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Vjollca Sadiraj
Georgia State University

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Keywords: risk aversion, probability transformation, calibration
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Department of Economics and Experimental Economics Center, Georgia State University,
14 Marietta St. NW, Atlanta, GA 30303
vsadiraj@gsu.edu
phone: 404-413-0193
fax: 404-413-0205

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1. Introduction

The first paradox to challenge expected utility theory was offered by Allais (1953). The Allais patterns violate the independence axiom, which gives the expected utility functional its idiosyncratic feature of linearity in probabilities. In order to avoid the Allais paradox, theories of decision under risk that relax the independence axiom were developed (see Starmer, 2000 for an accessible presentation).

The idea of representing risk aversion with nonlinear probability transformations originated in the psychology literature about mid-twentieth century (Preston and Baratta, 1948; Edwards, 1954) and entered the economics literature in late the seventies (Handa, 1977; Kahneman and Tversky, 1979; Quiggin, 1982). Some early models of probability weighting (Handa, 1977; Kahneman and Tversky, 1979) were shown to violate first order stochastic dominance. Subsequent models with rank dependence of prizes avoid that problem (Tversky and Kahneman, 1992; Quiggin, 1993). Further development and applications of rank dependent models, and other alternatives to expected utility, have continued to the present. Wakker (2010) provides a comprehensive presentation of the literature. Rank dependent utility models have been relatively successful in explaining several behavioral ‘anomalies’ that have been observed in the laboratory and in the field, which accounts for their recent widespread use in field applications.

This paper, however, is concerned with the implications of probabilistic sensitivity and rank dependence for risk aversion. I argue that attributing local (with respect to probabilities) risk aversion to attitudes towards probabilities can produce paradoxical risk aversion. This calls into question the ability of rank dependent utility models to rationalize risk aversion at small and large probabilities.

2. Risk Aversion as Attitude toward Probabilities: An Example of the Paradox

In order to provide an intuition for the main result in the paper, this section presents an illustrative example of risk-avoiding choices that provide a challenge to modeling risk preferences with probabilistic attitudes.

Suppose that an individual is offered a choice between two prospects represented by urn S_g and urn R_g. Urn S_g contains 100 balls in the following composition: 10 white balls, g green balls and b black balls. Each white ball pays \$1000, each green ball pays \$4000 and each black ball pays \$0. Urn R_g is constructed from urn S_g by replacing 10 white balls with 5 green balls and 5 black balls. Table 1 summarizes all the information on prizes and compositions of balls for the two urns.

Table 1. Illustration of composition of urns S_g and R_g ; *g* is from {0,5,…, 90}

Ball Prizes	**\$4000**	**\$1000**	**\$0**	
Ball Colors	Green	White	Black	Total
Ball composition in urn S_g	g	10	b	100
Ball composition in urn R_g	g+5	0	b+5	100

Thus, the decision problem the individual is facing is the choice between urn S_g that offers prizes \$4K, \$1K and \$0 with probabilities g/100, 10/100 and b/100 and a relatively riskier urn R_g that offers only extreme prizes, \$4K and \$0 with probabilities (g+5)/100 and (b+5)/100.[1]

[1] Formally, for $p=(g+5)/100$ and $n=10$, urn R_g is prospect $\{\$4K,0;\ p,1-p\}$ whereas urn S_g is prospect $\{\$4K,\$1K,0;\ p-1/2n,1/n,1-p-1/2n\}$. All prospects R and S considered in this paper are of these types.

(1K and 4K, respectively, denote \$1000 and \$4000.) Figure 1 illustrates, in the probability triangle, pairs of two prospects offered by urns S_g and R_g for different combinations of green and black balls.

What might be the decision of our individual? Cox, Sadiraj, Vogt and Dasgupta (2012) reports that in their experiments the majority of subjects, who were asked to make similar choices as our individual, were either indifferent or preferred the safer prospect, S_g over the riskier prospect R_g for all compositions of green and black balls that they faced. So suppose that our individual is like the majority of subjects in the Cox et al. experiments in that he weakly prefers urn S_g to urn R_g for all g from {0,5,…,90}.[2] Note that, in the probability triangle in Figure 1, such choices are expected wherever linear (but not necessarily parallel) indifference curves are steeper than the 45^o line.

What are the implications for rank dependent models if these risk avoiding choices are attributed (partly) to sensitivity towards probabilities? The implications are potentially extreme distortions of probabilities that produce paradoxical risk aversion at large probabilities as illustrated below. If $v(\cdot)$ denotes the value function defined over the prizes[3], normalized such that $v(0)=0$, and $f(\cdot)$ denotes the transformation of decumulative probabilities then it can be shown (see the proof of Proposition A.1 in the Appendix) that indifference or preference for urn S_g over R_g when the number of green balls is from $\{g, g+5, ,..., g+5k\}$ for some feasible integer k reveals that

(1) $$f\left(\frac{g+5(k+2)}{100}\right) - f\left(\frac{g+5(k+1)}{100}\right) \geq q^{k+1}\left[f\left(\frac{g+5}{100}\right) - f\left(\frac{g}{100}\right)\right]$$

where $q = -1 + v(4K)/v(1K)$. Inequality (1) shows that the slope of the transformation of decumulative probabilities increases geometrically and therefore severe underweighting of

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[2] The supposition that the safer prospect is weakly preferred for all g from {0, 5, ...,90} is made here for simplicity of exposition; section 3 provides general results for cases when the weak preference for the safer urn is observed only for some subset of {0,5,…,90}.

[3] In this paper the value function $v(.)$ is defined over prizes. For the terminal wealth model , as in rank dependent utility model, $v(y) = u(w+y)$; for the income model, as in cumulative prospect theory, $v(y) = u(y)$.

probabilities is expected when $q > 1$.[4] An implication of such preferences is for example $f(0.5) \le (q^{10}-1)/(q^{20}-1)$. If, consistent with Abdellaoui et al. (2007), we set $v(4K)/v(1K) \ge 3$ then the last inequality implies $f(0.5) \le 0.00098$. This extreme probability distortion and sub-additivity of $v(\cdot)$ imply preference for a sure amount of \$20 over a prospect that offers \$20,000 or 0 with equal probability.[5]

2.b Payoff Scale Invariance

The above example uses payoffs of \$4000, \$1000, and \$0. Inspection of statement (1) reveals that the only way in which the valuation of prizes enters the inequalities is through $q = [\upsilon(4K)/\upsilon(1K)] - 1$. Hence, irrespective of the size of payoffs (whether they are very large or very small or moderate in size), weak preference for urn S_g over a range of green balls implies the same paradoxical risk aversion for any pair of payoffs with the same ratio of valuation $q(>1)$. For example, the prizes can be \$40 (for a green ball), \$10 (for a white ball) and \$0 (for a black ball) and risk aversion implications are similar for corresponding prizes that involve millions as long as the valuation of \$40 (or \$40 million) is more than twice the valuation of \$10 (or \$10 million). Cox et al. (2012) report an experiment conducted in Magdeburg where payoffs \$40/\$10/0 were used. Arguably, at these small stakes the utility should be approximately linear and therefore q is expected to be larger than 2. The estimated percentage of subjects who revealed weak preference for the three outcome lottery S_i over lottery R_i, (i=1,…,9) is 65% which is similar to the 72% figure reported for another experiment conducted in Calcutta with payoffs 400/80/0 in rupees.

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[4] The supposition that the value of \$4K is more than twice the value of \$1K is consistent with estimates reported in Abdellaoui, Bleichrodt and L'Haridon (2008). This paper offers an appealing preference-based methods for measuring utility of (positive and negative) prizes under prospect theory; reported measurements of utility were robust to probabilities used in elicitation. In Abdellaoui et al. (2008) the reported estimated (mean) power exponent on the gain domain is 0.86. Since 4K and 1K are within the range of payoffs in their study, our numerical illustrations will build on the value of 4K being at least 3 times the value of K, which is satisfied for the power estimates in their study.

[5] Verify that $v(20000)f(0.5) \le 1000v(20)f(0.5) < v(20)$ where the first inequality follows from subadditivity of v(.) whereas the second inequality follows from f(0.5)<0.00098.

3. Paradoxical Probabilistic Sensitivity and Risk Aversion

This section contains the main result of the paper. It reports on implausible implications that follow from attributing risk-aversion (partly or wholly) to attitudes towards probabilities. All proofs are collected in the appendix.

Let $L=\{(x)_{j=1,..,n};(p)_{j=1,..,n}\}$ denote a prospect with n prizes: it pays x_j with probability p_j. As usual, let the outcomes be rank-ordered from best to worst. The rank dependent utility of prospect L is

(2) $$U(L)=\sum_{j=1...n} v(x_j)(f(P_j)-f(P_{j-1}))$$

where: $P_j=\Pr(x:x\geq x_j\}$; $f(\cdot)$ is the transformation of decumulative probability; and $v(\cdot)$ is a strictly increasing valuation function defined over prizes. Without any loss of generality the valuation of prize $0 is normalized to 0, $v(0)=0$. I assume that the function $v(\cdot)$ is subadditive over relatively large gains, which is consistent with the literature.

I am concerned herein with individual preferences over prospects $R_i=\{h,0;p_i\}$, and $S_i=\{h,m,0;p_i-1/2n,1/n\}$ where $p_i=i/2n$, $i=1,\cdots,2n-1$ and $n\in N$. In words, the three outcome lottery S_i pays h with probability $i/2n-1/2n$, m with probability $1/n$ (that does not depend on i) and 0 otherwise whereas the two outcome lottery R_i pays h with probability $i/2n$ and 0 otherwise. Figure 1 illustrates such pairs of lotteries in the probability triangle. Note that all two-outcome lotteries, R_i are on the hypotenuse whereas all three-outcome lotteries, S_i lie on a line parallel with the hypotenuse at a distance 1/n. Note also that lines that join lotteries R_i and S_i are parallel with slope 1. So, wherever in the probability triangle indifference curves are not flatter than the 45^o line lotteries S_i are weakly preferred to lotteries R_i.

The theorem below states implications of a risk avoiding pattern for the transformation of probabilities and paradoxical risk aversion that follow from it. Part (a) of the theorem offers an upper bound on the difference between the transformed probabilities of any $p\in(p_*,p^*)$ and p_* that follows from weak preference for S_i over R_i for all pairs $i\in(2np_*,2np^*)\cap N$; that is for all pairs i with R_i from a "connected" subset of the hypotenuse in the probability triangle. Part (b) states risk aversion implications that follow from part (a).

The following standard notation is used: $\succsim$ for weak preference, $\succ$ for strong preference, $\lceil x \rceil$ for the smallest integer larger than x, and $\lfloor x \rfloor$ for the largest integer smaller than x. In addition, hereafter whenever no confusion is expected symbols q and κ will be used as short notations for functions $q = v(h)/v(m) - 1$, $\kappa(q,t,s) = 1 + \sum_{j=0}^{t} q^{j+1} \Big/ \sum_{i=0}^{s} q^{-i}$.

Theorem: Let prizes $h > m > 0$ and the integer $n \in N$ be given. Suppose that $q > 1$ and $S_i \succsim R_i$, for all $p_i = i/2n \in (p_*, p^*)$. Then

a. $\forall p \in (p_*, p^*)$

(*) $$f(p) - f(p_*) \leq \frac{1}{\kappa}\left[f(p^*) - f(p_*) \right]$$

where $\kappa = \kappa\left(q, \lfloor 2np^* - 1 \rfloor - \lceil 2np \rceil, \lceil 2np \rceil - \lfloor 2np_* + 1 \rfloor\right)$.

b. $\forall \varepsilon \in (0, p^* - p_*)$, $\forall \mathrm{z} \in N$ and $\forall G \in N$ there exists $n^* \in N$ such that for all $n \geq n^*$

$$\{\mathrm{zG,z};\, p_*,\, p^* - p_*\} \succsim \{\mathrm{zG};\, p^* - \varepsilon\}$$

What does part (a) tell us? Suppose that for some interval $(p_*, p^*) \subset [0,1]$, the three outcome lottery, S_i is weakly preferred to the two outcome lottery R_i for all $p_i = i/2n \in (p_*, p^*)$ for some $n \in N$. Then statement (*) offers an upper bound on the difference between the transformed probability of any given $p \in (p_*, p^*)$ and p_*. Note that the smaller the upper bound the more flat the transformation of the probabilities, hence the more implausible the implication with respect to risk avoiding behavior. This leads to the result stated in part (b).

Part (b) of the theorem says that no matter how large the multiplier G is and how small ε is for sufficiently large $n \in N$ (that is for prospects S_i sufficiently close to the hypotenuse) weak preference for the safer prospect S_i for $p_i \in (p_*, p^*)$ implies that the agent would not be willing to trade-off prize z with probability $p^* - p_*$ for the alternative prize zG with probability $p^* - p_* - \varepsilon$ and 0 with probability ε.

The following examples provide numerical illustrations of parts (a) and (b) of the theorem for different intervals of probabilities where prospect S_i is preferred over prospect R_i.

Example 1: $S_i \succeq R_i$, for all $p_i = i/100 \in (0.5,1)$. In the probability triangle this means that the pattern of risk aversion holds for each pair *i* above the 45^o ray. An application of inequality (*) with $p = 0.75, p^* = 1$ and $p_* = 0.5$ reveals that $f(0.75) - f(0.5) \le [1 - f(0.5)]/K(q,24,24)$. If prizes *h* and *m* are such that $v(h) \ge 2.5v(m)$ then one has $f(0.75) - f(0.5) < 0.00004[1 - f(0.5)]$. What are risk avoiding implications of a probability transformation that is this flat on the interval [0.5, 0.75]? Our individual strictly prefers a lottery that offers prizes $20 and $500,000 with equal probability to the lottery that offers $500,000 with probability 0.75 and $0 otherwise.[6] Thus, our individual would be unwilling to trade-off $20 with probability 0.5 in exchange for $500,000 with probability 0.25 and 0 with probability 0.25, which is implausible risk aversion.

Example 2: $S_i \succeq R_i$, for all $p_i = i/100 \in (0, 0.5)$. In the probability triangle such *i* pairs of prospects are located below the 45^o line. For p=0.25 a direct application of statement (*) with $v(h) \ge 2.5v(m)$ reveals that $f(0.25) < 0.00004 f(0.5)$. Again severe probability distortions are implied. The last inequality implies preference for a prospect that pays $100 or $0, each with probability 0.5, over a prospect that offers $2.5 million or $0 with probabilities 0.25 and 0.75 respectively.

Example 3: $S_i \succeq R_i$, for all $p_i = i/100 \in (0.2, 0.8)$. In terms of the example in section 2 our individual weakly prefers having two white balls in the urn over replacing them with one black (increasing this way the number of black balls by one) and one green ball (increasing this way the number of green balls by one); he has this preference whenever the number of black balls in the safer urn is between 20 and 70. If $v(h) \ge 2.5v(m)$ an implication of this preference for rank dependent models is preference for a prospect that pays $1.9 million with probability 0.2, $10 with probability 0.6 or \$0 with probability 0.2 over a prospect that pays \$1.9 million or \$0 each with probability 0.5. So, our agent would not be willing to exchange \$10 with probability 0.6 for increasing the probability of getting \$1.9 million from 0.2 to 0.5 when that exchange increases the probability of prize \$0 from 0.2 to 0.5.

Table 2 reports further numerical illustrations of probability distortions and paradoxical risk aversion. For example, if one prefers the three outcome lottery for all $p_i = i/20 \in (0,1)$ then according to rank dependent models his perception of probability 0.1 must be no larger than 0.29×10^{-5} (row n=10 and column “f(0.1)<”), which is an implausible prediction. Figures in the bottom part of Table 2 show numerical illustrations of paradoxical risk aversion. For example, preference for the three outcome lottery for all $p_i = i/20 \in (0,1)$ implies that \$100 for sure is preferred to 34 million with probability 0.1 or 0 otherwise (see row n=10, p=0.1 column in Table 2). Or preference for the three outcome lottery for all $p_i = i/50 \in (0, 0.5)$ implies that the even odds prospect with prizes \$100 or \$0 is preferred to \$0.10 billion with probability 0.1 or 0 otherwise (see row n=25, most right G column in Table 2).

4. Implausibility of Modeling Risk Aversion as Attitude towards Probabilities

Previous literature has focused on the inability of expected utility theory (EUT) to rationalize some postulated patterns of choices. Allais (1953) introduced patterns of choices under risk that (if observed) refute expected utility. Allais’ critique was directed at the linearity in probabilities property of the EUT functional. Rabin’s (2000) patterns of risk aversion were directed at the nonlinearity in payoffs property of the EUT functional; they call into question the ability of the expected utility of terminal wealth model to rationalize risk aversion at large stakes and at small stakes.

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[6] Indeed, let 0.5M denote \$500,000 and use subadditivity of $v(.)$ and the upper bound on $f(0.75) - f(0.5)$ to verify that $v(0.5M)(f(0.75) - f(0.5)) \leq 25000v(20)\left(f(0.75) - f(0.5)\right) < v(20)(1 - f(0.5))$. Then rearrange terms in the last inequality to get $v(0.5M)f(0.75) < v(0.5M)f(0.5) + v(20)(1 - f(0.5))$.

This paper introduces a pattern of risk aversion that, if explained by probabilistic risk attitudes, implies implausible risk aversion. It is shown how attributing risk aversion to attitude towards small probabilities can produce extreme probability distortions that have paradoxical implications. Rank Dependent Utility and Cumulative Prospect Theory that are not vulnerable to the Allais paradox are vulnerable to the paradoxical risk aversion that follows from nonlinear transformation of probabilities, the very property that makes them accommodate the Allais paradox. This paper argues that rank dependent utility models have problems rationalizing risk aversion for both small and large probabilities. This is true whether the size of payoffs is small (as like in most laboratory experiments) or large.

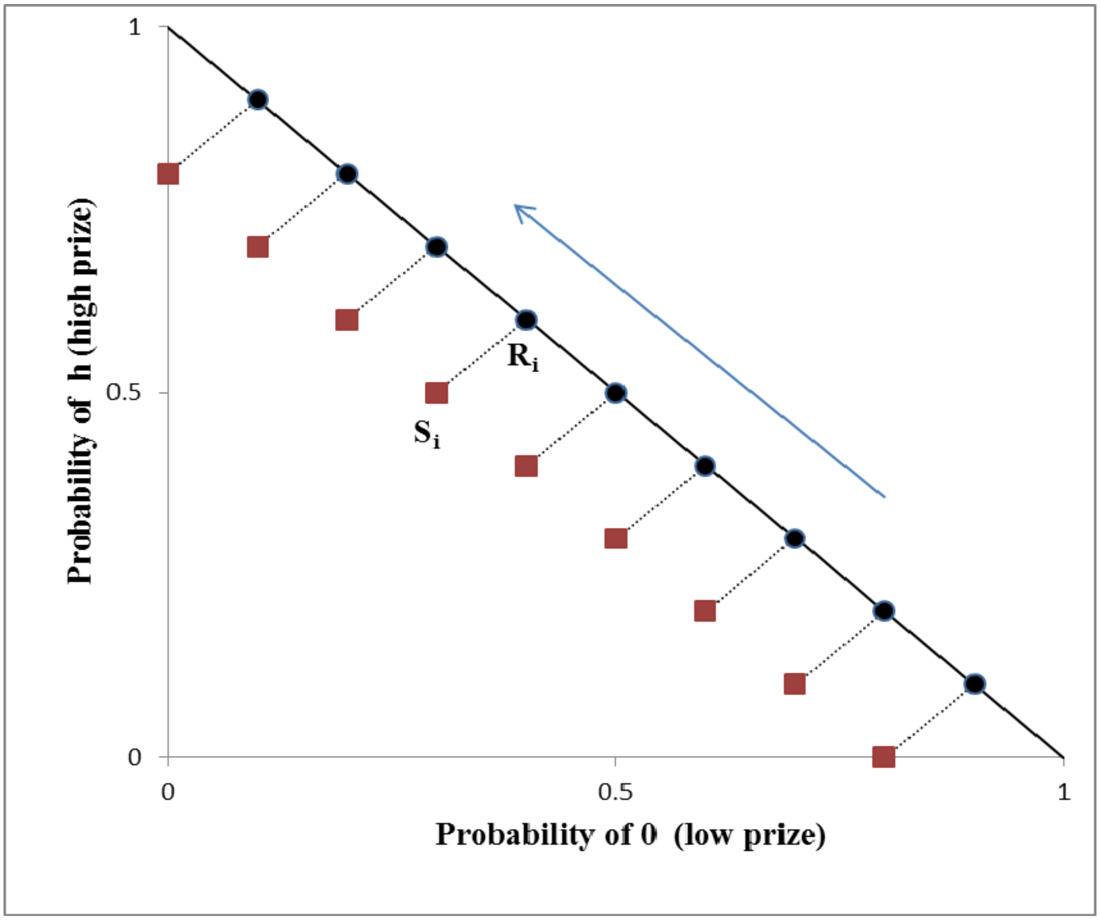

Figure 1. Illustration of pairs of lotteries (S_i, R_i) in the probability triangle

Table 2. Numerical Illustrations of Implications of Risk Aversion Pattern in the supposition on the Theorem for $v(h)/v(m) \geq 3$

Distortions of Probabilities					
$k_* / 2n = 1/2n$	$k^* / 2n = 1 - 1/2n$			$k^* / 2n = 0.5 - 1/2n$	
n	$k^* / 2n$	$f(0.1) <$	$f(0.5) <$	$k^* / 2n$	$f(0.1) <$
5	0.9	0.00098	0.03031	0.4	0.03226 $f(0.5)$
10	0.95	0.29×10^{-5}	0.00098	0.45	0.00294 $f(0.5)$
25	0.98	0.28×10^{-13}	0.30×10^{-7}	0.48	0.93×10^{-6} $f(0.5)$
50	0.99	0.81×10^{-27}	0.89×10^{-15}	0.49	0.91×10^{-12} $f(0.5)$
Paradoxical Risk Aversion					
$k_* / 2n = 1/2n$	$100 \succ \{G, 0; p, 1-p\}$			$\{100, 0; 0.5, 0.5\} \succ \{G, 0; 0.1, 0.9\}$	
n	$k^* / 2n$	$p = 0.1$ **G**	$p = 0.5$ **G**	$k^* / 2n$	**G**
5	0.9	102,000	3,300	0.4	3,100
10	0.95	0.34×10^{8}	102,000	0.45	34,100
25	0.98	0.36×10^{16}	0.33×10^{10}	0.48	0.10×10^{9}
50	0.99	0.12×10^{30}	0.11×10^{18}	0.49	0.11×10^{15}

Appendix: Proof of the Theorem

We first prove one proposition and one lemma. Then we use these two results to prove the main theorem.

For any given $n \in N$ and for any given two integers $k_*, k^* \in N$ such that $1 \le k_* < k^* \le 2n-1$, let $\Psi(k_*, k^*)$ denote the following finite set $\{k_*/2n, (k_*+1)/2n, ..., k^*/2n\}$. Recall that for given $h > m > 0$, $q = v(h)/v(m) - 1$ and for given $n \in N$, $S_i = \{h, m, 0; (i-1)/2n, 1/n, 1-(i+1)/2n\}$ and $R_i = \{h, 0; i/2n, 1-i/2n\}$.

Proposition A.1: Let prizes $h > m > 0$ and $n \in N$ be given. Suppose that $q > 1$ and $S_i \succeq R_i$, for all $p_i \in \Psi(k_*, k^*)$. Then $\forall \mathrm{k} \in \Psi(k_*, k^*)$

(a.1) $$f(\frac{k}{2n}) \le \frac{\Gamma}{1+\Gamma} f(\frac{k^*+1}{2n}) + \frac{1}{1+\Gamma} f(\frac{k_*-1}{2n})$$

where $\Gamma = \sum_{j=0}^{k-k_*} q^{-j} / \sum_{j=0}^{k^*-k} q^{j+1}$

Proof. If $S_i \succeq R_i$ for some $i \in \{k_*, \ldots, k^*\}$ then according to statement (2) in the text the agent has revealed

(a.2) $$v(h) f(\frac{i-1}{2n}) + v(m)\left(f(\frac{i+1}{2n}) - f(\frac{i-1}{2n}) \right) \ge v(h) f(\frac{i}{2n}), \ i = k_*, \ldots, k^*$$

Adding and subtracting $v(m) f(i/2n)$ on the left-hand-side of the above inequality, rearranging terms and using notation $q = v(h)/v(m) - 1$, one has

(a.3) $$f(\frac{i+1}{2n}) - f(\frac{i}{2n}) \ge q\left(f(\frac{i}{2n}) - f(\frac{i-1}{2n}) \right), \ i = k_*, \ldots, k^*$$

If $S_j \succeq R_j$ for all $j = i, ..., i+t$ where $i, i+t \in \{k_*, \ldots, k^*\}$ then apply inequality (a.3) t times to get

(a.4) $$f(\frac{i+t+1}{2n}) - f(\frac{i+t}{2n}) \ge q\left(f(\frac{i+t}{2n}) - f(\frac{i+t-1}{2n}) \right) \ge ... \ge q^{t+1}\left(f(\frac{i}{2n}) - f(\frac{i-1}{2n}) \right)$$

To complete the proof it suffices to show that $\forall k \in \{k_*, ..., k^*\}$, (a.5) $$f(\frac{k}{2n})-f(\frac{k_*-1}{2n})\leq\left(f(\frac{k}{2n})-f(\frac{k-1}{2n})\right)\sum_{j=0}^{k-k_*}q^{-j}$$

and

(a.6) $$f(\frac{k^*+1}{2n})-f(\frac{k}{2n})\geq\left(f(\frac{k}{2n})-f(\frac{k-1}{2n})\right)\sum_{j=0}^{k^*-k}q^{j+1}$$

because the last two inequalities imply that

$$\frac{1}{\sum_{j=0}^{k-k_*}q^{-j}}\left(f(\frac{k}{2n})-f(\frac{k_*-1}{2n})\right)\leq\frac{1}{\sum_{j=0}^{k^*-k}q^{j+1}}\left(f(\frac{k^*+1}{2n})-f(\frac{k}{2n})\right);$$

rearrange terms and use notation $\Gamma=\sum_{j=0}^{k-k_*}q^{-j}/\sum_{j=0}^{k^*-k}q^{j+1}$ to obtain statement (a.1)

Inequality (a.5) follows directly from inequality (a.4) and some rearrangement of terms

$$f(\frac{k}{2n})-f(\frac{k_*-1}{2n})=\sum_{i=k_*}^{k}\left(f(\frac{i}{2n})-f(\frac{i-1}{2n})\right)\leq\left(f(\frac{k}{2n})-f(\frac{k-1}{2n})\right)\sum_{j=0}^{k-k_*}q^{-j}$$

Similarly, for inequality (a.6) verify that

$$f(\frac{k^*+1}{2n})-f(\frac{k}{2n})=\sum_{i=k+1}^{k^*+1}\left(f(\frac{i}{2n})-f(\frac{i-1}{2n})\right)\geq\left(f(\frac{k}{2n})-f(\frac{k-1}{2n})\right)\sum_{j=0}^{k^*-k}q^{j+1}$$

Lemma A.1. If $q>1$ then $\lim_{t\to\infty}\kappa(q,t,s)=\infty$

Proof. It follows from $\kappa(q,t,s)=1+\sum_{j=0}^{t}q^{j+1}/\sum_{i=0}^{s}q^{-i}=\frac{q^{2+t+s}-1}{q^{1+s}-1}>q^{1+t}$

Proof of the Theorem

Part a. Suppose that $S_i\succeq R_i$, for all $p_i=i/2n\in(p_*,p^*)$ for some $n\in N$. Let $p\in(p_*,p^*)$ be given. Let k_* denote the largest integer smaller than $2np_*+1$, i.e. $k_*=\lfloor 2np_*+1\rfloor$. Let k denote the smallest integer larger than 2np, that is $k=\lceil 2np\rceil$ and k^* denote the largest integer smaller than $2np^*$-1, that is

$k^* = \lfloor 2np^* - 1 \rfloor$. By the supposition in the theorem and the construction of k_* and k^* one has $S_i \succeq R_i$, for all $p_i \in \Psi(k_*, k^*)$. By Proposition A.1 and construction of k one has

$$f(p) \le f(\frac{k}{2n}) \le \frac{\Gamma}{1+\Gamma} f(\frac{k^*+1}{2n}) + \frac{1}{1+\Gamma} f(\frac{k_*-1}{2n}) \le \frac{\Gamma}{1+\Gamma} f(p^*) + \frac{1}{1+\Gamma} f(p_*)$$

Subtract $f(p_*)$ from both sides of the last inequality and note that $(1+\Gamma)/\Gamma = \kappa$ to complete the proof

$$f(p) - f(p_*) \le \frac{\Gamma}{1+\Gamma}\left[f(p^*) - f(p_*)\right] = \frac{1}{\kappa}\left[f(p^*) - f(p_*)\right]$$

where $\kappa = \kappa(q, k^* - k, k - k_*)$..

Part b. For any given $\varepsilon \in (0, p^* - p_*)$, $G \in N$ and $q > 1$ take $n^* \ge \left(1 + \ln G / \ln q^2\right)/\varepsilon$. Take any $n \in N$ such that $n \ge n^*$ and suppose that $S_i \succeq R_i$, for all $p_i = i/2n \in (p_*, p^*)$. As above construct $k = \lceil 2np \rceil$, $p = p^* - \varepsilon$ and $k^* = \lfloor 2np^* - 1 \rfloor, k_* = \lfloor 2np^* + 1 \rfloor$. Apply Lemma A.1 and inequality $n \ge n^*$ to get

$$\kappa(q, k^* - k, k - k_*) \ge q^{k^* - k + 1} \ge q^{2(n\varepsilon - 1)} > G\,.$$

Next applying subaditivity of $v(.)$, the last inequality and inequality (*) in part (a) of the theorem one has

$$\begin{aligned} v(zG)(f(p^* - \varepsilon) - f(p_*)) &\le Gv(z)(f(p^* - \varepsilon) - f(p_*)) \le \kappa v(z)(f(p^* - \varepsilon) - f(p_*)) \\ &\le v(z)\left[f(p^*) - f(p_*)\right] \end{aligned}$$

Hence, $v(zG) f(p^* - \varepsilon) \le v(zG) f(p_*) + v(z)\left[f(p^*) - f(p_*)\right]$, which completes the proof

$$\{\mathrm{zG}, z; p_*, p^* - p_*\} \succeq \{\mathrm{zG}; p^* - \varepsilon\}.$$

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