The Valuation of an Equity-Linked Life Insurance Using the Theory of Indifference Pricing

Jungmin Choi

Abstract—**This study addresses the valuation problem of an equity-linked term life insurance in two mortality models - a deterministic mortality and a stochastic mortality. For each case, the Hamilton-Jacobi-Bellman (HJB) Partial Differential Equation (PDE) for the corresponding utility function is derived with a continuous time model, and the principle of equivalent utility is applied to obtain a PDE for the indifferent price of the premium when an exponential utility function is employed. Numerical examples are performed with Gompertz's law of mortality for the deterministic model and with a mean-reverting Brownian Gompertz (MRBG) process for the stochastic model.**

Index Terms—**Equity-Linked Life Insurance, indifference pricing, stochastic mortality.**

I. Introduction

THIS study focuses on the valuation problem of an equity-linked term life insurance using the theory of *indifference pricing*. In addition to a mortality risk like any other life insurance product, an equity-linked term life insurance has a market risk from the underlying asset.

In recent years, insurers have offered more flexible life insurance products that combine the death benefit coverage with an investment component, to compete with other forms of the policy holder's savings, for example, mutual funds or banks. An equity-linked life insurance product can offer a benefit from the performance of an underlying asset by defining the death benefit to depend on the account value of the underlying asset. The pricing and hedging problem of the equity-linked life insurance has been investigated extensively, and it is well summarized by Melnikov and Romanyuk [12]. As mentioned in their article, insurance firms do not consider the mortality risk in valuation of the policies nor adopt adequate mortality rates. This leads to an overpricing or underpricing of the premiums and the burden falls on the customers or the firm itself. Young [19] considered the same problem using the theory of indifference pricing when the mortality rate is computed using Gompertz's law of mortality. The purpose of this study is to extend their idea to include a stochastic mortality rate.

Indifference pricing, also known as reservation pricing or private valuation, is a method of pricing financial derivatives with regard to a utility function. It is one of the pricing tools in incomplete financial markets, and it uses the principle of equivalent utility. Utility functions are widely used for problems in pricing and hedging of financial derivatives, see [8] and [15], for example. Utility indifference pricing was first introduced by Hodges and Neuberger [6] when they considered transaction costs in replicating contingent claims. The main idea of indifference pricing is that by comparing the maximal expected utilities with and without a contingent claim, one can find a value of the price function which is indifferent to the existence of the contingent claim. This idea was used to price insurance risks in a dynamic financial market setting by Young and Zariphopoulou [20] using an exponential utility. The same idea was extended by Young [19] to study an equity-indexed life insurance, and the derived PDE for premiums and reserves generalized the Black-Scholes equation by including a nonlinear term reflecting the nonhedgeable mortality risk. We will adopt their model to study the indifference pricing of an equity-indexed life insurance with two different mortality risk models: a deterministic and a stochastic. Indifference pricing is also used in pricing problems in an incomplete market. The valuation of options in a stochastic volatility model for stock price using indifference pricing was studied by Sircar and Sturm [18] and Kumar [7].

Stochastic mortality became important especially for the mortality contingent claim. In [14], Milevsky and Promislow studied the pricing problem of an option to annuitize when considering stochastic mortality rates and stochastic interest rates. They also studied how to hedge an option to annuitize using pure endowments, default free bonds, and life insurance contracts. Variable annuities under stochastic mortality were also considered by Ballotta and Haberman [1]. Pricing, reserving and hedging of a guaranteed annuity option (GAO) valuation problem was studied when mortality risk was incorporated via a stochastic model of the underlying hazard rates. Assuming a stochastic mortality that is independent of the financial risk, a general pricing model was proposed, and the Monte Carlo method was used for the estimation of the value of GAO. Their stochastic mortality model was also used by Piscopo and Haberman [16], who considered a Guaranteed Lifelong Withdrawal Benefits (GLWB) contract under the hypothesis of a predetermined withdrawal amount. The valuation approach was based on the decomposition of the product into living and death benefits, and a no arbitrage model was used to derive the valuation formula, with a fixed mortality and a stochastic mortality.

Indifference pricing of mortality contingent claims was investigated by Ludkovsky and Young [10] with both stochastic hazard rates in the population mortality and the stochastic interest rates. The resulting PDEs were linear for pure endowments and temporary life annuities in a continuous time model, and it was found that the price-per-risk increases as more contracts are sold. A study of the indifference pricing of a traditional life insurance and pension products portfolio with stochastic mortality was presented by Delong [5], when a financial market consists of a risk-free asset with a constant rate of return and a risky asset whose price is driven by a Levy process. He applied techniques from stochastic control theory to solve the optimization problems.

In this paper, we consider the pricing problem of an equity-

Jungmin Choi is with the department of Mathematics at East Carolina University, Greenville, North Carolina, USA. (email: choiju@ecu.edu)

(Advance online publication: 26 November 2016)

linked term life insurance using the theory of indifference pricing, when the mortality of the insured is described by two different mortality models: a deterministic mortality and a stochastic mortality. The remainder of this paper is organized as follows. In Section II, we present the pricing PDE of an equity-indexed term life insurance using an exponential utility in the continuous-time model with a deterministic mortality function, as described in [19]. The numerical solutions of the PDE with Gompertz's law of mortality and the sensitivities of the indifference prices with respect to various parameters are also provided. The pricing PDE and numerical examples with a stochastic mortality model are given in Section III. We adopt the MRBG process to model the mortality risk. Finally, Section IV summarizes our findings and outlines future works to extend our model.

II. Indifference pricing of an Equity-Indexed Life Insurance with a deterministic mortality

A. The Financial Market: Merton's Model

We present the classical model by Merton [13] which investigates the optimal investment strategies of an individual with an initial wealth, who seeks to maximize the expected utility of the terminal wealth. The investor has the opportunity to trade between a risky asset (stock) and a risk-free asset (U.S. treasury bond). The price of the risky asset S_s for some time $s > t$, with a fixed time t, follows

$$\begin{cases} dS_s &= \mu S_s ds + \sigma S_s dB_s, \\ S_t &= S > 0, \end{cases}$$

where B_s is a standard Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with a filtration $\mathcal{F}$ and a probability measure $\mathbf{P}$. The rate of return μ and the volatility σ are positive constants.

The price of the risk-free bond $\bar{P}_s$ for some time $s > t$ follows

$$d\bar{P}_s = r\bar{P}_s ds,$$

where r is a constant rate of return (or force of return) of the risk-free bond, and we assume $\mu > r > 0$.

Let w be the initial wealth of the insurer at time t, and W_s be the wealth of the insurer at time s in $[t, T]$, where T is the terminal time. Suppose the insurer trades dynamically between the stock and the bond. Let π_s be the amount of wealth invested in the stock at time s. Then the amount invested in the bond is $\pi_s^b = W_s - \pi_s$, and the dynamics of the wealth process becomes

$$\begin{aligned} dW_s &= \pi_s^b \left(\frac{d\bar{P}_s}{\bar{P}_s}\right) + \pi_s \left(\frac{dS_s}{S_s}\right) \\ &= (W_s - \pi_s) r ds + \pi_s(\mu ds + \sigma dB_s). \end{aligned}$$

Hence we have

$$\begin{cases} dW_s &= (rW_s + (\mu - r)\pi_s)ds + \sigma\pi_s dB_s, t \le s \le T, \\ W_t &= w. \end{cases}$$

B. Expected Utility Without the insurance risk

Suppose the investor wants to maximize the expected utility of the terminal wealth, and define the value function $V(t, w)$ as

$$V(t, w) = \sup_{\pi_t \in \mathcal{A}} E[u(W_T)|W_t = w],$$

where $\mathcal{A}$ is the set of admissible policies, and $u : \mathcal{R} \to \mathcal{R}$ is a utility function, which is increasing, concave, and smooth. We will use an exponential utility function to derive a PDE for the indifference price.

It has been shown in [2] that V satisfies the following HJB equation:

$$\begin{cases} V_t + \max_{\pi_t}[(\mu - r)\pi_t V_w + \frac{1}{2}\sigma^2\pi_t^2 V_{ww}] + rwV_w = 0, \\ V(T, w) = u(w). \end{cases}$$

Since the maximum function is quadratic in π_t and the concavity of the utility function u is inherited to the value function, the maximum exists and we have the optimal investment process

$$\pi_t^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{V_w(w, t)}{V_{ww}(w, t)}.$$

This gives a closed form PDE for V:

$$\begin{cases} V_t + rwV_w - \frac{(\mu - r)^2}{2\sigma^2} \cdot \frac{V_w^2}{V_{ww}} &= 0, \\ V(T, w) &= u(w). \end{cases} \tag{1}$$

One of the advantages to considering an exponential utility function is that we can find the closed form solution to (1). Suppose $u(w) = -\frac{1}{\gamma}e^{-\gamma w}$, for some $\gamma > 0$, then we obtain the solution $V(t, w)$ to be

$$V(t, w) = -\frac{1}{\gamma} \exp\left[-\gamma w e^{r(T-t)} - \frac{(\mu - r)^2}{2\sigma^2}(T - t)\right]. \tag{2}$$

We also can find the corresponding optimal strategy

$$\pi_t^*(t, w) = \frac{\mu - r}{\sigma^2} \cdot \frac{e^{-r(T-t)}}{\gamma},$$

which is not stochastic and independent of w. It is generally observed when considering exponential utility. Since the absolute risk aversion for the exponential utility function is measured by a constant γ, $(-u''(w)/u'(w) = \gamma)$, one can observe that as the investor's risk aversion (γ) increases, the amount of money invested in the risky asset (π_t^*) decreases [20].

C. Expected Utility with the insurance risk

The insurer has an opportunity to insure a person whose age is $x + t$ at time t. The death benefit of this life insurance is defined to be $G = \max(A_0, A_\tau)$, where $\tau < T$ is the time of death of the policy holder, A_0 is the initial account value of the underlying mutual fund at the time when the contract is made, and A_s is the account value at time s. This insurance policy is an equity-indexed product since it is tied to an account value through the function G. Suppose the insurer charges an insurance fee to hedge the market risk, and we assume that it is deducted from the account value as an ongoing fraction, α[1]. The dynamics of A_s follow

$$dA_s = (\mu - \alpha)A_s ds + \sigma A_s dB_s,$$

where B_s is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbf{P})$, and μ (rate of return) and σ (volatility) are constants. Suppose

[1]For simplicity, we assume it is a fixed constant.

the insurer wants to maximize the expected utility of the terminal wealth, and define the value function

$$U(t,w,A) = \sup_{\pi_t \in \mathcal{A}} E\{u(W_T)|W_t = w, A_t = A\},$$

where $\mathcal{A}$ is the set of admissible policies, and $u : \mathcal{R} \to \mathcal{R}$ is a utility function, which is increasing, concave, and smooth.

The wealth process should follow

$$\begin{cases} dW_s &= [rW_s + (\mu - r)\pi_s]ds + \sigma\pi_s dB_s, \\ W_t &= w, \\ W_{\tau+} &= W_{\tau-} - G_\tau, \quad \text{if } \tau < T. \end{cases}$$

The HJB equation for U can be obtained as follows (see Appendix)

$$\begin{cases} U_t + rwU_w + (\mu-\alpha)AU_A + \frac{1}{2}\sigma^2A^2U_{AA} \\ +\lambda_x(t)[V(w-G,t) - U] \\ +\max_{\pi_t}[(\mu-r)\pi_tU_w + \sigma^2\pi_tAU_{wA} \\ +\frac{1}{2}\sigma^2\pi_t^2U_{ww}] = 0, \\ U(T,w,A) = u(w), \end{cases} \tag{3}$$

where $\lambda_x(t)$ is the force of mortality of a person aged x at time t.

The corresponding optimal strategy π_t^* is

$$\pi_t^* = -\frac{\mu - r}{\sigma^2}\cdot\frac{U_w}{U_{ww}} - A\frac{U_{wA}}{U_{ww}}.$$

Suppose we use the exponential utility $u(w) = -\frac{1}{\gamma}e^{-\gamma w}$, for some $\gamma > 0$. Because of the nature of the exponential utility, we propose the solution of (3) to be in the form of $U(t,w,A) = V(t,w)\cdot\phi(t,A)$ [19]. Then the optimal strategy becomes

$$\pi_t^* = -\frac{\mu - r}{\sigma^2}\cdot\frac{V_w}{V_{ww}} - A\frac{V_w}{V_{ww}}\cdot\frac{\phi_A}{\phi}.$$

Let $U = V\cdot\phi$ in (3). From (2), we also have

$$\frac{V_w^2}{V_{ww}} = V,$$

and

$$V(t,w-G) = V(t,w)\exp[\gamma Ge^{r(T-t)}].$$

Then we obtain the PDE for ϕ from (3):

$$\begin{cases} \phi_t + (r-\alpha)A\phi_A + \frac{1}{2}\sigma^2A^2(\phi_{AA} - \frac{\phi_A^2}{\phi}) \\ +\lambda_x(t)(e^{\gamma Ge^{r(T-t)}} - \phi) = 0, \\ \phi(T,A) = 1. \end{cases} \tag{4}$$

By introducing $\eta(t,A)$ as $\phi(t,A) = e^{\eta(t,A)}$, we have

$$\begin{cases} \eta_t + (r-\alpha)A\eta_A + \frac{1}{2}\sigma^2A^2\eta_{AA} \\ +\lambda_x(t)(e^{\gamma Ge^{r(T-t)} - \eta} - 1) = 0, \\ \eta(T,A) = 0. \end{cases} \tag{5}$$

Now let $P(t,A)$ be the indifference price, that is, the minimum premium the insurer should have in exchange for insuring the person whose age is $x+t$ at time t for a term life insurance which expires at time T. Then $P(t,A)$ should solve

$$V(t,w) = U(t,w+P,A) = V(t,w+P)\phi(t,A),$$

and using the closed form of V in (2), we have a formula for $P(t,A)$ with respect to η as

$$P(t,A) = \frac{1}{\gamma}e^{-r(T-t)}\cdot\eta(t,A).$$

TABLE I
PARAMETER VALUES

Age at inception x	50
Risk free interest rate r	0.08
Volatility of the risky asset σ	0.2
Insurance fee α	0.001
Term of policy T	15 years

Using this relationship in (5), we can derive a PDE for $P(t,A)$

$$\begin{cases} -rP + P_t + (r-\alpha)AP_A + \frac{1}{2}\sigma^2A^2P_{AA} \\ = \frac{1}{\gamma}e^{-r(T-t)}\lambda_x(t)[1 - e^{-\gamma e^{r(T-t)}\cdot(P-G)}], \\ P(T,A) = 0. \end{cases} \tag{6}$$

D. Numerical Example and Sensitivity Analysis

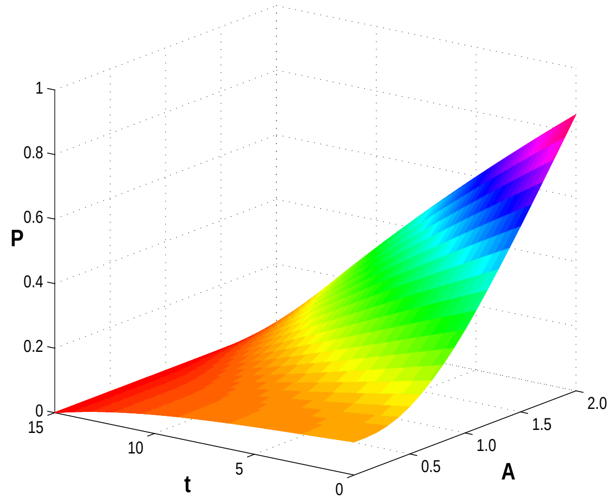

Fig. 1. Price with respect to time and account value

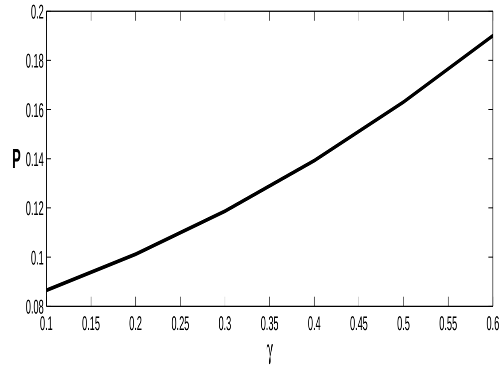

Fig. 2. The relationship between γ and $P(0, A_0)$

In this section, we solve (6) assuming the mortality $\lambda_x(t)$ follows Gompertz's law of mortality

$$\lambda_x(t) = B\cdot C^{x+t},$$

with $B = 1.164\times10^{-5}$ and $C = 1.1096$. These parameter estimations were obtained in [11] using 1959-1999 mortality data for Sweden. To solve (6) numerically, we use the finite

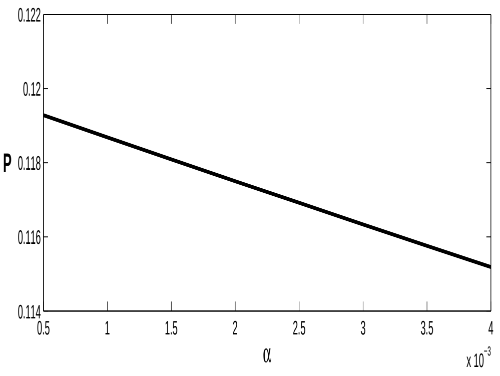

Fig. 3. The relationship between insurance fee and $P(0, A_0)$

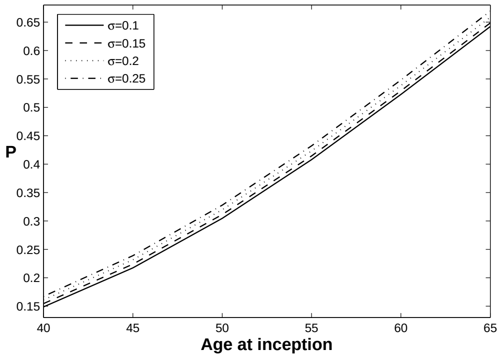

Fig. 4. The relationship between volaility and $P(0, A_0)$

difference method, particularly, backward difference scheme for the time t and the central difference scheme for the account value A. We set the minimum account value to be zero and the maximum account value to be 2, with the initial account value (A_0) of 1.

We assume the PDE holds on the boundary since the domain is truncated. For the boundary condition when the account value is maximum ($A = 2$), we assume the linearity of the premium P in terms of the account value, namely, we set $P_{AA} = 0$, and solve

$$-rP+P_t+(r-\alpha)AP_A = \frac{1}{\gamma}e^{-r(T-t)}\lambda[1-e^{-\gamma e^{r(T-t)}\cdot(P-G)}].$$

When the account value is minimum ($A = 0$), we solve

$$-rP + P_t = \frac{1}{\gamma}e^{-r(T-t)}\lambda[1 - e^{-\gamma e^{r(T-t)}\cdot(P-G)}].$$

A typical solution for the premium function P is plotted in Figure 1 with respect to the time and the account value, using the parameters in Table I for the base case. From the plot, we can observe that P increases as the account value A increases, which is expected since the higher the account value is, the higher the death benefit $G = \max(A_0, A_\tau)$ will be. The insurer should receive more premium for a higher death benefit. The premium P decreases as the time t increases,which is a common trend for an equity linked financial derivative (for example, the theta, the rate of change of the derivative with respect to the time t, is negative for a European call option).

Figure 2 shows the relationship between the risk aversion γ and the premium P. As observed in other studies of mortality contingent claims ([10], [19]), when the risk aversion increases, the indifference price of the premium also increases. Figure 3 shows the impact of the insurance fee α on the premium P. It is clear that they should have a negative relationship, since if one has to pay more insurance fees, the price of the product at time zero should decrease. A positive relationship between the volatility σ of the risky asset and the premium P is reflected in Figure 4, which is consistent with a general financial theory that the financial product is more expensive when the volatility is high. The premium P is plotted against the age at inception x for various volatilities in the same figure, and we can observe that if a person opted to purchase the life insurance product at later dates (as x increases), he should pay a higher price for the benefit. This can be justified as follows: when x increases, the mortality of the person increases, and, hence, the price of the life insurance product should also increase.

III. Indifference Pricing of an Equity-Indexed Life Insurance with a Stochastic Mortality

A. Pricing PDE with stochastic mortality

Now we consider a stochastic model for the force of mortality for an individual or a set of individuals of the same age. We adopt the model proposed by Ludkovsky and Young [10] and assume the force of mortality λ follows a diffusion process as

$$d\lambda_s = \overline{\mu}(s, \lambda_s)ds + \overline{\sigma}(s)\lambda_s dB_s^\lambda, \tag{7}$$

where B_s^λ is a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ which is independent of B_s in the previous section. The volatility $\overline{\sigma}$ is a nonzero continuous function of time s bounded below by a positive constant κ on $[0, T]$. The drift $\overline{\mu}(s, \lambda)$ is a continuous function of s and λ which is positive for all s in $[0, T]$. We will use the mean-reverting Brownian Gompertz model in [14] for the numerical examples.

Suppose the account value A_s and the wealth process W_s are defined as in Section II. The insurer agrees to pay $G_\tau = \max(A_0, A_\tau)$ upon death at $\tau < T$ given a person aged x at $t = 0$ purchased the life insurance product. Suppose the insurer wants to maximize the expected utility of the terminal wealth, and define the value function

$$\overline{U}(t, w, A, \lambda) = \sup_{\pi_t \in \mathcal{A}} \{u(W_T)|W_t = w, A_t = A, \lambda_t = \lambda\},$$

where $\mathcal{A}$ is the set of admissible policies, and $u : \mathcal{R} \to \mathcal{R}$ is a utility function, which is increasing, concave, and smooth.

The HJB equation for $\overline{U}$ can be obtained as follows (see Appendix)

$$\left\{\begin{array}{l} \overline{U}_t + rw\overline{U}_w + (\mu-\alpha)A\overline{U}_A + \frac{1}{2}\sigma^2A^2\overline{U}_{AA} \\ +\lambda[V(t, w-G) - \overline{U}] + \overline{\mu}\lambda\overline{U}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2\overline{U}\lambda\lambda \\ + \max_\pi[(\mu-r)\pi\overline{U}_w + \sigma^2\pi A\overline{U}_{wA} \\ +\frac{1}{2}\sigma^2\pi^2\overline{U}_{ww}] = 0, \\ \overline{U}(T, w, A) = u(w). \end{array}\right. \tag{8}$$

Since the optimized terms are the same as in (3), it will assume the same optimal strategy

$$\pi_t^* = -\frac{\mu - r}{\sigma^2}\cdot\frac{U_w}{U_{ww}} - A\frac{U_{wA}}{U_{ww}}.$$

Following the idea in Section II and using the same exponential utility $u(w) = -\frac{1}{\gamma}e^{-\gamma w}$, we assume the solution of (8) to be in the form of

$$\overline{U}(t, w, A, \lambda) = V(t, w) \cdot \overline{\phi}(t, A, \lambda).$$

Then the optimal strategy becomes

$$\pi_t^* = -\frac{\mu - r}{\sigma^2} \cdot \frac{V_w}{V_{ww}} - A\frac{V_w}{V_{ww}} \cdot \frac{\phi_A}{\phi}.$$

Let $\overline{U} = V \cdot \overline{\phi}$ in (8). After collecting V terms and $\overline{\phi}$ terms, (8) becomes

$$\begin{array}{l} V[\overline{\phi}_s + (r-\alpha)A\overline{\phi}_A + \frac{1}{2}\sigma^2 A^2 \overline{\phi}_{AA} \\ +\lambda[\exp(\gamma G e^{r(T-t)}) - \overline{\phi}] \\ +\overline{\mu}\lambda\overline{\phi}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2\overline{\phi}_{\lambda\lambda} - \frac{1}{2}\sigma^2 A^2 \frac{\overline{\phi}_A^2}{\overline{\phi}}] \\ +\overline{\phi}[V_s + rwV_w - \frac{(\mu-r)^2}{2\sigma^2}V] = 0. \end{array}$$

The multiple to $\overline{\phi}$ in the last term is zero because of (1), hence we obtain the PDE for $\overline{\phi}$

$$\left\{ \begin{array}{l} \overline{\phi}_t + (r-\alpha)A\overline{\phi}_A + \frac{1}{2}\sigma^2 A^2(\overline{\phi}_{AA} - \frac{\overline{\phi}_A^2}{\overline{\phi}}) \\ +\overline{\mu}\lambda\overline{\phi}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2\overline{\phi}_{\lambda\lambda} + \lambda(e^{\gamma G e^{r(T-t)}} - \overline{\phi}) = 0, \\ \overline{\phi}(T, A, \lambda) = 1. \end{array} \right. \tag{9}$$

To eliminate the nonlinear term $\frac{\overline{\phi}_A^2}{\overline{\phi}}$ in (4), let us define $\overline{\eta}(t, A, \lambda)$ by

$$\overline{\phi}(t, A, \lambda) = e^{\overline{\eta}(t, A, \lambda)}.$$

Then we have a PDE for $\overline{\eta}(t, A, \lambda)$:

$$\left\{ \begin{array}{l} \overline{\eta}_t + (r-\alpha)A\overline{\eta}_A + \frac{1}{2}\sigma^2 A^2 \overline{\eta}_{AA} + \overline{\mu}\lambda\overline{\eta}_\lambda \\ +\frac{1}{2}\overline{\sigma}^2\lambda^2(\overline{\eta}_\lambda^2 + \overline{\eta}_{\lambda\lambda}) + \lambda(e^{\gamma G e^{r(T-t)} - \overline{\eta}} - 1) = 0, \\ \overline{\eta}(T, A, \lambda) = 0. \end{array} \right. \tag{10}$$

Now let $\overline{P}(t, A, \lambda)$ be the indifference price, that is, the minimum premium the insurer should have in exchange for a term life insurance which expires at time T. Then $\overline{P}(t, A, \lambda)$ should solve

$$V(t, w) = U(t, w + \overline{P}, A, \lambda) = V(t, w + \overline{P})\overline{\phi}(t, A, \lambda),$$

and using the closed form of V in (2), we have a formula for $\overline{P}(t, A, \lambda)$ with respect to $\overline{\eta}$ as

$$P(t, A, \lambda) = \frac{1}{\gamma}e^{-r(T-t)} \cdot \overline{\eta}(t, A, \lambda).$$

Using this relationship in (10), we can derive a PDE for $\overline{P}(t, A, \lambda)$

$$\left\{ \begin{array}{l} -r\overline{P} + \overline{P}_t + (r-\alpha)A\overline{P}_A + \frac{1}{2}\sigma^2 A^2 \overline{P}_{AA} \\ +\overline{\mu}\lambda\overline{P}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2(\gamma e^{r(T-r)}\overline{P}_\lambda^2 + \overline{P}_{\lambda\lambda}) \\ = \dfrac{1}{\gamma}e^{-r(T-t)}\lambda[1 - e^{-\gamma e^{r(T-t)}\cdot(\overline{P}-G)}], \\ \overline{P}(T, A, \lambda) = 0. \end{array} \right. \tag{11}$$

B. Numerical Example

For numerical examples that solves (11), we use the following MRBG process proposed in [14]:

$$d\lambda_s = \left(g + \frac{1}{2}\overline{\sigma}^2 + \kappa(gs + \ln\lambda_0 - \ln\lambda_s)\right)\lambda_s ds + \overline{\sigma}\lambda_s dB_s^\lambda,$$

with $\kappa = 0.5$. The process $\ln\lambda_s$ follows an Ornstein-Uhlenbeck model with a linear drift g. The parameter values in Table II are used to obtain the solutions unless noted otherwise.

TABLE II
PARAMETER VALUES

Risk free interest rate r	0.08
Volatility of the risky asset σ	0.2
Insurance fee α	0.001
Term of policy T	10 years
Volatility of the force of mortality $\overline{\sigma}$	0.2
Force of mortality at inception $\overline{\lambda}$	0.003
Gompertz parameter g	0.1

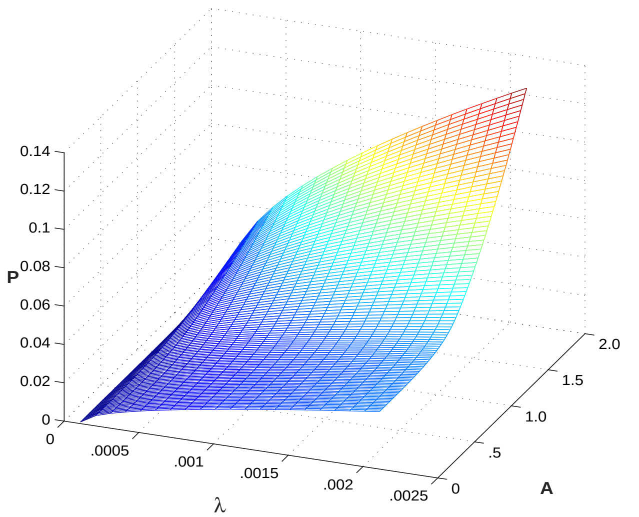

Fig. 5. 3D plot of the solution $P(0, A, \lambda)$

The PDE for the premium $P(t, A, \lambda)$ is solved in the domain $[0, T] \times [0, 2] \times [0, 0.025]$ with an initial account value (A_0) 1, using the backward in time finite difference method. As in the previous section, we assume the PDE holds on the boundary, and assuming linearity when $A = A_{max}$ and $\lambda = \lambda_{max}$. The boundary conditions imposed are

1) $A = 0$:

$$\begin{array}{l} -r\overline{P} + \overline{P}_t + +\overline{\mu}\lambda\overline{P}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2(\gamma e^{r(T-r)}\overline{P}_\lambda^2 + \overline{P}_{\lambda\lambda}) \\ = \dfrac{1}{\gamma}e^{-r(T-t)}\lambda[1 - e^{-\gamma e^{r(T-t)}\cdot(\overline{P}-G)}] \end{array}$$

2) $A = A_{max}$:

$$\begin{array}{l} -r\overline{P} + \overline{P}_t + (r-\alpha)A\overline{P}_A \\ +\overline{\mu}\lambda\overline{P}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2(\gamma e^{r(T-r)}\overline{P}_\lambda^2 + \overline{P}_{\lambda\lambda}) \\ = \dfrac{1}{\gamma}e^{-r(T-t)}\lambda[1 - e^{-\gamma e^{r(T-t)}\cdot(\overline{P}-G)}] \end{array}$$

3) $\lambda = 0$:

$$-r\overline{P} + \overline{P}_t + (r-\alpha)A\overline{P}_A + \frac{1}{2}\sigma^2 A^2 \overline{P}_{AA} = 0$$

4) $\lambda = \lambda_{max}$:

$$\begin{array}{l} -r\overline{P} + \overline{P}_t + (r-\alpha)A\overline{P}_A + \frac{1}{2}\sigma^2 A^2 \overline{P}_{AA} \\ +\overline{\mu}\lambda\overline{P}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2(\gamma e^{r(T-r)}\overline{P}_\lambda^2) \\ = \dfrac{1}{\gamma}e^{-r(T-t)}\lambda[1 - e^{-\gamma e^{r(T-t)}\cdot(\overline{P}-G)}] \end{array}$$

A typical solution for the premium function $P(0, A, \lambda)$ is given in Figure 5 at the time of inception ($t = 0$). It

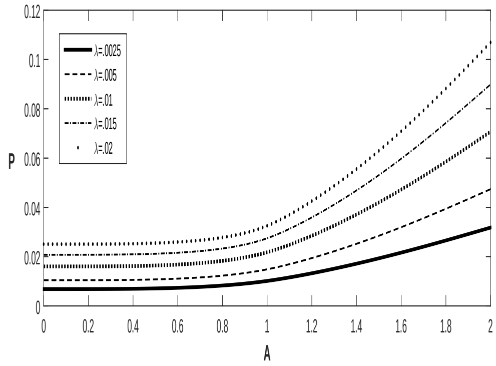

Fig. 6. The relationship between A and $P(0, A, \lambda)$ for various values of λ

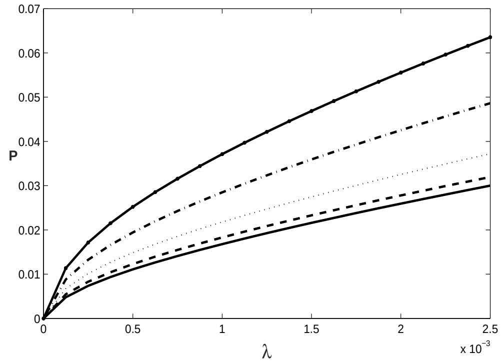

Fig. 7. The relationship between λ and $P(0, A, \lambda)$ for various values of A

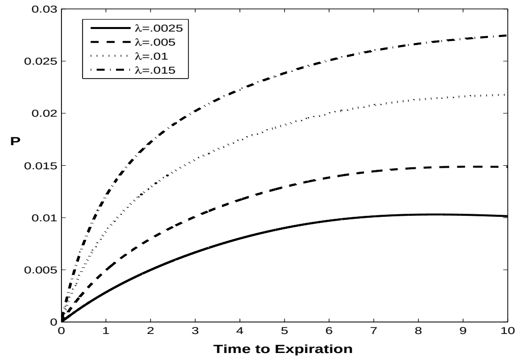

Fig. 8. The relationship between time to expiration $T - t$ and $P(T - t, 1.0, \lambda)$ for various values of λ

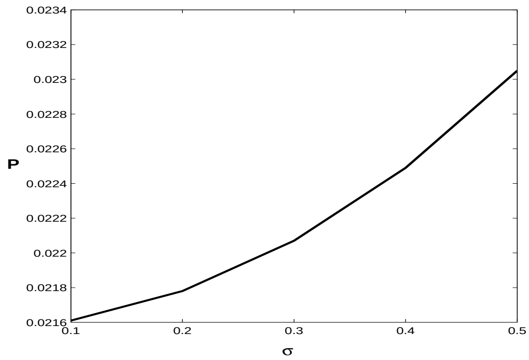

Fig. 9. The plot of $P(0, 1, 0.01)$ with respect to σ

shows that P is an increasing function in A and λ, which is expected since the price of a life insurance product should increase when the account value increases or the mortality rate increases. We can observe this more clearly in Figures 6 and 7. The premium function P is plotted against A for various λ in Figure 6. The premium P is an increasing convex function in the account value A for various values of λ; the rate of change is more significant with a higher value of A. It is clearer with a higher force of mortality λ. The premium function P is plotted against λ for various values of A in Figure 7. The premium P is an increasing concave function in λ; the rate of change is more significant with a smaller value of λ. It is clearer with a higher value of A. As in the deterministic mortality model, the premium function P has a positive relationship with the time to expiration $(T-t)$ in Figure 8. Similar trends can be observed in other literature, for example, in the description of the death benefit in the study by Piscopo and Haberman [16].

To see the effect of the volatility of the force of mortality in (7), the value of $P(0, 1, 0.01)$ is plotted for different values of $\overline{\sigma}$. We observe that as the volatility increases, the premium also increases in Figure 9, which reflects the common phenomenon in the financial markets that the price of a mortality contingent product increases when there is more risk in mortality. The effects of the risk aversion rate γ and the insurance fee α on the premium P is similar as in the case with a deterministic mortality model in Section II. The premium $P(0, 1, 0.01)$ with respect to the risk aversion rates γ is plotted in Figure 10, and we observe that the premium increases as the risk aversion rate increases. The premium $P(0, 1, 0.01)$ with respect to the insurance fee α is plotted in Figure 11, which shows that the premium of a life insurance product should decrease when the policy holder pays a higher insurance fee α.

IV. Conclusion

We have considered the valuation problem of an equity-indexed term life insurance with two different mortality models, a deterministic mortality and a stochastic mortality. For the financial market, we employ Merton's model and use an exponential utility to obtain HJB equations for the utility functions with and without the life insurance risks. By using the theory of equivalent utility, we derive the PDEs for the indifference price of the premium for both mortality models. The PDE with a deterministic mortality is solved numerically using Gompertz' law of mortality, while the MRBG process is adopted for the stochastic mortality case. The derived PDEs are not simple and closed form solutions cannot be found, but straightforward applications of the finite difference method with proper boundary conditions produce solutions that are reasonable for a life insurance contract. The sensitivity analysis shows that the models are appropriate to explain the premiums of the equity-indexed term life insurance.

Future research should consider the effect of stochastic interest rates, since it is unreasonable to assume the risk-free rate is constant for a long period of time. We can also apply the indifference pricing theory to variable annuity products exposed to similar risks, for example, with a Guaranteed Minimum Death Benefit option or a Guaranteed Lifelong Withdrawal Benefit option.

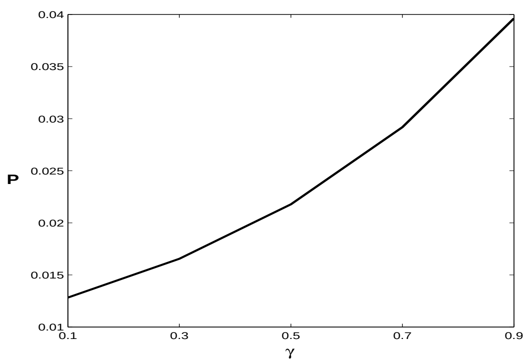

Fig. 10. The plot of $P(0, 1, 0.01)$ with respect to γ

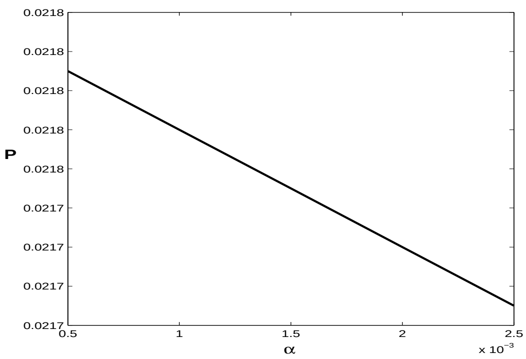

Fig. 11. The plot of $P(0, 1, 0.01)$ with respect to α

References

[1] Ballotta, L., and S. Haberman, "The Fair Valuation Problem of Guaranteed Annuity Options: The Stochastic Mortality Environment Case." Insurance: Mathematics and Economics, 38:195-214, 2006.

[2] Bjork, T. *Arbitrage Theory in Continuous Time*. Oxford University Press, 1998.

[3] Carmona, R. *Indifference Pricing*. Princeton University Press, 2009.

[4] Choi, J., and M. Gunzburger, "Option Pricing in the Presence of Random Arbitrage Return." International Journal of Computer Mathematics, 86(6): 1068-1081.

[5] Delong, L., "Indifference Pricing of a Life Insurance Portfolio with Systematic Mortality Risk in a Market with an Asset Driven by a Levy Process." Scadinavian Actuarial Journal, 1:1-26, 2009.

[6] Hodges, S. D., and A. Neuberger, "Optimal Replication of Contingent Claims under Transaction Costs." Review of Futures Markets, 8:222-239, 1989.

[7] Kumar, R., "Effect of Volatility Clustering on Indifference Pricing of Options by Convex Risk Measures." Applied Mathematical Finance 22(1):63-82, 2015, doi:10.1080/1350486X.2014.949805.

[8] Qiang Li, and Lap Keung Chu, "A Multi-stage Financial Hedging Strategy for a Risk-averse Firm with Contingent Payment," IAENG International Journal of Applied Mathematics, vol. 45, no. 1, pp71-76, 2015

[9] Loeve, M. *Probabiliity Theory*. New York: Springer, 1977.

[10] Ludkovski, M. and V.R. Young, "Indifference Pricing of Pure Endowments and Life Annuities under Stochastic Hazard and Interest Rates." Insurance: Mathematics and Economics, 42:14-30, 2008.

[11] Melnikov, A. and Y. Romaniuk, "Evaluating the Performance of Gompertz, Makeham and Lee-Carter mortality models for risk management with unit-linked contracts." Insurance: Mathematics and Economics, 39:310-329, 2006.

[12] Melnikov, A. and Y. Romaniuk, "Efficient Hedging and Pricing of Eruity-Linked Life Insurance Contracts on Several Risky Assets." International Journal of Theoretical and Applied Finance, 11(3): 295-323, 2008.

[13] Merton, R. C., "Lifetime Portfolio Selection under uncertainty: the Continuous-time Case." The Review of Economics and Statistics, 51(3): 247-257, 1969.

[14] Milevsky, M. A., and S. D. Promislow, "Mortality Derivatives and the Option to Annuitise." Insurance: Mathematics and Economics, 29:299-318, 2001.

[15] Mukupa G. M., and Offen E. R., "The Impact of Utility Functions on The Equilibrium Equity Premium In A Production Economy With Jump Diffusion," IAENG International Journal of Applied Mathematics, vol. 45, no. 2, pp120-127, 2015

[16] Piscopo, G., and S. Haberman, "The Valuation of Guaranteed Lifelong Withdrawal Benefit Options in Variable Annuity Contracts and the Impact of Mortality Riak." North American Actuarial Journal, 15(1): 59-76, 2011.

[17] Sithole, T. Z., S. Haberman, and R. J. Verrall, "An Investigation into Parametric Models for Mortality Projections, with Applications to Immediate Annuitants' and Life Office Pensioners' data." Insurance: Mathematics and Economics, 27: 285-312, 2001.

[18] Sircar, S. and S. Sturm, "From Smile Asymptotics to Market Risk Measures." Mathematical Finance, 10:259-276, 2012. doi:10.1111/mafi.12015.

[19] Young, V. R., "Equity-Indexed Life Insurance: Pricing and Reserving Using the Principle of Equivalent Utility." North American Actuarial Journal, 7(1): 68-86, 2003.

[20] Young, V. R., and T. Zariphopoulou, "Pricing Dynamic Insurance Risks Using the Principle of Equivalent Utility." Scandinavian Actuarial Journal, 4:246-279, 2002.

Appendix A
HJB equation for U with the insurance risk

Here we derive the HJB equation for U. Assume that the insurer follows an arbitrary investment policy $\{\pi_s\}$ between t and $t+h$, and after $t+h$, the insurer follows the optimal investment policy $\{\pi_s^*\}$. If the insured aged $x+t$ survives until time $t+h$, the contract goes on. If the insured aged $x+t$ dies before $t+h$, the insurer pays G_{t+h}, and continues under V, the value function without the claim. Thus

$$\begin{aligned} &U(t,w,A) \geq \\ &E[t+h, U(W_{t+h}, A_{t+h})|W_t = w, A_t = A] \cdot_h p_{x+t} \\ &+E[V(t+h, W_{t+h} - G_{t+h})|W_t = w] \cdot_h q_{x+t}, \end{aligned} \tag{12}$$

where ${}_hp_{x+t}$ is the probability of a person aged $x+t$ survives until $x+t+h$, and ${}_hq_{x+t} = 1 -_h p_{x+t}$. This will have an equality if and only if the investment policy is optimal between t and $t+h$. Assuming U and V are smooth enough to have all the derivatives, we have

$$U(t+h, W_{t+h}, A_{t+h}) = U(t,w,A) + \int_t^{t+h} dU, \tag{13}$$

where dU is the differential of U. Using Itô's formula,

$$\begin{aligned} dU = \quad & [U_s + U_w(rw + (\mu - r)\pi) + (\mu - \alpha)AU_A \\ & + \tfrac{1}{2}\sigma^2\pi^2 U_{ww} + \tfrac{1}{2}\sigma^2 A^2 U_{AA} + \sigma^2\pi A U_{wA}]ds \\ & + \sigma\pi U_w dB + \sigma A U_A dB. \end{aligned}$$

The right hand side of (13) becomes $U(t,w,A) + \int_t^{t+h} \mathcal{L}_1 U ds + \int_t^{t+h} \sigma\pi U_w dB + \int_t^{t+h} \sigma A U_A dB$,

where

$$\begin{aligned} \mathcal{L}_1 U = \quad & U_s + U_w(rw + (\mu - r)\pi) + (\mu - \alpha)AU_A + \\ & \tfrac{1}{2}\sigma^2\pi^2 U_{ww} + \tfrac{1}{2}\sigma^2 A^2 U_{AA} + \sigma^2\pi A U_{wA}, \end{aligned}$$

and taking expectations yields the last two integrals zero.

Similarly, for the value function without the claim V, we have

$$V(t+h, w_{t+h}) = V(t,w) + \int_t^{t+h} \mathcal{L}_2 V + \int_t^{t+h} \sigma\pi V_w dB ds,$$

where $\mathcal{L}_2 V = V_s + (rw + (\mu - r)\pi)V_w + \frac{1}{2}\sigma^2\pi^2 V_{ww}$.

Now (12) becomes

$$\begin{array}{l} U(t,w,A) \geq E^{t,w,A}[U(t,w,A) + \int_t^{t+h} \mathcal{L}_1 U ds]_h p_{x+t} \\ +E^{w,t}[V(t,w-G) + \int_t^{t+h} \mathcal{L}_2 V ds]_h q_{x+t}, \end{array}$$

or

$$\begin{array}{l} U(t,w,A) \cdot_h q_{x+t} \geq E^{t,w,A}[\int_t^{t+h} \mathcal{L}_1 U ds] \cdot_h p_{x+t} \\ +E^{w,t}[V(t,w-G) + \int_t^{t+h} \mathcal{L}_2 V ds]_h q_{x+h}. \end{array}$$

If we divide both sides by h and take limit $h \to 0$, then

$$\lambda_x(t) \cdot U \geq \mathcal{L}_1 U + V(w-G,t) \cdot \lambda_x(t),$$

since $\frac{{}_h q_{x+t}}{h} \to \lambda_x(t)$, the force of mortality of a person aged x at time t, and ${}_h q_{x+t} \to 0$ as $h \to 0$.

If the investment policy is optimal, we have an equality, which gives the following HJB equation of U;

$$\left\{ \begin{array}{l} U_t + rwU_w + (\mu-\alpha)AU_A + \frac{1}{2}\sigma^2 A^2 U_{AA} \\ +\lambda_x(t)[V(w-G,t) - U] \\ +\max_\pi[(\mu-r)\pi U_w + \sigma^2\pi A U_{wA} + \frac{1}{2}\sigma^2\pi^2 U_{ww}] = 0, \\ U(T,w,A) = u(w). \end{array} \right. \tag{14}$$

Appendix B
HJB equation for $\overline{U}$ with the insurance risk

The derivation of HJB equation for $\overline{U}$ is similar to the process for U. With the same assumption in the previous section, we have

$$\begin{array}{l} \overline{U}(t,w,A,\lambda) \geq \\ E[\overline{U}(t+h, W_{t+h}, A_{t+h}, \lambda_{t+h}) | W_t = w, A_t = A, \\ \lambda_t = \lambda] \cdot_h p_{x+t} \\ +E[V(t+h, W_{t+h} - G_{t+h}) | W_t = w] \cdot_h q_{x+t}. \end{array} \tag{15}$$

This will have an equality if and only if the investment policy is optimal between t and $t+h$. Assuming $\overline{U}$ and V are smooth enough to have all the derivatives, we have

$$\overline{U}(t+h, W_{t+h}, A_{t+h}, \lambda_{t+h}) = \overline{U}(t,w,A,\lambda) + \int_t^{t+h} d\overline{U}, \tag{16}$$

where $d\overline{U}$ is the differential of $\overline{U}$. Using Itô's formula,

$$\begin{array}{rl} d\overline{U} = & [\overline{U}_s + \overline{U}_w(rw + (\mu-r)\pi) + (\mu-\alpha)A\overline{U}_A \\ & +\overline{\mu}\lambda\overline{U}_\lambda + \frac{1}{2}\sigma^2\pi^2\overline{U}_{ww} \\ & +\frac{1}{2}\sigma^2 A^2 \overline{U}_{AA} + \sigma^2\pi A\overline{U}_{wA} + \frac{1}{2}\overline{\sigma}^2\lambda^2\overline{U}_{\lambda\lambda}]ds \\ & +(\sigma\pi\overline{U}_w + \sigma A\overline{U}_A)dB + \overline{\sigma}\lambda\overline{U}_\lambda dB^\lambda. \end{array}$$

The right hand side of (16) becomes

$$\begin{array}{l} \overline{U}(t,w,A,\lambda) + \int_t^{t+h} \mathcal{L}_3\overline{U} ds + \int_t^{t+h} \sigma\pi\overline{U}_w dB \\ + \int_t^{t+h} \sigma A\overline{U}_A dB + \int_t^{t+h} \overline{\sigma}\lambda\overline{U}_\lambda dB^\lambda, \end{array}$$

where

$$\begin{array}{rl} \mathcal{L}_3\overline{U} = & \overline{U}_s + \overline{U}_w(rw + (\mu-r)\pi) + (\mu-\alpha)A\overline{U}_A \\ & +\overline{\mu}\lambda\overline{U}_\lambda + \frac{1}{2}\sigma^2\pi^2\overline{U}_{ww} \\ & +\frac{1}{2}\sigma^2 A^2\overline{U}_{AA} + \sigma^2\pi A\overline{U}_{wA} + \frac{1}{2}\overline{\sigma}^2\lambda^2\overline{U}_{\lambda\lambda}, \end{array}$$

and taking expectations yields the last three integrals zero.

Now (15) becomes

$$\begin{array}{l} \overline{U}(t,w,A,\lambda) \geq \\ E^{t,w,A,\lambda}[\overline{U}(t,w,A,\lambda) + \int_t^{t+h} \mathcal{L}_3\overline{U} ds]_h p_{x+t} \\ +E^{t,w}[V(t,w-G) + \int_t^{t+h} \mathcal{L}_2 V ds]_h q_{x+t}, \end{array}$$

or

$$\begin{array}{l} \overline{U}(t,w,A,\lambda) \cdot_h q_{x+t} \geq \\ E^{t,w,A,\lambda}[\int_t^{t+h} \mathcal{L}_3\overline{U} ds] \cdot_h p_{x+t} \\ +E^{t,w}[V(t,w-G) + \int_t^{t+h} \mathcal{L}_2 V ds]_h q_{x+h}. \end{array}$$

If we divide both sides by h and take limit $h \to 0$, then

$$\lambda \cdot \overline{U} \geq \mathcal{L}_3\overline{U} + V(t,w-G) \cdot \lambda.$$

If the investment policy is optimal, we have an equality, which gives the following HJB equation of $\overline{U}$;

$$\left\{ \begin{array}{l} \overline{U}_t + rw\overline{U}_w + (\mu-\alpha)A\overline{U}_A + \frac{1}{2}\sigma^2 A^2\overline{U}_{AA} \\ +\lambda[V(w-G,t) - \overline{U}] + \overline{\mu}\lambda\overline{U}_\lambda + \frac{1}{2}\overline{\sigma}^2\lambda^2\overline{U}\lambda\lambda \\ +\max_\pi[(\mu-r)\pi\overline{U}_w + \sigma^2\pi A\overline{U}_{wA} \\ +\frac{1}{2}\sigma^2\pi^2\overline{U}_{ww}] = 0, \\ \overline{U}(w,A,T) = u(w). \end{array} \right. \tag{17}$$