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Block Principal Pivoting Algorithm for VGLCP: A Block Principal Pivoting Algorithm for the Vertical Generalized Linear Complementarity Problem (VGLCP) with a Vertical Block P-matrix

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Definitions and Background

An $m \times n$ matrix N, with $m \geq n$, is said to be of type $(m_1, \ldots, m_n)$ if it is partitioned row-wise into n blocks such that the $j-th$ block, N^j , is of dimension $m_j \times n$ and $m = \sum_{j=1}^{n} m_j$. That is:

$$N = \begin{bmatrix} N^1 \\ \vdots \\ N^n \end{bmatrix}$$

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$$N = \begin{bmatrix} N^1 \\ \vdots \\ N^n \end{bmatrix}$$

The vectors $w \in R^m$ and $q \in R^m$ are also partitioned to conform to the entries in the block, N^j of N:

$$w = \begin{bmatrix} w^1 \\ \vdots \\ w^n \end{bmatrix}, \qquad q = \begin{bmatrix} q^1 \\ \vdots \\ q^n \end{bmatrix}$$

where w^j, q^j are $m_j \times 1$ vectors.

Definitions and Background

The vertical generalized linear complementarity problem: Given a vertical block matrix N of type $(m_1, m_2, ..., m_n)$ and a vector $q \in R^m$, find vectors $w \in R^m, z \in R^n$ such that

$$w = Nz + q \tag{1}$$

$$w \geq 0, z \geq 0 \tag{2}$$

$$z_j \prod_{i=1}^{m_j} w_i^j = 0 \quad (j = 1, \dots, n) \tag{3}$$

We will denote this problem by VGLCP(q, N).

Definitions and Background

Consider the following sets:

$$K_1 = \{1, \ldots, m_1\}, \qquad K_i = \left\{1 + \sum_{t=1}^{i-1} m_t, \ldots, \sum_{t=1}^{i} m_t\right\}, \quad i = 2, \ldots, n.$$

The complementarity conditions can be reformulated as follows: Find vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^m$ such that

$$z_i \prod_{j \in K_i} w_j = 0, \quad i = 1, \ldots, n. \tag{4}$$

The Algorithm

Complementary Basic Solutions (CBS) for the VGLCP(q, N)
A CBS associated the VGLCP (q, N) consists of basic variables and nonbasic variables.

The nonbasic variables have zero values and the basic variables are obtained by solving the linear system (1) above and fixing the values of the nonbasic variables at zero.

Furthermore, the basic and nonbasic variables are chosen in such a way that the complementarity condition (3) or (4) holds.

The Algorithm

Let $x = [z \; w]^T$ be a CBS.
Denote the basic variables by z_F and w_T; and the nonbasic variables by z_G and w_R where the sets F, G, T, and R are defined as follows:

$$
\begin{aligned}
&F \subseteq \{1, \ldots, n\} \\
&G = \{1, \ldots, n\} - F \\
&F \cap G = \emptyset \\
&T \subseteq \{1, \ldots, m\} \\
&R = \{1, \ldots, m\} - T \\
&T \cap R = \emptyset
\end{aligned}
$$

The Algorithm

The basic variables, z_F and w_T, are obtained by solving:

$$\begin{bmatrix} I_T & -N_{TF} \\ 0 & -N_{RF} \end{bmatrix} \begin{bmatrix} w_T \\ z_F \end{bmatrix} = \begin{bmatrix} q_T \\ q_R \end{bmatrix} \tag{5}$$

and the nonbasic variables are obtained by setting

$$z_G = 0, \qquad w_R = 0 \tag{6}$$

The matrix in (5) is called a basis matrix.

The Algorithm

Theorem

If N is an $m \times n$ vertical block P-matrix of type $(m_1, \ldots, m_n)$, then the basis matrix in (5) is nonsingular.

Principal Pivoting Algorithms

Let $x = [z\ w]^T$ be a CBS and F, G, T and R the associated sets. Allowed principal pivot operations:

(i) A basic variable $z_i, i \in F$, is exchanged with a nonbasic variable w_j, with $j \in K_i$, and the sets are updated as follows:

$$F = F \setminus \{i\}, \quad G = G \cup \{i\},$$
$$T = T \cup \{j\}, \quad R = R \setminus \{j\}$$

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Principal Pivoting Algorithms

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$$F = F \setminus \{i\}, \quad G = G \cup \{i\},$$
$$T = T \cup \{j\}, \quad R = R \setminus \{j\}$$

(ii) A basic variable $w_j, j \in K_i$, is exchanged with a nonbasic variable $z_i, i \in G$, and the sets are updated as below:

$$F = F \cup \{i\}, \quad G = G \setminus \{i\},$$
$$T = T \setminus \{j\}, \quad R = R \cup \{j\}$$

Principal Pivoting Algorithms

(iii) A basic variable $w_j, j \in K_i$, is exchanged with a nonbasic variable $w_s, s \in K_i$, and the sets T and R are updated by:

$$T = T \setminus \{j\} \cup \{s\}, \quad R = R \setminus \{s\} \cup \{j\}$$

and the sets F and G are not updated.

Step 0: (Initialization)

Start with the associated sets:

$F = \emptyset, \qquad G = \{1, \ldots, n\}, \quad T = \{1, \ldots, m\}, \quad R = \emptyset.$

The corresponding CBS is $x = [z \; w]^T = [0 \; q]^T$.

The Block Principal Pivoting Algorithm

Step 0: (Initialization)
Start with the associated sets:
$F = \emptyset, \qquad G = \{1, \ldots, n\}, \quad T = \{1, \ldots, m\}, \quad R = \emptyset.$
The corresponding CBS is $x = [z \ w]^T = [0 \ q]^T$.

Step 1: If $x \geq 0$, terminate with x as the solution of VGLCP. Otherwise, go to Step 2.

The Algorithm

Step 2: Update the sets F, G, T, R by (i) and (ii) as given below:

(i) Define the following sets:

$$\bar{F} = \{i \in F\colon z_i < 0\} \tag{7}$$
$$\bar{T}_1 = \{\min\{j \in K_i\colon w_j < 0\}\colon i \in G\} \tag{8}$$
$$\bar{T}_2 = \{\min\{j \in K_i\colon w_j < 0\}\colon i \in F \setminus \bar{F}\} \tag{9}$$
$$\hat{F} = \{i \in G\colon j \in \bar{T}_1\} \tag{10}$$
$$\bar{R}_1 = \{j \in R \cap K_i\colon i \in \bar{F}\} \tag{11}$$
$$\bar{R}_2 = \{s \in R \cap K_i\colon i \in F \setminus \bar{F} \text{ and } j \in \bar{T}_2\} \tag{12}$$

(ii) Update the sets F, G, T and R by:

$$F = F \setminus \bar{F} \cup \hat{F} \tag{13}$$

$$G = \{1, \ldots, n\} \setminus F \tag{14}$$

$$T = T \setminus (\bar{T}_1 \cup \bar{T}_2) \cup (\bar{R}_1 \cup \bar{R}_2) \tag{15}$$

$$R = \{1, \ldots, m\} \setminus T \tag{16}$$

The Algorithm

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$$F = F \setminus \bar{F} \cup \hat{F} \tag{13}$$

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$$R = \{1, \ldots, m\} \setminus T \tag{16}$$

Step 3: Use the updated sets to solve equations (5) and (6) and obtain a new CBS $x = [z \quad w]^T$. Return to Step 1.

Example

Let N be a veertical block P-matrix of type (2, 2, 2):

$$N = \begin{bmatrix} 4 & 1 & 3 \\ 5 & 2 & 3 \\ 3 & 4 & 1 \\ 4 & 8 & -4 \\ 1 & 3 & 4 \\ 1 & 6 & 7 \end{bmatrix}, \qquad q = \begin{bmatrix} -2 \\ -3 \\ -1 \\ -5 \\ -9 \\ -3 \end{bmatrix} \tag{17}$$

Then $n = 3, \quad m = 6, \quad m_1 = m_2 = m_3 = 2, \quad K_1 = \{1, 2\}$
$K_2 = \{3, 4\}, \quad K_3 = \{5, 6\}$

Example

Solution in two iterations given by:

$w_1 = 3.16$, $w_2 = 3.43$, $w_3 = 5.39$, $w_4 = 0$, $w_5 = 0$, $w_6 = 13.70$, $z_1 = 0$, $z_2 = 1.27$, $z_3 = 1.30$.

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Remark: A single principal pivoting algorithm developed by Ebiefung et al [2] gets the same solution in 6 iterations.

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Remark: A single principal pivoting algorithm developed by Ebiefung et al [2] gets the same solution in 6 iterations.

Problem: Cycling/finite convergence

Credits

Collaborators:

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