumerate various classes of semilattices. In this respect, one of our main results is a new definition of the Catalan numbers.

If $F\colon X^2 \to X$ is an associative, idempotent, and commutative operation, then the equivalence relation $\sim$ on X defined by (3.2) reduces actually to the identity relation on X. Therefore, we conclude the following characterization from Theorem 3.8.

Theorem 3.9 (see [91]). *Let* $F\colon X^2 \to X$ *be an operation and let* $\leq$ *be a total order on* X*. Then* F *is associative, idempotent, commutative, and* $\leq$*-preserving if and only if the following conditions hold.*

(a) (X, F) *is a binary semilattice for which every ideal is convex for* $\leq$*.*

(b) F *is internal for* $\leq$*.*

Theorem 3.9 is of particular interest as it enables us to easily construct commutative ordered bands. For instance, consider the semilattice order $\preceq$ on X_3 defined by $1 \prec 2$, $3 \prec 2$, and $1 \parallel 3$. Then its associated semilattice $(X, \curlyvee)$ is ordered for $\leq_3$ by Theorem 3.9 (see Figure 3.2).

We now provide an alternative characterization of the class of ordered commutative bands. To this extent, we first introduce some definitions related to semilattice orders.

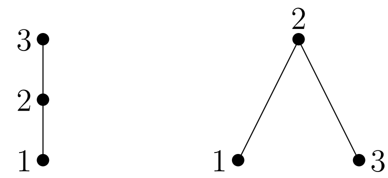

Figure 3.2: A semilattice that is ordered for $\leq_3$.

Definition 3.10 (see [36])**.** Let $(X, \leq)$ be a chain. We say that a semilattice order $\preceq$ on X has the *convex-ideal property* (*CI-property* for short) *for* $\leq$ if for every $a, b, c \in X$,

$$a \leq b \leq c \quad \Rightarrow \quad b \preceq a \curlyvee c. \tag{3.4}$$

We say that $\preceq$ is *internal for* $\leq$ if for every $a, b, c \in X$,

$$a < b < c \quad \Rightarrow \quad (a \neq b \curlyvee c \quad \text{and} \quad c \neq a \curlyvee b). \tag{3.5}$$

We say that $\preceq$ is *nondecreasing for* $\leq$ and that the semilattice $(X, \curlyvee)$ is *nondecreasing for* $\leq$ if $\preceq$ has the CI-property and is internal for $\leq$.

The terminology introduced in Definition 3.10 is justified in Lemmas 3.13 and 3.14, and Theorem 3.24. Note that conditions (3.4) and (3.5) are self-dual with respect to the total order $\leq$, that is, if $\leq^d$ is the dual order of $\leq$, then a semilattice order $\preceq$ on X has the CI-property (resp., is internal) for $\leq$ if and only if it has the CI-property (resp., is internal) for $\leq^d$.

Remark 3.11. Let $(X, \leq)$ be a chain, $\preceq$ be a semilattice order on X, and $P \subseteq X$ be such that the restriction $\preceq|_P$ of $\preceq$ to P is a total order. Then $\preceq|_P$ is nondecreasing for $\leq|_P$ if and only if it is single-peaked for $\leq|_P$.

It follows from Remark 3.11 that those total orders that are nondecreasing for a given total order $\leq$ are exactly the single-peaked ones.

Remark 3.12. Condition (3.4) is clearly equivalent to

$$a < b < c \quad \Longrightarrow \quad b \preceq a \curlyvee c,$$

but the partial order $\preceq$ cannot be replaced by its asymmetric part in (3.4). Indeed, if $\preceq$ is the partial semilattice order on X_3 defined by $1 \prec 2$, $3 \prec 2$, and $1 \parallel 3$, then $\preceq$ satisfies (3.4) for $\leq_3$ but $2 \not\prec 1 \curlyvee 3$.

The following lemma is a generalization of Proposition 1.5 for semilattice orders.

Lemma 3.13 (see [36])**.** *Let $(X, \leq)$ be a totally ordered set and $\preceq$ be a semilattice order on X. The following conditions are equivalent.*

(i) The semilattice order $\preceq$ has the CI-property for $\leq$.

(ii) Every ideal of $(X, \preceq)$ is a convex subset of $(X, \leq)$.

(iii) Every principal ideal of $(X, \preceq)$ is a convex subset of $(X, \leq)$.

(iv) If $x' \prec x$ or $x' \parallel x$ then x is an upper bound or a lower bound of $(x']_{\preceq}$ in $(X, \leq)$.

Proof. Let us first show that $(i) \Rightarrow (ii)$. Let I be an ideal of $(X, \preceq)$ and let $a, c \in I$. For every $b \in X$ such that $a \leq b \leq c$ we have $b \preceq a \curlyvee c$ by the CI-property. If follows that $b \in I$ since I is an ideal of $(X, \preceq)$ that contains a and c. The implication $(ii) \Rightarrow (iii)$ is obvious. Now, let us show that $(iii) \Rightarrow (i)$. Let $a \leq b \leq c$ in X. By (iii), the ideal $(a \curlyvee c]_{\preceq}$ is convex in $(X, \leq)$. Since it contains a and c, it also contains b. It follows that $b \preceq a \curlyvee c$. Finally, the equivalence $(iii) \Leftrightarrow (iv)$ is obvious. $\square$

Now we give equivalent formulations of the internality property (3.5) for semilattice orders.

Lemma 3.14 (see [36])**.** *Let $(X, \leq)$ be a chain and $\preceq$ be a join-semilattice order on X. The following conditions are equivalent.*

(i) The order $\preceq$ is internal for $\leq$.

(ii) The join operation of $\preceq$ is internal.

(iii) There are no $a, b, c \in X$ such that $a < b < c$ and $a \curlyvee b = b \curlyvee c \in \{a, c\}$.

Moreover, if any of the conditions $(i) - (iii)$ is satisfied, then there are no pairwise incomparable elements $a < b < c$ of X such that $a \curlyvee b = b \curlyvee c$.

Proof. Let us first show that $(i) \Rightarrow (ii)$. For any $a < b$ in X, we cannot have $a \curlyvee b < a < b$ or $a < b < a \curlyvee b$, since this would contradict internality of $\preceq$ for $\leq$. It follows that $a \curlyvee b \in [a, b]_{\leq}$.

Now, let us show that $(ii) \Rightarrow (iii)$. Let $a < b < c$ in X. If $a \curlyvee b = b \curlyvee c = a$, then $b \curlyvee c \notin [b, c]_{\leq}$. If $a \curlyvee b = b \curlyvee c = c$, then $a \curlyvee b \notin [a, b]_{\leq}$.

Finally, let us show that $(iii) \Rightarrow (i)$. We show the contrapositive. Assume that there are $a < b < c$ such that $a = b \curlyvee c$. Then $a \curlyvee b = b \curlyvee c = a$. Similarly, if $c = a \curlyvee b$ then $b \curlyvee c = a \curlyvee b = c$.

Now, assume that any of the conditions $(i) - (iii)$ is satisfied, and that $a < b < c$ are pairwise incomparable elements of X. If $a \curlyvee b$ and $b \curlyvee c$ are equal to a common element d, it follows from (ii) that $d \in \,]a, b[_{\leq} \cap \,]b, c[_{\leq} = \varnothing$, a contradiction. $\square$

Corollary 3.15 (see [36])**.** *Let $(X, \leq)$ be a chain and $F \colon X^2 \to X$ be an operation. Then F is associative, symmetric, and internal if and only if F is the join operation of a semilattice order that is internal for $\leq$.*

Remark 3.16. If $\preceq$ is a semilattice order that is internal for a total order $\leq$, there might be incomparable elements $a < b < c$ such that $a \curlyvee c = b \curlyvee c$. Consider for instance $X = \{a, b, c, d, e\}$ with $a < e < b < d < c$ and the semilattice order $\preceq$ defined by $a \,||\, b$, $a \,||\, c$, $b \,||\, c$, $a \curlyvee b = e$ and $e \curlyvee c = d$. Then $\preceq$ is internal for $\leq$ and $a \curlyvee c = b \curlyvee c$.

Remark 3.17. The join operation $\curlyvee$ of a semilattice order $\preceq$ that has the CI-property for a total order $\leq$ need not be $\leq$-preserving. For instance, if $\preceq$ is the semilattice order on X_3 defined by $2 \preceq 1$, $3 \preceq 1$ and $2 \,||\, 3$, then $\preceq$ has the CI-property for $\leq_3$ but $2 \curlyvee 2 = 2 >_3 1 = 2 \curlyvee 3$. This example also shows that the CI-property for $\leq$ does not imply internality for $\leq$.

Conversely, the join operation $\curlyvee$ of a semilattice order $\preceq$ that is internal for a total order $\leq$ need not be $\leq$-preserving. For instance, if $\leq$ is the total order on X_3 defined by $1 < 3 < 2$, then $\leq$ is internal for $\leq_3$ but $3 = 3 \vee 1 >_3 2 \vee 1 = 2$. This example also shows that internality for $\leq$ does not imply CI-property for $\leq$.

Definition 3.18 (see [36]). A partial order $\preceq$ on X is said to have the *linear filter property* if every of its filters is totally ordered.

Remark 3.19. Partial orders that have the linear filter property are also called *forests* [28].

The following lemma characterizes the linear filter property.

Lemma 3.20 (see [36]). *A partial order on X has the linear filter property if and only if no pair $\{a, b\}$ of incomparable elements of X has a lower bound.*

Proof. (Necessity) If $\preceq$ is a partial order on X that has the linear filter property and if there is a pair $\{a, b\}$ of incomparable elements of X that has a lower bound c, then $[c)_{\preceq}$ is a filter that is not totally ordered.

(Sufficiency) Obvious. □

The following proposition characterizes tree semilattices and binary semilattices in terms of the linear filter property.

Proposition 3.21. *Let $\preceq$ be a semilattice order on X. The following conditions hold.*

(a) $(X, \preceq)$ is a tree semilattice if and only if $\preceq$ has the linear filter property.

(b) Assume that $X = X_n$ for some integer $n \geq 1$. Then $(X, \preceq)$ is a binary semilattice if and only if $\preceq$ has the linear filter property and there are no pairwise incomparable elements a, b, c of X such that $a \curlyvee b = b \curlyvee c$.

Proof. Condition (a) follows from Fact 3.1 and Lemma 3.20. So let us show condition (b). Suppose first that $(X, \preceq)$ is a binary semilattice. By condition (a) we have that $\preceq$ has the linear filter property. Now, suppose to the contrary that X has three incomparable elements a, b, c such that $a \curlyvee b = b \curlyvee c$. Then, there are elements $a', b', c' \in X$ such that $a \preceq a'$, $b \preceq b'$, $c \preceq c'$, and that are covered by $a \curlyvee b = b \curlyvee c$, which contradicts the definition of a binary semilattice. The converse implication essentially follows from condition (a). □

The following proposition constitutes the last step towards the alternative characterization of the class of ordered commutative bands.

Proposition 3.22 (see [36]). *Let $(X, \leq)$ be a totally ordered set and $\preceq$ be a semilattice order on X. If $\preceq$ is nondecreasing for $\leq$, then it has the linear filter property.*

Proof. We prove the contrapositive. Assume that $\preceq$ does not have the linear filter property. By Lemma 3.20, there are incomparable elements a, b in X that have a lower bound c. Let us set $d = a \curlyvee b$ and assume that $a < b$. If $d < a$ or $b < d$, then $\preceq$ is not internal for $\leq$. Thus, we have $a < d < b$. If $c < d < b$ ($a < d < c$, respectively), then $\preceq$ does not have the CI-property for $\leq$ since $d \npreceq c \curlyvee b = b$ ($d \npreceq a \curlyvee c = a$, respectively). □

Remark 3.23. The converse of Proposition 3.22 does not hold. On the one hand, Remark 3.17 shows an instance of a semilattice order that has the linear filter property but that is not internal for a given total order $\leq$. On the other hand, if $\preceq$ is the semilattice order on X_4 defined by $1 \preceq 3$, $4 \preceq 3$, $3 \preceq 2$, and $1 \,||\, 4$ then $\preceq$ has the linear filter property for $\leq_4$, but does not have the CI-property for $\leq_4$.

The following theorem follows from Theorem 3.9, Lemmas 3.13 and 3.14, and Propositions 3.21 and 3.22.

Theorem 3.24 (see [36]). *Let $(X,\leq)$ be a totally ordered set and $\curlyvee\colon X^2\to X$ be a semilattice operation. The following conditions are equivalent.*

(i) $\curlyvee$ is $\leq$-preserving.

(ii) The order $\preceq$ associated with $\curlyvee$ is nondecreasing for $\leq$.

The following result is a direct consequence of Theorem 3.24.

Theorem 3.25 (see [36]). *Let $(X,\leq)$ be a totally ordered set and $F\colon X^2\to X$ be an operation. The following conditions are equivalent.*

(i) F is a $\leq$-preserving semilattice operation.

(ii) F is the join operation of a semilattice order $\preceq$ on X that is nondecreasing for $\leq$.

The following result, which provides a characterization of the class of orderable commutative bands, follows from Theorem 3.7.

Theorem 3.26 (see [92]). *An operation $F\colon X^2\to X$ is associative, idempotent, commutative, and order-preservable if and only if (X,F) is a binary semilattice.*

In the rest of this section we suppose that $X=X_n$ and we state the latter characterization in terms of properties of the Hasse graph of $(X_n,\preceq)$. We also prove that the number of semilattice operations that are $\leq_n$-preserving is the nth Catalan number, providing yet another realization of the sequence of Catalan numbers. Finally, given a binary semilattice order $\preceq$ on X_n, we consider the problem of constructing the total orders on X_n for which $\preceq$ is nondecreasing.

Recall that a *tree* is a connected undirected simple graph without cycle. An ordered pair (G,r) is a *rooted tree* if G is a tree and r is a vertex of G. If $\{u,v\}$ is an edge of a rooted tree (G,r), we say that v is a *child of* u (or that u is *the parent of* v) if the unique path from r to v contains u. In what follows, by *binary tree* we mean a rooted tree in which every vertex has at most two children. A *binary forest* is a graph whose connected components are binary trees.

From Proposition 3.21(b) it follows that the binary semilattices $(X_n,\preceq)$ are exactly those semilattices $(X_n,\preceq)$ whose Hasse graph are binary trees rooted at the top element of $(X_n,\preceq)$.

Lemma 3.27 (see [36]). *Let $(X_n,\preceq)$ be a semilattice. If $\preceq$ has the CI-property for $\leq_n$, then the following conditions are equivalent.*

(i) The order $\preceq$ is internal for $\leq_n$.

(ii) If x' is a child of x, then $x=\min\{z\colon z>_n y \text{ for all } y\in(x']_{\preceq}\}$ or $x=\max\{z\colon z<_n y \text{ for all } y\in(x']_{\preceq}\}$.

(iii) If x_1 and x_2 are two children of a vertex x in the Hasse graph of $(X_n,\preceq)$, then there are $i\neq j$ in $\{1,2\}$ such that x is an upper bound of $(x_i]_{\preceq}$ and a lower bound of $(x_j]_{\preceq}$ in $(X_n,\leq_n)$.

Proof. Let us first show that $(i) \Rightarrow (ii)$. By Lemma 3.13, we have that $(x']_{\preceq}$ is convex in $(X_n, \leq_n)$ with x as lower bound or upper bound. Assume that x is a lower bound of $(x']_{\preceq}$ (the other case can be dealt with similarly). If x has only one child, then by CI-property we have $x = \max\{z\colon z <_n y \text{ for all } y \in (x']_{\preceq}\}$. If x has two children x' and x'', then we obtain by internality that $y <_n x <_n z$ for every $y \preceq x''$ and $z \preceq x'$ (the case where $z <_n x <_n y$ can be dealt with similarly). It follows that $x = \max\{z\colon z <_n y \text{ for all } y \in (x']_{\preceq}\}$ and $x = \min\{z\colon z >_n y \text{ for all } y \in (x'']_{\preceq}\}$. The implication $(ii) \Rightarrow (iii)$ is obvious. Finally, let us show that $(iii) \Rightarrow (i)$. We prove that $\preceq$ satisfies condition (iii) of Lemma 3.14. Let x_1 and x_2 be incomparable elements, and assume that $x_1 <_n x_2$. Let x_1' and x_2' be the children of $x_1 \curlyvee x_2$ such that $x_1 \preceq x_1'$, and $x_2 \preceq x_2'$. We obtain by (iii) that $x_1 \curlyvee x_2$ is an upper bound in $(X_n, \leq_n)$ of $(x_1']_{\preceq}$ and a lower bound in $(X_n, \leq_n)$ of $(x_2']_{\preceq}$, which shows that $x_1 <_n x_1 \curlyvee x_2 <_n x_2$. □

The next result follows directly from Lemma 3.27.

Corollary 3.28 (see [36]). *If $\preceq$ is a semilattice order that is nondecreasing for $\leq_n$, then its top element r has only one child in the Hasse graph of $(X_n, \preceq)$ if and only if $r \in \{1, n\}$.*

As stated in the next result, a similar equivalence as in Lemma 3.27 holds for semilattice orders that satisfy the linear filter property.

Lemma 3.29 (see [36]). *Let $\preceq$ be a semilattice order on X_n that has the linear filter property. Then, conditions (i) and (iii) of Lemma 3.27 are equivalent.*

Proof. Let us first show that $(i) \Rightarrow (iii)$. By internality, we know that x lies between x_1 and x_2 in $(X_n, \leq_n)$. Assume that $x_1 <_n x <_n x_2$ (the case $x_2 <_n x <_n x_1$ is obtained by symmetry). By the linear filter property we have $(x_1]_{\preceq} \cap (x_2]_{\preceq} = \varnothing$. Also, by the internality condition, there are no $y, z \in X_n$ such that $\{y, z\} <_n \{x\}$ (resp. $\{y, z\} >_n \{x\}$), $y \in (x_1]_{\preceq}$, and $z \in (x_2]_{\preceq}$. It follows that x is an upper bound of $(x_1]_{\preceq}$ and a lower bound of $(x_2]_{\preceq}$ in $(X_n, \leq_n)$.

The proof of the implication $(iii) \Rightarrow (i)$ is the same as in Lemma 3.27. □

Lemma 3.30 (see [36]). *Let $(X_n, \leq_n)$ be a finite chain and $\preceq$ be a semilattice order on X_n with top element r.*

(a) If $\preceq$ has the CI-property for $\leq_n$, and if r has only one child in the Hasse graph of $(X_n, \preceq)$, then $r \in \{1, n\}$.

(b) If $\preceq$ is internal for $\leq_n$, and if r is either 1 or n, then r has only one child in the Hasse graph of $(X_n, \preceq)$.

Proof. Let us first show condition (a). If x is the child of r, then $(x]_{\preceq}$ is a convex subset of $(X_n, \leq_n)$ with $n-1$ elements.

Now, let us show condition (b). We prove the contrapositive. Assume that $\preceq$ is internal for $\leq_n$ and that x_1 and x_2 are two children of r in $(X_n, \preceq)$. By internality, we know that x lies in between x_1 and x_2 in $(X_n, \leq)$, which shows that $x \notin \{1, n\}$. □

The following result follows immediately from Lemmas 3.27, 3.29, and 3.30.

Corollary 3.31 (see [36]). *Let $(X_n, \preceq)$ be a semilattice order with top element r. Assume that $\preceq$ is internal for $\leq_n$, and has the CI-property for $\leq_n$ or the linear filter property. If r has two children x_1, x_2 in the Hasse graph of $(X_n, \preceq)$, then 1 and n are incomparable. Moreover, if $1 \preceq x_1$ and $n \preceq x_2$, then $(x_1]_{\preceq} = \{1, 2, \ldots, r-1\}$ and $(x_2]_{\preceq} = \{r+1, r+2, \ldots, n\}$.*

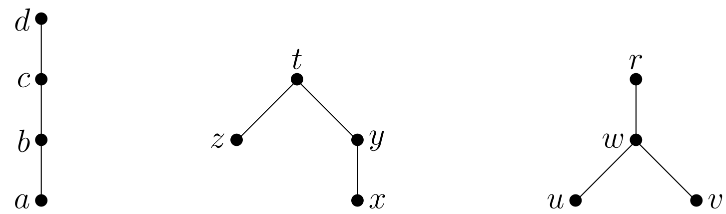

Figure 3.3: Hasse diagrams of semilattices that are nondecreasing for $\leq_4$.

The following theorem provides an alternative characterization of the class of finite ordered commutative bands.

Theorem 3.32 (see [36]). *Let $(X_n, \leq_n)$ be a finite totally ordered set and $\preceq$ be a semilattice order on X_n. The following conditions are equivalent.*

(i) The order $\preceq$ is nondecreasing for $\leq_n$.

(ii) $(X_n, \preceq)$ is a binary semilattice that satisfies condition (ii) of Lemma 3.27.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Lemmas 3.14 and 3.27 and Propositions 3.21 and 3.22.

Now, let us show that $(ii) \Rightarrow (i)$. By Lemma 3.13 we obtain that $\preceq$ has the CI-property for $\leq_n$. It follows by Lemma 3.27 that $\preceq$ is internal for $\leq_n$. □

Recall that two undirected graphs $G = (V, E)$ and $G' = (V', E')$ are said to be *isomorphic*, and we write $G \simeq G'$, if there exists a bijection $\phi\colon V' \to V$ such that

$$\{x, y\} \in E' \quad \Leftrightarrow \quad \{\phi(x), \phi(y)\} \in E, \qquad x, y \in V'.$$

The bijection ϕ is then called an *isomorphism* from G' to G. It is called an *automorphism* of G if $G' = G$.

Example 3.33. Let $\preceq$ be a semilattice order that is nondecreasing for $\leq_4$. According to Theorem 3.32, its Hasse graph is isomorphic to one of the binary trees depicted in Figure 3.3, and $\preceq$ is one of the orders defined by the following labellings in Figure 3.3: $(u, v, w, r) \in \{(1, 3, 2, 4), (2, 4, 3, 1)\}$, or

$$(x, y, z, t) \in \{(3, 4, 1, 2), (4, 3, 1, 2), (1, 2, 4, 3), (2, 1, 4, 3)\},$$

or

$$(a, b, c, d) \in \{(1, 2, 3, 4), (2, 1, 3, 4), (2, 3, 1, 4), (2, 3, 4, 1), \\ (3, 4, 2, 1), (3, 2, 1, 4), (3, 2, 4, 1), (4, 3, 2, 1)\}.$$

Observe that a finite semilattice $(X_n, \preceq)$ has a neutral element e if and only if e is a lower bound of X_n. The following result follows from the latter observation and Theorem 3.32.

Corollary 3.34 (see [36]). *Let $\curlyvee\colon X_n^2 \to X_n$ be a $\leq_n$-preserving semilattice operation. Then, $\curlyvee$ has a neutral element if and only if its associated order is a total order that is single-peaked for $\leq_n$.*

Theorem 3.32 enables us to give the isomorphism types of semilattices that are nondecreasing for $\leq_n$.

Corollary 3.35 (see [36]). *The isomorphism types of semilattices that are nondecreasing for $\leq_n$ and the isomorphism types of semilattices that have the linear filter property and are internal for $\leq_n$ coincide, and are the binary semilattices.*

Proof. It follows from Lemma 3.14, Proposition 3.21, and Theorem 3.32 that any semilattice that is nondecreasing for $\leq_n$, or that has the linear filter property and is internal for $\leq_n$ is a binary semilattice. Since any semilattice that is nondecreasing for $\leq_n$ is internal for $\leq_n$ and has the linear filter property, it suffices to show that if G is a binary tree with n vertices, then there is a labeling of the vertices turning G into the Hasse graph of a semilattice that is nondecreasing for $\leq_n$. We proceed by induction on $n \geq 1$. For the induction step, if the root r of G has only one child, then we define a labeling of the vertices of G by labeling r with n, and labeling the vertices of $G - r$ with $1, \ldots, n-1$ using induction hypothesis. If r has two children x_1 and x_2, let C_i be the connected component of $G - r$ that contains x_i for $i \in \{1, 2\}$. We define a labeling of G by labeling r by $|C_1|+1$, and by labeling the vertices of C_1 and C_2 by $1, \ldots, |C_1|$ and $|C_1|+2, \ldots, n$, respectively, using induction hypothesis. □

For every integer $n \geq 0$, let $\omega(n)$ be the number of semilattice orders on X_n that are nondecreasing for $\leq_n$. As a convention, we set $\omega(0) = 1$. The following result proves that $\omega(n)$ is the nth Catalan number (see, e.g., [95]).

Proposition 3.36 (see [36]). *The sequence $(\omega(n))_{n\geq 0}$ satisfies the recurrence relation*

$$\omega(n) \;=\; \sum_{i=1}^{n} \omega(n-i)\omega(i-1), \qquad n \geq 1. \tag{3.6}$$

It follows that $\omega(n)$ is the nth Catalan number $\frac{(2n)!}{n!(n+1)!}$ for every $n \geq 0$.

Proof. Let $\preceq$ be a semilattice order on X_n that is nondecreasing for $\leq_n$. By Theorem 3.32, we know that $(X_n, \preceq)$ is a binary semilattice. Let r be the top element of its Hasse graph, and set $X' = X_n \setminus \{r\}$. By Corollary 3.28, if $r \in \{1, n\}$, then $\preceq|_{X'}$ is one of the $\omega(n-1)$ semilattice orders that are nondecreasing for $\leq|_{X'}$. By Corollaries 3.28 and 3.31, if $r \notin \{1, n\}$, then $\preceq|_{X'}$ is the union of one of the $\omega(r-1)$ semilattices orders on $[1, r-1]_{\leq_n|_{[1,r-1]}}$ that are nondecreasing for $\leq_n|_{[1,r-1]}$ with one of the $\omega(n-r)$ semilattice orders on $[r+1, n]_{\leq_n|_{[r+1,n]}}$ that are nondecreasing for $\leq_n|_{[r+1,n]}$. □

Definition 3.37. A *planted binary tree* is a tuple (V, E, r, c) such that (V, E) is a binary tree with root $r \in V$, and $c\colon V \setminus \{r\} \to \{L, R\}$ is a map such that $c(x) \neq c(y)$ for every $x \neq y$ that have the same parent.

The two following results are direct consequences of Theorem 3.32. Proposition 3.39 provides a bijection between the set of semilattice orders that are nondecreasing for $\leq_n$ and the set of planted binary trees with n vertices. It gives an alternative proof of Proposition 3.36, since it is known that the number of planted binary trees with n vertices is the nth Catalan number (see, e.g., [95, 96]).

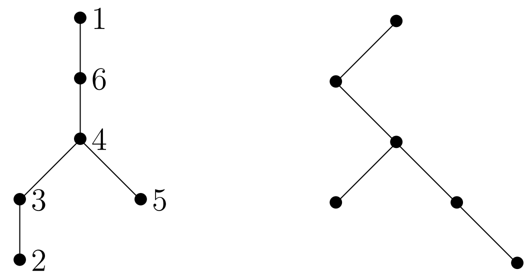

Figure 3.4: $(X_6, \preceq)$ and $f(\preceq)$

Lemma 3.38 (see [36]). *Let $\preceq$ be a nondecreasing semilattice order for $\leq_n$ with top element r, and let $c_\preceq$ be the map defined on $X_n \setminus \{r\}$ by*

$$c_\preceq(y) = \begin{cases} L & \text{if the parent } x \text{ of } y \text{ satisfies } x < y, \\ R & \text{if the parent } x \text{ of } y \text{ satisfies } x > y. \end{cases}$$

Then, the map c turns the Hasse graph of $(X_n, \preceq)$ to a planted binary tree.

Proposition 3.39 (see [36]). *For any semilattice order $\preceq$ that is nondecreasing for $\leq_n$, set*

$$f(\preceq) \;=\; (H_\preceq, c_\preceq),$$

where $H_\preceq$ is the rooted Hasse graph of $(X_n, \preceq)$, and $c_\preceq$ is the map defined in Lemma 3.38. Then, the map f is a bijection between the semilattice orders that are nondecreasing for $\leq_n$ and the ordered rooted binary trees with n vertices.

Figure 3.4 is an illustration of Proposition 3.39 in which we draw any vertex x such that $c(x) = L$ (resp. $c(x) = R$) on the left (resp. on the right) of their parent vertex.

Remark 3.40. We deduce from Proposition 3.39 that semilattice orders which are nondecreasing for $\leq_n$ are in one-to-one correspondence with the C-posets defined in [7]. In particular, Theorem 3.5 in [7] gives a characterization of the semilattice orders which are nondecreasing for $\leq_n$ in terms of permutation patterns avoidance. These observations are due to the anonymous reviewer of [36].

Note that the proof of Propositions 3.36 and 3.39 give two ways to construct all the semilattice orders that are nondecreasing for $\leq_n$. These results also count the number of semilattice orders on X_n that have the CI-property and are internal for $\leq_n$. The Hasse graph of these semilattices are binary trees verifying condition (ii) of Theorem 3.32.

Let $\preceq$ be a semilattice order on X_n. By Theorem 3.26, we have that $(X_n, \curlyvee)$ is orderable if and only if it is a binary semilattice. Now, assuming that $\preceq$ is a binary semilattice order, the family of total orders $\leq$ on X_n for which $\preceq$ is nondecreasing can be constructed by recursion using the following result.

Proposition 3.41 (see [36]). *Let $(X_n, \preceq)$ be a binary semilattice with top element r and let G be its Hasse graph. Let C_1 and C_2 be the connected components of $G - r$, with the convention that $C_2 = \varnothing$ if r has only one child. The following conditions are equivalent.*

(i) The order $\preceq$ is nondecreasing for $\leq_n$.

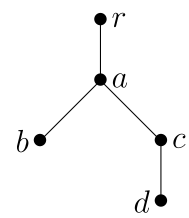

Figure 3.5: Semilattice $(X, \preceq)$ whose Hasse graph is a binary tree

(ii) There exist total orders $\leq_1$ on C_1 and $\leq_2$ on C_2 such that

(a) the order $\preceq_{C_i}$ is nondecreasing for $\leq_i$ for every $1 \leq i \leq 2$,

(b) the total order $\leq$ is obtained by adding r as the top of $\leq_1$ and the bottom of $\leq_2$, or conversely.

Proof. Let us first show that $(ii) \Rightarrow (i)$. Since $\preceq_{C_1}$ and $\preceq_{C_2}$ have the CI-property for $\leq_1$ and $\leq_2$, respectively, it follows by Lemma 3.13 that $\preceq$ has the CI-property for $\leq$. Similarly, we obtain by Lemma 3.14 (ii) that $\preceq$ is internal for $\leq$.

The proof of the implication $(i) \Rightarrow (ii)$ is obtained by an easy induction on n, using Lemma 3.30 and Corollary 3.31 in the induction step. □

The following corollary is obtained from Proposition 3.41 by an easy induction on n.

Corollary 3.42 (see [36]). *Let $(X_n, \preceq)$ be a binary semilattice, and let L be the number of minimal elements in $(X_n, \preceq)$. The number of total orders for which $\preceq$ is nondecreasing is equal to 2^{n-L}.*

Example 3.43. The eight total orders on $X = \{a, b, c, d, r\}$ for which the semilattice order $\preceq$ depicted in Figure 3.5 is nondecreasing are

$$\begin{array}{ll} r < b < a < c < d, & r < b < a < d < c, \\ r < c < d < a < b, & r < d < c < a < b, \end{array}$$

and their dual orders.

It follows from Corollary 3.35 that the number $\tau(n)$ of isomorphism types of semilattices that are $\leq_n$-preserving is equal to the number $A001190(n+1)$ of unlabeled rooted binary trees (see [94]; in such a tree, no order is specified on the children of a parent vertex).

Corollary 3.44 (see [94, A001190]). *The number $\tau(n)$ of isomorphism types of semilattices that are nondecreasing for $\leq_n$ satisfies $\tau(0) = 1$, $\tau(1) = 1$ and*

$$\tau(2n) = \sum_{i=0}^{n-1} \tau(i)\tau(2n-1-i)$$

$$\tau(2n+1) = \sum_{i=0}^{n-1} \tau(i)\tau(2n-i) + \frac{\tau(n)}{2}\left(\tau(n)+1\right)$$

for all $n \geq 1$.

Chapter 4

Quasitrivial semigroups

We let $\mathcal{Q}$ be the class of associative and quasitrivial operations $F\colon X^2 \to X$. We will often denote this class by $\mathcal{Q}_n$ if $X = X_n$ for some integer $n \geq 1$. Also, when considering enumeration problems, we often denote the class of associative and quasitrivial operations on X_0 by $\mathcal{Q}_0$. Although the class $\mathcal{Q}$ has been completely characterized (see Theorem 4.1 below), its structure can be investigated by classifying its elements into subclasses. The purpose of this chapter is to define and analyze such classifications by considering natural equivalence relations. The case where X is finite also raises the interesting problem of enumerating the corresponding equivalence classes.

The outline of this chapter is as follows. In Section 4.1, we essentially recall a descriptive characterization of the class $\mathcal{Q}$. In Section 4.2, we introduce and investigate classifications of the elements of $\mathcal{Q}$ by defining three natural equivalence relations. One of these classifications is simply obtained by considering orbits (conjugacy classes) defined by letting the group[1] of permutations on X act on $\mathcal{Q}$. We also focus on the finite case, where we enumerate the equivalence classes defined by each of these equivalence relations. In Section 4.3, we investigate the operations of $\mathcal{Q}$ that are order-preserving for some total order on X. In particular, we characterize the above-mentioned orbits that contain at least one such order-preserving operation. We also elaborate on the finite case, where the enumeration problems give rise to new integer sequences. In Section 4.4, we examine further subclasses of $\mathcal{Q}$ by considering additional properties: commutativity, anticommutativity, and bisymmetry. Most of the contributions presented in this chapter stem from [24, 25, 31, 34].

4.1 Characterizations of quasitrivial semigroups

Given a weak order $\precsim$ on X, the *maximum* on X for $\precsim$ is the partial commutative binary operation $\max_{\precsim}$ defined on

$$X^2 \setminus \{(x,y) \in X^2 : x \sim y,\ x \neq y\},$$

by $\max_{\precsim}(x,y) = y$ whenever $x \precsim y$. If $\precsim$ reduces to a total order, then clearly the operation $\max_{\precsim}$ is defined everywhere on X^2. The *minimum* on X for $\precsim$, denoted by $\min_{\precsim}$, is defined dually.

The following theorem provides a descriptive characterization of the class $\mathcal{Q}$. As recently observed in [1], this characterization can be derived from Theorem 3.3. A recent discussion and

[1] Recall that a *group* (X,F) is a monoid with neutral element $e \in X$ such that for any $x \in X$ there exists a unique $y \in X$ such that $F(x,y) = F(y,x) = e$.

a direct elementary proof can be found in [25].

Theorem 4.1 (see [72]). *We have $F \in \mathcal{Q}$ if and only if there exists a weak order $\precsim$ on X such that*

$$F|_{A\times B} \;=\; \begin{cases} \pi_1|_{A\times B} \;\; \textit{or} \;\; \pi_2|_{A\times B}, & \textit{if } A = B, \\ \max_{\precsim}|_{A\times B}, & \textit{if } A \neq B, \end{cases} \qquad \forall A, B \in X/{\sim}. \tag{4.1}$$

It is not difficult to see that the weak order $\precsim$ mentioned in Theorem 4.1 is unique. The following proposition provides a way to construct it.

Proposition 4.2 (see [25]). *The weak order $\precsim$ mentioned in Theorem 4.1 is uniquely determined from F and is defined by*

$$x \precsim y \quad \Leftrightarrow \quad F(x,y) = y \;\; \textit{or} \;\; F(y,x) = y, \qquad x, y \in X. \tag{4.2}$$

If $X = X_n$, then we also have the equivalence

$$x \precsim y \quad \Leftrightarrow \quad |F^{-1}[x]| \;\leq\; |F^{-1}[y]|, \qquad x, y \in X. \tag{4.3}$$

Moreover, we have

$$|F^{-1}[x]| \;=\; 2 \times |\{z \in X_n : z \prec x\}| + |\{z \in X_n : z \sim x\}|, \qquad x \in X_n. \tag{4.4}$$

Proof. Let us first show that the binary relation $\precsim$ defined on X by (4.2) is a weak order on X. Indeed, this relation is clearly total. Let us show that it is transitive. Let $x, y, z \in X$ be pairwise distinct and such that $x \precsim y$ and $y \precsim z$. Let us assume for instance that $F(x,y) = y$ and $F(z,y) = z$ (the other three cases can be dealt with similarly). Then we have $F(x,z) = z$ and hence $x \precsim z$. Indeed, otherwise we would have $x = F(x,z) = F(x, F(z,y)) = F(F(x,z),y) = F(x,y) = y$, a contradiction.

Now, assume that $X = X_n$ and let us show that (4.4) holds. By quasitriviality, only points of the form (x,z) or (z,x), with $z \in X_n$, may have the same value as (x,x).

- If $z \prec x$, then $F(x,z) = F(z,x) = x = F(x,x)$.
- If $x \prec z$, then $F(x,z) = F(z,x) = z \neq F(x,x)$.
- If $z \sim x$ and $z \neq x$, then either $F(x,z) = \pi_1(x,z)$ or $F(x,z) = \pi_2(x,z)$. In the first case, we have $F(x,z) = x = F(x,x) \neq z = F(z,x)$. The other case is similar.

Finally, let us show that (4.3) holds. Let $x, y \in X_n$ such that $x \precsim y$. We clearly have

$$|\{z \in X_n : z \prec x\}| \;\leq\; |\{z \in X_n : z \prec y\}|$$

and

$$|\{z \in X_n : z \precsim x\}| \;\leq\; |\{z \in X_n : z \precsim y\}|.$$

By (4.4), we then immediately have $|F^{-1}[x]| \leq |F^{-1}[y]|$. The (contrapositive of the) reverse implication can be proved similarly. $\square$

Remark 4.3. Condition (4.3) was equivalently stated in [24] in terms of F-degrees, where the F-degree of an element $z \in X_n$ is the natural integer $\deg_F(z) = |F^{-1}[z]| - 1$.

In what follows, the weak order $\precsim$ on X defined by (4.2) from any $F \in \mathcal{Q}$ will henceforth be denoted by $\precsim_F$.

From the properties of the maximum operation in (4.1), we can observe the following fact. Recall first that an element $a \in X$ is said to be an *annihilator* of an operation $F\colon X^2 \to X$ if $F(a,x) = F(x,a) = a$ for every $x \in X$.

Fact 4.4 (see [25]). *If $F\colon X^2 \to X$ is of the form* (4.1) *for the weak order $\precsim_F$ on X, then F has a neutral element $e \in X$ (resp. an annihilator element $a \in X$) if and only if the weakly ordered set $(X, \precsim_F)$ has a unique minimal element denoted by $\bot$ (resp. a unique maximal element denoted by $\top$). In this case we have $e = \bot$ (resp. $a = \top$).*

Remark 4.5. If $F\colon X^2 \to X$ is of the form (4.1) for some weak order $\precsim$ on X, then, by replacing $\precsim$ with its dual relation $\precsim^d$, we see that F is again of the form (4.1), except that the maximum operation is changed to the minimum operation. Thus, choosing the maximum or the minimum operation is just a matter of convention.

The following immediate corollary provides an alternative characterization of the class $\mathcal{Q}$ that does not make use of the concept of weak order.

Corollary 4.6 (see [34]). *We have $F \in \mathcal{Q}$ if and only if there exists a total order $\leq$ on X, a partition of X into nonempty $\leq$-convex sets $\{C_i : i \in I\}$, and a map $\varepsilon\colon I \to \{1,2\}$ such that*

$$F(x,y) \;=\; \begin{cases} \pi_{\varepsilon(i)}(x,y), & \textit{if } \exists\, i \in I \textit{ such that } x,y \in C_i, \\ \max_{\leq}(x,y), & \textit{otherwise}, \end{cases} \qquad \forall x,y \in X.$$

An operation $F\colon X^2 \to X$ of the form given in Corollary 4.6 is called an *ordinal sum* [72] of projections on the totally ordered set $(X, \leq)$. Such an operation is illustrated in Figure 4.1. Combining Theorem 4.1 and Corollary 4.6, we can easily see that any $F \in \mathcal{Q}$ is an ordinal sum of projections on $(X, \leq)$ if and only if $\leq$ extends $\precsim_F$.

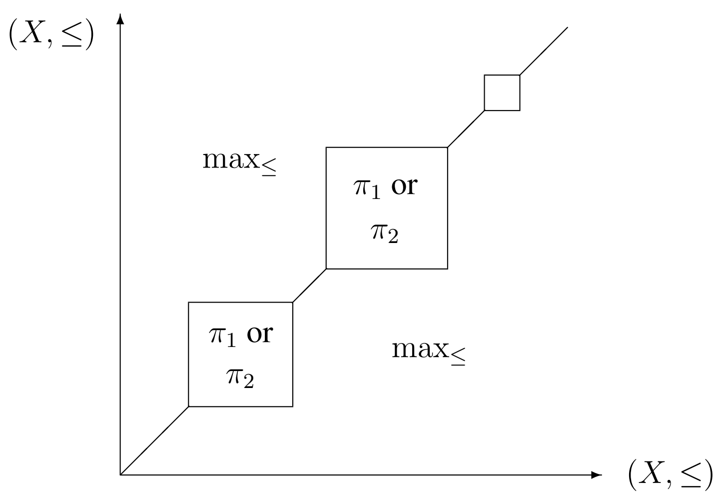

Figure 4.1: An ordinal sum of projections

Recall that the *kernel* of an operation $F\colon X^2 \to X$ is the equivalence relation

$$\ker(F) \;=\; \{\{(x,y),(u,v)\}\colon F(x,y) \;=\; F(u,v)\}.$$

Let us now assume that $X = X_n$. Define the *contour plot* of any operation $F\colon X_n^2 \to X_n$ by the undirected graph $\mathcal{C}_F = (X_n^2, E)$, where

$$E = \{\{(x,y),(u,v)\} : (x,y) \neq (u,v) \text{ and } F(x,y) = F(u,v)\}.$$

That is, E is the non-reflexive part of $\ker(F)$. We observe that, for any $z \in X_n$ such that $F^{-1}[z] \neq \varnothing$, the subgraph of $\mathcal{C}_F$ induced by $F^{-1}[z]$ is a complete connected component of $\mathcal{C}_F$. It is also clear that $\mathcal{C}_F$ has exactly $|F(X_n^2)|$ connected components. In particular, $\mathcal{C}_F$ has n connected components for every $F \in \mathcal{Q}_n$.

We can always represent the contour plot of any operation $F\colon X_n^2 \to X_n$ by fixing a total order on X_n. For instance, using the usual total order $\leq_6$ on X_6, in Figure 4.2 (left) we represent the contour plot of an operation $F\colon X_6^2 \to X_6$. To simplify the representation of the connected components, we omit edges that can be obtained by transitivity. In this representation we also assign a number to any element $(x,x) \in X_6^2$. This number is actually the value of $F(x,x)$. The weak order $\precsim$ on X_6 obtained from (4.3) is such that $3 \sim 4 \prec 2 \prec 1 \sim 5 \sim 6$. In Figure 4.2 (right) we represent the contour plot of F by using a total order $\leq$ on X_6 that extends $\precsim$. We then obtain an ordinal sum of projections on $\leq$, which finally shows that $F \in \mathcal{Q}_6$ and that $\precsim_F = \precsim$.

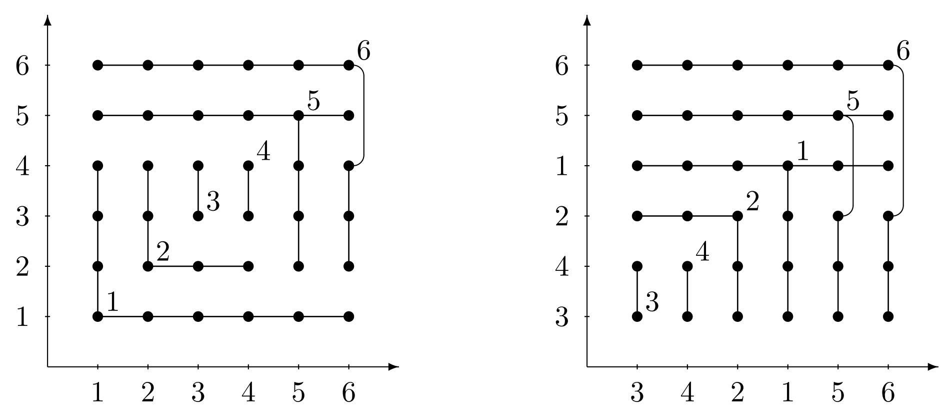

Figure 4.2: An operation $F \in \mathcal{Q}_6$ (left) and its ordinal sum representation (right)

This example clearly illustrates the following simple test to check whether a given operation $F\colon X_n^2 \to X_n$ is associative and quasitrivial. First, use condition (4.3) to construct the unique weak order $\precsim$ on X_n from the preimage sequence $|F^{-1}|$. Then, extend this weak order to a total order $\leq$ on X_n and check if F is an ordinal sum of projections on $\leq$. This test can be easily performed in $O(n^2)$ time.

For any integer $n \geq 0$, we denote by $\gamma(n)$ the number of operations $F \in \mathcal{Q}_n$ (i.e., the number of quasitrivial semigroups on an n-element set). As a convention, we set $\gamma(0) = 1$. Also, for any integer $n \geq 0$, we denote by

- $\gamma_e(n)$ the number of operations $F \in \mathcal{Q}_n$ that have neutral elements,
- $\gamma_a(n)$ the number of operations $F \in \mathcal{Q}_n$ that have annihilator elements,
- $\gamma_{ae}(n)$ the number of operations $F \in \mathcal{Q}_n$ that have distinct neutral and annihilator elements.

As a convention, we set $\gamma_e(0) = \gamma_a(0) = \gamma_{ae}(0) = 0$. Also, by definition we have $\gamma_e(1) = \gamma_a(1) = 1$ and $\gamma_{ae}(1) = 0$. Theorem 4.7 and Proposition 4.8 below provide explicit formulas for these sequences. The first few values of these sequences are shown in Table 4.1.

Theorem 4.7 (see [25]). *For any integer $n \geq 0$, we have the closed-form expression*

$$\gamma(n) \;=\; \sum_{i=0}^{n} 2^i \sum_{k=0}^{n-i} (-1)^k \binom{n}{k} \left\{ n-k \atop i \right\} (i+k)!\,, \qquad n \geq 0. \tag{4.5}$$

Moreover, the sequence $(\gamma(n))_{n\geq 0}$ satisfies the recurrence equation

$$\gamma(n+1) \;=\; (n+1)\,\gamma(n) + 2\sum_{k=0}^{n-1} \binom{n+1}{k} \gamma(k)\,, \qquad n \geq 0,$$

with $\gamma(0) = 1$. Furthermore, its EGF is given by $\hat{\Gamma}(z) = 1/(z + 3 - 2e^z)$.

Proof. We clearly have $\gamma(0) = 1$. Now, let $n \geq 1$, let $F \in \mathcal{Q}_n$, and let k be the number of maximal elements of X_n for $\precsim_F$. By Theorem 4.1, we have $F|_{(\max_{\precsim_F} X_n)^2} = \pi_i|_{(\max_{\precsim_F} X_n)^2}$ for some $i \in \{1, 2\}$. Moreover, the restriction of F to $(X_n \setminus \max_{\precsim_F} X_n)^2$ is associative and quasitrivial. Thus, it follows that the sequence $\gamma(n)$ satisfies the recurrence equation

$$\gamma(n) \quad = \quad n\,\gamma(n-1) + 2\sum_{k=2}^{n} \binom{n}{k} \gamma(n-k), \qquad n \geq 1.$$

The expression of the EGF of $(\gamma(n))_{n\geq 0}$ follows then straightforwardly.

Let us now establish Eq. (4.5). It is enough to show that the EGF of the sequence $(\tilde{\gamma}(n))_{n\geq 0}$ defined by $\tilde{\gamma}(0) = 1$ and

$$\tilde{\gamma}(n) \;=\; \sum_{i=0}^{n} 2^i \sum_{k=0}^{n-i} (-1)^k \binom{n}{k} \left\{ n-k \atop i \right\} (i+k)!\,, \qquad n \geq 1,$$

is exactly $\hat{\Gamma}(z)$.

For any integer $i \geq 0$, consider the sequences $(f_n^i)_{n\geq 0}$ and $(g_n^i)_{n\geq 0}$ defined by

$$f_n^i \;=\; (-1)^n (n+i)!,$$

and $g_n^i = \left\{ n \atop i \right\}$. Define also the sequence $(h_n^i)_{n\geq 0}$ by the binomial convolution of $(f_n^i)_{n\geq 0}$ and $(g_n^i)_{n\geq 0}$, that is,

$$h_n^i \;=\; \sum_{k=0}^{n} \binom{n}{k} (-1)^k (i+k)! \left\{ n-k \atop i \right\}.$$

Observing that $\left\{ n-k \atop i \right\} = 0$ if $n - k < i$, we see that

$$\tilde{\gamma}(n) \;=\; \sum_{i=0}^{n} 2^i h_n^i\,, \qquad n \geq 0. \tag{4.6}$$

Let $\hat{F}_i(z)$, $\hat{G}_i(z)$, and $\hat{H}_i(z)$ be the EGFs of the sequences $(f_n^i)_{n\geq 0}$, $(g_n^i)_{n\geq 0}$, and $(h_n^i)_{n\geq 0}$, respectively. It is known (see, e.g., [53]) that $\hat{F}_i(z) = i!(z+1)^{-i-1}$ and $\hat{G}_i(z) = (e^z-1)^i/i!$. We then have

$$\hat{H}_i(z) \;=\; \hat{F}_i(z)\hat{G}_i(z) \;=\; \frac{(e^z-1)^i}{(z+1)^{i+1}}\,.$$

Since $h_n^i = D_z^n\hat{H}_i(z)|_{z=0}$, using (4.6) we obtain

$$\tilde{\gamma}(n) \;=\; D_z^n\,\frac{1-\left(2\frac{e^z-1}{z+1}\right)^{n+1}}{z+3-2e^z}\bigg|_{z=0} \;=\; D_z^n\,\frac{1}{z+3-2e^z}\bigg|_{z=0} \;=\; (D_z^n\,\hat{\Gamma})(0).$$

This means that the EGF of $(\tilde{\gamma}(n))_{n\geq 0}$ is given by $\hat{\Gamma}(z)$. This completes the proof. □

Proposition 4.8 (see [25]). *For any integer $n\geq 2$, we have $\gamma_e(n) = \gamma_a(n) = n\,\gamma(n-1)$ and $\gamma_{ae}(n) = n(n-1)\,\gamma(n-2)$.*

Proof. Let us first show how we can construct an arbitrary associative and quasitrivial operation $F\colon X_n^2\to X_n$ having a neutral element. There are n ways to choose the neutral element e in X_n. Then we observe that the restriction of F to $(X_n\setminus\{e\})^2$ is still an associative and quasitrivial operation, so we have $\gamma(n-1)$ possible choices to construct this restriction. This shows that $\gamma_e(n) = n\,\gamma(n-1)$. Using the same reasoning, we also obtain $\gamma_a(n) = n\,\gamma(n-1)$ and $\gamma_{ea}(n) = n(n-1)\,\gamma(n-2)$. □

n	$\gamma(n)$	$\gamma_e(n)$	$\gamma_a(n)$	$\gamma_{ea}(n)$
0	1	0	0	0
1	1	1	1	0
2	4	2	2	2
3	20	12	12	6
4	138	80	80	48
5	1 182	690	690	400
6	12 166	7 092	7 092	4 140
OEIS	A292932	A292933	A292933	A292934

Table 4.1: First few values of $\gamma(n)$, $\gamma_e(n)$, $\gamma_a(n)$, and $\gamma_{ea}(n)$

Let us now present a result that will be useful as we continue.

Proposition 4.9 (see [34]). *For any two operations $F\colon X_n^2\to X_n$ and $G\colon X_n^2\to X_n$, we have $|F^{-1}| = |G^{-1}|$ if and only if $\mathcal{C}_F\simeq\mathcal{C}_G$.*

Proof. (Sufficiency) Trivial.

(Necessity) Recall that the *order* of a graph is simply the number of its vertices. Thus, by definition, $|F^{-1}|$ is the nondecreasing n-element sequence of the orders of the connected components of $\mathcal{C}_F$. If $|F^{-1}| = |G^{-1}|$, then it is not difficult to construct a bijection $\phi\colon X_n^2\to X_n^2$ that maps a connected component of $\mathcal{C}_F$ to a connected component of $\mathcal{C}_G$ of the same order. Since all these connected components are complete subgraphs, we obtain that $\mathcal{C}_F\simeq\mathcal{C}_G$. □

4.2 Classifications of quasitrivial semigroups

It is a fact that the class $\mathcal{Q}$ is generally very huge. In the finite case, the size of the class $\mathcal{Q}_n$ (a sequence recorded in the OEIS as Sloane's A292932; see [94]) becomes very large as n grows (see Theorem 4.7).[2] It is then natural to classify the elements of $\mathcal{Q}$ by considering relevant equivalence relations on this class. Before introducing such relations, let us recall some basic definitions.

Let $\mathfrak{S}$ be the group of permutations on X. We will often denote this group by $\mathfrak{S}_n$ if $X = X_n$ for some integer $n \geq 1$. For any operation $F\colon X^2 \to X$ and any permutation $\sigma \in \mathfrak{S}$, the *σ-conjugate of F* is the operation $F_\sigma\colon X^2 \to X$ defined by

$$F_\sigma(x,y) \;=\; \sigma(F(\sigma^{-1}(x),\sigma^{-1}(y))), \qquad x,y \in X.$$

A *conjugate of F* is a σ-conjugate of F for some $\sigma \in \mathfrak{S}$.

Clearly, the map $\psi\colon \mathfrak{S} \times \mathcal{Q} \to \mathcal{Q}$ defined by $\psi(\sigma, F) = F_\sigma$ is a group action. We then can define

- the *orbit* of $F \in \mathcal{Q}$ by $\mathrm{orb}(F) = \{F_\sigma : \sigma \in \mathfrak{S}\}$,
- the *stabilizer subgroup* of $\mathfrak{S}$ for $F \in \mathcal{Q}$ by $\mathrm{stab}(F) = \{\sigma \in \mathfrak{S} : F_\sigma = F\}$.

We can readily see that, for any $\sigma \in \mathfrak{S}$, we have $F \in \mathcal{Q}$ if and only if $F_\sigma \in \mathcal{Q}$. Moreover, using (4.2) we see that, for any $\sigma \in \mathfrak{S}$ and any $F \in \mathcal{Q}$ we have

$$x \precsim_F y \quad\Leftrightarrow\quad \sigma(x) \precsim_{F_\sigma} \sigma(y)\,, \qquad x,y \in X. \tag{4.7}$$

Now, let q be the identity relation on $\mathcal{Q}$. We also introduce relations p, s, r on $\mathcal{Q}$ as follows. For any $F, G \in \mathcal{Q}$, we write

$$\begin{aligned} F\,p\,G, &\quad \text{if} \quad {\precsim_F} = {\precsim_G} \\ F\,s\,G, &\quad \text{if} \quad {\precsim_F} \simeq {\precsim_G} \\ F\,r\,G, &\quad \text{if} \quad G \in \mathrm{orb}(F)\,. \end{aligned}$$

It is clear that each of the relations above is an equivalence relation and hence it partitions $\mathcal{Q}$ into equivalence classes. Moreover, we clearly have that $q \subseteq p \subseteq s$. Using (4.7), we also see that $q \subseteq r \subseteq s$. Furthermore, we observe that p and r are not comparable in general. Indeed, if $F = \pi_1$ and $G = \pi_2$ on X, then we have $F\,p\,G$ and $\neg(F\,r\,G)$. Similarly, if $F = \max_\leq$ and $G = \min_\leq$ for some total order $\leq$ on X, then we have $F\,r\,G$ and $\neg(F\,p\,G)$.

The following proposition provides further properties of the relations introduced above. Let us first investigate the conjunction of relations p and r.

We observe that, given an operation $F \in \mathcal{Q}$, any permutation $\sigma \in \mathfrak{S}$ for which $\sigma(x) \sim_F x$ for all $x \in X$ is an automorphism of $(X, \precsim_F)$. We say that such an automorphism is *trivial*. It is easy to prove by induction that all automorphisms of $(X, \precsim_F)$ are trivial whenever X is finite.

Lemma 4.10 (see [34]). *Let $F \in \mathcal{Q}$ and $\sigma \in \mathfrak{S}$. Consider the following four assertions.*

(i) $F\,p\,F_\sigma$.

[2] In fact, we have $|\mathcal{Q}_n| \sim \frac{1}{2\lambda+1}\, n!\, \lambda^{n+2}$ as $n \to \infty$, where λ (≈ 1.71) is the inverse of the unique positive zero of the real function $x \mapsto x + 3 - 2e^x$.

(ii) $F = F_\sigma$.

(iii) $\sigma(x) \sim_F x$ *for every* $x \in X$.

(iv) σ *is an automorphism of* $(X, \precsim_F)$.

Then we have (iii) $\Rightarrow$ *(ii)* $\Rightarrow$ *(i)* $\Leftrightarrow$ *(iv). The implication (iv)* $\Rightarrow$ *(iii) holds if and only if all automorphisms of* $(X, \precsim_F)$ *are trivial. The latter condition holds for instance if* X *is finite.*

Proof. The equivalence $(i) \Leftrightarrow (iv)$ is straightforward (simply use (4.7)). Also, the implications $(iii) \Rightarrow (iv)$ and $(ii) \Rightarrow (i)$ are obvious. Now, let us show that $(iii) \Rightarrow (ii)$. Let $x, y \in X$. Suppose first that $x \sim_F y$. By (4.7) and conditions (iii) and (iv), we have

$$x \ \sim_F \ \sigma^{-1}(x) \ \sim_F \ \sigma^{-1}(y) \ \sim_F \ y.$$

Hence, by Theorem 4.1 there exists $i \in \{1, 2\}$ such that

$$\begin{aligned} F(\sigma^{-1}(x), \sigma^{-1}(y)) \ &= \ \pi_i(\sigma^{-1}(x), \sigma^{-1}(y)) \ = \ \sigma^{-1}(\pi_i(x, y)) \\ &= \ \sigma^{-1}(F(x, y)), \end{aligned}$$

that is, $F_\sigma(x, y) = F(x, y)$. We proceed similarly if $x \prec_F y$ or $y \prec_F x$.

The last part of the lemma is trivial. □

Proposition 4.11 (see [34])**.** *We have* $p \vee r = p \circ r = s$. *If* X *is finite, we also have* $p \wedge r = q$.

Proof. Let us prove the first two identities. Since $p \circ r \subseteq p \vee r$, it is enough to show that $s \subseteq p \circ r$. Let $F, G \in \mathcal{Q}$ such that $F \, s \, G$. That is, there exists $\sigma \in \mathfrak{S}$ such that

$$x \precsim_G y \quad \Leftrightarrow \quad \sigma(x) \precsim_F \sigma(y), \qquad x, y \in X.$$

Using (4.7), we then see that

$$x \precsim_F y \quad \Leftrightarrow \quad \sigma^{-1}(x) \precsim_G \sigma^{-1}(y) \quad \Leftrightarrow \quad x \precsim_{G_\sigma} y, \qquad x, y \in X,$$

which means that $\precsim_F = \precsim_{G_\sigma}$. Therefore we have $F \, p \, G_\sigma \, r \, G$, from which we derive that $s \subseteq p \circ r$.

To prove the last identity, we only need to show that $p \wedge r \subseteq q$. Let $F, G \in \mathcal{Q}$ such that $F \, p \, G$ and $F \, r \, G$. By Lemma 4.10, we have $F = G$, that is, $F \, q \, G$. □

Remark 4.12. We can easily construct operations $F \in \mathcal{Q}$ for which $(X, \precsim_F)$ has nontrivial automorphisms. Consider for instance the operation $F = \max_\leq$ on $X = \mathbb{Z}$, where $\leq$ is the usual order on $\mathbb{Z}$, and take $\sigma(x) = x + 1$. Then, we have $F \in \mathcal{Q}$ and $\sigma \in \mathfrak{S}$. Also, conditions (i), (ii), and (iv) of Lemma 4.10 hold but condition (iii) fails to hold. Now, define the operation $F \colon \mathbb{Z}^2 \to \mathbb{Z}$ by

$$F(x, y) \ = \ \begin{cases} x, & \text{if } (x, y) \in \{0, 1\}^2, \\ y, & \text{if } (x, y) \in \bigcup_{m \in \mathbb{Z} \setminus \{0\}} \{2m, 2m + 1\}^2, \\ \max_\leq(x, y), & \text{otherwise,} \end{cases}$$

where $\leq$ is the usual order on $\mathbb{Z}$. Take also $\sigma(x) = x - 2$. Then, again we have $F \in \mathcal{Q}$ and $\sigma \in \mathfrak{S}$. Also, conditions (i) and (iv) of Lemma 4.10 hold but condition (iii) fails to hold. Moreover, we have $F(0, 1) = 0 \neq 1 = F_\sigma(0, 1)$, which shows that condition (ii) fails to hold, and hence that $p \wedge r \neq q$.

Proposition 4.13 (see [34]). *For any $\sigma \in \mathfrak{S}$, the map $\tilde{\sigma}\colon \mathcal{Q}/p \to \mathcal{Q}/p$ defined by $\tilde{\sigma}(F/p) = F_\sigma/p$ is a (well-defined) permutation.*

Proof. Let $\sigma \in \mathfrak{S}$. For any $F, G \in \mathcal{Q}$, by (4.7) we have $F\,p\,G$ if and only if $F_\sigma\,p\,G_\sigma$, which shows that $\tilde{\sigma}$ is well defined and injective. Now, for any $F \in \mathcal{Q}$, we have $\tilde{\sigma}(F_{\sigma^{-1}}/p) = F/p$, which shows that $\tilde{\sigma}$ is also surjective. □

In the rest of this section we restrict ourselves to the finite case when $X = X_n$ for some integer $n \geq 1$. This assumption will enable us to enumerate the equivalence classes for each of the equivalence relations introduced above. For any integer $n \geq 1$, define $\delta(n) = |\mathcal{Q}_n/p|$, $\gamma(n) = |\mathcal{Q}_n/q|$, $\mu(n) = |\mathcal{Q}_n/r|$, and $\nu(n) = |\mathcal{Q}_n/s|$. The first few values of these sequences are given in Table 4.2.

n	$\delta(n)$	$\gamma(n)$	$\mu(n)$	$\nu(n)$
1	1	1	1	1
2	3	4	3	2
3	13	20	7	4
4	75	138	17	8
5	541	1 182	41	16
6	4 683	12 166	99	32
OEIS	A000670	A292932	A001333	A011782

Table 4.2: First few values of $\delta(n)$, $\gamma(n)$, $\mu(n)$, and $\nu(n)$

By definition, $\delta(n)$ is the number of weak orders on X_n, or equivalently, the number of totally ordered partitions of X_n (Sloane's A000670). Thus we have $\delta(n) = p(n)$. Also, we clearly have $\gamma(n) = |\mathcal{Q}_n|$ and this number was computed in Theorem 4.7 (Sloane's A292932). Let us now investigate the numbers $\mu(n)$ and $\nu(n)$.

For any $F \in \mathcal{Q}_n$, we set $k = |X_n/{\sim_F}|$ and let $C_1, \ldots, C_k$ denote the elements of $X_n/{\sim_F}$ ordered by the relation induced by $\precsim_F$, that is, $C_1 \prec_F \cdots \prec_F C_k$ (where $C_i \prec_F C_j$ means that we have $x \prec_F y$ for all $x \in C_i$ and all $y \in C_j$). Also, we set $n_i = |C_i|$ for $i = 1, \ldots, k$. We then define the *signature* of F as the k-tuple $(n_1, \ldots, n_k)$ and we denote it by $\mathrm{sgn}(F)$.

It is clear that the number of possible signatures in $\mathcal{Q}_n$ is precisely the number of totally ordered partitions of a set of n unlabeled items (Sloane's A011782), that is,

$$\sum_{k=1}^{n} \sum_{\substack{n_1,\ldots,n_k \geq 1 \\ n_1+\cdots+n_k=n}} 1 ~=~ 2^{n-1}\,.$$

It follows that this number is also the number of weak orders on X_n that are defined up to an isomorphism. Thus, we have $\nu(n) = 2^{n-1} = p_{iso}(n)$ for all $n \geq 1$.

We actually have the following more general result.

Proposition 4.14 (see [34]). *For any $F, G \in \mathcal{Q}_n$, the following assertions are equivalent.*

(i) $F\,s\,G$.

(ii) $\mathcal{C}_F \simeq \mathcal{C}_G$.

(iii) $|F^{-1}| = |G^{-1}|$.

(iv) $\mathrm{sgn}(F) = \mathrm{sgn}(G)$.

Proof. The equivalence $(i) \Leftrightarrow (iv)$ is straightforward. Also, the equivalence $(ii) \Leftrightarrow (iii)$ is a special case of Proposition 4.9. Finally, let us show that $(i) \Leftrightarrow (iii)$. Clearly, $|F^{-1}| = |G^{-1}|$ holds if and only if there exists $\sigma \in \mathfrak{S}_n$ such that $|F^{-1}[x]| = |G^{-1}[\sigma(x)]|$ for every $x \in X_n$. The claimed equivalence then immediately follows from condition (4.3). $\square$

The following proposition provides explicit expressions (bounded above by $n!$) for $|\mathrm{stab}(F)|$ and $|\mathrm{orb}(F)|$ for any $F \in \mathcal{Q}_n$. In particular, it shows that $|\mathrm{orb}(F)|$ is precisely the number of ways to partition X_n into k subsets of sizes $n_1, \ldots, n_k$.[3]

Proposition 4.15 (see [34]). *For any $F \in \mathcal{Q}_n$, we have*

$$|\mathrm{stab}(F)| \;=\; \prod_{i=1}^{k} n_i! \quad \textit{and} \quad |\mathrm{orb}(F)| \;=\; \binom{n}{n_1, \ldots, n_k}.$$

Proof. The formula for $|\mathrm{stab}(F)|$ immediately follows from Lemma 4.10. By the classical orbit-stabilizer theorem, we have $|\mathrm{orb}(F)| \times |\mathrm{stab}(F)| = |\mathfrak{S}_n|$ for every $F \in \mathcal{Q}_n$. This immediately proves the formula for $|\mathrm{orb}(F)|$. $\square$

Recall that $\mu(n)$ is the number of orbits in $\mathcal{Q}_n$ under the action of $\mathfrak{S}_n$. Burnside's lemma then immediately provides the formula

$$\mu(n) \;=\; \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} |\mathcal{Q}_n^{\sigma}|, \qquad n \geq 1,$$

where $\mathcal{Q}_n^{\sigma} = \{F \in \mathcal{Q}_n : F_\sigma = F\}$.

The following proposition provides much simpler explicit expressions for $\mu(n)$ and shows that the corresponding sequence is known as Sloane's A001333, where we have set $\mu(0) = 1$.

Lemma 4.16. *For every $F \in \mathcal{Q}_n$, there exists $\sigma \in \mathfrak{S}_n$ such that F_σ is an ordinal sum of projections on $\leq_n$.*

Proof. By Corollary 4.6, F is an ordinal sum of projections on some total order $\leq$ that extends $\precsim_F$. Take $\sigma \in \mathfrak{S}_n$ such that

$$\sigma(x) \leq_n \sigma(y) \quad \Leftrightarrow \quad x \leq y, \qquad x, y \in X_n.$$

We then immediately see that $\leq_n$ extends $\precsim_{F_\sigma}$. Hence F_σ is an ordinal sum of projections on $\leq_n$. $\square$

Proposition 4.17. *The sequence $(\mu(n))_{n \geq 0}$ satisfies the linear recurrence equation*

$$\mu(n+2) \;=\; 2\mu(n+1) + \mu(n),$$

with $\mu(0) = 1$ and $\mu(1) = 1$. Its GF is $M(z) = (1-z)/(1-2z-z^2)$. Moreover we have

$$\mu(n) \;=\; \tfrac{1}{2}(1+\sqrt{2})^n + \tfrac{1}{2}(1-\sqrt{2})^n \;=\; \textstyle\sum_{k \geq 0} \binom{n}{2k} 2^k.$$

[3] Recall that $k = |X_n/\!\sim_F|$ and that $n_i = |C_i|$ for $i = 1, \ldots, k$.

Proof. We clearly have $\mu(0) = \mu(1) = 1$. Now let $n \geq 2$. By Lemma 4.16 the number $\mu(n)$ is nothing other than the number of ordinal sums of projections on $(X_n, \leq_n)$ defined up to conjugation by trivial automorphisms. If F is such an ordinal sum, then the restriction of F to $X'^2 = (X_n \setminus C_k)^2$ is an ordinal sum of projections on $(X', \leq_n|_{X'})$. Since there are two possible projections on C_k whenever $n_k \geq 2$, it follows that the sequence $(\mu(n))_{n\geq 0}$ must satisfy the recurrence equation

$$\mu(n) \; = \; \mu(n-1) + 2\sum_{i=2}^{n} \mu(n-i), \qquad n \geq 2.$$

From this recurrence equation, we immediately derive the claimed one. The rest of the proposition follows straightforwardly. □

Remark 4.18. We observe that an alternative expression for $\mu(n)$ is given by

$$\mu(n) \; = \; \sum_{k=1}^{n} \; \sum_{\substack{n_1,\ldots,n_k\geq 1\\ n_1+\cdots+n_k=n}} \; \prod_{\substack{i=1\\ n_i\geq 2}}^{k} 2\,, \qquad n \geq 1.$$

Indeed, by Lemma 4.16 the number $\mu(n)$ is precisely the number of ordinal sums of projections on $(X_n, \leq_n)$ defined up to conjugation by trivial automorphisms. To compute this number, we need to consider all the unordered partitions of X_n and count twice each subset containing at least two elements (because the two projections are to be considered for each such set). Actually, the product provides the exact number of orbits in $\mathcal{Q}_n$ corresponding to the signature $(n_1, \ldots, n_k)$.

Figure 4.3 provides the contour plots of the $\gamma(3) = 20$ operations of $\mathcal{Q}_3$, when X_3 is endowed with $\leq_3$. These operations are organized in a 7×6 array. Those in the first column consist of the $\mu(3) = 7$ ordinal sums of projections on $(X_3, \leq_3)$ defined up to conjugation by trivial automorphisms. Each of the rows represents an orbit and contains all the possible conjugates of the leftmost operation (we omit the duplicates). In turn, the orbits are grouped into $\nu(3) = 4$ different signatures. Also, all these 20 operations are grouped into $\delta(3) = 13$ weak orders (represented by rounded boxes).

Proposition 4.13 can also be easily illustrated in Figure 4.3 as follows. Any permutation $\sigma \in \mathfrak{S}_3$ that maps F to F_σ can be extended to a permutation of the corresponding rounded boxes (within the same signature).

We end this section by a discussion on the concept of preimage sequence. We know from Proposition 4.14 that the preimage sequence of any operation $F \in \mathcal{Q}_n$ contains the same information as its signature. Also, it was shown in Proposition 4.2 that

$$|F^{-1}[x]| \; = \; 2 \times |\{z \in X_n : z \prec_F x\}| + |\{z \in X_n : z \sim_F x\}|, \qquad x \in X_n. \tag{4.8}$$

From the latter identity we can actually derive the following formula:

$$|F^{-1}| \; = \; (\underbrace{n_1}_{n_1}, \underbrace{2\,n_1 + n_2}_{n_2}, \ldots, \underbrace{2\textstyle\sum_{i<k} n_i + n_k}_{n_k}). \tag{4.9}$$

Conversely, the signature $\operatorname{sgn}(F) = (n_1, \ldots, n_k)$ can be obtained immediately by considering the absolute frequencies of the sequence $|F^{-1}|$. That is, if $d_1, \ldots, d_k$ represent the distinct values of

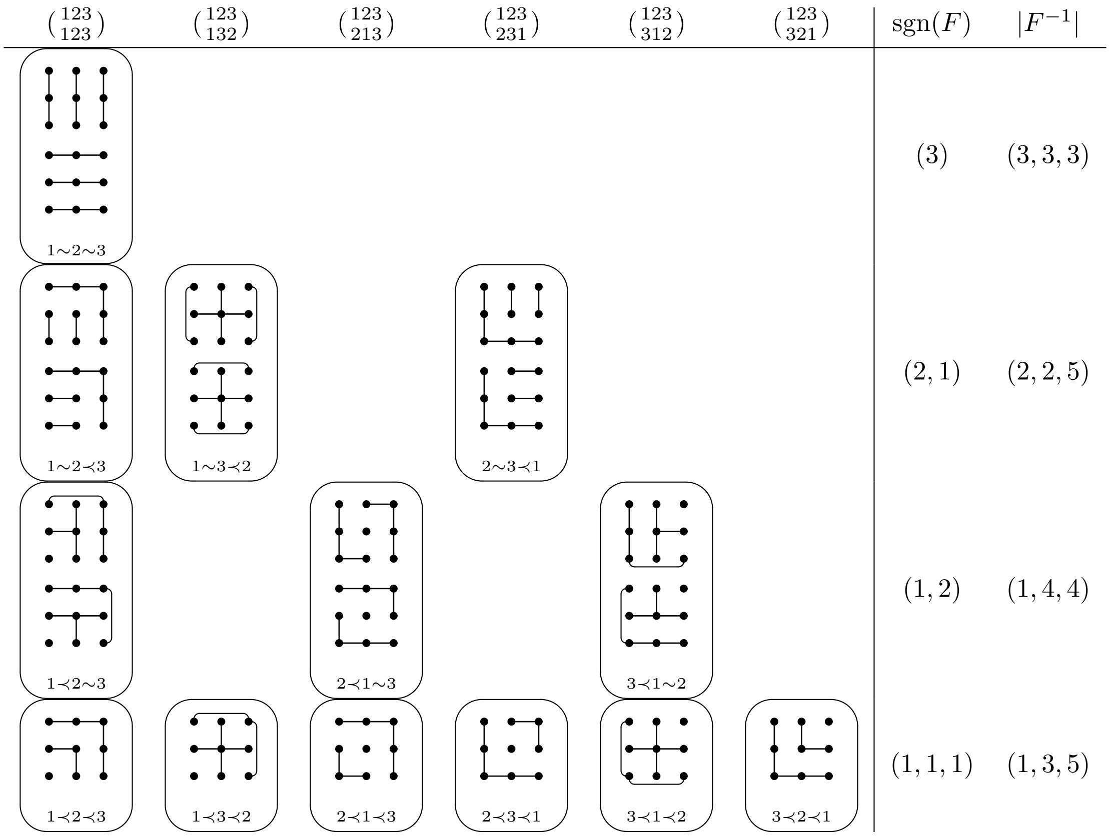

Figure 4.3: Classifications of the 20 associative and quasitrivial operations on $(X_3, \leq_3)$

the sequence $|F^{-1}|$ in increasing order, then n_i is the number of times d_i occurs in $|F^{-1}|$.[4] To give an example, let $F \in \mathcal{Q}_9$ be such that $\operatorname{sgn}(F) = (1,2,2,1,3)$. Then

$$|F^{-1}| \ = \ (1,4,4,8,8,11,15,15,15).$$

The following proposition solves the natural question of finding necessary and sufficient conditions for a nondecreasing n-sequence $(c_1, \ldots, c_n)$ to be the preimage sequence of an operation $F \in \mathcal{Q}_n$.

Proposition 4.19. *Let $c = (c_1, \ldots, c_n)$ be a nondecreasing n-sequence. Then there exists $F \in \mathcal{Q}_n$ such that $|F^{-1}| = c$ if and only if*

$$c_i \ = \ \min\{j : c_j = c_i\} + \max\{j : c_j = c_i\} - 1, \qquad i = 1, \ldots, n. \tag{4.10}$$

Proof. (Necessity) Replacing F with one of its conjugates if necessary, we can assume that $|F^{-1}[1]| \leq \cdots \leq |F^{-1}[n]|$. By (4.3), we then have $1 \precsim \cdots \precsim n$. For every $i \in \{1, \ldots, n\}$, define

$$\begin{aligned} p_i &= \min\{j : |F^{-1}[j]| = |F^{-1}[i]|\}, \\ q_i &= \max\{j : |F^{-1}[j]| = |F^{-1}[i]|\}. \end{aligned}$$

[4]In particular, we observe that, for any $F \in \mathcal{Q}_n$, the number of distinct values in the sequence $|F^{-1}|$ is exactly $|X_n/{\sim_F}|$.

By (4.8), we then have $|F^{-1}[i]| = 2(p_i - 1) + (q_i - p_i + 1) = p_i + q_i - 1$.

(Sufficiency) Let $c = (c_1, \ldots, c_n)$ be a nondecreasing n-sequence satisfying the stated condition and let $n_1, \ldots, n_k$ be the absolute frequencies of this sequence. Take any $F \in \mathcal{Q}_n$ such that $\operatorname{sgn}(F) = (n_1, \ldots, n_k)$. By definition, for any $\ell \in \{1, \ldots, k\}$, all the components of c corresponding to frequency n_ℓ are equal to the number

$$(\textstyle\sum_{i<\ell} n_i + 1) + (\sum_{i<\ell} n_i + n_\ell) - 1 \;=\; 2\sum_{i<\ell} n_i + n_\ell.$$

Equation (4.9) then shows that $|F^{-1}| = c$.

Let us provide an alternative proof that does not make use of (4.9). We proceed by induction on n. The result clearly holds for $n = 1$. Suppose that it holds for any $\ell \leq n - 1$ and let us prove that it still holds for n.

Let $c = (c_1, \ldots, c_n)$ be a nondecreasing n-sequence satisfying the stated condition. If $c_1 = c_n$, then we can take $F = \pi_1$ or $F = \pi_2$ on X_n. If $c_1 < c_n$, then let $\ell = \max\{j : c_j < c_n\}$. By the induction hypothesis, there exists $F_\ell \in \mathcal{Q}_\ell$ such that $|F_\ell^{-1}| = (c_1, \ldots, c_\ell)$. Now, let $F\colon X_n^2 \to X_n$ be defined by

$$F(x, y) \;=\; \begin{cases} F_\ell(x, y), & \text{if } x, y \in X_\ell, \\ \pi_1(x, y), & \text{if } x, y \in X_n \setminus X_\ell, \\ \max(x, y), & \text{otherwise.} \end{cases}$$

Then it is not difficult to see that $F \in \mathcal{Q}_n$ and that $|F^{-1}| = c$. $\square$

Remark 4.20. There are quasitrivial operations $F\colon X_n^2 \to X_n$ that are not associative and whose preimage sequences $|F^{-1}|$ satisfy condition (4.10). The operation $F\colon X_3^2 \to X_3$ whose contour plot is shown in Figure 4.4 could serve as an example here.

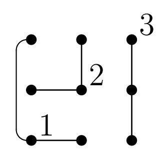

Figure 4.4: A quasitrivial operation $F\colon X_3^2 \to X_3$ that is not associative

4.3 Order-preserving operations

In this section we provide characterizations of the operations $F \in \mathcal{Q}$ that are order-preservable (see Proposition 4.23 below). To this extent, we first provide a characterization of the operations $F \in \mathcal{Q}$ that are $\leq$-preserving for some given total order $\leq$ on X. The latter characterization follows from Proposition 1.9 and Theorems 3.8 and 4.1.

Proposition 4.21 (see [25])**.** *Let $\leq$ be a total order on X and let $F \in \mathcal{Q}$. Then F is $\leq$-preserving if and only if $\precsim_F$ is single-plateaued for $\leq$.*

Example 4.22. Let us consider the operation $F\colon X_4^2 \to X_4$ whose contour plot is depicted in Figure 4.5 (left). We can see that this operation is of the form (4.1), where $\precsim_F$ is the weak order on X_4 defined by $3 \prec_F 2 \sim_F 4 \sim_F 1$; see Figure 4.5 (center). Since the weak order $\precsim_F$ is single-plateaued for $\leq_4$ (see Figure 1.7 (right)) we conclude that F is $\leq_4$-preserving by Proposition 4.21.

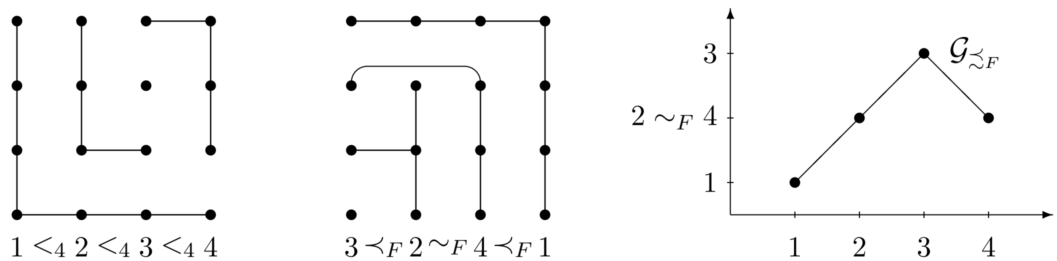

Figure 4.5: Example 4.22

Now, we are able to prove the following proposition.

Proposition 4.23 (see [34]). *For any $F \in \mathcal{Q}$, the following assertions are equivalent.*

(i) F is order-preservable.

(ii) There exists $\sigma \in \mathfrak{S}$ such that F_σ is order-preservable.

(iii) F_σ is order-preservable for every $\sigma \in \mathfrak{S}$.

(iv) $\precsim_F$ is 2-quasilinear.

If $X = X_n$ for some integer $n \geq 1$, then any of the assertions above is equivalent to any of the following ones.

(v) There exists $\sigma \in \mathfrak{S}_n$ such that F_σ is $\leq_n$-preserving.

(vi) $n_2, \ldots, n_k \in \{1, 2\}$, where $(n_1, \ldots, n_k) = \mathrm{sgn}(F)$.

(vii) Every integer strictly greater than c_1 occurs at most two times in $|F^{-1}| = (c_1, \ldots, c_n)$.

Moreover, the equivalence among (i), (ii), (iii), and (v) holds for any operation $F\colon X^2 \to X$.

Proof. The equivalences $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ are straightforward. Also, the implication $(v) \Rightarrow (ii)$ and the equivalence $(iv) \Leftrightarrow (vi)$ are obvious. Moreover, the equivalence $(i) \Leftrightarrow (iv)$ follows from both Propositions 4.21 and 1.15. Furthermore, the equivalence $(iv) \Leftrightarrow (vii)$ follows from (4.3). Let us now show that $(i) \Rightarrow (v)$. Let $\leq$ be a total order X_n for which F is $\leq$-preserving. Take $\sigma \in \mathfrak{S}_n$ such that

$$\sigma(x) \leq_n \sigma(y) \quad \Leftrightarrow \quad x \leq y, \qquad x, y \in X_n.$$

Now let $x, x', y, y' \in X_n$ such that $x \leq_n x'$ and $y \leq_n y'$. We then have $\sigma^{-1}(x) \leq \sigma^{-1}(x')$ and $\sigma^{-1}(y) \leq \sigma^{-1}(y')$. Since F is $\leq$-preserving, we have

$$F(\sigma^{-1}(x), \sigma^{-1}(y)) \ \leq \ F(\sigma^{-1}(x'), \sigma^{-1}(y')),$$

that is, $F_\sigma(x, y) \leq_n F_\sigma(x', y')$. $\square$

Corollary 4.24 (see [34]). *Let $c = (c_1, \ldots, c_n)$ be a nondecreasing n-sequence. Then there exists an order-preservable operation $F \in \mathcal{Q}_n$ such that $|F^{-1}| = c$ if and only if both Eq.* (4.10) *and assertion (vii) of Proposition 4.23 hold.*

Proof. This result immediately follows from Propositions 4.19 and 4.23. □

For any $F, G \in \mathcal{Q}$ such that $\mathrm{sgn}(F) = \mathrm{sgn}(G)$, by Proposition 4.14 we have $\precsim_F \simeq \precsim_G$, and hence $\precsim_F$ is 2-quasilinear if and only if so is $\precsim_G$. By Proposition 4.23, it follows that F is order-preservable if and only if so is G. In particular, for any $\sigma \in \mathfrak{S}$, we have that F is order-preservable if and only if so is F_σ, as also mentioned in Proposition 4.23. This observation justifies the following terminology. For any order-preservable operation $F \in \mathcal{Q}$, we say that its signature $\mathrm{sgn}(F)$ and orbit $\mathrm{orb}(F)$ are *order-preservable*.

Let us assume for the rest of this section that $X = X_n$ for some integer $n \geq 1$. It is not difficult to see by inspection that all the signatures in $\mathcal{Q}_n$ are order-preservable when $n \leq 3$. For $n = 4$, only the signature $(1, 3)$ is not order-preservable. It consists of two non-order-preservable orbits and corresponds to the preimage sequence $(1, 5, 5, 5)$. Figure 4.6 (left) shows the contour plot of one of the eight non-order-preservable operations in $\mathcal{Q}_4$.

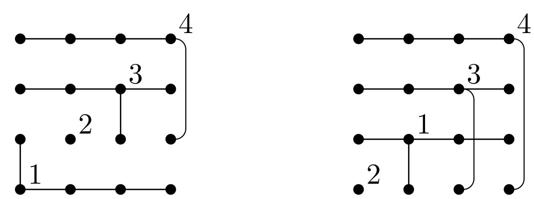

Figure 4.6: A non-order-preservable operation in $\mathcal{Q}_4$ (left) and its ordinal sum representation (right)

Let us now consider enumeration problems. For any integer $n \geq 0$, we denote by $\xi(n)$ the number of associative, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$. Also, we denote by

- $\xi_e(n)$ the number of associative, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ that have neutral elements,
- $\xi_a(n)$ the number of associative, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ that have annihilator elements,
- $\xi_{ae}(n)$ the number of associative, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ that have distinct neutral and annihilator elements.

As a convention, we set $\xi(0) = \xi_e(0) = \xi_a(0) = \xi_{ae}(0) = 0$. Propositions 4.25 and 4.26 below provide explicit formulas for these sequences. The first few values of these sequences are shown in Table 4.3.

Proposition 4.25 (see [25]). *The sequence $(\xi(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$\xi(n+2) - 2\,\xi(n+1) - 2\,\xi(n) \;=\; 2\,, \qquad n \geq 0,$$

with $\xi(0)=0$ and $\xi(1)=1$, and we have

$$\begin{aligned} 3\,\xi(n)+2 &= \tfrac{2+\sqrt{3}}{2}(1+\sqrt{3})^n+\tfrac{2-\sqrt{3}}{2}(1-\sqrt{3})^n \\ &= \textstyle\sum_{k\geq 0} 3^k\big(2\binom{n}{2k}+3\binom{n}{2k+1}\big)\,, \qquad n\geq 0. \end{aligned}$$

Moreover, its GF is given by $\Xi(z)=z(z+1)/(2z^3-3z+1)$.

Proof. We clearly have $\xi(0)=0$ and $\xi(1)=1$. So let us assume that $n\geq 2$. If $F\colon X_n^2\to X_n$ is an associative, quasitrivial, and $\leq_n$-preserving operation, then by Proposition 4.21 it is of the form (4.1) for some weak order $\precsim$ on X_n that is single-plateaued for $\leq_n$. By Lemma 1.30, either $\max_{\precsim} X_n = X_n$ or $\max_{\precsim} X_n=\{1\}$ or $\max_{\precsim} X_n=\{n\}$ or $\max_{\precsim} X_n=\{1,n\}$. In the first case we have to consider the two projections $F=\pi_1$ and $F=\pi_2$. In the three latter cases it is clear that the restriction of F to $(X_n\setminus\max_{\precsim} X_n)^2$ is associative, quasitrivial, and $\leq_n'$-preserving, where $\leq_n'$ is the restriction of $\leq_n$ to $X_n\setminus\max_{\precsim} X_n$. Also, in the last case we have to consider the two projections $F|_{\{1,n\}^2}=\pi_1|_{\{1,n\}^2}$ and $F|_{\{1,n\}^2}=\pi_2|_{\{1,n\}^2}$. It follows that the number $\xi(n)$ of associative, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2\to X_n$ satisfies the following second order linear equation

$$\xi(n) \;=\; 2+\xi(n-1)+\xi(n-1)+2\xi(n-2), \qquad n\geq 2.$$

The claimed expressions of $\xi(n)$ and the GF of $(\xi(n))_{n\geq 0}$ follow straightforwardly. $\square$

Proposition 4.26 (see [25]). *The sequence $(\xi_e(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$\xi_e(n+2)-2\,\xi_e(n+1)-2\xi_e(n) \;=\; 0\,, \qquad n\geq 0,$$

with $\xi_e(0)=0$ and $\xi_e(1)=1$, and we have

$$\xi_e(n) \;=\; \tfrac{\sqrt{3}}{6}(1+\sqrt{3})^n-\tfrac{\sqrt{3}}{6}(1-\sqrt{3})^n \;=\; \textstyle\sum_{k\geq 0}\binom{n}{2k+1}3^k\,, \qquad n\geq 0.$$

Moreover, its GF is given by $\Xi_e(z)=-z/(2z^2+2z-1)$. Furthermore, for any integer $n\geq 1$, we have $\xi_a(n)=2\xi(n-1)$, $\xi_{ae}(n)=2\xi_e(n-1)$, and $\xi_a(0)=\xi_{ae}(0)=0$.

Proof. The formula describing the sequence $(\xi_e(n))_{n\geq 0}$ is obtained by following the same steps as in the proof of Proposition 4.25, except that in this case we always have $\max_{\precsim} X_n\neq X_n$. As for the sequence $(\xi_a(n))_{n\geq 0}$ we note that $\max_{\precsim} X_n$ must be either $\{1\}$ or $\{n\}$ and that the restriction of F to $(X_n\setminus\max_{\precsim} X_n)^2$ is associative, quasitrivial, and $\leq_n'$-preserving, where $\leq_n'$ is the restriction of $\leq_n$ to $X_n\setminus\max_{\precsim} X_n$. We proceed similarly for the sequence $(\xi_{ae}(n))_{n\geq 0}$. $\square$

Now, for any integer $n\geq 0$, let $\gamma_{\mathrm{op}}(n)$ be the number of order-preservable operations $F\in\mathcal{Q}_n$. Let also $\mu_{\mathrm{op}}(n)$ be the number of order-preservable orbits in $\mathcal{Q}_n$ and let $\nu_{\mathrm{op}}(n)$ be the number of order-preservable signatures in $\mathcal{Q}_n$. By convention, we set $\gamma_{\mathrm{op}}(0)=\mu_{\mathrm{op}}(0)=\nu_{\mathrm{op}}(0)=1$. It is easy to see that $\nu_{\mathrm{op}}(n)=q_{\mathrm{iso}}(n)$ for every integer $n\geq 0$. The next three propositions provide explicit expressions for these sequences. Also, the first few values are given in Table 4.4.[5]

[5]Note that the sequences A000071 is a shifted version of $(\nu_{\mathrm{op}}(n))_{n\geq 0}$. More precisely, we have $\nu_{\mathrm{op}}(n)=$ A163271$(n+2)$ for every integer $n\geq 1$.

n	$\xi(n)$	$\xi_e(n)$	$\xi_a(n)$	$\xi_{ae}(n)$
0	0	0	0	0
1	1	1	0	0
2	4	2	2	2
3	12	6	8	4
4	34	16	24	12
5	94	44	68	32
6	258	120	188	88
OEIS	A293005	A002605	A293006	A293007

Table 4.3: First few values of $\xi(n)$, $\xi_e(n)$, $\xi_a(n)$, and $\xi_{ae}(n)$

Proposition 4.27 (see [34]). *The sequence $(\gamma_{\mathrm{op}}(n))_{n\geq 0}$ satisfies the second order linear recurrence equation*

$$\gamma_{\mathrm{op}}(n+2) ~=~ 2+(n+2)\,\gamma_{\mathrm{op}}(n+1)+(n+2)(n+1)\,\gamma_{\mathrm{op}}(n), \qquad n\geq 1,$$

with $\gamma_{\mathrm{op}}(1)=1$ and $\gamma_{\mathrm{op}}(2)=4$, and we have

$$\gamma_{\mathrm{op}}(n) ~=~ n!\,F_n+2\sum_{k=0}^{n-1}\frac{n!}{(n+1-k)!}\,F_k\,, \qquad n\geq 1,$$

where F_n is the nth Fibonacci number. Moreover, its EGF is given by

$$\widehat{\Gamma}_{\mathrm{op}}(z)=(2e^z-1-2z-z^2)/(1-z-z^2).$$

Proof. The proof is similar to that of Proposition 1.28. □

Proposition 4.28 (see [34]). *We have $\mu_{\mathrm{op}}(n)=2^n-1$ for any $n\geq 1$.*

Proof. We clearly have $\mu_{\mathrm{op}}(1)=1$ and $\mu_{\mathrm{op}}(2)=3$. To compute $\mu_{\mathrm{op}}(n)$ for $n\geq 3$, we proceed exactly as in the proof of Proposition 4.17, except that here we have $n_k\in\{1,2,n\}$. It follows that the sequence $\mu_{\mathrm{op}}(n)$ satisfies the second order linear recurrence equation

$$\mu_{\mathrm{op}}(n) ~=~ 2+\mu_{\mathrm{op}}(n-1)+2\,\mu_{\mathrm{op}}(n-2), \qquad n\geq 3.$$

The explicit expression for $\mu_{\mathrm{op}}(n)$ then follows immediately. □

4.4 Commutative, anticommutative, and bisymmetric operations

Recall that an operation $F\colon X^2\to X$ is said to be *bisymmetric* (or *medial*) [4, 63] if

$$F(F(x,y),F(u,v)) ~=~ F(F(x,u),F(y,v)), \qquad x,y,u,v\in X.$$

In that case, the groupoid (X,F) is also said to be *medial*. It is known [4] that any associative and commutative operation $F\colon X^2\to X$ is bisymmetric. Conversely, a bisymmetric operation

n	$\gamma_{\mathrm{op}}(n)$	$\mu_{\mathrm{op}}(n)$	$\nu_{\mathrm{op}}(n)$
0	1	1	1
1	1	1	1
2	4	3	2
3	20	7	4
4	130	15	7
5	1 052	31	12
6	10 214	63	20
OEIS	A307006	A255047	A000071

Table 4.4: First few values of $\gamma_{\mathrm{op}}(n)$, $\mu_{\mathrm{op}}(n)$, and $\nu_{\mathrm{op}}(n)$

is in general neither associative nor symmetric (for instance, consider the operation $F\colon \mathbb{R}^2 \to \mathbb{R}$ defined by $F(x,y) = \frac{1}{3}x + \frac{2}{3}y$ for any $x, y \in \mathbb{R}$). Thus, the class of medial groupoids generalizes the class of commutative semigroups. Medial groupoids have been extensively investigated in algebra (see, e.g., [59–63]). Also, the bisymmetry property for binary real operations was first studied by Aczél [2, 3]. Since then, it has been investigated in the theory of functional equations, especially in characterizations of mean functions (see, e.g., [4, 5, 48, 52]).

In this final section, we investigate the subclasses of $\mathcal{Q}$ defined by each of the following three properties: commutativity, anticommutativity, and bisymmetry. As far as commutative or anticommutative operations are concerned, we have the following two propositions. We first recall the following result.

Lemma 4.29 (see [76])**.** *If $F\colon X^2 \to X$ is quasitrivial, commutative, and $\leq$-preserving for some total order $\leq$ on X, then F is associative.*

Proposition 4.30 (see [34])**.** *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) $F \in \mathcal{Q}$ *and is commutative.*

(ii) F *is quasitrivial, order-preservable, and commutative.*

(iii) $F = \max_{\leq'}$ *for some total order* $\leq'$ *on* X.

If $X = X_n$ for some integer $n \geq 1$, then any of the assertions (i)–(iii) above is equivalent to any of the following ones.

(iv) $F \in \mathcal{Q}_n$ *and* $|\mathrm{orb}(F)| = n!$.

(v) $F \in \mathcal{Q}_n$ *and* $\mathrm{sgn}(F) = (1, \ldots, 1)$.

(vi) F *is quasitrivial and satisfies* $|F^{-1}| = (1, 3, 5, \ldots, 2n-1)$.

(vii) F *is associative, idempotent, order-preservable, commutative, and has a neutral element.*

Moreover, there are exactly $n!$ commutative operations $F \in \mathcal{Q}_n$.

Proof. The equivalence $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ follows from Theorem 4.1 and Lemma 4.29. We have $(ii) \Rightarrow (vii)$ by Fact 4.4 and Lemma 4.29. Moreover, we have $(vii) \Rightarrow (ii)$ by Theorem 3.8. Also, it is clear that $(iii) \Rightarrow (vi)$.

Let us now show by induction on n that $(vi) \Rightarrow (iii)$. The result clearly holds for $n = 1$. Suppose that it holds for some $n \geq 1$ and let us show that it still holds for $n+1$. Assume that $F\colon X_{n+1}^2 \to X_{n+1}$ is quasitrivial and that $|F^{-1}| = (1,3,5,\ldots,2n+1)$. Let $\leq'$ be the unique total order on X_{n+1} defined by $x \leq' y$ if and only if $|F^{-1}[x]| \leq |F^{-1}[y]|$ and let $z = \max_{\leq'} X_{n+1}$. Clearly, the operation $F' = F|_{(X_{n+1}\setminus\{z\})^2}$ is quasitrivial and $|F'^{-1}| = (1,3,5,\ldots,2n-1)$. By induction hypothesis we have $F' = \max_{\leq^\star}$, where $\leq^\star$ is the restriction of $\leq'$ to $X_{n+1}\setminus\{z\}$. Since $|F^{-1}[z]| = 2n+1$ we necessarily have $F = \max_{\leq'}$.

Also, the equivalence $(iv) \Leftrightarrow (v)$ immediately follows from Proposition 4.15. Finally, the equivalence $(iii) \Leftrightarrow (v)$ is trivial. The rest of the statement is straightforward. □

The following result follows from Remark 1.7, Theorem 4.1, and Proposition 4.21.

Corollary 4.31 (see [25]). *Let $\leq$ be a total order on X. An operation $F \in \mathcal{Q}$ is commutative and $\leq$-preserving if and only if $F = \max_{\leq'}$ for some total order $\leq'$ on X that is single-peaked for $\leq$.*

We can associate with any operation $F \in \mathcal{Q}$ a commutative operation $F^S \in \mathcal{Q}$ such that $\leq_{F^S}$ extends $\precsim_F$. In that case, we say that F^S is a *symmetrization of* F. Of course, a symmetrization is not necessarily unique. For instance, the operation $\pi_1\colon X_2^2 \to X_2$ has the two symmetrization $F_1^S, F_2^S \in \mathcal{Q}$ defined by the following conditions:

- $F_1^S(x,x) = F_2^S(x,x) = x$ for any $x \in X_2$,
- $F_1^S(1,2) = F_1^S(2,1) = 1$ and $F_2^S(1,2) = F_2^S(2,1) = 2$.

Actually, an operation $F \in \mathcal{Q}$ has a unique symmetrization if and only if $\precsim_F$ is a total order.

Now, let $\leq$ be a total order on X. It is clear that a symmetrization of an operation $F \in \mathcal{Q}$ is not necessarily $\leq$-preserving. The next proposition characterizes the class of operations $F \in \mathcal{Q}$ for which there exists a symmetrization that is $\leq$-preserving.

Proposition 4.32. *Let $F \in \mathcal{Q}$ and $\leq$ be a total order on X. Then F has a symmetrization that is $\leq$-preserving if and only if $\precsim_F$ is existentially single-peaked for $\leq$.*

Proof. This follows from Corollary 4.31. □

The following proposition will be useful for the characterization of the class of associative, quasitrivial, and anticommutative operations $F\colon X^2 \to X$.

Proposition 4.33 (see [31]). *An operation $F\colon X_n^2 \to X_n$ is quasitrivial, $\leq_n$-preserving, and satisfies $|F^{-1}| = (n,\ldots,n)$ if and only if $F = \pi_i$ for some $i \in \{1,2\}$.*

Proof. (Necessity) Since F is quasitrivial we know that $F(1,n) \in \{1,n\}$. Suppose that $F(1,n) = n = F(n,n)$ (the other case is similar). Since F is $\leq_n$-preserving, we have $F(x,n) = n$ for all $x \in X_n$. Since $|F^{-1}[n]| = n$, it follows that $F(n,y) = y$ for all $y \in X_n$. In particular, we have $F(n,1) = 1 = F(1,1)$, and by $\leq_n$-preservation we obtain $F(x,1) = 1$ for all $x \in X_n$. Finally, since $|F^{-1}[1]| = n$, it follows that $F(1,y) = y$ for all $y \in X_n$. Thus, since F is $\leq_n$-preserving, we have

$$y ~=~ F(1,y) ~\leq_n~ F(x,y) ~\leq_n~ F(n,y) ~=~ y, \quad x,y \in X_n,$$

which shows that $F = \pi_2$.

(Sufficiency) Obvious. □

Proposition 4.34 (see [34]). *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) $F \in \mathcal{Q}$ *and is anticommutative.*

(ii) F *is quasitrivial, order-preservable, and anticommutative.*

(iii) $F = \pi_1$ *or* $F = \pi_2$.

If $X = X_n$ *for some integer* $n \geq 1$*, then any of the assertions (i)–(iii) above is equivalent to any of the following ones.*

(iv) $F \in \mathcal{Q}_n$ *and* $|\mathrm{orb}(F)| = 1$.

(v) $F \in \mathcal{Q}_n$ *and* $\mathrm{sgn}(F) = (n)$.

(vi) F *is quasitrivial, order-preservable, and satisfies* $|F^{-1}| = (n, \ldots, n)$.

Proof. The equivalence $(i) \Leftrightarrow (iii)$ follows from Theorem 4.1. Also, the implications $(iii) \Rightarrow ((ii)$ and $(vi))$ are obvious. Let us show that $(ii) \Rightarrow (iii)$. Let $\leq$ be a total order on X for which F is $\leq$-preserving. Let $x, y \in X$ such that $x < y$. Since F is quasitrivial and anticommutative, it follows that $F|_{\{x,y\}^2} = \pi_1|_{\{x,y\}^2}$ or $F|_{\{x,y\}^2} = \pi_2|_{\{x,y\}^2}$. Suppose that $F|_{\{x,y\}^2} = \pi_1|_{\{x,y\}^2}$ (the other case is similar). Let $z \in X \setminus \{x, y\}$ and let us show that $F|_{\{x,z\}^2} = \pi_1|_{\{x,z\}^2}$. We then have the following discussion of cases.

- If $x < z < y$, then $x = F(x, x) \leq F(x, z) \leq F(x, y) = x$. We then have $F(x, z) = x$ and hence $F(z, x) = z$.
- If $y < z$, then $y = F(y, x) \leq F(z, x) \in \{x, z\}$. We then have $F(z, x) = z$ and hence $F(x, z) = x$.
- The case $z < x$ is similar to the previous one.

Therefore, we have $F|_{\{x,z\}^2} = \pi_1|_{\{x,z\}^2}$. Similarly, we can show that $F|_{\{u,v\}^2} = \pi_1|_{\{u,v\}^2}$ for any $u, v \in X$. Now, let us show that $(vi) \Rightarrow (iii)$. By Proposition 4.23, there exists $\sigma \in \mathfrak{S}_n$ such that F_σ is $\leq_n$-preserving. Clearly, F_σ is quasitrivial. Also, by Proposition 4.9 (using σ to define the graph isomorphism) we have that $|F_\sigma^{-1}| = (n, \ldots, n)$. Thus, we conclude the result by Fact 2.2 and Proposition 4.33. Finally, the equivalences $(iii) \Leftrightarrow (iv) \Leftrightarrow (v)$ follow from Proposition 4.15. □

Let us now investigate those operations in $\mathcal{Q}$ that are bisymmetric. The following theorem provides a characterization of the class of bisymmetric and quasitrivial operations $F\colon X^2 \to X$.

Theorem 4.35 (see [63]). *An operation $F\colon X^2 \to X$ is bisymmetric and quasitrivial if and only if there exists a subset $Y \subseteq X$ such that the following conditions hold.*

(i) $F|_{Y^2} = \pi_i|_{Y^2}$ *for some* $i \in \{1, 2\}$.

(ii) $F|_{(X \setminus Y)^2}$ *is associative, quasitrivial, and commutative.*

(iii) Any $x \in X \setminus Y$ *is an annihilator for* $F|_{(\{x\} \bigcup Y)^2}$.

Theorem 4.35 is of particular interest as it enables us to easily construct bisymmetric and quasitrivial operations $F\colon X^2 \to X$. For instance, consider the chain $(X_5, \leq_5)$ together with the operation $F\colon X_5^2 \to X_5$ defined by the following conditions:

- $F|_{\{1,2,3\}^2} = \pi_1|_{\{1,2,3\}^2}$,
- $F|_{\{4,5\}^2} = \max_{\leq_5}|_{\{4,5\}^2}$,
- for any $x \in \{1,2,3\}$, $G(x,4) = G(4,x) = 4$ and $G(x,5) = G(5,x) = 5$.

Then the operation $F\colon X_5^2 \to X_5$ is bisymmetric and quasitrivial by Theorem 4.35.

Remark 4.36. Let (X, F) be a semigroup. Recall that an *ideal* of (X, F) [21] is a non-empty subset $Y \subseteq X$ such that for any $x \in X$ and any $y \in Y$ we have $F(x,y), F(y,x) \in Y$. So let $Y \subseteq X$ be an ideal of (X, F). Then we can define the *Rees congruence* $\sim$ on (X, F) [88] by

$$x \sim y \quad \Leftrightarrow \quad x = y \quad \text{or} \quad x, y \in Y, \qquad x, y \in X.$$

The semigroup $(X/{\sim}, \tilde{F})$ is then called the *Rees factor semigroup* of (X, F) modulo Y [88]. Now, let (U, G) and (V, H) be two semigroups. Then (X, F) is said to be an *ideal extension* of (U, G) by (V, H) [79] if U is an ideal of (X, F) and the Rees factor semigroup $(X/{\sim}, \tilde{F})$ is isomorphic to (V, H).

Now, assume that the semigroup (X, F) is also quasitrivial and medial. Moreover, assume that (X, F) is neither a left zero semigroup nor a right zero semigroup. Then by Theorem 4.35, there exists a subset $Y \subseteq X$ such that $X \setminus Y$ is an ideal of (X, F), $(X \setminus Y, F|_{(X\setminus Y)^2})$ is a semigroup, and $(\{x\} \bigcup Y, F|_{(\{x\}\bigcup Y)^2})$ is a semigroup for any $x \in X \setminus Y$. It is then easy to see that the Rees factor semigroup $(X/{\sim}, \tilde{F})$ is isomorphic to $(\{x\} \bigcup Y, F|_{(\{x\}\bigcup Y)^2})$ for any $x \in X \setminus Y$. Thus, for any $x \in X \setminus Y$ we have that (X, F) is an ideal extension of $(X \setminus Y, F|_{(X\setminus Y)^2})$ by $(\{x\} \bigcup Y, F|_{(\{x\}\bigcup Y)^2})$.

In what follows, we provide alternative characterizations of the class of bisymmetric and quasitrivial operations $F\colon X^2 \to X$.

The following corollary, which follows from Theorem 4.35, shows that any bisymmetric and quasitrivial operation $F\colon X^2 \to X$ is associative. Here we provide a direct proof [24] that does not make use of Theorem 4.35.

Corollary 4.37 (see [63]). *If an operation $F\colon X^2 \to X$ is bisymmetric and quasitrivial, then it is associative.*

Proof. Let $x, y, z \in X$. By quasitriviality we have $F(x,z) \in \{x, z\}$. If $F(x,z) = x$, then

$$F(F(x,y),z) \;=\; F(F(x,y),F(z,z)) \;=\; F(F(x,z),F(y,z)) \;=\; F(x,F(y,z)).$$

If $F(x,z) = z$, then

$$F(F(x,y),z) \;=\; F(F(x,y),F(x,z)) \;=\; F(F(x,x),F(y,z)) \;=\; F(x,F(y,z)).$$

This shows that F is associative. □

Now, we introduce the notion of disconnected level set which will be useful in our alternative characterizations.

Definition 4.38 (see [31])**.** Let $\leq$ be a total order on X. We say that an operation $F\colon X^2 \to X$ has

- a *$\leq$-disconnected level set* if there exist $x, y, u, v, s, t \in X$, with $(x, y) < (u, v) < (s, t)$, such that $F(x, y) = F(s, t) \neq F(u, v)$.
- a *horizontal (resp. vertical) $\leq$-disconnected level set* if there exist $x, y, z, u \in X$, with $x < y < z$, such that $F(x, u) = F(z, u) \neq F(y, u)$ (resp. $F(u, x) = F(u, z) \neq F(u, y)$).

Thus, for any total order $\leq$ on X, an operation $F\colon X^2 \to X$ has no $\leq$-disconnected level set if and only if for any $x, y \in X$ the class of (x, y) for $\ker(F)$ is convex for $\leq$. For instance, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4.7 has a $\leq_3$-disconnected level set since $F(1, 1) = F(2, 3) = 1 \neq 2 = F(2, 2)$. Also, it has a horizontal and a vertical $\leq_3$-disconnected level set since $F(1, 3) = F(3, 3) = 3 \neq 1 = F(2, 3)$ and $F(3, 1) = F(3, 3) = 3 \neq 2 = F(3, 2)$.

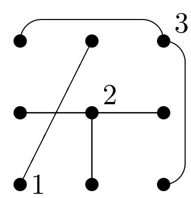

Figure 4.7: An operation on X_3 that has $\leq_3$-disconnected level sets

Fact 4.39 (see [31])**.** *Let $\leq$ be a total order on X. If $F\colon X^2 \to X$ has a horizontal or vertical $\leq$-disconnected level set, then it has a $\leq$-disconnected level set.*

Remark 4.40. We observe that, for any total order $\leq$ on X, an operation $F\colon X^2 \to X$ having a $\leq$-disconnected level set need not have a horizontal or vertical $\leq$-disconnected level set. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4.8 has a $\leq_3$-disconnected level set since $F(1, 1) = F(2, 3) = 1 \neq 2 = F(2, 2)$ but it has no horizontal or vertical $\leq_3$-disconnected level set.

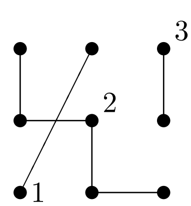

Figure 4.8: An idempotent operation on X_3

Lemma 4.41 (see [31])**.** *Let $\leq$ be a total order on X. If $F\colon X^2 \to X$ is quasitrivial, then it has a $\leq$-disconnected level set if and only if it has a horizontal or vertical $\leq$-disconnected level set.*

Proof. (Necessity) Suppose that F has a $\leq$-disconnected level set and let us show that it has a horizontal or vertical $\leq$-disconnected level set. By assumption, there exist $x, y, u, v, s, t \in X$, with $(x,y) < (u,v) < (s,t)$, such that $F(x,y) = F(s,t) \neq F(u,v)$. Since F is quasitrivial, we have $F(x,y) \in \{x,y\}$. Suppose that $F(x,y) = x$ (the other case is similar). Also, since F is quasitrivial, we have $s = x$ or $t = x$. If $s = x$, then $u = x$ and thus F has a vertical $\leq$-disconnected level set. Otherwise, if $t = x$ and $s \neq x$, then $y \leq x$. If $y = x$, then $v = x$ and thus F has a horizontal $\leq$-disconnected level set. Otherwise, if $y < x$, then considering the point $(s,y) \in X^2$, we get $(x,y) < (s,y) < (s,x)$ and $F(x,y) = F(s,x) = x \neq F(s,y) \in \{s,y\} = \{F(s,s), F(y,y)\}$, which shows that F has either a horizontal or a vertical $\leq$-disconnected level set.

(Sufficiency) This follows from Fact 4.39. □

Remark 4.42. (a) For any quasitrivial operation $F\colon X^2 \to X$ and any $x \in X$, consider the sets

$$L_x^h(F) = \{y \in X\colon F(y,x) = x\} \quad \text{and} \quad L_x^v(F) = \{y \in X\colon F(x,y) = x\}.$$

Clearly, for any total order $\leq$ on X, a quasitrivial operation $F\colon X^2 \to X$ has no $\leq$-disconnected level set if and only if for any $x \in X$, the sets $L_x^h(F)$ and $L_x^v(F)$ are convex for $\leq$.

(b) It is not difficult to see that for any total order $\leq$ on X, a quasitrivial operation $F\colon X^2 \to X$ has no $\leq$-disconnected level set if and only if for any $x \in X$, the class of (x,x) for $\ker(F)$ is convex for $\leq$.

Fact 4.43 (see [31]). *Let $\leq$ be a total order on X. If $F\colon X^2 \to X$ is $\leq$-preserving then it has no $\leq$-disconnected level set.*

Proposition 4.44 (see [31]). *Let $\leq$ be a total order on X. If $F\colon X^2 \to X$ is quasitrivial, then it is $\leq$-preserving if and only if it has no $\leq$-disconnected level set.*

Proof. (Necessity) This follows from Fact 4.43.

(Sufficiency) Suppose that F has no $\leq$-disconnected level set and let us show by contradiction that F is $\leq$-preserving. Suppose for instance that there exist $x, y, z \in X$ with $y < z$ such that $F(x,y) > F(x,z)$. By quasitriviality we see that $x \notin \{y,z\}$. Suppose for instance that $x < y < z$ (the other cases are similar). By quasitriviality we have $F(x,y) = y$ and $F(x,z) = x = F(x,x)$, and hence by Lemma 4.41, F has a $\leq$-disconnected level set, a contradiction. □

Remark 4.45. We cannot relax quasitriviality into idempotency in Proposition 4.44. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4.9 is idempotent and has no $\leq_3$-disconnected level set. However it is not $\leq_3$-preserving.

Now, assume that $X = X_n$ for some integer $n \geq 1$. The following results about the annihilator of a quasitrivial operation $F\colon X_n^2 \to X_n$ will be useful in order to characterize the class of bisymmetric and quasitrivial operations by means of preimage sequences.

Lemma 4.46 (see [31]). *If $F\colon X_n^2 \to X_n$ is quasitrivial, then $|F^{-1}[x]| \leq 2n-1$ for all $x \in X_n$.*

Proof. This follows from the quasitriviality of F. □

The following result was mentioned in [24, Section 2] without proof.

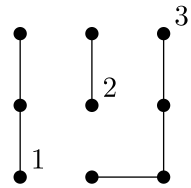

Figure 4.9: An idempotent operation on X_3

Proposition 4.47 (see [31]). *Let $F\colon X_n^2 \to X_n$ be a quasitrivial operation and let $a \in X_n$. Then a is an annihilator of F if and only if $|F^{-1}[a]| = 2n-1$.*

Proof. (Necessity) By definition of an annihilator, we have $F(x,a) = F(a,x) = a$ for all $x \in X_n$. Thus, we have $|F^{-1}[a]| \geq 2n-1$ and hence by Lemma 4.46 we conclude that $|F^{-1}[a]| = 2n-1$.

(Sufficiency) This follows from the quasitriviality of F. □

Remark 4.48. We observe that Proposition 4.47 no longer holds if we relax quasitriviality into idempotency. Indeed, the operation $F\colon X_3^2 \to X_3$ whose contour plot is depicted in Figure 4.10 is idempotent and the element $a = 1$ is the annihilator of F. However, $|F^{-1}[1]| = 7 > 5$.

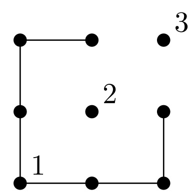

Figure 4.10: An idempotent operation with an annihilator on X_3

Lemma 4.49 (see [31]). *Let $F\colon X_n^2 \to X_n$ be a quasitrivial and $\leq_n$-preserving operation and let $a \in X_n$. If a is an annihilator of F, then $a \in \{1,n\}$.*

Proof. We proceed by contradiction. Suppose that $a \in X_n \setminus \{1,n\}$. Since F is quasitrivial, we have $F(1,n) \in \{1,n\}$. Suppose that $F(1,n) = 1 = F(1,1)$ (the other case is similar). Then

$$1 \;=\; F(1,1) \;\leq_n\; F(1,a) \;\leq_n\; F(1,n) \;=\; 1,$$

and hence $F(1,a) = 1$ which contradicts the fact that a is an annihilator. □

Proposition 4.50 (see [31]). *Let $F\colon X_n^2 \to X_n$ be quasitrivial and $\leq_n$-preserving. Then F is bisymmetric if and only if there exists $\ell \in \{1,\ldots,n\}$ such that*

$$|F^{-1}| \;=\; (\underbrace{\ell,\ldots,\ell}_{\ell},2\ell+1,2\ell+3,\ldots,2n-1). \tag{4.11}$$

Proof. (Necessity) This follows from Proposition 4.2, Theorem 4.35, and Corollary 4.37.

(Sufficiency) We proceed by induction on n. The result clearly holds for $n = 1$. Suppose that it holds for some $n \geq 1$ and let us show that it still holds for $n+1$. Assume that $F\colon X_{n+1}^2 \to X_{n+1}$ is quasitrivial, $\leq_{n+1}$-preserving, and satisfies

$$|F^{-1}| = (\underbrace{\ell, \ldots, \ell}_{\ell}, 2\ell+1, 2\ell+3, \ldots, 2n+1).$$

for some $\ell \in \{1, \ldots, n+1\}$. If $\ell = n+1$ then by Proposition 4.33 we have that $F = \pi_1$ or $F = \pi_2$ and hence F is clearly bisymmetric. Otherwise, if $\ell \in \{1, \ldots, n\}$ then, by the form of the preimage sequence of F, there exists an element $a \in X_{n+1}$ such that $|F^{-1}[a]| = 2n+1$. Using Proposition 4.47 we have that a is an annihilator of F. Moreover, by Lemma 4.49, we have $a \in \{1, n+1\}$. Suppose that $a = n+1$ (the other case is similar). Then, $F' = F|_{X_n^2}$ is clearly quasitrivial, $\leq_n$-preserving, and satisfies (4.11). Thus, by induction hypothesis, F' is bisymmetric. Since $a = n+1$ is the annihilator of F, we necessarily have that F is bisymmetric. $\square$

We now have the following proposition which provides characterizations of the class of bisymmetric and quasitrivial operations.

Proposition 4.51 (see [31, 34])**.** *Let $F\colon X^2 \to X$ be an operation. The following assertions are equivalent.*

(i) *F is bisymmetric and quasitrivial.*

(ii) *$F \in \mathcal{Q}$ and $\precsim_F$ is quasilinear.*

(iii) *$F \in \mathcal{Q}$ and is $\leq$-preserving for every total order $\leq$ on X that extends $\precsim_F$.*

(iv) *$F \in \mathcal{Q}$ and has no $\leq$-disconnected level set for every total order $\leq$ on X that extends $\precsim_F$.*

If $X = X_n$ for some integer $n \geq 1$, then any of the assertions (i)–(iv) above is equivalent to any of the following ones.

(v) *$F \in \mathcal{Q}_n$ and there exists $\ell \in \{1, \ldots, n\}$ such that*

$$\operatorname{sgn}(F) = (\ell, \underbrace{1, \ldots, 1}_{n-\ell}).$$

(vi) *$F \in \mathcal{Q}_n$ and satisfies* (4.11) *for some $\ell \in \{1, \ldots, n\}$.*

(vii) *F is quasitrivial, order-preservable, and satisfies* (4.11) *for some $\ell \in \{1, \ldots, n\}$.*

Proof. The implication $(i) \Rightarrow (ii)$ follows from Theorem 4.35. Also, the implication $(ii) \Rightarrow (iii)$ follows from Propositions 1.18 and 4.21. Moreover, the implication $(iii) \Rightarrow (iv)$ follows from Proposition 4.44. Now, let us show that $(iv) \Rightarrow (i)$. We proceed by contradiction. Suppose that F has no $\leq$-disconnected level set for any total order $\leq$ on X that extends $\precsim_F$. Suppose also that there exist pairwise distinct $a, b, c \in X$, such that $a \prec_F b \sim_F c$. Fix a total order $\leq'$ on X that extends $\precsim_F$. Suppose that $a <' b <' c$ (the other case is similar). If $F|_{[b]^2_{\sim_F}} = \pi_1|_{[b]^2_{\sim_F}}$, then

$$F(a, c) = F(c, c) = c \neq b = F(b, c),$$

which contradicts the fact that F has no $\leq'$-disconnected level set. The case where $F|_{[b]^2_{\precsim}} = \pi_2|_{[b]^2_{\precsim}}$ is similar. The equivalence $(ii) \Leftrightarrow (v)$ and the implication $(vi) \Rightarrow (vii)$ follow from Theorem 4.1. Also, the implication $(iv) \Rightarrow (vii)$ follows from Theorem 4.1 and Propositions 1.18 and 4.21. Moreover, the implication $(ii) \Rightarrow (vi)$ follows from Theorem 4.1.

Let us now prove that $(vii) \Rightarrow (i)$. By Proposition 4.23, there exists $\sigma \in \mathfrak{S}_n$ such that F_σ is $\leq_n$-preserving. Clearly, F_σ is quasitrivial. Also, by Proposition 4.9 (using σ to define the graph isomorphism) we have that

$$|F_\sigma^{-1}| \;=\; (\underbrace{\ell,\ldots,\ell}_{\ell}, 2\ell+1, 2\ell+3, \ldots, 2n-1).$$

Now, using Proposition 4.50 it follows that F_σ is bisymmetric and quasitrivial, and hence so is F. □

Remark 4.52. We observe that Proposition 1.18 can also be easily established by using Theorem 4.1 and Propositions 4.21 and 4.51.

The following proposition follows from Corollary 4.37 and Propositions 4.21 and 4.51.

Proposition 4.53. *Let $\leq$ be a total order on X. An operation $F\colon X^2 \to X$ is bisymmetric, quasitrivial, and $\leq$-preserving if and only if F is of the form* (4.1) *for some quasilinear weak order $\precsim$ on X that is single-plateaued for $\leq$.*

Let us now consider enumeration problems. For any integer $n \geq 0$ we denote by $\chi(n)$ the number of bisymmetric and quasitrivial operations $F\colon X_n^2 \to X_n$. We also denote by $\chi_e(n)$ (resp. $\chi_a(n)$) the number of bisymmetric and quasitrivial operations $F\colon X_n^2 \to X_n$ that have neutral elements (resp. annihilator elements). By convention, we set $\chi(0) = \chi_e(0) = \chi_a(0) = 0$. Proposition 4.54 provides explicit formulas for these sequences. The first few values of these sequences are shown in Table 4.5.[6]

Proposition 4.54. *The sequence $(\chi(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$\chi(n+1) - (n+1)\chi(n) \;=\; 2, \qquad n \geq 1,$$

with $\chi(0) = 0$ and $\chi(1) = 1$, and we have the closed-form expression

$$\chi(n) \;=\; 2r(n) - n! \;=\; n!\left(2\sum_{i=1}^{n}\frac{1}{i!} - 1\right), \qquad n \geq 1.$$

Moreover, its EGF is given by $\hat{X}(z) = (2e^z - z - 2)/(1-z)$. Furthermore, for any integer $n \geq 1$ we have $\chi_e(n) = n!$, with $\chi_e(0) = 0$. Also, for any integer $n \geq 2$ we have $\chi_a(n) = \chi(n) - 2$, with $\chi_a(0) = 0$ and $\chi_a(1) = 1$.

Proof. It is not difficult to see that the number of bisymmetric and quasitrivial operations on X_n is given by

$$\chi(n) \;=\; 2r(n) - n! \;=\; n!\left(2\sum_{i=1}^{n}\frac{1}{i!} - 1\right), \qquad n \geq 1.$$

[6] Note that the sequence A000142 differs from $(\chi_e(n))_{n\geq 0}$ only at $n = 0$.

Indeed, since $F\colon X_n^2 \to X_n$ is bisymmetric and quasitrivial, we have by Proposition 4.51 that F is of the form (4.1) for some quasilinear weak order $\precsim_F$ on X_n. Since $F|_{(\min_{\precsim_F} X_n)^2} = \pi_1|_{(\min_{\precsim} X_n)^2}$ or $\pi_2|_{(\min_{\precsim} X_n)^2}$, we have to count twice the number of k-element subsets of X_n, for every $k \in \{1,\ldots,n\}$. However, the number of total orders on X_n should be counted only once (indeed, by Propositions 4.30 and 4.51 there is a one-to-one correspondence between total orders and bisymmetric, commutative, and quasitrivial operations on X_n). Hence, $\chi(n) = 2r(n) - n!$. The claimed linear recurrence equation and the EGF of $(\chi(n))_{n\geq 0}$ follow straightforwardly. Using Proposition 4.51, we observe that the sequence $(\chi_e(n))_{n\geq 0}$, with $\chi_e(0) = 0$, gives the number of total orders on X_n. Finally, regarding the sequence $(\chi_a(n))_{n\geq 0}$, we observe that $\max_{\precsim_F} X_n \neq X_n$ whenever $n \geq 2$. □

n	$\chi(n)$	$\chi_e(n)$	$\chi_a(n)$
0	0	0	0
1	1	1	1
2	4	2	2
3	14	6	12
4	58	24	56
5	292	120	290
6	1754	720	1752
OEIS	A296943	A000142	A296944

Table 4.5: First few values of $\chi(n)$, $\chi_e(n)$, and $\chi_a(n)$

For any integer $n \geq 0$ we denote by $\theta(n)$ the number of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$. We also denote by $\theta_e(n)$ (resp. $\theta_a(n)$) the number of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ that have neutral elements (resp. annihilator elements). By convention, we set $\theta(0) = \theta_e(0) = \theta_a(0) = 0$. Proposition 4.55 provides explicit formulas for these sequences. The first few values of these sequences are shown in Table 4.6.

Proposition 4.55. *The sequence $(\theta(n))_{n\geq 0}$ satisfies the linear recurrence equation*

$$\theta(n+1) \;=\; 2\theta(n) + 2, \qquad n \geq 1,$$

with $\theta(0) = 0$ and $\theta(1) = 1$, and we have the closed-form expression

$$\theta(n) \;=\; 3 \cdot 2^{n-1} - 2, \qquad n \geq 1.$$

Moreover, its GF is given by $\Theta(z) = z(z+1)/(2z^2-3z+1)$. Furthermore, for any integer $n \geq 1$ we have $\theta_e(n) = 2^{n-1}$ with $\theta_e(0) = 0$. Also, for any integer $n \geq 2$ we have $\theta_a(n) = \theta(n) - 2$ with $\theta_a(0) = 0$ and $\theta_a(1) = 1$.

Proof. It is not difficult to see that the number of bisymmetric, quasitrivial, and $\leq_n$-preserving operations on X_n is given by

$$\theta(n) \;=\; 2u(n) - 2^{n-1} \;=\; 3 \cdot 2^{n-1} - 2, \qquad n \geq 1.$$

Indeed, since $F\colon X_n^2 \to X_n$ is bisymmetric, quasitrivial, and $\leq_n$-preserving, we have by Proposition 4.53 that F is of the form (4.1) for some quasilinear weak order $\precsim_F$ on X_n that is single-plateaued for $\leq_n$. Since $F|_{(\min_{\precsim_F} X_n)^2} = \pi_1|_{(\min_{\precsim_F} X_n)^2}$ or $\pi_2|_{(\min_{\precsim_F} X_n)^2}$, we have to count twice the number of k-element subsets of X_n, for every $k \in \{1,\ldots,n\}$. However, the number of total orders on X_n that are single-peaked for $\leq_n$ should be counted only once (indeed, by Propositions 4.30 and 4.53 there is a one-to-one correspondence between total orders that are single-peaked for $\leq_n$ and bisymmetric, commutative, quasitrivial, and $\leq_n$-preserving operations on X_n). Hence, $\theta(n) = 2u(n) - 2^{n-1}$. The claimed linear recurrence equation and the GF of $(\theta(n))_{n\geq 0}$ follow straightforwardly. Using Fact 4.4 and Proposition 4.53, we observe that the sequence $(\theta_e(n))_{n\geq 0}$, with $\theta_e(0) = 0$, gives the number of total orders on X_n that are single-peaked for $\leq_n$. Finally, regarding the sequence $(\theta_a(n))_{n\geq 0}$, we observe that $\max_{\precsim_F} X_n \neq X_n$ whenever $n \geq 2$. □

Remark 4.56. We observe that an alternative characterization of the class of bisymmetric, quasitrivial, and $\leq_n$-preserving operations $F\colon X_n^2 \to X_n$ was obtained in [65]. Also, the explicit expression of $\theta(n)$ as stated in Proposition 4.55 was independently obtained in [65] by means of a totally different approach.

n	$\theta(n)$	$\theta_e(n)$	$\theta_a(n)$
0	0	0	0
1	1	1	1
2	4	2	2
3	10	4	8
4	22	8	20
5	46	16	44
6	94	32	92
OEIS	A296953	A131577	A296954

Table 4.6: First few values of $\theta(n)$, $\theta_e(n)$, and $\theta_a(n)$

Example 4.57. We show in Figure 4.11 the $\chi(3) = 14$ bisymmetric and quasitrivial operations on X_3. Among these operations, $\chi_e(3) = 6$ have neutral elements, $\chi_a(3) = 12$ have annihilator elements, and $\theta(3) = 10$ are $\leq_3$-preserving.

For any $F, G \in \mathcal{Q}$ such that $\mathrm{sgn}(F) = \mathrm{sgn}(G)$, by Proposition 4.51 we have that F is bisymmetric if and only if so is G. Thus, for any bisymmetric operation in $\mathcal{Q}$, we can say that its signature and orbit are *bisymmetric*.

For any integer $n \geq 0$, let $\mu_{\mathrm{b}}(n)$ be the number of bisymmetric orbits in $\mathcal{Q}_n$ and let $\nu_{\mathrm{b}}(n)$ be the number of bisymmetric signatures in $\mathcal{Q}_n$. By convention, we set $\mu_{\mathrm{b}}(0) = \nu_{\mathrm{b}}(0) = 1$.

Proposition 4.58. *We have* $\mu_{\mathrm{b}}(n) = 2n - 1$ *and* $\nu_{\mathrm{b}}(n) = n$ *for any* $n \geq 1$.

Proof. We clearly have $\mu_{\mathrm{b}}(1) = 1$. To compute $\mu_{\mathrm{b}}(n)$ for $n \geq 2$, we proceed exactly as in the proof of Proposition 4.17, except that here we have $n_k \in \{1, n\}$. It follows that the sequence $\mu_{\mathrm{b}}(n)$ satisfies the first order linear recurrence equation

$$\mu_{\mathrm{b}}(n) \;=\; 2 + \mu_{\mathrm{b}}(n-1), \qquad n \geq 2.$$

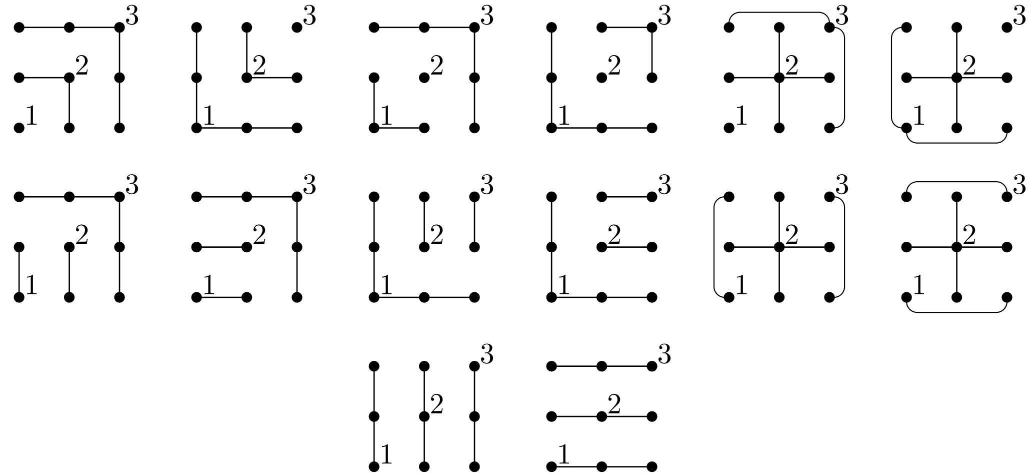

Figure 4.11: The 14 bisymmetric and quasitrivial operations on X_3

The explicit expression for $\mu_{\mathrm{b}}(n)$ then follows immediately.

Let us now consider the sequence $\nu_{\mathrm{b}}(n)$. We clearly have $\nu_{\mathrm{b}}(1) = 1$. Let $n \geq 2$. We know by Proposition 4.14 that $\nu_{\mathrm{b}}(n)$ is also the number of quasilinear weak orders on X_n that are defined up to an isomorphism. Thus, proceeding as in the proof of Proposition 1.28, we see that the sequence $\nu_{\mathrm{b}}(n)$ satisfies the first order linear recurrence equation

$$\nu_{\mathrm{b}}(n) \ = \ 1 + \nu_{\mathrm{b}}(n-1), \qquad n \geq 2.$$

The explicit expression for $\nu_{\mathrm{b}}(n)$ then follows immediately. □

Part II

Idempotent n-ary semigroups

Chapter 5

Quasitrivial n-ary semigroups

In this chapter we characterize the class of associative and quasitrivial n-ary operations $F\colon X^n \to X$ and show that all these operations are reducible to binary associative operations (Section 5.1). In particular, we provide necessary and sufficient conditions that ensure the existence of a unique and quasitrivial binary reduction (Section 5.2). In the case when X is finite, we also provide several enumeration results that explicitly determine the sizes of the corresponding classes of associative and quasitrival n-ary operations in terms of the size of the underlying set X. As a by-product, these enumeration results led to several integer sequences that were previously unknown in the Sloane's On-Line Encyclopedia of Integer Sequences (OEIS, see [94]). These results are further refined in the case of bisymmetric and symmetric operations (Section 5.3). Most of the contributions presented in this chapter stem from [22, 26, 33].

Throughout the rest of this manuscript, X is a non-empty set and $n \geq 2$ is an integer. Also, for every integer $k \geq 1$ we denote the finite set $\{1, \ldots, k\}$ by X_k.

5.1 Motivating results

In this section we first recall the result of Ackerman [1] which states that almost every quasitrivial n-ary semigroup is reducible to a semigroup. Then, using a result of Dudek and Mukhin [43], we show that any quasitrivial n-ary semigroup is reducible to a semigroup. Finally, when the underlying set is finite, we present some geometric results for quasitrivial n-ary operations.

An n-ary operation $F\colon X^n \to X$ is said to be *associative* if

$$\begin{aligned} &F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\ &\qquad = F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}), \end{aligned}$$

for all $x_1, \ldots, x_{2n-1} \in X$ and all $1 \leq i \leq n-1$. The pair (X, F) is then called an *n-ary semigroup*. This notion actually stems back to Dörnte [44] and has led to the concept of n-ary group,[1] which was first studied by Post [82]. The study of the classes of n-ary semigroups and n-ary groups has gained an increasing interest since then (see, e.g., [23, 37, 38, 40–43, 51, 66, 68–70, 75]).

In [43] the authors investigated associative n-ary operations that are determined by binary associative operations. An n-ary operation $F\colon X^n \to X$ is said to be *reducible to* an associative

[1] Recall that an n-ary group is an n-ary semigroup (X, F) such that for any $i \in \{1, \ldots, n\}$ and any $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y \in X$ there exists a unique $z \in X$ such that $F(x_1, \ldots, x_{i-1}, z, x_{i+1}, \ldots, x_n) = y$.

binary operation $G\colon X^2 \to X$ if there are $G^m\colon X^{m+1} \to X$ ($m = 1, \ldots, n-1$) such that $G^{n-1} = F$, $G^1 = G$, and

$$G^m(x_1, \ldots, x_{m+1}) \;=\; G^{m-1}(x_1, \ldots, x_{m-1}, G(x_m, x_{m+1})), \qquad m \geq 2.$$

In that case, F and (X, F) are said to be the *n-ary extensions* of G and (X, G), respectively. Moreover, G and (X, G) are said to be *binary reductions* of F and (X, F), respectively. Also, for simplicity's sake, we often say that (X, F) is reducible to (X, G).

It is easy to see that the n-ary extension of a semigroup is an n-ary semigroup. However, there are n-ary semigroups that are not constructed this way. For instance, if $n \geq 3$ is odd, then the operation $F\colon \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x_1, \ldots, x_n) \;=\; \sum_{i=1}^{n} (-1)^{i-1} x_i, \qquad x_1, \ldots, x_n \in \mathbb{R},$$

is associative but is not reducible to a binary associative operation (see, e.g., [75]).

An n-ary operation $F\colon X^n \to X$ is said to be

- *quasitrivial*[2] if $F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}$ for all $x_1, \ldots, x_n \in X$.
- *idempotent* if $F(x, \ldots, x) = x$ for all $x \in X$.
- *symmetric* if $F(x_1, \ldots, x_n)$ is invariant under any permutation of $x_1, \ldots, x_n$.

Remark 5.1. Quasitrivial n-ary operations are exactly those n-ary operations that preserve all unary relations.

Recall that a *neutral element* for $F\colon X^n \to X$ is an element $e_F \in X$ such that

$$F((i-1) \cdot e_F, x, (n-i) \cdot e_F) \;=\; x,$$

for all $x \in X$ and all $i \in \{1, \ldots, n\}$. When the meaning is clear from the context, we may drop the index F and denote a neutral element for F by e. Here and throughout, for any $m \in \{0, \ldots, n\}$ and any $x \in X$, the notation $m \cdot x$ stands for the m-tuple $x, \ldots, x$. For instance, we have

$$F(3 \cdot x, 0 \cdot y, 2 \cdot z) \;=\; F(x, x, x, z, z).$$

Throughout the rest of this manuscript we also denote the set of neutral elements for an operation $F\colon X^n \to X$ by E_F. An n-ary semigroup that has a neutral element is called an *n-ary monoid*.

The quest for conditions under which an associative n-ary operation is reducible to an associative binary operation gained an increasing interest since the pioneering work of Post [82] (see, e.g., [1, 23, 39, 43, 66, 70, 74, 75]). For instance, Dudek and Mukhin [43] proved that an associative operation $F\colon X^n \to X$ is reducible to an associative binary operation if and only if one can *adjoin to X a neutral element e for F*; that is, there is an n-ary associative operation $F^*\colon\ (X \cup \{e\})^n \to X \cup \{e\}$ such that e is a neutral element for F^* and $F^*|_{X^n} = F$. In this case, a binary reduction G_e of F can be defined by

$$G_e(x, y) \;=\; F^*(x, (n-2) \cdot e, y) \quad x, y \in X.$$

[2]Quasitrivial operations are also called conservative operations [83]. This property has been extensively used in the classification of constraint satisfaction problems into complexity classes (see, e.g, [12] and the references therein).

However, it is usually difficult to see whether one can adjoin to X a neutral element for an associative operation $F\colon X^n \to X$. Therefore, it is interesting to investigate other necessary and sufficient conditions under which an associative n-ary operation is reducible to an associative binary operation. In this respect, Ackerman [1] also investigated reducibility criteria for n-ary associative and quasitrivial operations. We first need to introduce the concept of reducibility to ternary associative operations.

Definition 5.2 (see [1]). An operation $F\colon X^n \to X$ is said to be *reducible to* a ternary associative operation $H\colon X^3 \to X$ if n is odd and there are $H^m\colon X^{m+3} \to X$ ($m = 0, \ldots, n-3$ even) such that $H^{n-3} = F$, $H^0 = H$, and

$$H^m(x_1, \ldots, x_{m+3}) \;=\; H^{m-2}(x_1, \ldots, x_m, H(x_{m+1}, x_{m+2}, x_{m+3})), \qquad m \geq 2.$$

In this case, H is said to be a *ternary reduction of* F.

Theorem 5.3 (see [1]). *Let $F\colon X^n \to X$ be an associative and quasitrivial operation.*

(a) F is reducible to an associative and quasitrivial binary operation $G\colon X^2 \to X$ whenever n is even.

(b) F is reducible to an associative and quasitrivial ternary operation $H\colon X^3 \to X$ whenever n is odd.

(c) If $n = 3$ and F is not reducible to an associative binary operation $G\colon X^2 \to X$, then there exist $a_1, a_2 \in X$ with $a_1 \neq a_2$ such that

- *$F|_{(X\setminus\{a_1,a_2\})^3}$ is reducible to an associative binary operation.*
- *a_1 and a_2 are neutral elements for F.*

From Theorem 5.3 (c) it follows that if an associative and quasitrivial operation $F\colon X^n \to X$ is not reducible to an associative binary operation $G\colon X^2 \to X$, then n is odd and there exist distinct $a_1, a_2 \in X$ that are neutral elements for F.

However, Theorem 5.3 (c) supposes the existence of a ternary associative and quasitrivial operation $H\colon X^3 \to X$ that is not reducible to an associative binary operation, and Ackerman did not provide any example of such an operation. In what follows, we show that there is no associative and quasitrivial n-ary operation that is not reducible to an associative binary operation (Corollary 5.6). Hence, for any associative and quasitrivial operation $F\colon X^n \to X$ one can adjoin a neutral element to X.

Throughout the rest of this manuscript we denote the set of all constant n-tuples over X by $\Delta_X^n = \{(n \cdot y)\colon\ y \in X\}$.

As we will see, every associative and quasitrivial operation $F\colon X^n \to X$ is reducible to an associative binary operation. To show this, we will make use of the following auxiliary result.

Lemma 5.4 (see [43]). *If $F\colon X^n \to X$ is associative and has a neutral element $e \in X$, then F is reducible to the associative operation $G_e\colon X^2 \to X$ defined by*

$$G_e(x, y) = F(x, (n-2) \cdot e, y), \qquad x, y \in X. \tag{5.1}$$

Moreover, e is the neutral element of G_e.

Using this result, we can also show that all reductions of an associative operation $F\colon X^n \to X$ obtained from neutral elements are conjugate to each other. For instance, the ternary sum on $\mathbb{Z}_2$ has two neutral elements, namely 0 and 1. By Lemma 5.4 it is reducible to the operations $G_0, G_1\colon \mathbb{Z}_2^2 \to \mathbb{Z}_2$ defined by $G_0(x,y) = x + y \pmod 2$ and $G_1(x,y) = x + y + 1 \pmod 2$. It is easy to see that the semigroups $(\mathbb{Z}_2, G_0)$ and $(\mathbb{Z}_2, G_1)$ are isomorphic.

Proposition 5.5 (see [26]). *Let $F\colon X^n \to X$ ($n \geq 3$) be an associative operation such that $E_F \neq \varnothing$. If $e_1, e_2 \in E_F$, then (X, G_{e_1}) and (X, G_{e_2}) are isomorphic.*

Proof. The definition of neutral elements as well as the associativity of F ensure that the map $\psi\colon X \to X$ defined by

$$\psi(x) \; = \; F(e_2, x, (n-2)\cdot e_1),$$

is a bijection and that $\psi^{-1}(x) = F((n-2)\cdot e_2, x, e_1)$. We then have

$$\begin{aligned} & G_{e_2}(\psi(x), \psi(y)) \\ & = \; F(F(e_2, x, (n-2)\cdot e_1), (n-2)\cdot e_2, F(e_2, y, (n-2)\cdot e_1)) \\ & = \; F(F(e_2, x, (n-2)\cdot e_1), F((n-1)\cdot e_2, y), (n-2)\cdot e_1) \\ & = \; F(F(e_2, x, (n-2)\cdot e_1), y, (n-2)\cdot e_1) \\ & = \; F(e_2, F(x, (n-2)\cdot e_1, y), (n-2)\cdot e_1) \\ & = \; \psi(G_{e_1}(x,y)), \end{aligned}$$

which completes the proof. □

The following corollary follows from Theorem 5.3 and Lemma 5.4.

Corollary 5.6 (see [22]). *Every associative and quasitrivial operation $F\colon X^n \to X$ is reducible to an associative binary operation.*

Theorem 5.3(c) states that a ternary associative and quasitrivial operation $H\colon X^3 \to X$ must have two neutral elements, whenever it is not reducible to a binary operation. In particular, we can show that two distinct elements $a_1, a_2 \in X$ are neutral elements for H if and only if they are neutral elements for the restriction $H|_{\{a_1,a_2\}^3}$ of H to $\{a_1, a_2\}^3$. Indeed, the condition is obviously necessary, while its sufficiency follows from the Lemma 5.7 below.

Lemma 5.7 (see [22]). *Let $H\colon X^3 \to X$ be an associative and quasitrivial operation.*

(a) If $a_1, a_2 \in X$ are two distinct neutral elements for $H|_{\{a_1,a_2\}^3}$, then

$$H(a_1, a_1, x) \; = \; H(x, a_1, a_1) \; = \; x \; = \; H(x, a_2, a_2) \; = \; H(a_2, a_2, x), \qquad x \in X.$$

(b) If $a_1, a_2 \in X$ are two distinct neutral elements for $H|_{\{a_1,a_2\}^3}$, then both a_1 and a_2 are neutral elements for H.

Proof. (a) Let $x \in X$. We only show that $H(a_1, a_1, x) = x$, since the other equalities can be shown similarly. Clearly, the equality holds when $x \in \{a_1, a_2\}$. So let $x \in X \setminus \{a_1, a_2\}$ and, for a contradiction, suppose that $H(a_1, a_1, x) = a_1$. By the associativity and quasitriviality of H, we then have

$$\begin{aligned} a_1 \; &= \; H(a_1, a_1, x) \; = \; H(a_1, H(a_1, a_2, a_2), x) \\ &= \; H(H(a_1, a_1, a_2), a_2, x) \; = \; H(a_2, a_2, x) \in \{a_2, x\}, \end{aligned}$$

which contradicts the fact that a_1, a_2 and x are pairwise distinct.

(b) Suppose to the contrary that a_1 is not a neutral element for H (the other case can be dealt with similarly). By Lemma 5.7(a) we have that $H(a_1, a_1, y) = H(y, a_1, a_1) = y$ for all $y \in X$. By assumption, there exists $x \in X \setminus \{a_1, a_2\}$ such that $H(a_1, x, a_1) = a_1$. We have two cases to consider.

- If $H(a_2, x, a_2) = x$, then by Lemma 5.7(a) we have that

$$\begin{aligned} H(x, a_2, a_1) &= H(H(x, a_1, a_1), a_2, a_1) \;=\; H(x, a_1, H(a_1, a_2, a_1)) \\ &= H(x, a_1, a_2) \;=\; H(H(a_1, a_1, x), a_1, a_2) \\ &= H(a_1, H(a_1, x, a_1), a_2) \;=\; H(a_1, a_1, a_2) \;=\; a_2. \end{aligned}$$

  Also, by Lemma 5.7(a) we have that

$$\begin{aligned} x &= H(x, a_1, a_1) \;=\; H(H(a_2, x, a_2), a_1, a_1) \\ &= H(a_2, H(x, a_2, a_1), a_1) \;=\; H(a_2, a_2, a_1) \;=\; a_1, \end{aligned}$$

  which contradicts the fact that $x \neq a_1$.

- If $H(a_2, x, a_2) = a_2$, then by Lemma 5.7(a) we have that

$$\begin{aligned} H(x, x, a_2) &= H(x, H(a_2, a_2, x), a_2) \\ &= H(x, a_2, H(a_2, x, a_2)) \;=\; H(x, a_2, a_2) \;=\; x, \end{aligned}$$

  and

$$\begin{aligned} H(a_1, x, x) &= H(a_1, H(x, a_1, a_1), x) \\ &= H(H(a_1, x, a_1), a_1, x) \;=\; H(a_1, a_1, x) \;=\; x. \end{aligned}$$

  By Lemma 5.7(a) we also have that

$$\begin{aligned} x &= H(x, a_2, a_2) \;=\; H(H(a_1, x, x), a_2, a_2) \\ &= H(a_1, H(x, x, a_2), a_2) \;=\; H(a_1, x, a_2) \\ &= H(a_1, H(x, a_1, a_1), a_2) \;=\; H(H(a_1, x, a_1), a_1, a_2) \\ &= H(a_1, a_1, a_2) \;=\; a_2, \end{aligned}$$

  which contradicts the fact that $x \neq a_2$. □

We now present some geometric considerations regarding quasitrivial operations. Recall that the *preimage* of an element $x \in X$ under an operation $F\colon X^n \to X$ is denoted by $F^{-1}[x]$. When X is finite, i.e. $X = X_k$, we also define the *preimage sequence of* F as the nondecreasing k-element sequence of the numbers $|F^{-1}[x]|$, $x \in X_k$. We denote this sequence by $|F^{-1}|$.

Recall that the *kernel* of an operation $F\colon X^n \to X$ is the equivalence relation

$$\ker(F) \;=\; \{\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\} \colon F(x_1, \ldots, x_n) = F(y_1, \ldots, y_n)\}.$$

The *contour plot* of $F\colon X_k^n \to X_k$ is the undirected graph $\mathcal{C}_F = (X_k^n, E)$, where E is the non-reflexive part of $\ker(F)$. We say that two tuples $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X_k^n$ are F-*connected* (or simply *connected*) if $\{(x_1, \ldots, x_n), (y_1, \ldots, y_n)\} \in \ker(F)$.

Lemma 5.8 (see [22]). *An operation $F\colon X_k^n \to X_k$ is quasitrivial if and only if it is idempotent and each $(x_1, ..., x_n) \in X_k^n \setminus \Delta_{X_k}^n$ is connected to some $(n \cdot x) \in \Delta_{X_k}^n$.*

Proof. Clearly, F is quasitrivial if and only if it is idempotent and for any $(x_1, ..., x_n) \in X_k^n \setminus \Delta_{X_k}^n$ there exists $i \in \{1, ..., n\}$ such that $F(x_1, ..., x_n) = x_i = F(n \cdot x_i)$. □

In the sequel we shall make use of the following two lemmas.

Lemma 5.9 (see [22]). *For each $x \in X_k$, the number of tuples $(x_1, \ldots, x_n) \in X_k^n$ with at least one component equal to x is given by $k^n - (k-1)^n$.*

Proof. Let $x \in X_k$. The set of tuples in X_k^n with at least one component equal to x is the set $X_k^n \setminus (X_k \setminus \{x\})^n$, and its cardinality is $k^n - (k-1)^n$ since $(X_k \setminus \{x\})^n \subseteq X_k^n$. □

Lemma 5.10 (see [22]). *Let $F\colon X_k^n \to X_k$ be a quasitrivial operation. Then, for each $x \in X_k$, we have $|F^{-1}[x]| \leq k^n - (k-1)^n$.*

Proof. Let $x \in X_k$. Since $F\colon X_k^n \to X_k$ is quasitrivial, it follows from Lemma 5.8 that the point $(n \cdot x)$ is at most connected to all $(x_1, ..., x_n) \in X_k^n$ with at least one component equal to x. By Lemma 5.9, we conclude that there are exactly $k^n - (k-1)^n$ such points. □

Recall that an element $z \in X$ is said to be an *annihilator* for $F\colon X^n \to X$ if

$$F(x_1, ..., x_n) \ = \ z,$$

whenever $z \in \{x_1, ..., x_n\}$.

Remark 5.11. A neutral element need not be unique when $n \geq 3$ (for instance, $F(x_1, x_2, x_3) \equiv x_1 + x_2 + x_3 \pmod 2$ on $X = \mathbb{Z}_2$). However, if an annihilator exists, then it is unique.

The following result is the counterpart of Proposition 4.47 for n-ary quasitrivial operations.

Proposition 5.12 (see [22]). *Let $F\colon X_k^n \to X_k$ be a quasitrivial operation and let $z \in X_k$. Then z is an annihilator if and only if $|F^{-1}[z]| = k^n - (k-1)^n$.*

Proof. (Necessity) If z is an annihilator, then we know that $F(i \cdot z, x_{i+1}, ..., x_n) = z$ for all $i \in \{1, \ldots, n\}$, all $x_{i+1}, ..., x_n \in X_k$ and all permutations of $(i \cdot z, x_{i+1}, ..., x_n)$. Thus, $(n \cdot z)$ is connected to $k^n - (k-1)^n$ points by Lemma 5.9. Finally, we get $|F^{-1}[z]| = k^n - (k-1)^n$ by Lemma 5.10.

(Sufficiency) If $|F^{-1}[z]| = k^n - (k-1)^n$, then by Lemmas 5.8 and 5.9 we have that $(n \cdot z)$ is connected to the $k^n - (k-1)^n$ points $(x_1, ..., x_n) \in X_k^n$ containing at least one component equal to z. Thus, we have $F(i \cdot z, x_{i+1}, ..., x_n) = z$ for all $i \in \{1, \ldots, n\}$, all $x_{i+1}, ..., x_n \in X_k$ and all permutations of $(i \cdot z, x_{i+1}, ..., x_n)$, which shows that z is an annihilator. □

Remark 5.13. By Proposition 5.12, if $F\colon X_k^n \to X_k$ is quasitrivial, then each element x such that $|F^{-1}[x]| = k^n - (k-1)^n$ is unique.

5.2 Criteria for unique reductions and some enumeration results

In this section we show that an associative and quasitrivial operation $F\colon X^n \to X$ is uniquely reducible to an associative and quasitrivial binary operation if and only if F has at most one neutral element (Theorem 5.26). We also show that every associative and quasitrivial operation has at most two neutral elements (Proposition 5.24). Moreover, we provide a characterization of the class of associative and quasitrivial operations that have exactly two neutral elements by means of binary reductions (Corollary 5.25). Finally, we enumerate the class of associative and quasitrivial n-ary operations, which leads to a previously unknown sequence in the OEIS [94] (Proposition 5.33).

Let us first recall a useful result from [33].

Lemma 5.14 (see [33]). *Assume that the operation $F\colon X^n \to X$ is associative and reducible to associative binary operations $G\colon X^2 \to X$ and $G'\colon X^2 \to X$. If G and G' are idempotent or have the same neutral element, then $G = G'$.*

Proof. Assume that G and G' are idempotent (the other case can be dealt with similarly). Then, for any $x, y \in X$ we have

$$G(x,y) \;=\; G^{n-1}((n-1)\cdot x, y) \;=\; F((n-1)\cdot x, y) \;=\; G'^{n-1}((n-1)\cdot x, y) \;=\; G'(x,y),$$

which shows that $G = G'$. □

From Lemma 5.14, we immediately get a necessary and sufficient condition that guarantees unique reductions for associative operations that have a neutral element.

Corollary 5.15 (see [22]). *Let $F\colon X^n \to X$ be an associative operation that is reducible to associative binary operations $G\colon X^2 \to X$ and $G'\colon X^2 \to X$ that have neutral elements. Then, $G = G'$ if and only if G and G' have the same neutral element.*

If $F\colon X^n \to X$ is an associative operation such that $E_F \neq \varnothing$, then for any $e \in E_F$ we have that F is reducible to the operation G_e defined by (5.1) (see Lemma 5.4). The following proposition shows that any binary reduction G of F is of the form (5.1) for some $e \in E_F$.

Proposition 5.16 (see [22]). *Let $F\colon X^n \to X$ be an associative operation and let R_F be the set of its binary reductions. If $E_F \neq \varnothing$, then for any $G \in R_F$, there exists $e \in E_F$ such that $G = G_e$. Moreover, the mapping $\sigma\colon E_F \to R_F$ defined by $\sigma(e) = G_e$ is a bijection. In particular, e is the unique neutral element for F if and only if G_e is the unique binary reduction of F.*

Proof. Suppose that $E_F \neq \varnothing$. By Lemma 5.4 we have $R_F \neq \varnothing$. So let $e \in E_F$ and $G \in R_F$. For any $x \in X$ we have

$$G(G^{n-2}((n-1)\cdot e), x) = F((n-1)\cdot e, x) = x = F(x, (n-1)\cdot e) = G(x, G^{n-2}((n-1)\cdot e)),$$

which shows that $G^{n-2}((n-1)\cdot e)$ is the neutral element for G. Also, since F is reducible to G we have that $G^{n-2}((n-1)\cdot e)$ is a neutral element for F. Thus, by Lemma 5.4 and Corollary 5.15 we have $G = G_{G^{n-2}((n-1)\cdot e)}$ which shows that the mapping $\sigma\colon E_F \to R_F$ defined by $\sigma(e) = G_e$ is surjective. Finally, the injectivity of σ follows from Lemma 5.4 and Corollary 5.15. □

As we will see in Proposition 5.24, the size of E_F, and thus of R_F, is at most 2 whenever F is quasitrivial.

Let $Q_1^2(X)$ denote the class of associative and quasitrivial operations $G\colon X^2 \to X$ that have exactly one neutral element, and let $A_1^2(X)$ denote the class of associative operations $G\colon X^2 \to X$ that have exactly one neutral element $e_G \in X$ and that satisfy the following conditions:

- $G(x,x) \in \{e_G, x\}$ for all $x \in X$,
- $G(x,y) \in \{x,y\}$ for all $(x,y) \in X^2 \setminus \Delta_X^2$,
- If there exists $x \in X \setminus \{e_G\}$ such that $G(x,x) = e_G$, then x is unique and we have $G(x,y) = G(y,x) = y$ for all $y \in X \setminus \{x, e_G\}$.

Note that $Q_1^2(X) = A_1^2(X) = X^{X^2}$ when $|X| = 1$. Also, it is not difficult to see that $Q_1^2(X) \subseteq A_1^2(X)$. Actually, we have that $G \in Q_1^2(X)$ if and only if $G \in A_1^2(X)$ and $|G^{-1}[e]| = 1$, where e is the neutral element for G.

The following straightforward proposition states, in particular, that any $G \in A_1^2(X) \setminus Q_1^2(X)$ gives rise to a semigroup which has a unique 2-element subsemigroup isomorphic to the additive semigroup on $\mathbb{Z}_2$.

Proposition 5.17 (see [22]). *Let $G\colon X^2 \to X$ be an operation. Then $G \in A_1^2(X) \setminus Q_1^2(X)$ if and only if there exists a unique pair $(x,y) \in X^2 \setminus \Delta_X^2$ such that the following conditions hold.*

(a) $(\{x,y\}, G|_{\{x,y\}^2})$ *is isomorphic to* $(\mathbb{Z}_2, +)$.

(b) $G|_{(X\setminus\{x,y\})^2}$ *is associative and quasitrivial.*

(c) Any $z \in X \setminus \{x,y\}$ *is an annihilator for* $G|_{\{x,y,z\}^2}$.

Remark 5.18. Let $G \in A_1^2(X) \setminus Q_1^2(X)$. By Proposition 5.17, we have that for any $z \in X \setminus \{x,y\}$ the semigroup (X,G) is an ideal extension of $(X \setminus \{x,y\}, G|_{(X\setminus\{x,y\})^2})$ by $(\{x,y,z\}, G|_{\{x,y,z\}^2})$.

Proposition 5.19 (see [22]). *Let $F\colon X^n \to X$ be an associative and quasitrivial operation. Suppose that $e \in X$ is a neutral element for F.*

(a) If n is even, then F is reducible to an operation $G \in Q_1^2(X)$.

(b) If n is odd, then F is reducible to the operation $G_e \in A_1^2(X)$.

Proof. (a) By Theorem 5.3(a) we have that F is reducible to an associative and quasitrivial binary operation $G\colon X^2 \to X$. Finally, we observe that $G^{n-2}((n-1)\cdot e)$ is the neutral element for G.

(b) By Lemma 5.4 we have that F is reducible to an associative operation $G_e\colon X^2 \to X$ of the form (5.1) and that e is also a neutral element for G_e. Since F is quasitrivial, it follows from (5.1) that $G_e(x,x) \in \{x,e\}$ for all $x \in X$. If $|X| = 2$, then the proof is complete. So suppose that $|X| > 2$ and let us show that $G_e(x,y) \in \{x,y\}$ for all $(x,y) \in X^2 \setminus \Delta_X^2$. Since e is a neutral element for G_e, we have that $G_e(x,e) = G_e(e,x) = x$ for all $x \in X \setminus \{e\}$. So suppose to the contrary that there are distinct $x,y \in X \setminus \{e\}$ such that $G_e(x,y) \notin \{x,y\}$. As G_e is a reduction of F and F is quasitrivial, we must have $G_e(x,y) = e$. But then, using the associativity of G_e, we have that

$$y = G_e(e,y) = G_e(G_e(x,y),y) = G_e(x, G_e(y,y)) \in \{G_e(x,y), G_e(x,e)\} = \{e,x\},$$

which contradicts the fact that x, y and e are pairwise distinct.

Now, suppose that there exists $x \in X \setminus \{e\}$ such that $G_e(x,x) = e$ and let $y \in X \setminus \{x,e\}$. Since

$$y \ = \ G_e(e,y) \ = \ G_e(G_e(x,x),y) \ = \ G_e(x,G_e(x,y)),$$

we must have $G_e(x,y) = y$. Similarly, we can show that $G_e(y,x) = y$.

To complete the proof, we only need to show that such an x is unique. Suppose to the contrary that there exists $x' \in X \setminus \{x,e\}$ such that $G_e(x',x') = e$. Since x, x' and e are pairwise distinct and

$$x' \ = \ G_e(e,x') \ = \ G_e(G_e(x,x),x') \ = \ G_e(x,G_e(x,x')),$$

and

$$x \ = \ G_e(x,e) \ = \ G_e(x,G_e(x',x')) \ = \ G_e(G_e(x,x'),x'),$$

we must have $x = G_e(x,x') = x'$, which yields the desired contradiction. □

Let us now state and prove some intermediate results. The following two lemmas were stated and proved in [42] for n-ary groups. Here we state and prove these lemmas for n-ary semigroups. The proofs we provide are using Lemma 5.4.

Lemma 5.20 (see [26]). *Let $F\colon X^n \to X$ be an associative operation and let $e \in E_F$. Then for any $x_1, \ldots, x_{n-1} \in X$ we have*

$$F(x_1,\ldots,x_{n-1},e) \ = \ F(x_1,\ldots,e,x_{n-1}) \ = \ \cdots \ = \ F(e,x_1,\ldots,x_{n-1}).$$

Moreover, for any $x \in X$ the restriction $F|_{(\{x\}\bigcup E_F)^n}$ is symmetric.

Proof. Let $x_1, \ldots, x_{n-1} \in X$ and let G_e be the reduction of F defined by (5.1). For any $i \in \{1,\ldots,n-1\}$ we have $G_e(x_i,e) = x_i = G_e(e,x_i)$, which proves the first part of the statement for $n = 2$. For $n \geq 3$ we have

$$F(x_1,\ldots,x_i,e,x_{i+1},\ldots,x_{n-1}) \ = \ G_e^{n-2}(x_1,\ldots,x_{i-1},G_e(x_i,e),x_{i+1},\ldots,x_{n-1}),$$

and the first part of the statement follows from the fact that each x_i commutes with e in G_e. The second part is a direct consequence of the first part. □

Lemma 5.21 (see [26]). *Let $F\colon X^n \to X$ be an associative operation such that $E_F \neq \varnothing$. Then F preserves E_F, i.e., $F(E_F^n) \subseteq E_F$.*

Proof. Let $e_1, \ldots, e_n \in E_F$ and let us show that $F(e_1,\ldots,e_n) \in E_F$. By Lemma 5.20 and associativity of F, for any $x \in X$ we have

$$\begin{aligned} &F((n-1)\cdot F(e_1,\ldots,e_n),x) \\ &\quad = \ F(F(e_1,(n-1)\cdot e_2),F(e_1,(n-1)\cdot e_3),\ldots,F(e_1,(n-1)\cdot e_n),x) \\ &\quad = \ F((n-1)\cdot e_1,x) \ = \ x. \end{aligned}$$

Similarly, for any $x \in X$ we can show that

$$F(i\cdot F(e_1,\ldots,e_n),x,(n-i-1)\cdot F(e_1,\ldots,e_n)) \ = \ x, \qquad i \in \{0,\ldots,n-2\},$$

and the proof is now complete. □

Combining Lemmas 5.20 and 5.21, we immediately derive the following result.

Corollary 5.22 (see [26]). *If (X,F) is an n-ary monoid, then $(E_F, F|_{E_F^n})$ is a symmetric n-ary monoid.*

We observe that the associative operation $F\colon \mathbb{Z}_2^n \to \mathbb{Z}_2$ defined by

$$F(x_1,\ldots,x_n) \equiv \sum_{i=1}^{n} x_i \pmod 2, \qquad x_1,\ldots,x_n \in \mathbb{Z}_2,$$

has 2 neutral elements, namely 0 and 1, when n is odd. Moreover, it is quasitrivial if and only if n is odd. This also illustrates the fact that an associative and quasitrivial n-ary operation that has 2 neutral elements does not necessarily have a quasitrivial reduction. Indeed, when n is odd, $G(x_1,x_2) \equiv x_1 + x_2 \pmod 2$ and $G'(x_1,x_2) \equiv x_1 + x_2 + 1 \pmod 2$ on $X = \mathbb{Z}_2$ are the two distinct reductions of F but neither is quasitrivial.

Clearly, if an associative operation $F\colon X^n \to X$ is reducible to an associative operation $G \in Q_1^2(X)$, then it is quasitrivial. The following proposition provides a necessary and sufficient condition for F to be quasitrivial when $G \in A_1^2(X) \setminus Q_1^2(X)$.

Proposition 5.23 (see [22]). *Let $F\colon X^n \to X$ be an associative operation. Suppose that F is reducible to an operation $G \in A_1^2(X) \setminus Q_1^2(X)$. Then F is quasitrivial if and only if n is odd.*

Proof. (Necessity) Let $x \in X \setminus \{e\}$ such that $G(x,x) = e$. If n is even, then $F(n \cdot x) = G^{\frac{n}{2}-1}(\frac{n}{2} \cdot G(x,x)) = e$, contradicting quasitriviality.

(Sufficiency) Since F is reducible to $G \in A_1^2(X) \setminus Q_1^2(X)$, we clearly have that $F(n \cdot x) = x$ for all $x \in X$ such that $G(x,x) = x$. So let $y \in X \setminus \{e\}$ such that $G(y,y) = e$. Let us show that y is a neutral element for F. So let $x \in X$, let $i \in \{1,\ldots n\}$, and let us show that $F((i-1)\cdot y, x, (n-i)\cdot y) = x$. Since n is odd, we have that $i-1$ and $n-i$ are both even or both odd and thus we have

$$F((i-1)\cdot y, x, (n-i)\cdot y) \in \{G^2(e,x,e), G^2(y,x,y)\} = \{x\},$$

which shows that y is a neutral element for F. Thus, F is idempotent. Finally, we conclude that $F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\}$ for any $x_1,\ldots,x_n \in X$ by Proposition 5.17 and Lemma 5.20. $\square$

It is not difficult to see that the operation $F\colon \mathbb{Z}_{n-1}^n \to \mathbb{Z}_{n-1}$ defined by

$$F(x_1,\ldots,x_n) \equiv \sum_{i=1}^{n} x_i \pmod{(n-1)}, \qquad x_1,\ldots,x_n \in \mathbb{Z}_{n-1},$$

is associative, idempotent, symmetric and has $n-1$ neutral elements. However, this number is much smaller for quasitrivial operations.

Proposition 5.24 (see [22]). *Let $F\colon X^n \to X$ be an associative and quasitrivial operation.*

(a) If n is even, then F has at most one neutral element.

(b) If n is odd, then F has at most two neutral elements.

Proof. (a) By Theorem 5.3(a) we have that F is reducible to an associative and quasitrivial binary operation $G\colon X^2 \to X$. Suppose that $e_1, e_2 \in X$ are two neutral elements for F. Since G is quasitrivial we have

$$\begin{aligned} e_2 &= F((n-1)\cdot e_1, e_2) \;=\; G(G^{n-2}((n-1)\cdot e_1), e_2) \\ &= G(e_1, e_2) \;=\; G(e_1, G^{n-2}((n-1)\cdot e_2)) \;=\; F(e_1, (n-1)\cdot e_2) \;=\; e_1. \end{aligned}$$

Hence, F has at most one neutral element.

(b) By Theorem 5.3(b) we have that F is reducible to an associative and quasitrivial ternary operation $H\colon X^3 \to X$. For a contradiction, suppose that $e_1, e_2, e_3 \in X$ are three distinct neutral elements for F. Since H is quasitrivial, it is not difficult to see that e_1, e_2, and e_3 are neutral elements for H. Also, by Proposition 5.19(b) we have that H is reducible to the operations $G_{e_1}, G_{e_2}, G_{e_3} \in A_1^2(X)$. In particular, we have

$$G_{e_1}(e_2,e_3) = G_{e_1}(G_{e_1}(e_1,e_2),e_3) = H(e_1,e_2,e_3) = G_{e_2}(G_{e_2}(e_1,e_2),e_3) = G_{e_2}(e_1,e_3)$$

and

$$H(e_1,e_2,e_3) \;=\; G_{e_3}(e_1, G_{e_3}(e_2,e_3)) \;=\; G_{e_3}(e_1,e_2).$$

Hence, $H(e_1,e_2,e_3) \in \{e_2,e_3\} \cap \{e_1,e_3\} \cap \{e_1,e_2\}$, which shows that e_1, e_2, e_3 are not pairwise distinct, and thus yielding the desired contradiction. □

Corollary 5.25 (see [22]). *Let $F\colon X^n \to X$ be an operation and let e_1 and e_2 be distinct elements of X. Then F is associative, quasitrivial, and has exactly the two neutral elements e_1 and e_2 if and only if n is odd and F is reducible to exactly the two operations $G_{e_1}, G_{e_2} \in A_1^2(X) \setminus Q_1^2(X)$.*

Proof. (Necessity) This follows from Propositions 5.16, 5.19, and 5.24 together with the observation that $G_{e_1}(e_2,e_2) = e_1$ and $G_{e_2}(e_2,e_2) = e_2$.

(Sufficiency) This follows from Propositions 5.16 and 5.23. □

We can now state and prove the main result of this section.

Theorem 5.26 (see [22]). *Let $F\colon X^n \to X$ be an associative and quasitrivial operation. The following assertions are equivalent.*

(i) Any binary reduction of F is idempotent.

(ii) Any binary reduction of F is quasitrivial.

(iii) F has at most one binary reduction.

(iv) F has at most one neutral element.

(v) $F((n-1)\cdot x, y) \;=\; F(x, (n-1)\cdot y)$ for any $x, y \in X$.

Proof. The implications $(i) \Rightarrow ((ii)$ and $(v))$ and $(v) \Rightarrow (iv)$ are straightforward. By Proposition 5.24 and Corollary 5.25 we also have the implications $((ii)$ or $(iii)) \Rightarrow (iv)$. Hence, to complete the proof, it suffices to show that $(iv) \Rightarrow ((i)$ and $(iii))$. First, we prove that $(iv) \Rightarrow (i)$. We consider the two possible cases.

If F has a unique neutral element e, then by Proposition 5.16 we have that $G = G_e$ is the unique reduction of F with neutral element e. For the sake of a contradiction, suppose that G is not idempotent. By Proposition 5.19 we then have that n is odd and $G \in A_1^2(X) \setminus Q_1^2(X)$.

So let $x \in X \setminus \{e\}$ such that $G(x,x) \neq x$. Since $G = G_e$, we must have $G(x,x) = e$. It is not difficult to see that $F(y,(n-1)\cdot x) = y = F((n-1)\cdot x, y)$ for all $y \in X$. Now, if there is $i \in \{2,\ldots,n-1\}$ such that

$$F((i-1)\cdot x, e, (n-i)\cdot x) \ = \ x,$$

then we have that $i-1$ and $n-i$ are both even or both odd (since n is odd), and thus

$$x \ = \ F((i-1)\cdot x, e, (n-i)\cdot x) \in \{G^2(x,e,x), G^2(e,e,e)\} \ = \ \{e\},$$

which contradicts our assumption that $x \neq e$. Hence, we have $F((i-1)\cdot x, e, (n-i)\cdot x) = e$ for all $i \in \{1,\ldots,n\}$.

Now, if $|X| = 2$, then the proof is complete since e and x are both neutral elements for F, which contradicts our assumption. So suppose that $|X| > 2$.

Since e is the unique neutral element for F, there exist $y \in X \setminus \{e,x\}$ and $i \in \{2,\ldots,n-1\}$ such that

$$F((i-1)\cdot x, y, (n-i)\cdot x) \ = \ x.$$

Again by the fact that n is odd, $i-1$ and $n-i$ are both even or both odd, and thus

$$x \ = \ F((i-1)\cdot x, y, (n-i)\cdot x) \in \{G^2(x,y,x), G^2(e,y,e)\} \ = \ \{G^2(x,y,x), y\}.$$

Since $G \in A_1^2(X) \setminus Q_1^2(X)$ we have that $G^2(x,y,x) = y$, which contradicts our assumption that $x \neq y$.

Now, suppose that F has no neutral element and that G is a reduction of F that is not idempotent. So let $x \in X$ such that $G(x,x) \neq x$, and let $y \in X \setminus \{x, G(x,x)\}$. By the quasitriviality of F we have $F((n-1)\cdot x, y) \in \{x,y\}$. On the other hand, by the quasitriviality (and hence idempotency) of F and the associativity of G we have

$$\begin{aligned}
F((n-1)\cdot x, y) &= F(F(n\cdot x), (n-2)\cdot x, y) \\
&= G(G^{n-2}(G^{n-1}(n\cdot x), (n-2)\cdot x), y) \\
&= G(G^{2n-3}((2n-2)\cdot x), y) \\
&= G(G^{n-2}((n-1)\cdot G(x,x)), y) \\
&= F((n-1)\cdot G(x,x), y) \in \{G(x,x), y\}.
\end{aligned}$$

Since $x, G(x,x)$, and y are pairwise distinct, it follows that $F((n-1)\cdot x, y) = y$, which implies that $G(G^{n-2}((n-1)\cdot x), y) = y$. Similarly, we can show that

$$G(y, G^{n-2}((n-1)\cdot x)) \ = \ y.$$

Also, it is not difficult to see that

$$G(G^{n-2}((n-1)\cdot x), G(x,x)) \ = \ G(x,x) \ = \ G(G(x,x), G^{n-2}((n-1)\cdot x)).$$

Furthermore, since F is idempotent and reducible to G, we also have that

$$G(G^{n-2}((n-1)\cdot x), x) \ = \ x \ = \ G(x, G^{n-2}((n-1)\cdot x)).$$

Thus $G^{n-2}((n-1)\cdot x)$ is a neutral element for G and therefore a neutral element for F, which contradicts our assumption that F has no neutral element.

As both cases yield a contradiction, we conclude that G must be idempotent. The implication $(iv) \Rightarrow (iii)$ is an immediate consequence of the implication $(iv) \Rightarrow (i)$ together with Lemma 5.14. Thus, the proof of Theorem 5.26 is now complete. □

Remark 5.27. We observe that an alternative necessary and sufficient condition for the quasitriviality of a binary reduction of an n-ary quasitrivial semigroup was also provided in [1].

Theorem 5.26 together with Corollary 5.6 imply the following result.

Corollary 5.28 (see [22]). *Let $F\colon X^n \to X$ be an operation. Then F is associative, quasitrivial, and has at most one neutral element if and only if it is reducible to an associative and quasitrivial operation $G\colon X^2 \to X$. In this case, G is defined by $G(x,y) = F(x,(n-1)\cdot y)$.*

Proposition 5.23 and Corollaries 5.25 and 5.28 are of particular interest as they enable us to easily construct n-ary associative quasitrivial operations that have exactly two neutral elements. For instance, consider the set $X_4 = \{1,2,3,4\}$ together with the operation $G\colon X_4^2 \to X_4$ defined by the following conditions:

- $(\{1,2\}, G|_{\{1,2\}^2})$ is isomorphic to $(\mathbb{Z}_2,+)$,
- $G(x,y) = x$ for any $x,y \in \{3,4\}$,
- for any $x \in \{1,2\}$, $G(x,3) = G(3,x) = 3$ and $G(x,4) = G(4,x) = 4$.

Then we have $G \in A_1^2(X_4) \setminus Q_1^2(X_4)$ by Proposition 5.17. Thus, for any integer $p \geq 1$, we have that the operation associated with any $(2p+1)$-ary extension of (X_4, G) is associative, quasitrivial, and has exactly 2 neutral elements (namely, 1 and 2) by Proposition 5.23 and Corollaries 5.25 and 5.28.

Given a weak order $\precsim$ on X, the n-ary maximum on X for $\precsim$ is the partial symmetric n-ary operation $\max^n_{\precsim}$ defined on

$$X^n \setminus \{(x_1,\ldots,x_n) \in X^n \colon |\max_{\precsim}\{x_1,\ldots,x_n\}| \geq 2\}$$

by $\max^n_{\precsim}(x_1,\ldots,x_n) = x_i$ where $i \in \{1,\ldots,n\}$ is such that $x_j \precsim x_i$ for all $j \in \{1,\ldots,n\}$. If $\precsim$ reduces to a total order, then the operation $\max^n_{\precsim}$ is defined everywhere on X^n. Also, the *projection operations* $\pi_1\colon X^n \to X$ and $\pi_n\colon X^n \to X$ are respectively defined by $\pi_1(x_1,\ldots,x_n) = x_1$ and $\pi_n(x_1,\ldots,x_n) = x_n$ for all $x_1,\ldots,x_n \in X$.

Corollary 5.28 together with Theorem 4.1 and Proposition 4.2 imply the following characterization of the class of quasitrivial n-ary semigroups with at most one neutral element. This result is the counterpart of Theorem 4.1 for quasitrivial n-ary semigroups.

Theorem 5.29 (see [22]). *Let $F\colon X^n \to X$ be an operation. Then F is associative, quasitrivial, and has at most one neutral element if and only if there exists a weak order $\precsim$ on X and a binary reduction $G\colon X^2 \to X$ of F such that*

$$G|_{A\times B} = \begin{cases} \pi_1|_{A\times B} \ \ \textit{or} \ \ \pi_2|_{A\times B}, & \textit{if } A = B, \\ \max^2_{\precsim}|_{A\times B}, & \textit{otherwise}, \end{cases} \qquad \forall A,B \in X/\sim. \tag{5.2}$$

Moreover, when $X = X_k$, then the weak order $\precsim$ is uniquely defined as follows:

$$x \precsim y \quad \Leftrightarrow \quad |G^{-1}[x]| \leq |G^{-1}[y]|, \qquad x,y \in X_k. \tag{5.3}$$

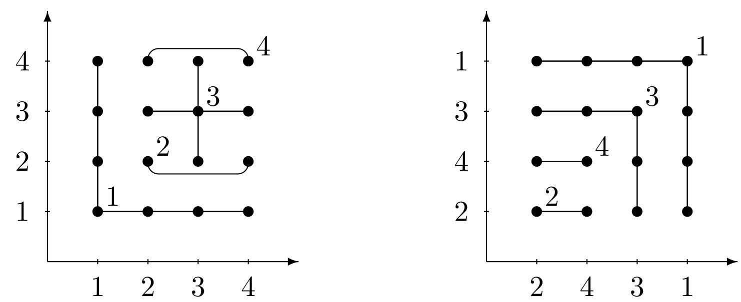

Figure 5.1: An associative and quasitrivial binary operation G on X_4

Now, let us illustrate Theorem 5.29 for binary operations by means of their contour plots. In Figure 5.1 (left), we represent the contour plot of an operation $G\colon X_4^2 \to X_4$ using the usual total order $\leq_4$ on X_4. It is not difficult to see that G is quasitrivial. To check whether G is associative, by Theorem 5.29, it suffices to show that G is of the form (5.2) where the weak order $\precsim$ is defined on X_4 by (5.3). In Figure 5.1 (right) we represent the contour plot of G using the weak order $\precsim$ on X_4 defined by (5.3). We observe that G is of the form (5.2) for $\precsim$ and thus by Theorem 5.29 it is associative.

Let $\leq$ be a total order on X. An operation $F\colon X^n \to X$ is said to be *$\leq$-preserving* if $F(x_1,\ldots,x_n) \leq F(x_1',\ldots,x_n')$, whenever $x_i \leq x_i'$ for all $i \in \{1,\ldots,n\}$. Some associative binary operations $G\colon X^2 \to X$ are $\leq$-preserving for any total order on X (e.g., $G(x,y) = x$ for all $x,y \in X$). However, there is no total order $\leq$ on X for which an operation $G \in A_1^2(X) \setminus Q_1^2(X)$ is $\leq$-preserving. A typical example is the binary addition modulo 2.

Proposition 5.30 (see [22]). *Suppose $|X| \geq 2$. If $G \in A_1^2(X) \setminus Q_1^2(X)$, then there is no total order $\leq$ on X that is preserved by G.*

Proof. Let $e \in X$ be the neutral element for G and let $x \in X \setminus \{e\}$ such that $G(x,x) = e$. Suppose to the contrary that there exists a total order $\leq$ on X such that G is $\leq$-preserving. If $x < e$, then $e = G(x,x) \leq G(x,e) = x$, which contradicts our assumption. The case $x > e$ yields a similar contradiction. $\square$

Remark 5.31. It is not difficult to see that any $\leq$-preserving operation $F\colon X^n \to X$ has at most one neutral element. Therefore, by Corollary 5.6 and Theorem 5.26 we conclude that any associative, quasitrivial, and $\leq$-preserving operation $F\colon X^n \to X$ is reducible to an associative, quasitrivial, and $\leq$-preserving operation $G\colon X^2 \to X$. For a characterization of the class of associative, quasitrivial, and $\leq$-preserving operations $G\colon X^2 \to X$, see Proposition 4.21.

We now provide several enumeration results that give the sizes of the classes of associative and quasitrivial operations that were considered above when $X = X_k$ for some integer $k \geq 1$.

For any integer $k \geq 1$, let $\gamma^n(k)$ denote the number of associative and quasitrivial n-ary operations on X_k. Recall that $\gamma(k)$ denotes the number of associative and quasitrivial binary operations on X_k for any integer $k \geq 1$ (see Section 4.1). For any integer $k \geq 1$, we have $|Q_1^2(X_k)| = \gamma_e(k)$ (see Section 4.1). Also, we denote by $a_e^2(k)$ the cardinality of $A_1^2(X_k)$. By definition, we have $a_e^2(1) = 1$.

Proposition 5.32 (see [22]). *For any integer $k \geq 2$, we have $a_e^2(k) = k\gamma(k-1)+k(k-1)\gamma(k-2)$.*

Proof. We already have that $Q^2_1(X_k) \subseteq A^2_1(X_k)$. Now, let us show how to construct an operation $G \in A^2_1(X_k) \setminus Q^2_1(X_k)$. There are k ways to choose the element $x \in X_k$ such that $G(x,x) = e$ and $G(x,y) = G(y,x) = y$ for all $y \in X_k \setminus \{x,e\}$. Then we observe that the restriction of G to $(X_k \setminus \{x\})^2$ belongs to $Q^2_1(X_k \setminus \{x\})$, so we have $\gamma_e(k-1)$ possible choices to construct this restriction. This shows that $a^2_e(k) = \gamma_e(k) + k\gamma_e(k-1)$. Finally, by Proposition 4.8 we conclude that $a^2_e(k) = k\gamma(k-1) + k(k-1)\gamma(k-2)$. □

For any integer $k \geq 1$ let $\gamma^n_1(k)$ (resp. $\gamma^n_0(k)$) denote the number of associative and quasitrivial n-ary operations that have exactly one neutral element (resp. that have no neutral element) on X_k. Also, for any integer $k \geq 1$, let $\gamma^n_2(k)$ denote the number of associative and quasitrivial n-ary operations that have two neutral elements on X_k. Clearly, $\gamma^n(1) = \gamma^n_1(1) = 1$ and $\gamma^n_2(1) = 0$. The following proposition provides explicit forms of the latter sequences. Table 5.1 below provides the first few values of all the previously considered sequences.[3]

Proposition 5.33 (see [22])**.** *For any integer $k \geq 1$ we have $\gamma^n_1(k) = \gamma_e(k)$ and $\gamma^n_0(k) = \gamma(k) - \gamma_e(k)$. Also, for any integer $k \geq 2$ we have*

$$\gamma^n_2(k) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \binom{k}{2}\gamma(k-2) & \text{if } n \text{ is odd.} \end{cases}$$

and

$$\gamma^n(k) = \begin{cases} \gamma(k) & \text{if } n \text{ is even} \\ \gamma(k) + \binom{k}{2}\gamma(k-2) & \text{if } n \text{ is odd.} \end{cases}$$

Proof. By Theorem 5.26 we have that the number of associative and quasitrivial n-ary operations that have exactly one neutral element (resp. that have no neutral element) on X_k is exactly the number of associative and quasitrivial binary operations on X_k that have a neutral element (resp. that have no neutral element). This number is given by $\gamma_e(k)$ (resp. $\gamma(k) - \gamma_e(k)$). Also, if n is even, then by Theorem 5.3(a) and Proposition 5.24(a) we conclude that $\gamma^n(k) = \gamma(k)$ and $\gamma^n_2(k) = 0$.

Now, suppose that n is odd. By Corollary 5.25 and Propositions 4.8 and 5.32 we have that $\gamma^n_2(k) = \frac{a^2_e(k) - \gamma_e(k)}{2} = \binom{k}{2}\gamma(k-2)$. Finally, by Proposition 5.24, Corollary 5.25, and Theorem 5.26 we have that $\gamma^n(k) = \gamma^n_0(k) + \gamma^n_1(k) + \gamma^n_2(k) = \gamma(k) + \binom{k}{2}\gamma(k-2)$. □

k	$\gamma^n_0(k)$	$\gamma^n_2(k)$	$\gamma^n(k)$	$a^2_e(k)$
1	0	0	1	1
2	2	1	5	4
3	8	3	23	18
4	58	24	162	128
5	492	200	1 382	1 090
6	5 074	2 070	14 236	11 232
OEIS	A308352	A308354	A308362	A308351

Table 5.1: First few values of $\gamma^n_0(k)$, $\gamma^n_2(k)$, $\gamma^n(k)$ and $a^2_e(k)$

[3] In view of Corollary 5.25, we only consider the case where n is odd for $\gamma^n_2(k)$ and $\gamma^n(k)$.

5.3 Bisymmetric and symmetric operations

In this section we refine our previous results to the subclasses of associative and quasitrivial n-ary operations that are symmetric and bisymmetric, respectively, and present further enumeration results accordingly.

We first recall and establish some auxiliary results.

Fact 5.34 (see [22]). *Suppose that $F\colon X^n \to X$ is associative and surjective. If it is reducible to an associative operation $G\colon X^2 \to X$, then G is surjective.*

Lemma 5.35 (see [33]). *Suppose that $F\colon X^n \to X$ is associative, symmetric, and reducible to an associative and surjective operation $G\colon X^2 \to X$. Then G is symmetric.*

Proof. For any $x, y \in X$ there exist $y_1, \ldots, y_{n-2} \in X$ and $z_1, \ldots, z_{n-2} \in X$ such that

$$\begin{aligned} G(x,y) &= G^2(x,y_1,z_1) = G^3(x,y_1,y_2,z_2) = \cdots \\ &= G^{n-1}(x,y_1,\ldots,y_{n-2},z_{n-2}) = F(x,y_1,\ldots,y_{n-2},z_{n-2}) \\ &= F(y_1,\ldots,y_{n-2},z_{n-2},x) = \cdots = G(y,x), \end{aligned}$$

which shows that G is symmetric. □

Proposition 5.36 (see [22]). *If $F\colon X^n \to X$ is associative, quasitrivial, and symmetric, then it is reducible to an associative, surjective, and symmetric operation $G\colon X^2 \to X$. Moreover, if $X = X_k$, then F has a neutral element.*

Proof. By Corollary 5.6, F is reducible to an associative operation $G\colon X^2 \to X$. By Fact 5.34 and Lemma 5.35, it follows that G is surjective and symmetric.

For the moreover part, we only have two cases to consider.

- If G is quasitrivial, then by Proposition 4.30 it follows that G has a neutral element, and thus F also has a neutral element.
- If G is not quasitrivial, then by Proposition 5.24 and Theorem 5.26 F has in fact two neutral elements. □

Proposition 5.37 (see [1]). *An operation $F\colon X^n \to X$ is associative, quasitrivial, symmetric, and reducible to an associative and quasitrivial operation $G\colon X^2 \to X$ if and only if there exists a total order $\leq'$ on X such that $F = \max^n_{\leq'}$.*

Proposition 5.38 (see [22]). *A quasitrivial operation $F\colon X_k^n \to X_k$ is associative, symmetric, and reducible to an associative and quasitrivial operation $G\colon X_k^2 \to X_k$ if and only if $|F^{-1}| = (1, 2^n - 1, \ldots, k^n - (k-1)^n)$.*

Proof. (Necessity) Since G is quasitrivial, it is surjective and hence by Lemma 5.35 it is symmetric. Thus, by Proposition 5.37 there exists a total order $\leq'$ on X such that $G(x,y) = \max_{\leq'}(x,y)$ for all $x, y \in X_k$. Hence $F = \max^n_{\leq'}$, which has an annihilator, and the proof of the necessity then follows by Proposition 5.12.

(Sufficiency) We proceed by induction on k. The result clearly holds for $k = 1$. Suppose that it holds for some $k \geq 1$ and let us show that it still holds for $k+1$. Assume that $F\colon X_{k+1}^n \to X_{k+1}$ is quasitrivial and that

$$|F^{-1}| = (1, 2^n - 1, \ldots, (k+1)^n - k^n).$$

Let $\leq'$ be the total order on X_{k+1} defined by

$$x \leq' y \text{ if and only if } |F^{-1}(x)| \leq |F^{-1}(y)|,$$

and let $z = \max_{\leq'}^{k+1}(1, \ldots, k+1)$. Clearly, $F' = F|_{(X_{k+1}\setminus\{z\})^n}$ is quasitrivial and $|F'^{-1}| = (1, 2^n - 1, \ldots, k^n - (k-1)^n)$. By induction hypothesis, we have that $F' = \max_{\leq^\star}^n$, where $\leq^\star$ is the restriction of $\leq'$ to $X_{k+1} \setminus \{z\}$. Since $|F^{-1}[z]| = (k+1)^n - k^n$ we necessarily have $F = \max_{\leq'}^n$ by Proposition 5.12. □

The following result provides characterizations of the class of symmetric quasitrivial n-ary semigroups that are n-ary extensions of quasitrivial semigroups.

Theorem 5.39 (see [22]). *Let $F\colon X^n \to X$ be an associative, quasitrivial, symmetric operation. The following assertions are equivalent.*

(i) F is reducible to an associative and quasitrivial operation $G\colon X^2 \to X$.

(ii) There exists a total order $\leq'$ on X such that F is $\leq'$-preserving.

(iii) There exists a total order $\leq'$ on X such that $F = \max_{\leq'}^n$.

Moreover, when $X = X_k$, each of the assertions (i) – (iii) is equivalent to each of the following assertions.

(iv) F has exactly one neutral element.

(v) $|F^{-1}| = (1, 2^n - 1, \ldots, k^n - (k-1)^n)$.

Furthermore, the total order $\leq'$ considered in assertions (ii) and (iii) is uniquely defined as follows:

$$x \leq' y \quad \textit{if and only if} \quad |G^{-1}[x]| \leq |G^{-1}[y]|, \qquad x, y \in X_k. \tag{5.4}$$

Moreover, there are $k!$ operations satisfying any of the conditions (i) – (v).

Proof. The implication $(i) \Rightarrow (iii)$ follows from Proposition 5.37. Also, the implication $(iii) \Rightarrow (ii)$ is obvious. Now, let us show that $(ii) \Rightarrow (i)$. By Corollary 5.6 we have that F is reducible to an associative operation $G\colon X^2 \to X$. Suppose to the contrary that G is not quasitrivial. From Theorem 5.26 and Proposition 5.24, it then follows that F has two neutral elements $e_1, e_2 \in X$, which contradicts Remark 5.31. The equivalence $(i) \Leftrightarrow (v)$ follows from Proposition 5.38. Also, the implication $(i) \Rightarrow (iv)$ follows from Theorem 5.26 and Proposition 5.36. Finally, the implication $(iv) \Rightarrow (i)$ follows from Lemma 5.4 and Theorem 5.26. The rest of the statement follows from Propositions 4.2 and 4.30. □

Now, let us illustrate Theorem 5.39 for binary operations by means of their contour plots. In Figure 5.2 (left), we represent the contour plot of an operation $G\colon X_4^2 \to X_4$ using the usual total order $\leq_4$ on X_4. In Figure 5.2 (right) we represent the contour plot of G using the total order $\leq'$ on X_4 defined by (5.4). We then observe that $G = \max_{\leq'}^2$, which shows by Theorem 5.39 that G is associative, quasitrivial, and symmetric.

Based on this example, we illustrate a simple test to check whether an operation $F\colon X_k^n \to X_k$ is associative, quasitrivial, symmetric, and has exactly one neutral element. First, construct the unique weak order $\precsim$ on X_k from the preimage sequence $|F^{-1}|$, i.e., $x \precsim y$ if $|F^{-1}[x]| \leq |F^{-1}[y]|$. Then, check if $\precsim$ is a total order and if F is the maximum operation for $\precsim$.

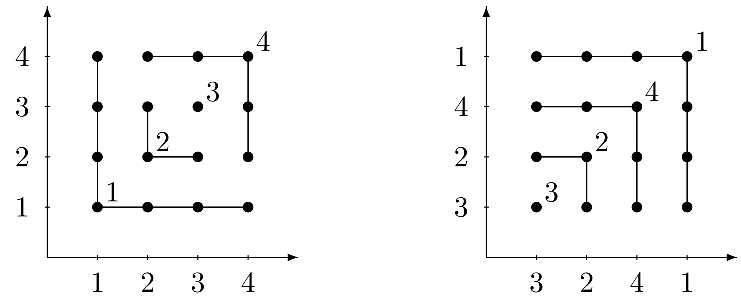

Figure 5.2: An associative, quasitrivial, and symmetric binary operation G on X_4

We denote the class of associative, quasitrivial, symmetric operations $G\colon X^2 \to X$ that have a neutral element $e \in X$ by $QS_1^2(X)$. Also, we denote by $AS_1^2(X)$ the class of symmetric operations $G\colon X^2 \to X$ that belong to $A_1^2(X)$. It is not difficult to see that $QS_1^2(X) \subseteq AS_1^2(X)$. In fact, $G \in QS_1^2(X)$ if and only if $G \in AS_1^2(X)$ and $|G^{-1}[e]| = 1$, where e is the neutral element for G.

For each integer $k \geq 2$, let $qs^n(k)$ denote the number of associative, quasitrivial, and symmetric n-ary operations on X_k. Also, denote by $as_1^2(k)$ the size of $AS_1^2(X_k)$. From Theorem 5.39 it follows that $qs^2(k) = |QS_1^2(X_k)| = k!$. Also, it is easy to check that $as_1^2(2) = 4$. The remaining terms of the sequence are given in the following proposition.

Proposition 5.40 (see [22]). *For every integer $k \geq 3$, $as_1^2(k) = qs^2(k) + kqs^2(k-1) = 2k!$.*

Proof. As observed $QS_1^2(X_k) \subseteq AS_1^2(X_k)$. So let us enumerate the operations in $AS_1^2(X_k) \setminus QS_1^2(X_k)$. There are k ways to choose the element $x \in X_k$ such that $G(x,x) = e$ and $G(x,y) = G(y,x) = y$ for all $y \in X_k \setminus \{x, e\}$. Moreover, the restriction of G to $(X_k \setminus \{x\})^2$ belongs to $QS_1^2(X_k \setminus \{x\})$, and we have $qs^2(k-1)$ possible such restrictions. Thus $as_1^2(k) = qs^2(k) + kqs^2(k-1)$. By Theorem 5.39 it then follows that $as_1^2(k) = k! + k(k-1)! = 2k!$. □

For any integer $k \geq 2$ let $qs_1^n(k)$ denote the number of associative, quasitrivial, and symmetric n-ary operations that have exactly one neutral element on X_k. Also, let $qs_2^n(k)$ denote the number of associative, quasitrivial, and symmetric n-ary operations that have two neutral elements on X_k.

Proposition 5.41 (see [22]). *For each integer $k \geq 2$, $qs_1^n(k) = qs^2(k) = k!$. Moreover, $qs_2^n(k) = \frac{k!}{2}$, and $qs^n(k) = \frac{3k!}{2}$.*

Proof. By Theorems 5.26 and 5.39 and Lemma 5.35 we have that the number of associative, quasitrivial, and symmetric n-ary operations that have exactly one neutral element on X_k is exactly the number of associative, quasitrivial, and symmetric binary operations on X_k. By Theorem 5.39 this number is given by $qs^2(k) = k!$. Also, by Corollary 5.25, Proposition 5.40, and Theorems 5.26 and 5.39, we have that $qs_2^n(k) = \frac{as_1^2(k) - qs^2(k)}{2} = \frac{k!}{2}$. Finally, by Theorems 5.26 and 5.39 and Propositions 5.24 and 5.36 we have that $qs^n(k) = qs_1^n(k) + qs_2^n(k) = \frac{3k!}{2}$. □

Now, we turn to the study of the class of bisymmetric, symmetric, and quasitrivial n-ary operations. As we will see these operations are all associative. First, let us recall several links between the class of associative binary operations and the class of bisymmetric binary operations.

Lemma 5.42 (see [63, 77, 93, 97]). *Let $G\colon X^2 \to X$ be an operation. The following assertions hold.*

(a) If G is bisymmetric and has a neutral element, then it is associative and symmetric.

(b) If G is associative and symmetric, then it is bisymmetric.

(c) If G is bisymmetric and quasitrivial, then it is associative.

In what follows, we establish similar links between the class of associative n-ary operations and the class of bisymmetric n-ary operations.

An operation $F\colon X^n \to X$ is said to be *bisymmetric* if

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \ = \ F(F(\mathbf{c}_1), \ldots, F(\mathbf{c}_n))$$

for all $n \times n$ matrices $[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n] = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T \in X^{n\times n}$.

Remark 5.43. Assume that $n = 3$. An operation $F\colon X^3 \to X$ is bisymmetric if

$$\begin{aligned} &F(F(x_{11}, x_{12}, x_{13}), F(x_{21}, x_{22}, x_{23}), F(x_{31}, x_{32}, x_{33})) \\ &\qquad = \ F(F(x_{11}, x_{21}, x_{31}), F(x_{12}, x_{22}, x_{32}), F(x_{13}, x_{23}, x_{33})), \qquad x_{11}, \ldots, x_{33} \in X. \end{aligned}$$

For instance, the operation $F\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = \frac{x+y+z}{3}$ is bisymmetric.

We now introduce a functional equation that will be useful as we continue.

Definition 5.44 (see [33]). We say that a operation $F\colon X^n \to X$ is *ultrabisymmetric* if

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \ = \ F(F(\mathbf{r}'_1), \ldots, F(\mathbf{r}'_n))$$

for all $n \times n$ matrices $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T, [\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T \in X^{n\times n}$, where $[\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T$ is obtained from $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T$ by exchanging two entries only.

Remark 5.45. Assume that $n = 3$. An operation $F\colon X^3 \to X$ is ultrabisymmetric if

$$F(F(x_{11}, x_{12}, x_{13}), F(x_{21}, x_{22}, x_{23}), F(x_{31}, x_{32}, x_{33}))$$

is invariant when replacing x_{ij} by x_{kl} for any $i, j, k, l \in \{1, 2, 3\}$. For instance, the operation $F\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x, y, z) = \frac{x+y+z}{3}$ is ultrabisymmetric.

Ultrabisymmetry seems to be a rather strong property. However, as shown in the next result, this property is satisfied by any operation that is bisymmetric and symmetric.

Proposition 5.46 (see [33]). *Let $F\colon X^n \to X$ be an operation. If F is ultrabisymmetric, then it is bisymmetric. The converse holds whenever F is symmetric.*

Proof. We immediately see that any ultrabisymmetric operation is bisymmetric (just apply ultrabisymmetry repeatedly to exchange the (i, j)- and (j, i)-entries for all $i, j \in \{1, \ldots, n\}$).

Now suppose that $F\colon X^n \to X$ is symmetric and bisymmetric. Then we have

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \ = \ F(F(\mathbf{r}'_1), \ldots, F(\mathbf{r}'_n)),$$

for all matrices $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T, [\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T \in X^{n\times n}$, where $[\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T$ is obtained from $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T$ by permuting the entries of any column or any row. By applying three times this property, we can easily exchange two arbitrary entries of the matrix. Indeed, exchanging the (i, j)- and (k, l)-entries can be performed through the following three steps: exchange the (i, j)- and (i, l)-entries in row i, exchange the (i, l)- and (k, l)-entries in column l, and exchange the (i, j)- and (i, l)-entries in row i. □

Remark 5.47. (a) The symmetry property is necessary in Proposition 5.46. Indeed, for any $k \in \{1, \ldots, n\}$, the kth projection operation $F\colon X^n \to X$ defined by $F(x_1, \ldots, x_n) = x_k$ is bisymmetric but not ultrabisymmetric.

(b) An ultrabisymmetric operation need not be symmetric. For instance, consider the operation $F\colon X^2 \to X$, where $X = \{a, b, c\}$, defined by $F(a, c) = a$ and $F(x, y) = b$ for every $(x, y) \neq (a, c)$. Clearly, this operation is not symmetric. However, it is ultrabisymmetric since $F(F(x, y), F(u, v)) = b$ for all $x, y, u, v \in X$.

(c) In [30] the author stated without proof that any ternary symmetric and bisymmetric operation is ultrabisymmetric. Here we provided a proof for n-ary operations.

Lemma 5.48 (see [33]). *If $F\colon X^n \to X$ is surjective (i.e., onto) and ultrabisymmetric, then it is symmetric.*

Proof. Let $x_1, \ldots, x_n \in X$. Then there exists a matrix $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T \in X^{n\times n}$ such that $x_i = F(\mathbf{r}_i)$ for $i = 1, \ldots, n$. By ultrabisymmetry,

$$F(x_1, \ldots, x_n) \ = \ F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n))$$

is symmetric in $x_1, \ldots, x_n$. □

Remark 5.49. We observe that if $F\colon X^n \to X$ is idempotent or quasitrivial or has a neutral element, then it is surjective.

Lemma 5.50 (see [33]). *If $F\colon X^n \to X$ is quasitrivial, then for any $x, y \in X$, there exists $k \in \{1, \ldots, n\}$ such that*

$$F((k-1)\cdot x, (n-k+1)\cdot y) \ = \ y \quad \textit{and} \quad F(k\cdot x, (n-k)\cdot y) \ = \ x.$$

Proof. We proceed by contradiction. Suppose that there exist $x, y \in X$, with $x \neq y$, such that for every $k \in \{1, \ldots, n\}$ we have

$$F((k-1)\cdot x, (n-k+1)\cdot y) \ = \ x \quad \text{or} \quad F(k\cdot x, (n-k)\cdot y) \ = \ y. \tag{5.5}$$

Using the fact that $F(n\cdot y) = y$ we see that only the second condition of (5.5) holds. When $k = n$ this gives $F(n\cdot x) = y$, a contradiction. □

Proposition 5.51 (see [33]). *If $F\colon X^n \to X$ is quasitrivial and ultrabisymmetric, then it is associative and symmetric.*

Proof. Symmetry immediately follows from Lemma 5.48 and Remark 5.49. Let us prove that associativity holds. Let $x_1, \ldots, x_{2n-1} \in X$ and let $i \in \{1, \ldots, n-1\}$. By Lemma 5.50 there exists $k \in \{1, \ldots, n\}$ such that

$$F((k-1)\cdot x_i, (n-k+1)\cdot x_{i+n}) \ = \ x_{i+n} \quad \text{and} \quad F(k\cdot x_i, (n-k)\cdot x_{i+n}) \ = \ x_i.$$

We then have

$$\begin{aligned}
& F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), x_{i+n}, \ldots, x_{2n-1}) \\
= \ & F(x_1, \ldots, x_{i-1}, F(x_i, \ldots, x_{i+n-1}), F((k-1)\cdot x_i, (n-k+1)\cdot x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}).
\end{aligned}$$

Replacing x_j with $F(n \cdot x_j)$ for all $j \in \{1, \ldots, 2n-1\} \setminus \{i, \ldots, i+n\}$ and then applying ultrabisymmetry repeatedly to exchange the $(n-1)$-tuples

$$(x_{i+1}, \ldots, x_{i+n-1}) \quad \text{and} \quad ((k-1) \cdot x_i, (n-k) \cdot x_{i+n}),$$

we see that the latter expression becomes

$$\begin{aligned} F(x_1, \ldots, x_{i-1}, F(k \cdot x_i, (n-k) \cdot x_{i+n}), F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}) \\ = F(x_1, \ldots, x_i, F(x_{i+1}, \ldots, x_{i+n}), x_{i+n+1}, \ldots, x_{2n-1}). \end{aligned}$$

This shows that F is associative. □

Remark 5.52. Ultrabisymmetry cannot be relaxed into bisymmetry in Proposition 5.51. For instance, the ternary operation $F\colon X^3 \to X$ defined by $F(x, y, z) = y$ is quasitrivial and bisymmetric, but it is neither associative nor symmetric. This example also shows that the result stated in Lemma 5.42(c) cannot be extended to n-ary operations.

Proposition 5.53 (see [33]). *If $F\colon X^n \to X$ is associative and symmetric, then it is ultrabisymmetric.*

Proof. Let $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T, [\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T \in X^{n \times n}$, where $[\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T$ is obtained from $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T$ by exchanging the (i, j)- and (k, l)-entries for some $i, j, k, l \in \{1, \ldots, n\}$. We only need to prove that

$$F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) = F(F(\mathbf{r}'_1), \ldots, F(\mathbf{r}'_n)).$$

Permuting the rows of $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T$ if necessary (this is allowed by symmetry), we may assume that $k = i + 1$. Denote by $x_{i,j}$ (resp. $x_{k,l}$) the (i, j)-entry (resp. (k, l)-entry) of $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T$.

Using associativity and symmetry, we see that there exist $p, q \in \{1, \ldots, n\}$, with $p \neq j$ and $q \neq l$, such that

$$\begin{aligned} & F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_n)) \\ & = F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_{i-1}), F(x_{i,p}, \ldots, x_{i,j}), F(x_{k,l}, \ldots, x_{k,q}), F(\mathbf{r}_{k+1}), \ldots, F(\mathbf{r}_n)) \\ & = F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_{i-1}), x_{i,p}, F(\ldots, x_{i,j}, F(x_{k,l}, \ldots, x_{k,q})), F(\mathbf{r}_{k+1}), \ldots, F(\mathbf{r}_n)) \\ & = F(F(\mathbf{r}_1), \ldots, F(\mathbf{r}_{i-1}), x_{i,p}, F(\ldots, F(x_{i,j}, x_{k,l}, \ldots), x_{k,q}), F(\mathbf{r}_{k+1}), \ldots, F(\mathbf{r}_n)). \end{aligned}$$

This shows that F is ultrabisymmetric since the latter expression is symmetric in $x_{i,j}$ and $x_{k,l}$. □

Remark 5.54. It was already shown in [68] that any associative and symmetric operation is bisymmetric.

Corollary 5.55 (see [33]). *If $F\colon X^n \to X$ is quasitrivial, then it is associative and symmetric if and only if it is ultrabisymmetric.*

Proof. The statement immediately follows from Propositions 5.51 and 5.53. □

Remark 5.56. If $F\colon X^n \to X$ is ultrabisymmetric but not quasitrivial, then it need not be associative (e.g., $F(x, y, z) = 2x + 2y + 2z$ when $X = \mathbb{R}$).

Corollary 5.57 (see [33]). *If $F\colon X^n \to X$ is quasitrivial and symmetric, then it is associative if and only if it is bisymmetric.*

Proof. The statement immediately follows from Propositions 5.46, 5.51, and 5.53. □

From Corollary 5.57 we immediately derive the following theorem, which is an important and surprising result.

Theorem 5.58 (see [33]). *In Theorem 5.39 we can replace associativity with bisymmetry.*

We end this section by investigating bisymmetric operations that have neutral elements. It was already shown in [68] that the latter operations are associative and symmetric. Here, we provide an alternative proof that makes use of ultrabisymmetry [33].

Proposition 5.59 (see [68]). *If $F\colon X^n \to X$ is bisymmetric and has a neutral element, then it is associative and symmetric.*

Proof. Let e be a neutral element of F. Let us first prove symmetry. Let $x_1,\ldots,x_n \in X$, let $i,j \in \{1,\ldots,n\}$, and let $[\mathbf{c}_1 \ \cdots \ \mathbf{c}_n] = [\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T \in X^{n\times n}$ be defined as

$$\mathbf{r}_k \ = \ \begin{cases} ((j-1)\cdot e, x_i, (n-j)\cdot e), & \text{if } k=i \\ ((i-1)\cdot e, x_j, (n-i)\cdot e), & \text{if } k=j \\ ((k-1)\cdot e, x_k, (n-k)\cdot e), & \text{otherwise.} \end{cases}$$

By bisymmetry we have

$$\begin{aligned} F(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) \ &= \ F(F(\mathbf{r}_1),\ldots,F(\mathbf{r}_n)) \ = \ F(F(\mathbf{c}_1),\ldots,F(\mathbf{c}_n)) \\ &= \ F(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n). \end{aligned}$$

This shows that F is symmetric.

Let us now show that F is associative by using ultrabisymmetry (which follows from bisymmetry and symmetry by Proposition 5.46). Let $x_1,\ldots,x_{2n-1} \in X$, let $i \in \{1,\ldots,n-1\}$, and let $[\mathbf{r}_1 \ \cdots \ \mathbf{r}_n]^T, [\mathbf{r}'_1 \ \cdots \ \mathbf{r}'_n]^T \in X^{n\times n}$ be defined as

$$\mathbf{r}_k \ = \ \begin{cases} (x_k, (n-1)\cdot e), & \text{if } k<i \\ (x_i,\ldots,x_{i+n-1}), & \text{if } k=i \\ (x_{k+n-1}, (n-1)\cdot e), & \text{if } k>i \end{cases}$$

and

$$\mathbf{r}'_k \ = \ \begin{cases} (x_k, (n-1)\cdot e), & \text{if } k<i+1 \\ (x_{i+1},\ldots,x_{i+n}), & \text{if } k=i+1 \\ (x_{k+n-1}, (n-1)\cdot e), & \text{if } k>i+1. \end{cases}$$

Using ultrabisymmetry, we then have

$$\begin{aligned} &F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}) \ = \ F(F(\mathbf{r}_1),\ldots,F(\mathbf{r}_n)) \\ &\quad = \ F(F(\mathbf{r}'_1),\ldots,F(\mathbf{r}'_n)) \ = \ F(x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1}). \end{aligned}$$

This shows that F is associative. □

Corollary 5.60 (see [33]). *Assume that $F\colon X^n \to X$ has a neutral element. Then the following assertions are equivalent.*

(i) F *is bisymmetric.*

(ii) F *is associative and symmetric.*

(iii) F *is ultrabisymmetric.*

Proof. We have $(i) \Rightarrow (ii)$ by Proposition 5.59. We have $(ii) \Rightarrow (iii)$ by Proposition 5.53. Finally we have $(iii) \Rightarrow (i)$ by Proposition 5.46. $\square$

Remark 5.61. If $F\colon X^n \to X$ is bisymmetric and does not have a neutral element, then it need not be associative nor symmetric (e.g., $F(x, y, z) = x + 2y + 3z$ when $X = \mathbb{R}$). If $F\colon X^n \to X$ is ultrabisymmetric and does not have a neutral element, then it need not be associative (e.g., $F(x, y, z) = 2x + 2y + 2z$ when $X = \mathbb{R}$).

Chapter 6

Towards idempotent n-ary semigroups

As observed in Corollary 5.6, all the quasitrivial associative n-ary operations are reducible to associative binary operations. On the other hand, the associative idempotent ternary operation $F\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $F(x,y,z) = x - y + z$ is neither quasitrivial nor reducible (see, e.g., [98] or more recently [75]).

In this chapter, we are interested in studying conditions under which an idempotent n-ary semigroup is reducible to a semigroup. The observation above lead us to investigate certain subclasses of idempotent n-ary semigroups containing the quasitrivial ones, for instance by requiring the condition

$$F(x_1,\ldots,x_n) \in \{x_1,\ldots,x_n\},$$

to hold on at least some subsets of X^n. More precisely, in Section 6.1 we state our main results and prove them in Section 6.2. In Section 6.3 we study alternative subclasses of idempotent n-ary semigroups containing the quasitrivial ones. In particular, using some results of Chapter 5, we show that every n-ary semigroup in our new class of idempotent n-ary semigroups is reducible to a binary semigroup. In Section 6.4, we provide an alternative proof of the latter result that does not make use of any result from Chapter 5. Most of the contributions presented in this chapter stem from [26].

6.1 Main results

In this section we introduce a new subclass of idempotent n-ary semigroups and state our main results without proof. In particular, we show that each of these semigroups is built from a quasitrivial semigroup and an Abelian group whose exponent divides $n-1$. The proofs of the results stated in this section are deferred until Section 6.2.

For every set $S \subseteq \{1,\ldots,n\}$, let

$$D_S^n ~=~ \{(x_1,\ldots,x_n)\in X^n : \forall\, i,j\in S, x_i = x_j\}.$$

Also, for every $k\in\{1,\ldots,n\}$, let

$$D_k^n ~=~ \bigcup_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|\geq k}} D_S^n ~=~ \bigcup_{\substack{S\subseteq\{1,\ldots,n\}\\ |S|= k}} D_S^n.$$

Thus, the set D_k^n consists of those tuples of X^n for which at least k components are equal to each other. In particular, $D_1^n = X^n$ and $D_n^n = \{(x,\ldots,x) : x\in X\}$.

For every $k \in \{1, \ldots, n\}$, denote by $\mathcal{F}_k^n$ the class of those associative n-ary operations $F\colon X^n \to X$ that satisfy

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}, \qquad (x_1, \ldots, x_n) \in D_k^n.$$

We say that these operations are *quasitrivial on* D_k^n.

Thus defined, $\mathcal{F}_1^n$ is exactly the class of quasitrivial associative n-ary operations and $\mathcal{F}_n^n$ is exactly the class of idempotent associative n-ary operations. It follows directly from the definition of the classes $\mathcal{F}_k^n$ that $\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_n^n$ if $|X| \leq 2$. Therefore, throughout the rest of this chapter we assume that $|X| \geq 3$. Since the sets D_k^n are nested in the sense that $D_{k+1}^n \subseteq D_k^n$ for $1 \leq k \leq n-1$, the classes $\mathcal{F}_k^n$ clearly form a filtration, that is,

$$\mathcal{F}_1^n \subseteq \mathcal{F}_2^n \subseteq \cdots \subseteq \mathcal{F}_n^n.$$

Quite surprisingly, we have the following result, which shows that this filtration actually reduces to three nested classes only.

Proposition 6.1 (see [26])**.** *For every $n \geq 3$, we have $\mathcal{F}_1^n = \mathcal{F}_{n-2}^n$.*

We observe that the class $\mathcal{F}_1^n = \mathcal{F}_2^n = \cdots = \mathcal{F}_{n-2}^n$ was characterized in Chapter 5 (see Corollaries 5.25 and 5.28). Moreover, we showed that all its elements are reducible.

In this section, we provide a characterization of the class $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. We show that all of its elements are also reducible to binary associative operations. We give a full description of the possible reductions of the operations in this class.

Let us begin with a particular case and assume first that all the elements in X are neutral. Recall that a group (X, G) with neutral element e has *bounded exponent* if there exists an integer $m \geq 1$ such that $G^{m-1}(m \cdot x) = e$ for every $x \in X$ (with the usual convention that $G^0(x) = x$ for every $x \in X$). In that case, the *exponent* of the group is the smallest integer having this property.

Theorem 6.2 (see [26])**.** *Let $F\colon X^n \to X$ ($n \geq 3$) be an associative operation. Then $E_F = X$ if and only if (X, F) is the n-ary extension of an Abelian group whose exponent divides $n-1$.*

Abelian groups having bounded exponent play a central role in this first result, but also in the next theorems. We recall that Prüfer and Baer (see, e.g., [90]) showed that an Abelian group is of bounded exponent if and only if it is isomorphic to a direct sum of cyclic groups of bounded exponent (where the exponent refers to the direct sum).

Recall that an *n-ary groupoid* is a nonempty set equipped with an n-ary operation. Moreover, two n-ary groupoids (X, F_1) and (Y, F_2) are said to be *isomorphic* if there exists a bijection $\phi\colon X \to Y$ such that

$$\phi(F_1(x_1, \ldots, x_n)) = F_2(\phi(x_1), \ldots, \phi(x_n)), \qquad x_1, \ldots, x_n \in X.$$

In that case, the operations F_1 and F_2 are said to be *conjugate to each other*.

In order to state one of the main results of this chapter, we shall make use of the following classes of operations.

Definition 6.3 (see [26])**.** For any integer $m \geq 1$, let $\mathcal{H}_m$ be the class of binary operations $G\colon X^2 \to X$ such that there exists a subset $Y \subseteq X$ with $|Y| \geq 3$ for which the following assertions hold.

(a) $(Y, G|_{Y^2})$ is an Abelian group whose exponent divides m.

(b) $G|_{(X\setminus Y)^2}$ is associative and quasitrivial.

(c) Any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\}\bigcup Y)^2}$.

Note that $\mathcal{H}_1 = \varnothing$. As we will see, every operation in $\mathcal{H}_m$ is associative, and the set Y is unique. In fact, the family of classes $\mathcal{H}_m$ is the key for the characterization of the classes $\mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$.

Remark 6.4. Let $G \in \mathcal{H}_m$ for some $m \geq 2$. We can easily see that for any $x \in X \setminus Y$ the semigroup (X, G) is an ideal extension of $(X \setminus Y, G|_{(X\setminus Y)^2})$ by $(\{x\} \bigcup Y, G|_{(\{x\}\bigcup Y)^2})$.

Theorem 6.5 (see [26]). *Every $G \in \mathcal{H}_m$ is associative. If $G \in \mathcal{H}_{n-1}$, then its n-ary extension $F = G^{n-1}$ is in $\mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$. Conversely, for every $F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$ we have that $|E_F| \geq 3$, and the reductions of F are exactly the operations of the form G_e for $e \in E_F$ and they lie in $\mathcal{H}_{n-1}$.*

As an immediate corollary we solve the reducibility problem for operations in $\mathcal{F}^n_{n-1}$.

Corollary 6.6 (see [26]). *Every operation in $\mathcal{F}^n_{n-1}$ is reducible to an associative binary operation.*

Theorem 6.5 is of particular interest as it enables us to easily construct n-ary operations in $\mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$. For instance, for any integers $n \geq 3$ and $p \geq 1$, the operation of the cyclic group $(\mathbb{Z}_n, +)$ is in $\mathcal{H}_{np}$, and thus its $(np+1)$-ary extension is in $\mathcal{F}^{np+1}_{np} \setminus \mathcal{F}^{np+1}_1$ by Theorem 6.5.

To give another example, consider the chain $(X_5, \leq) = (\{1, 2, 3, 4, 5\}, \leq)$ together with the operation $G\colon X_5^2 \to X_5$ defined by the following conditions:

- $(\{1, 2, 3\}, G|_{\{1,2,3\}^2})$ is isomorphic to $(\mathbb{Z}_3, +)$,
- $G|_{\{4,5\}^2} = \vee|_{\{4,5\}^2}$, where $\vee\colon X_5^2 \to X_5$ is the maximum operation for $\leq$,
- for any $x \in \{1, 2, 3\}$, $G(x, 4) = G(4, x) = 4$ and $G(x, 5) = G(5, x) = 5$.

Then we have $G \in \mathcal{H}_{3p}$ for any integer $p \geq 1$ and so G^{3p} is in $\mathcal{F}^{3p+1}_{3p} \setminus \mathcal{F}^{3p+1}_1$.

Now we give a reformulation of Theorem 6.5 that is not based on binary reductions.

Theorem 6.7 (see [26]). *Suppose that $F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$. Then, setting $Y = E_F$, we have that $|Y| \geq 3$ and the following assertions hold.*

(a) $(Y, F|_{Y^n})$ is the n-ary extension of an Abelian group whose exponent divides $n - 1$.

(b) $F|_{(X\setminus Y)^n}$ is associative, quasitrivial, and has at most one neutral element.

(c) For all $x_1, \ldots, x_n \in X$ and $i \in \{1, \ldots, n-1\}$ such that $\{x_i, x_{i+1}\} \cap (X \setminus Y) = \{x\}$ we have

$$F(x_1, \ldots, x_n) = F(x_1, \ldots, x_{i-1}, x, x, x_{i+2}, \ldots, x_n).$$

Conversely, if an operation F satisfies these conditions for some $Y \subseteq X$ with $|Y| \geq 3$, then $F \in \mathcal{F}^n_{n-1} \setminus \mathcal{F}^n_1$ and $E_F = Y$.

Proposition 6.1 shows that all operations in $\mathcal{F}^n_{n-2}$ are quasitrivial. The examples we just presented show that there are operations in $\mathcal{F}^n_{n-1}$ that are not quasitrivial, for some $n \geq 3$ and some sets X. Theorem 6.5 entails necessary and sufficient conditions on the set X for such operations to exist. We first give a technical definition.

Definition 6.8 (see [26])**.** For any integer $m \geq 2$, let c_m denote the cardinality of the smallest Abelian group with at least three elements whose exponent divides m.

Proposition 6.9 (see [26])**.** *For every $n \geq 3$, we have $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \varnothing$ if and only if $|X| \geq c_{n-1}$.*

Corollary 6.10 (see [26])**.** *For any integer $n \geq 3$, let p be its least odd prime divisor if $n-1$ is not a power of 2; otherwise, set $p = 4$. The following assertions hold.*

(a) If n is even, then $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \varnothing$ if and only if $|X| \geq p$.

(b) If n is odd, then $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \varnothing$ if and only if $|X| \geq \min(4, p)$.

Remark 6.11. Proposition 6.9 and Corollary 6.10 are due to the anonymous reviewer of [21].

Finally, we observe that if $(X, \preceq)$ is a semilattice that is not a chain, then the n-ary operation $F\colon X^n \to X$ defined by $F(x_1, \ldots, x_n) = x_1 \curlyvee \ldots \curlyvee x_n$ is in $\mathcal{F}_n^n$. However, it is not in $\mathcal{F}_{n-1}^n$ since $F((n-1) \cdot x, y) \notin \{x, y\}$ whenever x and y are not comparable, i.e, $x \curlyvee y \notin \{x, y\}$. This example shows that in general the classes $\mathcal{F}_n^n$ and $\mathcal{F}_{n-1}^n$ are different. More precisely, we have the following result.

Proposition 6.12 (see [26])**.** *For every $n \geq 2$, we have $\mathcal{F}_n^n \setminus \mathcal{F}_{n-1}^n \neq \varnothing$ if and only if $|X| \geq 3$.*

6.2 Technicalities and proofs of the main results

In this section we provide the proofs of the results stated in Section 6.1. Let us start by providing the proof of Proposition 6.1.

Proof of Proposition 6.1. We only need to prove that $\mathcal{F}_{n-2}^n \subseteq \mathcal{F}_1^n$, and so we can assume that $n \geq 4$. Let $F \in \mathcal{F}_{n-2}^n$ and let us show by induction that for every $k \in \{1, \ldots, n\}$ we have

$$F(k \cdot x_1, x_{k+1}, \ldots, x_n) \in \{x_1, x_{k+1}, \ldots, x_n\}, \qquad x_1, x_{k+1}, \ldots, x_n \in X. \tag{6.1}$$

By the definition of $\mathcal{F}_{n-2}^n$, condition (6.1) holds for any $k \in \{n-2, n-1, n\}$. Let us now assume that it holds for some $k \in \{2, \ldots, n\}$ and let us show that it still holds for $k-1$. Using associativity and idempotency, we have

$$\begin{aligned} F((k-1) \cdot x_1, x_k, \ldots, x_n) &= F(F(n \cdot x_1), (k-2) \cdot x_1, x_k, \ldots, x_n) \\ &= F(k \cdot x_1, F((n-2) \cdot x_1, x_k, x_{k+1}), \ldots, x_n). \end{aligned}$$

By the induction hypothesis, the latter expression lies in $\{x_1, x_k, \ldots, x_n\}$. This completes the proof. □

Let us now prove Theorem 6.2.

Proof of Theorem 6.2. (Sufficiency) Obvious.

(Necessity) Suppose that $X = E_F$. Let $e \in E_F$ and $G_e\colon X^2 \to X$ be the corresponding reduction of F defined by (5.1). By Corollary 5.22, we have that F is symmetric. Thus, we have that G_e also is symmetric. Moreover, since G_e is a binary reduction of F and $E_F = X$, it follows that

$$G_e(G_e^{n-2}((n-1) \cdot x), y) \;=\; y \;=\; G_e(y, G_e^{n-2}((n-1) \cdot x)), \qquad x, y \in X,$$

which shows that $G_e^{n-2}((n-1) \cdot x) \in E_{G_e}$ for any $x \in X$. However, since $E_{G_e} = \{e\}$, we have that $G_e^{n-2}((n-1) \cdot x) = e$ for any $x \in X$. Thus, (X, G_e) is an Abelian group whose exponent divides $n-1$. □

The following result follows immediately from Theorem 6.2.

Corollary 6.13 (see [26]). *If (X,F) is an n-ary monoid, then $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$.*

Let us now prove Theorem 6.7. To this extent, we first state and prove some intermediate results. We have the following remarkable result, which characterizes the existence of a pair of neutral elements for $F\in\mathcal{F}_{n-1}^n$ by means of two identities.

Lemma 6.14 (see [26]). *Let $F\in\mathcal{F}_{n-1}^n$ and let $a,b\in X$ such that $a\neq b$. Then $a,b\in E_F$ if and only if $F((n-1)\cdot a,b)=b$ and $F(a,(n-1)\cdot b)=a$.*

Proof. (Necessity) Obvious.

(Sufficiency) For any $x\in X$, we have

$$\begin{aligned}F((n-1)\cdot a,x) &= F((n-2)\cdot a,F(a,(n-1)\cdot b),x)\\ &= F(F((n-1)\cdot a,b),(n-2)\cdot b,x) \;=\; F((n-1)\cdot b,x),\end{aligned}$$

which implies that $F((n-1)\cdot a,x)=F((n-1)\cdot b,x)=x$ for any $x\in X$. Indeed, for $x\in\{a,b\}$ this relation follows from idempotency, and for $x\notin\{a,b\}$ we have

$$F((n-1)\cdot a,x) \;=\; F((n-1)\cdot b,x)\in\{a,x\}\cap\{b,x\} \;=\; \{x\},$$

due to the definition of $\mathcal{F}_{n-1}^n$. Similarly, we can show that

$$F(x,(n-1)\cdot a) \;=\; x \;=\; F(x,(n-1)\cdot b),\qquad x\in X.$$

It follows from these relations, together with associativity of F that for any $k\in\{1,\ldots,n-2\}$, the maps $\psi_k,\xi_k\colon X\to X$ defined by

$$\begin{aligned}\psi_k(x) &= F(k\cdot a,x,(n-k-1)\cdot a)\\ \xi_k(x) &= F(k\cdot b,x,(n-k-1)\cdot b)\end{aligned}$$

are bijections with inverse maps ψ_{n-k-1} and ξ_{n-k-1}, respectively. It then follows that, for any $k\in\{1,\ldots,n-2\}$, we have $F(k\cdot a,x,(n-k-1)\cdot a)=\psi_k(x)=x$ for every $x\in X$. Indeed, for $x=a$, this relation follows from idempotency, and for $x\neq a$, we have $\psi_k(x)\in\{a,x\}$ and $\psi_k(x)\neq a$. Similarly, we can show that $F(k\cdot b,x,(n-k-1)\cdot b)=\xi_k(x)=x$ for every $x\in X$, which shows that $a,b\in E_F$. □

For any associative operation $F\colon X^n\to X$, we define the sequence $(F^q)_{q\geq 1}$ of $(qn-q+1)$-ary associative operations inductively by the rules $F^1=F$ and

$$F^q(x_1,\ldots,x_{qn-q+1}) \;=\; F^{q-1}(x_1,\ldots,x_{(q-1)n-q+1},F(x_{(q-1)n-q+2},\ldots,x_{qn-q+1})),$$

for any integer $q\geq 2$ and any $x_1,\ldots,x_{qn-q+1}\in X$. It is easy to see that F^q is idempotent whenever F is idempotent. Also, it was shown in [68] that F^q is symmetric whenever F is symmetric.

The following proposition shows that every tuple in X^n that violates the quasitriviality condition for $F\in\mathcal{F}_{n-1}^n$ belongs to E_F^n.

Proposition 6.15 (see [26]). *Let $F \in \mathcal{F}_{n-1}^n$. For any $a_1, \ldots, a_n \in X$ such that $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$, we have that $a_1, \ldots, a_n, F(a_1, \ldots, a_n) \in E_F$. Moreover, $F|_{(X \setminus E_F)^n}$ is quasitrivial.*

Proof. The case $n = 2$ is trivial. So assume that $n \geq 3$. Let us prove by induction on $k \in \{1, \ldots, n-1\}$ that for every $a_1, a_2, \ldots, a_{k+1} \in X$ the condition

$$F((n-k) \cdot a_1, a_2, \ldots, a_{k+1}) \notin \{a_1, \ldots, a_{k+1}\},$$

implies $a_1, \ldots, a_{k+1} \in E_F$. For $k = 1$, there is nothing to prove. We thus assume that the result holds true for a given $k \in \{1, \ldots, n-2\}$ and we show that it still holds for $k+1$. Now, consider elements $a_1, \ldots, a_{k+2}$ such that

$$F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}. \tag{6.2}$$

We first prove that $a_1, a_2 \in E_F$.

If $a_1 = a_2$, then $a_1, \ldots, a_{k+2} \in E_F$ by the induction hypothesis.

If $a_1 \neq a_2$, then we prove that $F((n-1) \cdot a_1, a_2) = a_2$ and $F(a_1, (n-1) \cdot a_2) = a_1$, which shows that $a_1, a_2 \in E_F$ by Lemma 6.14.

- For the sake of a contradiction, assume first that $F((n-1) \cdot a_1, a_2) = a_1$. Then, for $\ell \geq 1$ we have

$$\begin{aligned} & F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ = \; & F^{\ell+1}(((n-k-1) + \ell(n-2)) \cdot a_1, (\ell+1) \cdot a_2, \ldots, a_{k+2}). \end{aligned} \tag{6.3}$$

  Choosing $\ell = n-k-1$ and using idempotency of F, we obtain

$$F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \;=\; F^2((n-1) \cdot a_1, (n-k) \cdot a_2, a_3, \ldots, a_{k+2}).$$

  Since the left-hand side of this equation does not lie in $\{a_1, \ldots, a_{k+2}\}$ by (6.2), we obtain

$$F((n-k) \cdot a_2, a_3, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}.$$

  By the induction hypothesis, we have $a_2, \ldots, a_{k+2} \in E_F$. Then choosing $\ell = n-2$ in (6.3) and using idempotency and the fact that $a_2 \in E_F$, we obtain

$$\begin{aligned} & F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ = \; & F^{n-1}(((n-k-1) + (n-2)^2) \cdot a_1, (n-1) \cdot a_2, \ldots, a_{k+2}) \\ = \; & F^2((n-k) \cdot a_1, (n-1) \cdot a_2, a_3, \ldots, a_{k+2}) \\ = \; & F((n-k) \cdot a_1, a_3, \ldots, a_{k+2}). \end{aligned}$$

  By the induction hypothesis, we have $a_1 \in E_F$. We then have $F((n-1) \cdot a_1, a_2) = a_2 \neq a_1$, a contradiction.

- Assume now that $F(a_1, (n-1) \cdot a_2) = a_2$. Then, for $\ell \geq 1$ we have

$$\begin{aligned} & F((n-k-1) \cdot a_1, a_2, \ldots, a_{k+2}) \\ = \; & F^{\ell+1}((n-k-1+\ell) \cdot a_1, (\ell(n-2)+1) \cdot a_2, \ldots, a_{k+2}). \end{aligned}$$

For $\ell = k$, using idempotency and the fact that $k(n-2)+1 = n-k+(k-1)(n-1)$, we obtain

$$\begin{aligned}&F((n-k-1)\cdot a_1, a_2, \ldots, a_{k+2})\\ &\quad = \quad F^2((n-1)\cdot a_1, (n-k)\cdot a_2, a_3, \ldots, a_{k+2}).\end{aligned}$$

Thus, $F((n-k)\cdot a_2, a_3, \ldots, a_{k+2}) \notin \{a_1, \ldots, a_{k+2}\}$. By the induction hypothesis, we have $a_2, \ldots, a_{k+2} \in E_F$. It follows that $F(a_1, (n-1)\cdot a_2) = a_1 \neq a_2$, a contradiction.

Now, since $a_2 \in E_F$, it commutes with all other arguments of F by Lemma 5.20. Also, by (6.2) we have

$$F((n-k-1)\cdot a_1, a_3, \ldots, a_{k+2}, a_2) \notin \{a_1, \ldots, a_{k+2}\},$$

and thus $a_3 \in E_F$. Repeating this argument, we have that $a_1, \ldots, a_{k+2} \in E_F$.

It follows from the induction that if $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$, then $a_1, \ldots, a_n \in E_F$. Finally, we have $F(a_1, \ldots, a_n) \in E_F$ by Lemma 5.21. The second part is straightforward. □

In Proposition 5.24, it was shown that a quasitrivial n-ary semigroup cannot have more than two neutral elements. The next result shows that an operation in $\mathcal{F}^n_{n-1}$ is quasitrivial whenever it has at most two neutral elements.

Corollary 6.16 (see [26]). *An operation $F \in \mathcal{F}^n_{n-1}$ is quasitrivial if and only if $|E_F| \leq 2$.*

Proof. (Necessity) This follows from Proposition 5.24.

(Sufficiency) Suppose that F is not quasitrivial, i.e., there exist $a_1, \ldots, a_n \in X$ such that $F(a_1, \ldots, a_n) \notin \{a_1, \ldots, a_n\}$. Since F is idempotent, we must have $|\{a_1, \ldots, a_n\}| \geq 2$ and so $|\{a_1, \ldots, a_n, F(a_1, \ldots, a_n)\}| \geq 3$. We also have $\{a_1, \ldots, a_n, F(a_1, \ldots, a_n)\} \subseteq E_F$ by Proposition 6.15. Therefore we have $|E_F| \geq 3$. □

Proposition 6.17 (see [26]). *Let $F \in \mathcal{F}^n_{n-1}$ and suppose that $|E_F| \geq 3$. Then any element $x \in X \setminus E_F$ is an annihilator of $F|_{(\{x\}\bigcup E_F)^n}$. Moreover, $F|_{(X\setminus E_F)^n}$ is quasitrivial and has at most one neutral element.*

Proof. Let $x \in X \setminus E_F$ and $e \in E_F$ and let us show that $F(k\cdot x, (n-k)\cdot e) = x$ for any $k \in \{1, \ldots, n-1\}$. If $k = 1$, then this equality follows from the definition of a neutral element. Now, suppose that there exists $k \in \{2, \ldots, n-1\}$ such that $F(k\cdot x, (n-k)\cdot e) \neq x$. Since $x \in X \setminus E_F$, by Proposition 6.15 we must have $F(k\cdot x, (n-k)\cdot e) = e$. But then, using the associativity of F, we get

$$\begin{aligned}F((n-1)\cdot x, e) &= F((n-1)\cdot x, F(k\cdot x, (n-k)\cdot e))\\ &= F(k\cdot x, (n-k)\cdot e) \;=\; e,\end{aligned}$$

and we conclude by Lemma 6.14 that $x \in E_F$, which contradicts our assumption. Thus, we have

$$F(k\cdot x, (n-k)\cdot e) \;=\; x, \qquad k \in \{1, \ldots, n-1\}. \tag{6.4}$$

Now, let us show that $F(k\cdot x, e_{k+1}, \ldots, e_n) = x$ for any $k \in \{1, \ldots, n-1\}$ and any $e_{k+1}, \ldots, e_n \in E_F$. To this extent, we only need to show that

$$F(k\cdot x, e_{k+1}, \ldots, e_n) \;=\; F((k+1)\cdot x, e_{k+2}, \ldots, e_n),$$

for any $k \in \{1,\ldots,n-1\}$ and any $e_{k+1},\ldots,e_n \in E_F$. So, let $k \in \{1,\ldots,n-1\}$ and $e_{k+1},\ldots,e_n \in E_F$. Using (6.4) and the associativity of F we get

$$\begin{aligned} F(k\cdot x, e_{k+1},\ldots,e_n) &= F((k-1)\cdot x, F(2\cdot x,(n-2)\cdot e_{k+1}), e_{k+1},\ldots,e_n) \\ &= F(k\cdot x, F(x,(n-1)\cdot e_{k+1}), e_{k+2},\ldots,e_n) \\ &= F((k+1)\cdot x, e_{k+2},\ldots,e_n), \end{aligned}$$

which completes the proof by idempotency of F and Lemma 5.20. For the second part of the proposition, we observe that $F|_{(X\setminus E_F)^n}$ is quasitrivial by Proposition 6.15. Also, using (6.4) and the associativity of F, for any $x,y \in X\setminus E_F$ and any $e\in E_F$ we obtain

$$\begin{aligned} F((n-1)\cdot x, y) &= F((n-1)\cdot x, F(e,(n-1)\cdot y)) \\ &= F(F((n-1)\cdot x, e),(n-1)\cdot y) \;=\; F(x,(n-1)\cdot y), \end{aligned}$$

which shows that $F|_{(X\setminus E_F)^n}$ cannot have more than one neutral element. □

Proof of Theorem 6.5. It is easy to check that every $G\in\mathcal{H}_m$ is associative.

Now, we consider $G\in\mathcal{H}_{n-1}$ and define $F = G^{n-1}$. Then we have $E_F = Y$. Indeed, conditions (a) and (c) of Definition 6.3 imply directly that $Y\subseteq E_F$. Moreover if $x\notin Y$, then still by condition (c) we have $F((n-1)\cdot x, y) = x \neq y$ for $y\in Y$, so $x\notin E_F$.

Next, we show that $F((k-1)\cdot x, y, (n-k)\cdot x)\in\{x,y\}$ for every $x,y\in X$ and every $k\in\{1,\ldots,n\}$. If $x\in Y$, x is a neutral element, so this expression is equal to y. If $x\in X\setminus Y$, then either $y\in Y$ and this expression is equal to x (by condition (c)), or $y\in X\setminus Y$, and this expression is in $\{x,y\}$ (by condition (b)). Finally, $F\notin\mathcal{F}_1^n$ by Corollary 6.16, since $|E_F| = |Y|\geq 3$.

Now we prove the converse statement and consider $F\in\mathcal{F}_{n-1}^n\setminus\mathcal{F}_1^n$. Setting $Y = E_F$ we have $|Y|\geq 3$ by Corollary 6.16. By Propositions 5.5 and 5.16, every reduction of F reads G_e for some $e\in E_F$.

Finally, we show that $G_e\in\mathcal{H}_{n-1}$. We have that $(Y, G_e|_{Y^2})$ is an Abelian group whose exponent divides $n-1$ by Proposition 5.5 and Corollary 6.13. Also, we have that $G_e|_{(X\setminus Y)^2}$ is quasitrivial by Theorem 5.26 and Proposition 6.17. Finally, we have that any $x\in X\setminus Y$ is an annihilator for $G_e|_{(\{x\}\bigcup Y)^2}$ by Proposition 6.17. □

Proof of Corollary 6.6. This follows from Lemma 5.4, Corollary 5.6, and Proposition 6.15. □

Remark 6.18. In the proof of Corollary 6.6 we used Corollary 5.6 which is based on results obtained by Ackerman [1]. In Section 6.4 we provide an alternative proof of Corollary 6.6 that does not make use of Corollary 5.6.

Proof of Theorem 6.7. If $F\in\mathcal{F}_{n-1}^n\setminus\mathcal{F}_1^n$, then by Theorem 6.5 we have $|E_F|\geq 3$ and for every $e\in E_F$, G_e is in $\mathcal{H}_{n-1}$. Then $G_e|_{Y^2}$ is a reduction of $F|_{Y^n}$ and (a) holds true. Also $G_e|_{(X\setminus Y)^2}$ is a quasitrivial reduction of $F|_{(X\setminus Y)^n}$, so (b) holds true by Corollary 5.28. Finally, if $x_i, x_{i+1}\in X$ satisfy the conditions of (c), we have $G_e(x_i,x_{i+1}) = x = G_e(x,x)$, so that (c) holds true.

Let us now assume that an operation F satisfies conditions (a), (b) and (c). By (a), there exists an Abelian group (Y, G_Y) whose exponent divides $n-1$ such that $(Y, F|_{Y^n})$ is the n-ary extension of (Y, G_Y). We denote by e the neutral element of G_Y. We also define the operation $G\colon X^2\to X$ by $G(x,y) = F(x,(n-2)\cdot e, y)$ for every $x,y\in X$. We now show that G is in $\mathcal{H}_{n-1}$. It is easy to see that $G|_{Y^2} = G_Y$. Then, by condition (c), $G|_{(X\setminus Y)^2}(x,y) = F((n-1)\cdot x, y)$, so $G|_{(X\setminus Y)^2}$ is the unique quasitrivial reduction of $F|_{(X\setminus Y)^n}$ (see Corollary 5.28). Finally, condition (c) also implies that any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\}\bigcup Y)^2}$. Then by Theorem 6.5, G is associative and we have $G^{n-1} \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$. We conclude the proof by showing that $G^{n-1} = F$. To this aim we compare $G^{n-1}(x_1, \ldots, x_n)$ and $F(x_1, \ldots, x_n)$ for every $(x_1, \ldots, x_n) \in X^n$. We already showed that both expressions coincide if $(x_1, \ldots, x_n)$ belongs to Y^n or $(X \setminus Y)^n$. Otherwise, let us denote by $\sigma_1, \ldots, \sigma_r$ the integers such that $\sigma_1 < \cdots < \sigma_r$ and $\{x_1, \ldots, x_n\} \cap (X \setminus Y) = \{x_{\sigma_1}, \ldots, x_{\sigma_r}\}$. By condition (c) there exist integers $a_1, \ldots, a_r$ such that

$$F(x_1, \ldots, x_n) = F(a_1 \cdot x_{\sigma_1}, \ldots, a_r \cdot x_{\sigma_r}).$$

This expression is equal to $G^{r-1}(x_{\sigma_1}, \ldots, x_{\sigma_r})$ because $G|_{(X\setminus Y)^2}$ is a quasitrivial reduction of $F|_{(X\setminus Y)^n}$. Using condition (c) in Definition 6.3 for $G \in \mathcal{H}_{n-1}$ we get that this expression is equal to $G^{n-1}(x_1, \ldots, x_n)$. □

Proof of Proposition 6.9. (Necessity) If $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n \neq \varnothing$, then Theorem 6.5 implies that there is a subset $Y \subseteq X$ and an Abelian group (Y, G) whose exponent divides $n - 1$ and $|Y| \geq 3$. This shows that $|X| \geq |Y| \geq c_{n-1}$.

(Sufficiency) Assume that $|X| \geq c_{n-1}$. Then we choose a subset $Y \subseteq X$ such that $|Y| = c_{n-1} \geq 3$ and we endow Y with an operation G_Y such that (Y, G_Y) is an Abelian group whose exponent divides $n - 1$. Let us consider the operation $G\colon X^2 \to X$ defined by the conditions that any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\}\bigcup Y)^2}$, that $G|_{Y^2} = G_Y$, and that $G(x, y) = y$ for any $x, y \in X \setminus Y$. Then we have $G \in \mathcal{H}_{n-1}$ and so $G^{n-1} \in \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$ by Theorem 6.5, which concludes the proof. □

Proof of Corollary 6.10. By Proposition 6.9 it is sufficient to compute c_{n-1} in the two cases.

(a) The cyclic group of order p is an Abelian group with at least three elements whose exponent divides $n - 1$, hence $c_{n-1} \leq p$. On the other hand, let (Y, G) be any Abelian group with at least three elements whose exponent m divides $n - 1$. Let q be a prime divisor of m; then q divides $n - 1$, hence q is odd. From the definition of the exponent it follows that Y contains an element of order q, thus $|Y| \geq q$. Since q divides $n - 1$, we have $q \geq p$ by the minimality of p. Therefore, $|Y| \geq q \geq p$, which shows that $c_{n-1} \geq p$.

(b) If $p = 3$, then we can take the group $\mathbb{Z}_p$ as in the previous case; if $p \geq 5$, then we can take the group $\mathbb{Z}_2^2$ (with exponent 2 dividing $n - 1$) in order to see that $c_{n-1} \leq \min(4, p)$. Conversely, let (Y, G) be any Abelian group with at least three elements and with exponent m such that m divides $n - 1$. If m has an odd prime divisor q, then we can conclude that $|Y| \geq q \geq p \geq \min(4, p)$ just as in (a). If m has no odd prime divisors, then m is a power of 2, and then $|Y|$ is even, which together with $|Y| \geq 3$ implies that $|Y| \geq 4 \geq \min(4, p)$. Thus, we conclude that $c_{n-1} \geq \min(4, p)$.

□

Proof of Proposition 6.12. (Necessity) Obvious.

(Sufficiency) Let $Y \subseteq X$ such that $|Y| = 3$. We can endow Y with a semilattice order $\preceq$ such that $(Y, \preceq)$ is a semilattice that is not a chain. Let us consider the operation $G\colon X^2 \to X$ defined by the following conditions:

- $G|_{Y^2} = \curlyvee$, where $\curlyvee : Y^2 \to Y$ is the semilattice operation associated with $(Y, \preceq)$.
- $G(x, y) = x$ for any $x, y \in X \setminus Y$.

- Any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\} \bigcup Y)^2}$.

It is not difficult to see that G is associative and idempotent and that $G^{n-1} \in \mathcal{F}_n^n \setminus \mathcal{F}_{n-1}^n$. □

6.3 An alternative hierarchy

For any integer $k \geq 1$, let S_k^n be the set of n-tuples $(x_1, \ldots, x_n) \in X^n$ such that $|\{x_1, \ldots, x_n\}| \leq k$. Of course, we have $D_k^n \subseteq S_{n-k+1}^n$ for any $k \in \{1, \ldots, n\}$. Also, we have $S_k^n \subseteq S_{k+1}^n$ for any $k \in \{1, \ldots, n-1\}$. Now, denote by $\mathcal{G}_k^n$ the class of those associative n-ary operations $F\colon X^n \to X$ satisfying

$$F(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\}, \qquad (x_1, \ldots, x_n) \in S_k^n.$$

We say that these operations are *quasitrivial on* S_k^n.

It is not difficult to see that if $F \in \mathcal{G}_k^n$, then $F \in \mathcal{F}_{n-k+1}^n$. Actually, we have $\mathcal{G}_1^n = \mathcal{F}_n^n$ and $\mathcal{G}_n^n = \mathcal{F}_1^n$. These are the only classes when $n = 2$, and thus we assume throughout this section that $n \geq 3$. Due to Proposition 6.1, we have that $\mathcal{G}_n^n = \cdots = \mathcal{G}_3^n$ is exactly the class of quasitrivial associative n-ary operations, and hence we only need to consider operations in $\mathcal{G}_2^n$. The analog of Theorem 6.5 can then be stated as follows.

Theorem 6.19 (see [26]). *If n is odd and $G \in \mathcal{H}_2$, then its n-ary extension $F = G^{n-1}$ is in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$. Conversely, for every $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$ we have $|E_F| \geq 3$, n is odd, the reductions of F are exactly the operations of the form G_e for $e \in E_F$, and they lie in $\mathcal{H}_2$.*

Proof. If n is odd and $G \in \mathcal{H}_2$, then $n-1$ is even, and so $G \in \mathcal{H}_{n-1}$. Therefore by Theorem 6.5, $F = G^{n-1}$ is in $\mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n = \mathcal{F}_{n-1}^n \setminus \mathcal{G}_n^n$. We have shown in the proof of Theorem 6.5 that $E_F = Y$. In order to show $F \in \mathcal{G}_2^n$, we need to show that if $x_1, \ldots, x_n \in \{x, y\}$, then $F(x_1, \ldots, x_n) \in \{x, y\}$. If x or y is in $X \setminus Y$, this follows from Proposition 6.17. If $\{x, y\} \subsetneq Y$, then if k arguments are equal to x and $n-k$ are equal to y, $F(x_1, \ldots, x_n) = F(k \cdot x, (n-k) \cdot y)$ because $(Y, G|_{Y^2})$ is an Abelian group. Since n is odd, the parity of k and of $n-k$ are different. Since $(Y, G|_{Y^2})$ has exponent 2, this expression is equal to x (resp. y) when k is odd (resp. even).

Conversely, if $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n \subseteq \mathcal{F}_{n-1}^n \setminus \mathcal{F}_1^n$, then by Theorem 6.5, we have $|E_F| \geq 3$, all the reductions of F are exactly the operations G_e for $e \in E_F$ and they lie in $\mathcal{H}_{n-1}$. In particular, for any $e \in E_F$, we have that (E_F, G_e) is an Abelian group whose exponent divides $n-1$. However, since the neutral element is the only idempotent element of a group and since $G_e(e', e') \in \{e, e'\}$ for any $e, e' \in E_F$, it follows that $G_e(e', e') = e$ for any $e, e' \in E_F$, i.e., for any $e \in E_F$ we have that (E_F, G_e) is a group of exponent 2. (Recall that an element $x \in X$ is said to be *idempotent* for an operation $F\colon X^n \to X$ if $F(n \cdot x) = x$.) Therefore, we conclude that $(E_F, F|_{E_F^n})$ is the n-ary extension of an Abelian group of exponent 2. Also, since 2 divides $n-1$ we conclude that n is odd. □

Theorem 6.19 is particularly interesting as it enables us to construct easily n-ary operations in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$. For instance, consider the set $X_6 = \{1, 2, 3, 4, 5, 6\}$ together with the operation $G\colon X_6^2 \to X_6$ defined by the following conditions:

- $(\{1, 2, 3, 4\}, G|_{\{1,2,3,4\}^2})$ is isomorphic to $(\mathbb{Z}_2^2, +)$,
- $G|_{\{5,6\}^2} = \pi_1|_{\{5,6\}^2}$,

- for any $x \in \{1,2,3,4\}$, $G(x,5) = G(5,x) = 5$ and $G(x,6) = G(6,x) = 6$.

Then for any integer $p \geq 1$, we have that the operation associated with any $(2p+1)$-ary extension of $(\{1,2,3,4,5,6\}, G)$ is in $\mathcal{G}_2^{2p+1} \setminus \mathcal{G}_{2p+1}^{2p+1}$ by Theorem 6.19.

We now state a reformulation of Theorem 6.19 that does not make use of binary reductions. We omit the proof of this result as it is a straightforward adaptation of the proof of Theorem 6.7.

Theorem 6.20 (see [26]). *If an operation F is in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$, then n is odd and setting $Y = E_F$ we have $|Y| \geq 3$ and the following assertions hold.*

(a) $(Y, F|_{Y^n})$ *is the n-ary extension of an Abelian group of exponent 2.*

(b) $F|_{(X\setminus Y)^n}$ *is associative, quasitrivial, and has at most one neutral element.*

(c) For all $x_1, \ldots, x_n \in X$ and $i \in \{1, \ldots, n-1\}$ such that $\{x_i, x_{i+1}\} \cap (X \setminus Y) = \{x\}$ we have

$$F(x_1, \ldots, x_n) = F(x_1, \ldots, x_{i-1}, x, x, x_{i+2}, \ldots, x_n).$$

Conversely, if n is odd and F is an operation that satisfies these conditions for some $Y \subseteq X$ such that $|Y| \geq 3$, then $F \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$ and $E_F = Y$.

We end this section with the counterpart of Proposition 6.9 and Corollary 6.10 for operations in $\mathcal{G}_2^n \setminus \mathcal{G}_n^n$.

Corollary 6.21 (see [26]). *We have $\mathcal{G}_2^n \setminus \mathcal{G}_n^n \neq \varnothing$ if and only if n is odd and $|X| \geq 4$.*

Proof. (Necessity) By Theorem 6.19, we have that n is odd and there exists a subset $Y \subseteq X$ and an Abelian group (Y, G) of exponent 2 such that $|Y| \geq 3$. Since (Y, G) is of exponent 2 we have $|X| \geq |Y| \geq 4$.

(Sufficiency) Let $Y \subseteq X$ such that $|Y| = 4$. We can endow Y with an operation G_Y such that (Y, G_Y) is an Abelian group of exponent 2 that is isomorphic to $(\mathbb{Z}_2^2, +)$. Let us consider the operation $G\colon X^2 \to X$ defined by the following conditions:

- $G|_{Y^2} = G_Y$.
- $G|_{(X\setminus Y)^2} = \pi_2|_{(X\setminus Y)^2}$.
- Any $x \in X \setminus Y$ is an annihilator for $G|_{(\{x\}\bigcup Y)^2}$.

It is not difficult to see that $G \in \mathcal{H}_2$ (see Definition 6.3). Thus, we have $G^{n-1} \in \mathcal{G}_2^n \setminus \mathcal{G}_n^n$ by Theorem 6.19, which concludes the proof. □

6.4 An alternative proof of Corollary 6.6

We provide an alternative proof of Corollary 6.6 that does not use Corollary 5.6. To this extent, we first prove the following general result.

Proposition 6.22 (see [26]). *Let $F \in \mathcal{F}_n^n$. The following assertions are equivalent.*

(i) F is reducible to an associative and idempotent operation $G\colon X^2 \to X$.

(ii) $F((n-1)\cdot x,y)=F(x,(n-1)\cdot y)$ *for any* $x,y\in X$.

Proof. The implication $(i)\Rightarrow(ii)$ is straightforward. Now, let us show that $(ii)\Rightarrow(i)$. So, suppose that

$$F((n-1)\cdot x,y) \;=\; F(x,(n-1)\cdot y) \qquad x,y\in X, \tag{6.5}$$

and consider the operation $G\colon X^2\to X$ defined by $G(x,y)=F((n-1)\cdot x,y)$ for any $x,y\in X$. It is not difficult to see that G is associative and idempotent. Now, let $x_1,\ldots,x_n\in X$ and let us show that $G^{n-1}(x_1,\ldots,x_n)=F(x_1,\ldots,x_n)$. Using repeatedly (6.5) and the idempotency of F we obtain

$$\begin{aligned}
G^{n-1}(x_1,\ldots,x_n) &= F^{n-1}((n-1)\cdot x_1,(n-1)\cdot x_2,\ldots,(n-1)\cdot x_{n-1},x_n)\\
&= F^{n-1}((2n-3)\cdot x_1,x_2,(n-1)\cdot x_3,\ldots,(n-1)\cdot x_{n-1},x_n)\\
&= \cdots\\
&= F^{n-1}(((n-2)(n-1)+1)\cdot x_1,x_2,x_3,\ldots,x_{n-1},x_n)\\
&= F(x_1,\ldots,x_n),
\end{aligned}$$

which shows that F is reducible to G. □

Remark 6.23. Let $\leq$ be a total order on X. One of the main results of Kiss and Somlai [66, Theorem 4.8] is that every $\leq$-preserving operation $F\in\mathcal{F}_n^n$ is reducible to an associative, idempotent, and $\leq$-preserving binary operation. To this extent, they first show [66, Lemma 4.1] that any $\leq$-preserving operation $F\in\mathcal{F}_n^n$ satisfies

$$F((n-1)\cdot x,y) \;=\; F(x,(n-1)\cdot y) \qquad x,y\in X.$$

Thus, we conclude that [66, Theorem 4.8] is an immediate consequence of [66, Lemma 4.1] and Proposition 6.22 above.

The following result is the key for the alternative proof of Corollary 6.6.

Proposition 6.24 (see [26]). *Let* $F\in\mathcal{F}_{n-1}^n$. *The following assertions are equivalent.*

(i) F *is reducible to an associative and quasitrivial operation* $G\colon X^2\to X$.

(ii) F *is reducible to an associative and idempotent operation* $G\colon X^2\to X$.

(iii) $F((n-1)\cdot x,y)=F(x,(n-1)\cdot y)$ *for any* $x,y\in X$.

(iv) $|E_F|\leq 1$.

Proof. The equivalence $(i)\Leftrightarrow(ii)$ and the implication $(iii)\Rightarrow(iv)$ are straightforward. Also, the equivalence $(ii)\Leftrightarrow(iii)$ follows from Proposition 6.22. Now, let us show that $(iv)\Rightarrow(iii)$. So, suppose that $|E_F|\leq 1$ and suppose to the contrary that there exist $x,y\in X$ with $x\neq y$ such that $F((n-1)\cdot x,y)\neq F(x,(n-1)\cdot y)$. We have two cases to consider. If $F((n-1)\cdot x,y)=y$ and $F(x,(n-1)\cdot y)=x$, then by Lemma 6.14 we have that $x,y\in E_F$, which contradicts our assumption on E_F. Otherwise, if $F((n-1)\cdot x,y)=x$ and $F(x,(n-1)\cdot y)=y$, then we have

$$\begin{aligned}
x \;=\; F((n-1)\cdot x,y) \;&=\; F((n-1)\cdot x,F(n\cdot y))\\
&=\; F(F((n-1)\cdot x,y),(n-1)\cdot y) \;=\; F(x,(n-1)\cdot y) \;=\; y,
\end{aligned}$$

which contradicts the fact that $x\neq y$. □

Proof of Corollary 6.6. This follows from Lemma 5.4 and Proposition 6.24. □

Chapter 7

Symmetric idempotent n-ary semigroups

In this chapter, we study the class of symmetric idempotent n-ary semigroups. More precisely, we introduce the concept of strong semilattice of semigroups which is the most important semigroup construction in this chapter (Section 7.1). Then we show that with any symmetric idempotent n-ary semigroup (X, F) we can associate a binary band (X, B), that is in general not commutative. We study the properties of this band and in particular its semilattice decomposition (Section 7.2). Also, we show that the restriction of F to each subset of this decomposition reduces to the operation of a commutative group whose exponent divides $n-1$ (Section 7.3). Conversely, we show that given a binary band (X, B) such that the restriction of B to each subset of its semilattice decomposition is a right normal band operation, we can build in a unique way a symmetric idempotent n-ary semigroup (X, F) whose associated binary band is (X, B) (Section 7.3). These two constructions provide a structure theorem for symmetric idempotent n-ary semigroups. Finally, we show how to use this theorem to provide necessary and sufficient conditions that ensure the reducibility of any symmetric idempotent n-ary semigroup to a semigroup (Section 7.4). Most of the contributions presented in this chapter stem from [35].

7.1 Semilattices of semigroups

In this section, we introduce the most important semigroup constructions that we use in this chapter; for a background on these constructions, see for instance [21, 56, 80, 81, 87].

We observe that if (X, G) is a semilattice of semigroups (see p. 40), then we do not have a complete understanding of (X, G). Indeed, by (3.3) we only know that $G(x, y) \in X_{\alpha \curlyvee \beta}$ for any $(x, y) \in X_\alpha \times X_\beta$ but we do not know what is the exact value of $G(x, y)$. In order to get this information we need to introduce the concept of strong semilattice of semigroups [81].

Definition 7.1. Let $(X, G) = ((Y, \curlyvee), (X_\alpha, G_\alpha))$ be a semilattice of semigroups. Suppose that for any $\alpha, \beta \in Y$ such that $\alpha \preceq \beta$ there is a homomorphism $\varphi_{\alpha,\beta}\colon X_\alpha \to X_\beta$ such that the following conditions hold.

(a) The map $\varphi_{\alpha,\alpha}$ is the identity on X_α.

(b) For any $\alpha, \beta, \gamma \in Y$ such that $\alpha \preceq \beta \preceq \gamma$ we have $\varphi_{\beta,\gamma} \circ \varphi_{\alpha,\beta} = \varphi_{\alpha,\gamma}$.

(c) For any $x \in X_\alpha$ and any $y \in X_\beta$ we have $G(x, y) = G_{\alpha \curlyvee \beta}(\varphi_{\alpha,\alpha \curlyvee \beta}(x), \varphi_{\beta,\alpha \curlyvee \beta}(y))$.

Then (X,G) is said to be a *strong semilattice* $(Y,\curlyvee)$ *of semigroups* (X_α,G_α). In this case we write $(X,G)=((Y,\curlyvee),(X_\alpha,G_\alpha),\varphi_{\alpha,\beta})$ and we simply say that (X,G) is a strong semilattice of semigroups.

It is not difficult to see that any strong semilattice of semigroups is a semigroup [56].

Now, we introduce a class of bands that will be useful in the next section. A band (X,G) is said to be *right normal* if $G(G(x,y),z)=G(G(y,x),z)$ for any $x,y,z\in X$. If (X,G) is a right normal band, then the least semilattice congruence $\sim$ on (X,G) is defined by

$$x\sim y\quad\Leftrightarrow\quad G(y,x)=x\quad\text{and}\quad G(x,y)=y,\qquad x,y\in X.\tag{7.1}$$

The following proposition provides a characterization of right normal bands.

Proposition 7.2 (see [56])**.** *A band (X,G) is right normal if and only if it is a strong semilattice of right zero semigroups.*

To conclude this section we introduce a generalization of the concepts of semilattices of semigroups and strong semilattices of semigroups to n-ary semigroups.

Let $(Y,\curlyvee)$ be a semilattice. We denote the n-ary extension of $(Y,\curlyvee)$ by $(Y,\curlyvee^{n-1})$, where $\curlyvee^{n-1}\colon Y^n\to Y$ is defined by $\curlyvee^{n-1}(x_1,\ldots,x_n)=x_1\curlyvee\cdots\curlyvee x_n$ for any $x_1,\ldots,x_n\in Y$. Also, we say that the n-ary extension of a semilattice is an *n-ary semilattice*.

Recall that an equivalence relation $\sim$ on X is said to be a *congruence for $F\colon X^n\to X$* [98] if it is compatible with F, that is, $F(x_1,\ldots,x_n)\sim F(y_1,\ldots,y_n)$ for all $x_1,\ldots,x_n,y_1,\ldots,y_n\in X$ such that $x_i\sim y_i$ for each $i\in\{1,\ldots,n\}$. In that case, $\sim$ is also said to be a congruence on the n-ary groupoid (X,F). Also, we denote by $\tilde{F}$ the map induced by F on $X/\sim$, that is,

$$\tilde{F}([x_1]_\sim,\ldots,[x_n]_\sim)=[F(x_1,\ldots,x_n)]_\sim,\qquad x_1,\ldots,x_n\in X.$$

We say that a congruence $\sim$ on an n-ary groupoid (X,F) is an *n-ary semilattice congruence* if $(X/{\sim},\tilde{F})$ is an n-ary semilattice.

Remark 7.3. We observe that an alternative definition of semilattice congruences for ternary semigroups was already given in [58]. In this reference, a congruence $\sim$ on a ternary semigroup (X,F) is said to be a semilattice congruence if $(X/{\sim},\tilde{F})$ is a symmetric idempotent ternary semigroup.

Let $(Y,\curlyvee)$ be a semilattice and let $\{(X_\alpha,F_\alpha)\colon\alpha\in Y\}$ be a set of n-ary semigroups such that $X_\alpha\cap X_\beta=\varnothing$ for any $\alpha\neq\beta$. We say that an n-ary groupoid (X,F) is an *n-ary semilattice* $(Y,\curlyvee^{n-1})$ *of n-ary semigroups* (X_α,F_α) if $X=\bigcup_{\alpha\in Y}X_\alpha$, $F|_{X_\alpha^n}=F_\alpha$ for every $\alpha\in Y$, and

$$F(X_{\alpha_1}\times\cdots\times X_{\alpha_n})\subseteq X_{\alpha_1\curlyvee\cdots\curlyvee\alpha_n},\qquad\alpha_1,\ldots,\alpha_n\in Y.\tag{7.2}$$

In this case we write $(X,F)=((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha))$ and we simply say that (X,F) is an n-ary semilattice of n-ary semigroups.

Actually, we have the following characterization of the class of n-ary semilattices of n-ary semigroups.

Proposition 7.4. *An n-ary semigroup (X,F) is an n-ary semilattice of n-ary semigroups if and only if there exists an n-ary semilattice congruence on (X,F).*

Proof. (Necessity) If $(X,F)=((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha))$, then it is not difficult to see that the binary relation $\sim$ on X defined by

$$x\sim y\quad\Leftrightarrow\quad \exists\,\alpha\in Y \text{ such that } x,y\in X_\alpha,\qquad x,y\in X,$$

is an n-ary semilattice congruence on (X,F).

(Sufficiency) If $\sim$ is an n-ary semilattice congruence on (X,F), then it is not difficult to see that $(X,F)=((X/{\sim},\tilde{F}),([x]_\sim,F|_{[x]^n_\sim}))$. $\square$

As a consequence, we obtain the following result which is the counterpart of Corollary 3.2 for n-ary groupoids.

Corollary 7.5. *An n-ary groupoid (X,F) is an n-ary semilattice of n-ary semigroups if and only if there exists an n-ary semilattice congruence on (X,F) such that $([x]_\sim,F|_{[x]^n_\sim})$ is an n-ary semigroup for any $x\in X$.*

We observe that if (X,F) is an n-ary semilattice of n-ary semigroups, then we do not have a complete understanding of (X,F). Indeed, by (7.2) we only know that $F(x_1,\ldots,x_n)\in X_{\alpha_1\curlyvee\cdots\curlyvee\alpha_n}$ for any $(x_1,\ldots,x_n)\in X_{\alpha_1}\times\cdots\times X_{\alpha_n}$ but we do not know what is the exact value of $F(x_1,\ldots,x_n)$. In order to get this information we need to introduce the concept of strong n-ary semilattice of n-ary semigroups.

Definition 7.6 (see [35])**.** Let $(X,F)=((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha))$ be an n-ary semilattice of n-ary semigroups. Suppose that for any $\alpha,\beta\in Y$ such that $\alpha\preceq\beta$ there is a homomorphism $\varphi_{\alpha,\beta}\colon X_\alpha\to X_\beta$ such that the following conditions hold.

(a) The map $\varphi_{\alpha,\alpha}$ is the identity on X_α.

(b) For any $\alpha,\beta,\gamma\in Y$ such that $\alpha\preceq\beta\preceq\gamma$ we have $\varphi_{\beta,\gamma}\circ\varphi_{\alpha,\beta}=\varphi_{\alpha,\gamma}$.

(c) For any $(x_1,\ldots,x_n)\in X_{\alpha_1}\times\cdots\times X_{\alpha_n}$ we have

$$F(x_1,\ldots,x_n)=F_{\alpha_1\curlyvee\cdots\curlyvee\alpha_n}(\varphi_{\alpha_1,\alpha_1\curlyvee\cdots\curlyvee\alpha_n}(x_1),\ldots,\varphi_{\alpha_n,\alpha_1\curlyvee\cdots\curlyvee\alpha_n}(x_n)).$$

Then (X,F) is said to be a *strong n-ary semilattice* $(Y,\curlyvee^{n-1})$ *of n-ary semigroups* (X_α,F_α). In this case we write $(X,F)=((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha),\varphi_{\alpha,\beta})$ and we simply say that (X,F) is a strong n-ary semilattice of n-ary semigroups.

The next result shows that any strong n-ary semilattice of n-ary semigroups is an n-ary semigroup.

Proposition 7.7 (see [35])**.** *If $(X,F)=((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha),\varphi_{\alpha,\beta})$ is a strong n-ary semilattice of n-ary semigroups, then it is an n-ary semigroup.*

Proof. Let $x_1,\ldots,x_{2n-1}\in X$, let $i\in\{1,\ldots,n\}$, and let us consider

$$F(x_1,\ldots,x_{i-1},F(x_i,\ldots,x_{i+n-1}),x_{i+n},\ldots,x_{2n-1}).$$

If x_k belongs to X_{α_k} for $1\leq k\leq 2n-1$, then setting $\alpha'=\alpha_i\curlyvee\cdots\curlyvee\alpha_{i+n-1}$ we have

$$F(x_i,\ldots,x_{i+n-1})=F_{\alpha'}(\varphi_{\alpha_i,\alpha'}(x_i),\ldots,\varphi_{\alpha_{i+n-1},\alpha'}(x_{i+n-1}))\in X_{\alpha'}.$$

Then denoting the latter element by x and setting $\alpha = \alpha_1 \curlyvee \cdots \curlyvee \alpha_{i-1} \curlyvee \alpha' \curlyvee \alpha_{i+n} \curlyvee \cdots \curlyvee \alpha_{2n-1}$, we have that

$$\begin{aligned} & F(x_1 \ldots, x_{i-1}, x, x_{i+n}, \ldots, x_{2n-1}) \\ & \qquad = F_\alpha(\varphi_{\alpha_1,\alpha}(x_1), \ldots, \varphi_{\alpha_{i-1},\alpha}(x_{i-1}), \varphi_{\alpha',\alpha}(x), \varphi_{\alpha_{i+n},\alpha}(x_{i+n}), \ldots, \varphi_{\alpha_{2n-1},\alpha}(x_{2n-1})). \end{aligned}$$

Also, using the fact that $\varphi_{\alpha',\alpha}$ is a homomorphism and condition (b) of Definition 7.6, we have that

$$\varphi_{\alpha',\alpha}(x) = F_\alpha(\varphi_{\alpha_i,\alpha}(x_i), \ldots, \varphi_{\alpha_{i+n-1},\alpha}(x_{i+n-1})).$$

Thus, the associativity of F follows from the associativity of F_α and from the fact that α is independent of i. □

We say that an idempotent n-ary semigroup is an *n-ary band*. Also, we say that a symmetric idempotent n-ary semigroup is a *commutative n-ary band*. Examples of commutative n-ary bands are given by n-ary extensions of semilattices and n-ary extensions of Abelian groups whose exponents divide $n-1$. As the following example shows, there are also commutative n-ary bands that are not reducible to binary semigroups. This example can be checked by hands, by tedious computations. We will develop tools that will allow to check these properties and to build such examples very easily.

Example 7.8. Consider the set $X = \{1, 2, 3, 4\}$ together with the symmetric ternary operation $F\colon X^3 \to X$ defined by its level sets given (up to permutations) by $F^{-1}[1] = \{(1, 1, 1)\}$, $F^{-1}[2] = \{(2, 2, 2)\}$,

$$F^{-1}[3] = \{(1, 1, 2), (1, 1, 3), (1, 2, 4), (1, 3, 4), (2, 2, 3), (2, 3, 3), (2, 4, 4), (3, 3, 3), (3, 4, 4)\},$$

and

$$F^{-1}[4] \;=\; X^3 \setminus (F^{-1}[1] \cup F^{-1}[2] \cup F^{-1}[3]).$$

This operation defines a commutative ternary band and is not reducible to any binary operation.

7.2 The associated binary band

Throughout this section, we consider a commutative n-ary band (X, F). We associate with it a classical (binary) band and study its most important properties. In particular, we show that this associated band is right normal.

Definition 7.9 (see [35])**.** Let (X, F) be a symmetric n-ary semigroup. The binary operation $B_F\colon X^2 \to X$ associated with F is defined by

$$B_F(x, y) = F((n-1) \cdot x, y), \qquad x, y \in X.$$

For any $x \in X$, we also define the operation $\ell_x^F : X \to X$ by

$$\ell_x^F(y) = B_F(x, y), \qquad y \in X.$$

When there is no risk of confusion, we also denote these operations by B and ℓ_x, respectively. We now study elementary properties of these maps.

Let (Y, F_1) and (Z, F_2) be two n-ary groupoids. Recall that a map $\varphi\colon Y \to Z$ is said to be a *homomorphism* if

$$\varphi(F_1(x_1,\ldots,x_n)) = F_2(\varphi(x_1),\ldots,\varphi(x_n)), \qquad x_1,\ldots,x_n \in Y.$$

It is said to be an *endomorphism* if $(Y, F_1) = (Z, F_2)$. In that case, we say that φ is an endomorphism for F_1.

Proposition 7.10 (see [35]). *Let (X, F) be a commutative n-ary band. We have $\ell_x^2 = \ell_x$ for any $x \in X$. Also, we have*

$$\ell_x(F(x_1,\ldots,x_n)) = F(x_1,\ldots,\ell_x(x_i),\ldots,x_n), \qquad x, x_1,\ldots,x_n \in X, i \in \{1,\ldots,n\}. \quad (7.3)$$

Moreover, for any $x \in X$ the map ℓ_x is an endomorphism for F.

Proof. Let $x, y \in X$ and let us show that $\ell_x^2(y) = \ell_x(y)$. By associativity and idempotency of F we have

$$\begin{aligned} \ell_x^2(y) &= F((n-1)\cdot x, \ell_x(y)) \;=\; F((n-1)\cdot x, F((n-1)\cdot x, y)) \\ &= F(F(n\cdot x), (n-2)\cdot x, y) \;=\; F((n-1)\cdot x, y) \;=\; \ell_x(y). \end{aligned}$$

Now, let $x, x_1,\ldots,x_n \in X$ and let us show that (7.3) holds for $i = 1$. The other cases are obtained by the symmetry of F. Using the definition of ℓ_x and the associativity of F we have

$$\begin{aligned} \ell_x(F(x_1,\ldots,x_n)) &= F((n-1)\cdot x, F(x_1,\ldots,x_n)) \\ &= F(F((n-1)\cdot x, x_1), x_2,\ldots,x_n) \\ &= F(\ell_x(x_1), x_2,\ldots,x_n). \end{aligned}$$

Finally, let us show that for any $x \in X$ the map ℓ_x is an endomorphism for F. To do so, let $x, x_1,\ldots,x_n \in X$ and let us show that $\ell_x(F(x_1,\ldots,x_n)) = F(\ell_x(x_1),\ldots,\ell_x(x_n))$. Since $\ell_x^2 = \ell_x$, we have $\ell_x(F(x_1,\ldots,x_n)) = \ell_x^n(F(x_1,\ldots,x_n))$. Thus, applying (7.3) several times, we obtain

$$\ell_x(F(x_1,\ldots,x_n)) \;=\; F(\ell_x(x_1),\ldots,\ell_x(x_n)),$$

which concludes the proof. □

From the idempotency of F we derive that $\ell_x(x) = x$ for any $x \in X$. We then obtain the following corollary, that will be useful in the next section.

Corollary 7.11 (see [35]). *Let (X, F) be a commutative n-ary band. For any $x_1,\ldots,x_n \in X$ we have*

$$F(x_1,\ldots,x_n) \;=\; F(\ell_{F(x_1,\ldots,x_n)}(x_1),\ldots,\ell_{F(x_1,\ldots,x_n)}(x_n)).$$

Now, we show that the groupoid (X, B) associated with (X, F) is a right normal band.

Proposition 7.12 (see [35]). *Let (X, F) be a commutative n-ary band. We have*

$$\ell_x \circ \ell_y \;=\; \ell_y \circ \ell_x \;=\; \ell_{\ell_x(y)} \;=\; \ell_{\ell_y(x)}, \qquad x, y \in X.$$

In other words, the pair (X, B) is a right normal band.

Proof. Let $x, y, z \in X$ and let us show that $\ell_x(\ell_y(z)) = \ell_y(\ell_x(z))$. Using (7.3) we have

$$\ell_x(\ell_y(z)) \;=\; \ell_x(F((n-1)\cdot y, z)) \;=\; F((n-1)\cdot y, \ell_x(z)) \;=\; \ell_y(\ell_x(z)).$$

The same relation (applied $n-1$ times) yields

$$\ell_x(\ell_y(z)) = \ell_x(F((n-1)\cdot y, z)) = \ell_x^{n-1}(F((n-1)\cdot y, z)) = F((n-1)\cdot \ell_x(y), z) = \ell_{\ell_x(y)}(z).$$

The last relation is obtained by exchanging the roles of x and y.

Expressing these conditions for B, we have

$$B(B(x,y),z) = \ell_{\ell_x(y)}(z) = \ell_x(\ell_y(z)) = B(x, B(y,z)), \quad x,y,z \in X,$$

which shows that B is associative. Moreover, $B(x,x) = F(n\cdot x) = x$ for any $x \in X$, which shows that (X,B) is a band. Finally,

$$B(B(x,y),z) = \ell_x(\ell_y(z)) = \ell_y(\ell_x(z)) = B(B(y,x),z), \quad x,y,z \in X,$$

which shows that (X,B) is a right normal band. □

The following corollary follows from Propositions 7.10 and 7.12.

Corollary 7.13 (see [35])**.** *If (X,F) is a commutative n-ary band, then the pair $(\{\ell_x : x \in X\}, \circ)$ is a semilattice.*

Example 7.14. The binary band associated with the ternary band (X,F) defined in Example 7.8 is given by the following table:

B_F	1	2	3	4
1	1	3	3	4
2	4	2	3	4
3	4	3	3	4
4	4	3	3	4

Proposition 7.15 (see [35])**.** *Let (X,F) be a commutative n-ary band. For any $x_1,\ldots,x_n \in X$ we have*

$$\ell_{F(x_1,\ldots,x_n)} \;=\; \ell_{x_1} \circ \cdots \circ \ell_{x_n}.$$

Proof. Let $t \in X$. By the associativity and the symmetry of F we have

$$\begin{aligned}(\ell_{x_1} \circ \cdots \circ \ell_{x_n})(t) \;&=\; F^n((n-1)\cdot x_1, \ldots, (n-1)\cdot x_n, t)\\ &=\; F((n-1)\cdot F(x_1,\ldots,x_n), t) \;=\; \ell_{F(x_1,\ldots,x_n)}(t),\end{aligned}$$

which completes the proof. □

Throughout the rest of this chapter, T_X denotes the full transformation monoid of X, i.e., the semigroup of all maps from X to X, endowed with the composition of maps. The next results characterize the reducibility of a commutative n-ary band to a commutative (binary) band, i.e., a semilattice. In order to state them, we consider the map $\ell : X \to T_X$ defined by $\ell(x) = \ell_x$, for every $x \in X$.

Proposition 7.16 (see [35]). *Let (X,F) be a commutative n-ary band. The following assertions are equivalent.*

(i) The map ℓ is injective.

(ii) The n-ary band (X,F) is isomorphic to the n-ary extension of $(\{\ell_x\colon x\in X\},\circ)$.

(iii) The n-ary band (X,F) is the n-ary extension of a semilattice.

(iv) The band (X,B) is commutative.

(v) The bands (X,B) and $(\{\ell_x\colon x\in X\},\circ)$ are isomorphic.

Proof. Let us first show that $(i)\Rightarrow(ii)$. If ℓ is injective, then it induces a bijection from X to $(\{\ell_x\colon x\in X\},\circ)$. Also, Proposition 7.15 shows that ℓ is an isomorphism of n-ary semigroups. The implication $(ii)\Rightarrow(iii)$ follows from Corollary 7.13. Now, let us show that $(iii)\Rightarrow(iv)$. So, suppose that (X,F) is the n-ary extension of a semilattice and let $x,y\in X$. By Proposition 6.22 we have

$$B(x,y) \;=\; F((n-1)\cdot x,y) \;=\; F(x,(n-1)\cdot y)=B(y,x),$$

which shows that B is commutative. Now, let us show that $(iv)\Rightarrow(i)$. So, assume that B is commutative and that $\ell_x=\ell_y$ for some $x,y\in X$. Then we have $B(x,z)=B(y,z)$ for every $z\in X$, and thus

$$x \;=\; B(x,x) \;=\; B(y,x) \;=\; B(x,y) \;=\; B(y,y) \;=\; y,$$

which shows that ℓ is injective. The implication $(i)\Rightarrow(v)$ is immediate since ℓ is the left action associated with B. Moreover, since $(\{\ell_x : x\in X\},\circ)$ is commutative by Proposition 7.12, we have $(v)\Rightarrow(iv)$. □

In the same spirit, the map ℓ also enables us to characterize those n-ary bands (X,F) that reduce to commutative groups whose exponents divide $n-1$.

For a symmetric n-ary semigroup (X,F), it is not difficult to see that an element $e\in X$ is neutral for F if and only if $\ell_e=i_X$, where i_X is the identity map on X.

Proposition 7.17 (see [35]). *Let (X,F) be a commutative n-ary band. The following conditions are equivalent.*

(i) The band (X,F) is the n-ary extension of an Abelian group whose exponent divides $n-1$.

(ii) The map ℓ is constant.

(iii) The band (X,B) is a right zero semigroup.

(iv) $\ell(X)=\{i_X\}$.

Proof. Let us first show that $(i)\Rightarrow(ii)$. So, suppose that (X,F) is the n-ary extension of an Abelian group (X,G) whose exponent divides $n-1$ and let $x,y\in X$. We have

$$\ell_x(y) \;=\; F((n-1)\cdot x,y) \;=\; G^{n-1}((n-1)\cdot x,y) \;=\; y,$$

which shows that ℓ_x is the identity map. The equivalence $(ii)\Leftrightarrow(iv)$ is straightforward. Also, the equivalence $(ii)\Leftrightarrow(iii)$ follows directly from the definition of a right zero semigroup. Finally, let us show that $(ii)\Rightarrow(i)$. By (ii), every element $x\in X$ is neutral for F since $\ell_x=i_X$. Thus, we conclude that (X,F) is the n-ary extension of an Abelian group whose exponent divides $n-1$ by Theorem 6.2. □

7.3 Semilattice decomposition and induced group structures

In this section we provide a characterization of the class of commutative n-ary bands. In particular, we show that on each commutative n-ary band (X, F) there is an n-ary semilattice congruence such that the restriction of F to each equivalence class is reducible to the operation of an Abelian group whose exponent divides $n-1$.

Let (X, F) be a commutative n-ary band. When ℓ is not injective, it is natural to consider a quotient, and identify the elements of X that have the same image by ℓ. On the other hand, we have that the associated band (X, B) is a strong semilattice of right normal bands by Proposition 7.2. Also, we know that the smallest semilattice congruence $\sim$ on (X, B) is defined by (7.1).

The next proposition enables us to express $\sim$ in terms of ℓ and properties of F.

Proposition 7.18 (see [35]). *Let (X, F) be a commutative n-ary band, let $\sim$ be the smallest semilattice congruence on (X, B), and let $x, y \in X$. The following conditions are equivalent.*

(i) $x \sim y$.

(ii) $\ell_x = \ell_y$.

(iii) There exist $t, t' \in X$ such that $y = \ell_x(t)$ and $x = \ell_y(t')$.

(iv) We have $y = F(x, x_2, \ldots, x_n)$ and $x = F(y, y_2, \ldots, y_n)$ for some $x_i, y_i \in X$ ($i \in \{2, \ldots, n\}$).

Proof. Let us first show that $(i) \Rightarrow (ii)$. First note that (i) is equivalent to the conditions $\ell_x(y) = y$ and $\ell_y(x) = x$. By Proposition 7.12 we have

$$\ell_x(t) = \ell_{\ell_y(x)}(t) = (\ell_y \circ \ell_x)(t) \text{ and } \ell_y(t) = \ell_{\ell_x(y)}(t) = (\ell_x \circ \ell_y)(t), \qquad t \in X,$$

and we conclude by Proposition 7.12. Now, let us show that $(ii) \Rightarrow (iii)$. We observe that $y = \ell_y(y)$, so that (ii) implies $y = \ell_x(y)$, and similarly we have $x = \ell_y(x)$. The implication $(iii) \Rightarrow (iv)$ follows from the definition of ℓ. Finally, a direct computation, using associativity and idempotency of F shows that $(iv) \Rightarrow (i)$. □

For a right normal band (X, G), the decomposition of Proposition 7.2 can be given explicitly: denoting by ℓ the left action of the band on itself, the semilattice congruence $\sim$ on (X, G) is defined as in Proposition 7.18. The semilattice Y is then $X/\sim$ and the semigroups X_α are the equivalence classes for $\sim$. We then have $[x]_\sim \preceq_{\tilde{G}} [y]_\sim$ if and only if $[G(x, y)]_\sim = [y]_\sim$, which is also equivalent to $G(x, y) = y$ in this particular situation. The homomorphisms[1] are then defined by $\varphi_{[x]_\sim, [y]_\sim} = \ell_y|_{[x]_\sim}$.

The congruence $\sim$ was built using the binary band (X, B). We will now show that it also defines a decomposition of (X, F).

Proposition 7.19 (see [35]). *Let (X, F) be a commutative n-ary band and let $\sim$ be the smallest semilattice congruence on (X, B). Then $\sim$ is an n-ary semilattice congruence on (X, F). Moreover, we have $B_{\tilde{F}} = \tilde{B}$ and $(X/\sim, \tilde{F})$ is the n-ary extension of the semilattice $(X/\sim, \tilde{B})$.*

[1] Recall that homomorphisms between right zero semigroups are just mappings.

Proof. Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in X$ and suppose that $x_i \sim y_i$ for some $i \in \{1, \ldots, n\}$. By Propositions 7.15 and 7.18, we have

$$\ell_{F(x_1,\ldots,x_i,\ldots,x_n)} = \ell_{x_1} \circ \cdots \circ \ell_{x_i} \circ \cdots \circ \ell_{x_n} = \ell_{x_1} \circ \cdots \circ \ell_{y_i} \circ \cdots \circ \ell_{x_n} = \ell_{F(x_1,\ldots,y_i,\ldots,x_n)}.$$

By Proposition 7.18, we have $F(x_1, \ldots, x_i, \ldots, x_n) \sim F(x_1, \ldots, y_i, \ldots, x_n)$. Thus, $\sim$ is a congruence on (X, F). Now, for any $x, y \in X$ we have

$$B_{\tilde{F}}([x]_\sim, [y]_\sim) = \tilde{F}((n-1) \cdot [x]_\sim, [y]_\sim) = [F((n-1) \cdot x, y)]_\sim = [B(x, y)]_\sim = \tilde{B}([x]_\sim, [y]_\sim),$$

which shows that $B_{\tilde{F}} = \tilde{B}$. Also, since $\sim$ is a semilattice congruence for B, we have that $\tilde{B}$ is commutative. Thus, $B_{\tilde{F}}$ is commutative, and the result follows from Proposition 7.16. □

Now, since $\sim$ is a congruence for F, this operation restricts to each equivalence class. We now analyze the properties of this restriction.

Proposition 7.20 (see [35])**.** *Let (X, F) be a commutative n-ary band and let $\sim$ be the smallest semilattice congruence on (X, B). For any $x \in X$, $([x]_\sim, F|_{[x]^n_\sim})$ is the n-ary extension of an Abelian group whose exponent divides $n - 1$.*

Proof. Let $x \in X$. It is easy to see that $([x]_\sim, F|_{[x]^n_\sim})$ is a commutative n-ary band. Its associated binary band operation is given by $B_{F|_{[x]^n_\sim}}(y, z) = F|_{[x]^n_\sim}((n-1) \cdot y, z) = B_F(y, z)$, for every $y, z \in [x]_\sim$. It is thus the restriction of B_F to $[x]^2_\sim$. Since this restriction defines a right zero semigroup operation, we have that $([x]_\sim, B_{F|_{[x]^n_\sim}})$ is a right zero semigroup. The result then follows from Proposition 7.17. □

Recall that the strong semilattice decomposition of (X, B) defines a family of right zero semigroup homomorphisms $(\varphi_{[x]_\sim,[y]_\sim}, [x]_\sim \preceq_{\tilde{B}} [y]_\sim)$ defined by $\varphi_{[x]_\sim,[y]_\sim} = \ell_y|_{[x]_\sim}$. In the next proposition we study the compatibility of these maps with respect to the structure induced by F on the classes $[x]_\sim$ and $[y]_\sim$.

Proposition 7.21 (see [35])**.** *For every $x, y \in X$ such that $[x]_\sim \preceq_{\tilde{F}} [y]_\sim$, the map $\varphi_{[x]_\sim,[y]_\sim}$ is a homomorphism from $([x]_\sim, F|_{[x]^n_\sim})$ to $([y]_\sim, F|_{[y]^n_\sim})$.*

Proof. This follows from Proposition 7.10 and from the definition of the map $\varphi_{[x]_\sim,[y]_\sim}$. □

We can now state and prove a characterization of the class of commutative n-ary bands.

Theorem 7.22 (see [35])**.** *If (X, F) is a commutative n-ary band, then it is a strong n-ary semilattice of n-ary extensions of Abelian groups whose exponents divide $n - 1$. Conversely, any strong n-ary semilattice of n-ary extensions of Abelian groups whose exponents divide $n - 1$ is a commutative n-ary band.*

Proof. The first part of the theorem follows from the previous results. More precisely, if (X, F) is a symmetric n-ary band, then we can associate with it a right normal band (X, B) by Proposition 7.12. The semilattice decomposition of this band, associated with the semilattice congruence $\sim$ defined by (7.1) yields a semilattice $Y = X/\sim$ and a partition $X = \cup_{\alpha \in Y} X_\alpha$. Moreover, $\sim$ is an n-ary semilattice congruence for F and the n-ary groupoids $(X_\alpha, F|_{X^n_\alpha})$ are n-ary subsemigroups by Proposition 7.19. Also, by Proposition 7.21, the homomorphism from the class $[x]_\sim$ to the class $[y]_\sim$ (with $[x]_\sim \preceq_{\tilde{F}} [y]_\sim$) is defined by $\varphi_{[x]_\sim,[y]_\sim} = \ell_y|_{[x]_\sim}$. Conditions (a) and (b) of

Definition 7.6 follow from Proposition 7.2. Moreover, for any $\alpha \in Y$ we have that $(X_\alpha, F|_{X_\alpha^n})$ is the n-ary extension of an Abelian group whose exponent divides $n-1$ by Proposition 7.20. Finally, condition (c) of Definition 7.6 follows from Corollary 7.11 and Proposition 7.19.

Let us show the converse statement. So, suppose that $(X, F) = ((Y, \Upsilon^{n-1}), (X_\alpha, F_\alpha), \varphi_{\alpha,\beta})$ is a strong n-ary semilattice of n-ary extensions of Abelian groups whose exponents divide $n-1$. The associativity of F follows from Proposition 7.7. The idempotency of F follows from condition (a) of Definition 7.6 and from the idempotency of the n-ary operations F_α. Finally, the symmetry of F follows from condition (c) of Definition 7.6 and the symmetry of the n-ary operations F_α. $\square$

In view of this result, in order to build commutative n-ary bands, we have to consider Abelian groups whose exponents divide $n-1$, and build homomorphisms between the n-ary extensions of such groups. These homomorphisms are described in the next result.

Proposition 7.23 (see [35]). *Let (Y, G_1) and (Z, G_2) be two Abelian groups whose exponents divide $n-1$ and let (Y, F_1) and (Z, F_2) be the n-ary extensions of (Y, G_1) and (Z, G_2), respectively. For any group homomorphism $\psi : Y \to Z$ and any $g_2 \in Z$, the map $h : Y \to Z$ defined by*

$$h(x) \;=\; G_2(g_2, \psi(x)), \qquad x \in Y,$$

is a homomorphism of n-ary semigroups.

Conversely, every homomorphism from (Y, F_1) to (Z, F_2) is obtained in this way.

Proof. Let $x_1, \ldots, x_n \in Y$. Since ψ is a group homomorphism we have

$$\begin{aligned} h(F_1(x_1, \ldots, x_n)) &= G_2(g_2, \psi(F_1(x_1, \ldots, x_n))) \\ &= G_2(g_2, \psi(G_1^{n-1}(x_1, \ldots, x_n))) = G_2(g_2, G_2^{n-1}(\psi(x_1), \ldots, \psi(x_n))). \end{aligned}$$

Moreover, using the definition of h and the commutativity of G_2, we have

$$F_2(h(x_1), \ldots, h(x_n)) = G_2^{n-1}(h(x_1), \ldots, h(x_n)) = G_2^{2n-1}(n \cdot g_2, \psi(x_1), \ldots, \psi(x_n)).$$

Thus, the first part of the result follows from the fact that the exponent of (Z, G_2) divides $n-1$.

For the second part, we consider a homomorphism h of n-ary semigroups from (Y, F_1) to (Z, F_2) and we denote by e_1 and e_2 the neutral elements of (Y, G_1) and (Z, G_2), respectively. Also, we denote by i_1 the inverse of $h(e_1)$ with respect to G_2. Then the map $\psi \colon Y \to Z$ defined by

$$\psi(x) = G_2(i_1, h(x)), \qquad x \in Y,$$

is a group homomorphism from (Y, G_1) to (Z, G_2). Indeed, for any $x, y \in Y$ we have

$$\begin{aligned} \psi(G_1(x, y)) &= G_2(i_1, h(G_1(x, y))) \\ &= G_2(i_1, h(G_1^{n-1}(x, y, (n-2) \cdot e_1))) = G_2(i_1, h(F_1(x, y, (n-2) \cdot e_1))). \end{aligned}$$

Since h is a homomorphism, the latter expression equals

$$G_2(i_1, F_2(h(x), h(y), (n-2) \cdot h(e_1))) = G_2(i_1, G_2^{n-1}(h(x), h(y), (n-2) \cdot h(e_1))).$$

Since the exponent of (Z, G_2) divides $n-1$ we have $G_2^{n-3}((n-2) \cdot h(e_1)) = i_1$ and this shows that ψ is a group homomorphism. $\square$

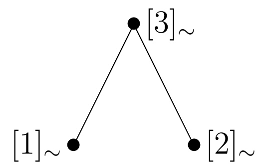

Figure 7.1: Hasse diagram of $(X/{\sim}, \preceq_{\tilde{F}})$

Using Theorem 7.22 we can now easily see that the ternary operation $F\colon X^3 \to X$ defined in Example 7.8 is a commutative ternary band operation. Indeed, F is clearly idempotent and symmetric. Now, let us show that F is associative. It is not difficult to see that the binary relation $\sim$ on X defined by

$$x \sim y \quad \Leftrightarrow \quad \ell_x = \ell_y, \qquad x, y \in X,$$

is a ternary semilattice congruence on the ternary groupoid (X, F). The Hasse diagram of $(X/{\sim}, \preceq_{\tilde{F}})$ is depicted in Figure 7.1. More precisely, we have $[1]_\sim = \{1\}$, $[2]_\sim = \{2\}$, and $[3]_\sim = \{3, 4\}$. Also, $([3]_\sim, F|_{[3]^3_\sim})$ is isomorphic to the ternary extension of $(\mathbb{Z}_2, +)$. Moreover, for any $[x]_\sim, [y]_\sim \in X/{\sim}$ such that $[x]_\sim \preceq_{\tilde{F}} [y]_\sim$, the maps $\ell_y|_{[x]_\sim}\colon [x]_\sim \to [y]_\sim$ are clearly homomorphisms. Also, for any $x \in X$ we have $\ell_x|_{[x]_\sim} = i_X|_{[x]_\sim}$. Furthermore, for any $(x, y, z) \in X^3 \setminus \Delta^3_X$ we have

$$F(x, y, z) \;=\; F|_{[3]^3_\sim}(\ell_3(x), \ell_3(y), \ell_3(z)).$$

Thus, (X, F) is a strong ternary semilattice of ternary extensions of Abelian groups whose exponents divide 2. Hence, (X, F) is a commutative ternary band by Theorem 7.22.

7.4 Reducibility of commutative n-ary bands

In this section, we use the structure theorem that we developed in the previous section in order to analyze the reducibility problem for commutative n-ary bands. More precisely, we provide necessary and sufficient conditions under which a commutative n-ary band (X, F) is reducible to a binary semigroup (X, G).

We recall that we associated with any commutative n-ary band (X, F) a binary band (X, B), a congruence $\sim$ and a triple $((Y, \Upsilon^{n-1}), (X_\alpha, F_\alpha), \varphi_{\alpha,\beta})$.

Proposition 7.24 (see [35]). *Let $F\colon X^n \to X$ be an associative, idempotent, and symmetric operation. If F is reducible to an associative operation $G\colon X^2 \to X$, then the following conditions hold.*

(i) G is surjective and symmetric.

(ii) For any $x \in X$, the map ℓ_x is an endomorphism for G.

(iii) For any $x, y \in X$, we have $\ell_{G(x,y)} = \ell_x \circ \ell_y = \ell_{B(x,y)}$.

(iv) The congruence $\sim$ associated with F is a congruence for G. Moreover, the associated operation $\tilde{G}$ on $X/{\sim}$ is equal to the quotient operation $\tilde{B}$.

Proof. Condition (i) follows from Fact 5.34 and Lemma 5.35. Let us show condition (ii). For any $x, y, z \in X$ we have

$$\begin{aligned}\ell_x(G(y,z)) = F((n-1)\cdot x, G(y,z)) &= G^{n-1}((n-1)\cdot x, G(y,z)) \\ &= G(G^{n-1}((n-1)\cdot x, y), z) = G(\ell_x(y), z).\end{aligned}$$

The result then follows from the symmetry of G and Proposition 7.10.

Now, let $x, y, z \in X$ and let us show condition (iii). Using the associativity and the symmetry of G we have

$$\begin{aligned}\ell_{G(x,y)}(z) = F((n-1)\cdot G(x,y), z) &= G^{n-1}((n-1)\cdot G(x,y), z) \\ &= G^{n-1}((n-1)\cdot x, G^{n-1}((n-1)\cdot y, z)) = \ell_x(\ell_y(z)).\end{aligned}$$

The result then follows from Proposition 7.12 and from the definition of B.

Finally, from (iii) and Proposition 7.18, we obtain $G(x,y) \sim B(x,y)$ for every $x, y \in X$. Since $\sim$ is a congruence for B, it is also a congruence for G. The quotient operation can be easily computed :

$$\tilde{G}([x]_\sim, [y]_\sim) \;=\; [G(x,y)]_\sim \;=\; [B(x,y)]_\sim \;=\; \tilde{B}([x]_\sim, [y]_\sim), \qquad x, y \in X,$$

and the proof is complete. □

It follows from Proposition 7.24 that if F is reducible to G, then G induces an operation $G|_{[x]^2_\sim}$ on every class $[x]_\sim$ of X. This operation is a reduction of $F|_{[x]^n_\sim}$. It is therefore natural to study the properties of this reduction.

Proposition 7.25 (see [35]). *If (X,F) is the n-ary extension of an Abelian group (X,G_1) whose exponent divides $n-1$, then every binary reduction G_2 of F is a group operation that is conjugate to G_1. In particular, the binary reductions of F are obtained applying* (5.1) *with any element e of X.*

Proof. This follows from Propositions 5.5 and 5.16. □

We are now able to analyze the reducibility of commutative n-ary bands.

Theorem 7.26 (see [35]). *Let $(X,F) = ((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha),\varphi_{\alpha,\beta})$ be a commutative n-ary band. Then F is reducible to a binary associative operation if and only if there exists a map $e\colon Y \to X$ such that the following conditions hold.*

(i) For every $\alpha \in Y$, $e(\alpha) = e_\alpha \in X_\alpha$.

(ii) For every $\alpha, \beta \in Y$ such that $\alpha \preceq \beta$, we have $\varphi_{\alpha,\beta}(e_\alpha) = e_\beta$.

Moreover, if (X,F) is the n-ary extension of a semigroup (X,G), then

$$(X,G) \;=\; ((Y,\curlyvee),(X_\alpha,G_\alpha),\varphi_{\alpha,\beta}),$$

where G_α is the reduction of F_α with respect to e_α.

Proof. Assume first that F is reducible to an operation $G\colon X^2 \to X$. By Proposition 7.24, we have that $(X,G) = ((Y,\curlyvee),(X_\alpha,G_\alpha),\varphi_{\alpha,\beta})$, where $Y = X/\sim$, the sets X_α are the equivalence classes of X, and the maps $\varphi_{\alpha,\beta}$ are given, for $\alpha = [x]_\sim \preceq_{\tilde{G}} \beta = [y]_\sim$, by $\varphi_{\alpha,\beta} = \ell_y|_{[x]_\sim}$. For any $y \in X$, the map ℓ_y is an endomorphism of (X,G) by Proposition 7.24. Thus, for any $\alpha \preceq_{\tilde{G}} \beta$, the map $\varphi_{\alpha,\beta}$ is a homomorphism from X_α to X_β. Now, let $x,y \in X$ such that $[x]_\sim \preceq_{\tilde{G}} [y]_\sim$. By Proposition 7.25, the restriction $G|_{[x]^2_\sim}$ is a reduction of the restriction $F|_{[x]^n_\sim}$, and is therefore associated with an element $e_{[x]_\sim} \in [x]_\sim$. Since $e_{[x]_\sim}$ is the unit of the group $([x]_\sim, G|_{[x]^2_\sim})$ and $\varphi_{[x]_\sim,[y]_\sim}$ is a group homomorphism, we have $\varphi_{[x]_\sim,[y]_\sim}(e_{[x]_\sim}) = e_{[y]_\sim}$, and conditions (i) and (ii) are satisfied.

Conversely, assume that (i) and (ii) are satisfied. For every $\alpha \in Y$, denote by G_α the reduction of F_α associated with e_α. If $\alpha \preceq \beta$, then $\varphi_{\alpha,\beta}$ is a homomorphism from (X_α,F_α) to (X_β,F_β). By Proposition 7.23 and condition (ii), $\varphi_{\alpha,\beta}$ is a group homomorphism. Conditions (a) and (b) of Definition 7.1 are then satisfied, and it follows that $((Y,\curlyvee),(X_\alpha,G_\alpha),\varphi_{\alpha,\beta})$ defines a semigroup (X,G) by condition (c) of the same definition. Finally, it remains to show that (X,F) is the n-ary extension of (X,G). We first observe that, when $\alpha,\alpha_1,\alpha_2 \in Y$ are such that $\alpha_1 \curlyvee \alpha_2 \preceq \alpha$, we have

$$G_\alpha(\varphi_{\alpha_1,\alpha}(x_1),\varphi_{\alpha_2,\alpha}(x_2)) \;=\; \varphi_{\alpha_1\curlyvee\alpha_2,\alpha}(G(x_1,x_2)). \tag{7.4}$$

This relation is obtained by decomposing $\varphi_{\alpha_i,\alpha}(x_i)$ as $\varphi_{\alpha_1\curlyvee\alpha_2,\alpha}(\varphi_{\alpha_i,\alpha_1\curlyvee\alpha_2}(x_i))$ in the left-hand side, for $i \in \{1,2\}$, using that $\varphi_{\alpha_1\curlyvee\alpha_2,\alpha}$ is a homomorphism, and finally using the definition of G. Then, if $x_i \in X_{\alpha_i}$ for every $i \in \{1,\ldots,n\}$, setting $\alpha = \alpha_1 \curlyvee \cdots \curlyvee \alpha_n$ we have by definition

$$F(x_1,\ldots,x_n) \;=\; F_\alpha(\varphi_{\alpha_1,\alpha}(x_1),\ldots,\varphi_{\alpha_n,\alpha}(x_n)).$$

Since F_α is the n-ary extension of G_α, we also have

$$F(x_1,\ldots,x_n) \;=\; G_\alpha^{n-1}(\varphi_{\alpha_1,\alpha}(x_1),\ldots,\varphi_{\alpha_n,\alpha}(x_n)).$$

Then using (7.4) a straightforward induction shows that we have

$$F(x_1,\ldots,x_n) \;=\; G_\alpha^{n-i}(\varphi_{\alpha_1\curlyvee\cdots\curlyvee\alpha_i,\alpha}(G^{i-1}(x_1,\cdots,x_i)),\varphi_{\alpha_{i+1},\alpha}(x_{i+1}),\ldots,\varphi_{\alpha_n,\alpha}(x_n)),$$

for every $i \in \{1,\ldots,n\}$. Considering $i = n$ leads to the desired result. □

Using Theorem 7.26 we can now easily see that the commutative ternary band (X,F) defined in Example 7.8 is not reducible to a semigroup. Indeed, we have

$$(X,F) \;=\; ((Y,\curlyvee^{n-1}),(X_\alpha,F_\alpha),\varphi_{\alpha,\beta}) \;=\; ((X/\sim,\tilde{F}),([x]_\sim,F|_{[x]^n_\sim}),\ell_y|_{[x]_\sim}),$$

by Theorem 7.22. Now, let $e_1,e_2\colon X/\sim \to X$ be the maps defined by $e_1([1]_\sim) = e_2([1]_\sim) = 1$, $e_1([2]_\sim) = e_2([2]_\sim) = 2$, $e_1([3]_\sim) = 3$, and $e_2([3]_\sim) = 4$. Then we have $\ell_3(1) = 4 = e_2([3]_\sim)$ and $\ell_3(2) = 3 = e_1([3]_\sim)$. Hence, (X,F) is not reducible to a semigroup by Theorem 7.26.

Conclusion

In this thesis, we provided characterizations and descriptions of various classes of idempotent n-ary semigroups. When the underlying set is finite, we also enumerated several of these classes. This study was conducted in two parts.

In Part I we essentially showed unexpected links between semigroup theory and social choice theory. More precisely, in Chapter 1 we carried out an in-depth mathematical study of single-peakedness and other related properties. In particular, we provided algebraic characterizations of these concepts. By doing so, we sometimes encountered difficulties to generalize those characterizations on finite sets to arbitrary sets. For instance, finding a very simple necessary and sufficient condition that ensures the existence of a total order on X for which a given weak order is single-plateaued (see Proposition 1.15) was a challenging problem. Then in Chapter 2 we provided alternative characterizations of the class of rectangular semigroups. In particular, when the underlying set is finite, we provided a necessary condition on a groupoid to be a rectangular semigroup in terms of preimage sequences (see Corollary 2.9). The main difficulty remaining in this chapter is to find necessary and sufficient conditions on a groupoid to be a rectangular semigroup in terms of preimage sequences (see Corollary 3.5 for semigroups). Finally, in Chapters 3 and 4 we characterized several classes of totally ordered idempotent semigroups in terms of the concepts established in the previous chapters. It turns out that single-peakedness is the key property that characterizes the preservation of a total order by a quasitrivial semigroup operation (see Proposition 4.21 and Corollary 4.31). Although we developed many algebraic tools throughout this first part, characterizations of the classes of weakly ordered bands and partially ordered bands still elude us. In view of these results, several questions emerge naturally and we list some of them below.

- Finding extensions of single-peakedness and other related properties to reference orders that are weak orders or partial orders constitutes a first approach to the characterizations of the classes of weakly ordered bands and partially ordered bands.

- The number of anticommutative bands and totally ordered commutative bands on X_n was provided in Propositions 2.11 and 3.36. In particular, we showed that the number of totally ordered commutative bands on X_n is exactly the nth Catalan number. Now, finding the numbers of bands and commutative bands on X_n remains challenging problems.

- We provided an in-depth study of the class of quasitrivial semigroups by classifying its elements into subclasses (for instance, by considering conjugacy classes). Such a classification for the classes of bands and commutative bands would be welcome.

In Part II we provided constructive descriptions of relevant classes of idempotent n-ary semigroups. More precisely, in Chapter 5 we showed that every quasitrivial n-ary semigroup is reducible to a semigroup and provided necessary and sufficient conditions for the reduction to be quasitrivial and unique. As a byproduct we were able to provide a constructive description of the class of quasitrivial n-ary semigroup based on binary reductions. In particular, we introduced the class of operations $A_1^2(X) \setminus Q_1^2(X)$ (see p. 92) and provided a very simple characterization of this class (see Proposition 5.17). It still remains to classify its elements into subclasses as previously done with the class of quasitrivial semigroups. Building on these results, in Chapter 6 we introduced and characterized hierarchical classes of idempotent n-ary semigroups that satisfy quasitriviality on certain subsets of the domain. In particular, we showed that each of these n-ary semigroups is reducible to a semigroup that is built from a quasitrivial semigroup and an Abelian group whose exponent divides $n-1$. It seems a challenging problem to provide characterizations of the latter class of semigroups and to classify its elements into subclasses. Finally, in Chapter 7 we provided a description of the class of symmetric idempotent n-ary semigroups based on the concepts of strong semilattices of right zero semigroups and Abelian groups whose exponents divide $n-1$. As symmetric idempotent n-ary semigroups are in general not reducible to semigroups, we also provided necessary and sufficient conditions that ensure the reducibility of any symmetric idempotent n-ary semigroup to a semigroup. Although we developed many algebraic tools throughout this second part, the characterization of the class of idempotent n-ary semigroups is still missing and seems to be a difficult problem. We list below several open questions and topics of current research.

- A study of the concept of semilattice congruence as defined in [58] (see Remark 7.3) on any idempotent n-ary semigroup would be compelling. In that case, the quotient n-ary semigroup is a symmetric idempotent n-ary semigroup.

- Finding an extension of the concept of rectangular semigroup to n-ary semigroups by requiring for instance the associative n-ary operation $F\colon X^n \to X$ to satisfy

$$F((n-1)\cdot x, F(y,(n-1)\cdot x)) \;=\; x, \qquad x,y\in X.$$

  It is not difficult to see that any such operation is idempotent. Moreover, in contrast to the binary case, n-ary semigroups satisfying the above equation are not exactly diagonal algebras [85] (see p. 33 for the binary case).

- Recall that an n-ary semigroup (X,F) is said to be *cancellative* if F is one-to-one in each variable; that is, for every $k\in\{1,\ldots,n\}$ and every $(x_1,\ldots,x_n),(x_1',\ldots,x_n')\in X^n$,

$$(x_i = x_i' \;\forall\, i\in\{1,\ldots,n\}\setminus\{k\} \text{ and } F(x_1,\ldots,x_n) \;=\; F(x_1',\ldots,x_n')) \Rightarrow x_k \;=\; x_k'.$$

  Cancellative n-ary semigroups have been studied by many authors (see, e.g., [23,37,38,40, 41,68,69,101]). For instance, whenever $n\geq 3$ is odd, the operation $F\colon \mathbb{R}^n\to\mathbb{R}$ defined by

$$F(x_1,\ldots,x_n) \;=\; \sum_{i=1}^{n}(-1)^i x_i\,, \qquad x_1,\ldots,x_n\in\mathbb{R},$$

  is associative, idempotent, and cancellative [101]. Finding a characterization of the class of idempotent cancellative n-ary semigroups constitutes a step towards the characterization of the class of idempotent n-ary semigroups.

- Finding the number of idempotent n-ary semigroups on X_k for any integer $k\geq 1$ constitutes a challenging problem.

Notation

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