

Cayley–Klein Poisson Homogeneous Spaces

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Abstract

The nine two-dimensional Cayley–Klein geometries are firstly reviewed by following a graded contraction approach. Each geometry is considered as a set of three symmetrical homogeneous spaces (of points and two kinds of lines), in such a manner that the graded contraction parameters determine their curvature and signature. Secondly, new Poisson homogeneous spaces are constructed by making use of certain Poisson–Lie structures on the corresponding motion groups. Therefore, the quantization of these spaces provides noncommutative analogues of the Cayley–Klein geometries. The kinematical interpretation for the semi-Riemannian and pseudo-Riemannian Cayley–Klein geometries is emphasized, since they are just Newtonian and Lorentzian spacetimes of constant curvature.

MSC: 17Bxx, 22Exx, 16Txx

Keywords: Riemannian geometries, Lorentzian spacetimes, contraction, curvature, deformation, Poisson–Lie groups, quantum groups

¹Based on the contribution presented at the “XXth International Conference on Geometry, Integrability and Quantization” held in Varna, Bulgaria, June 2–7, 2018

Proceedings of the Twentieth International Conference on Geometry, Integrability and Quantization, Ivalo M. Mladenov, Vladimir Pulov and Akira Yoshioka, eds. (Sofia: Avangard Prima, 2019), 161-183

1 Introduction

The family of orthogonal Cayley–Klein (CK) algebras is a distinguished set of real Lie algebras that can be obtained through a graded contraction procedure from $\mathfrak{so}(N + 1)$ [22]. The CK family depends on N real contraction parameters κ_i ($i = 1, \dots, N$) and is denoted $\mathfrak{so}_{\kappa_1, \dots, \kappa_N}(N + 1)$. The relevant fact is that the CK algebra contains both semisimple and non-semisimple Lie algebras which share common geometrical and algebraical properties. The sign of the parameters κ_i determine a specific real form $\mathfrak{so}(p, q)$, and when (at least) one of the parameters vanishes the CK algebra becomes a non-semisimple one. Independently of the κ_i values, all the CK algebras have the same number of algebraically independent Casimir invariants [24], so that they have the same rank (even for the most contracted case with all $\kappa_i = 0$) and they are also known as quasi-simple orthogonal algebras. From this viewpoint they can be regarded as the “closest” contracted algebras to the semisimple ones.

The “Cayley–Klein” terminology is due to the appearance of the corresponding Lie groups $\mathrm{SO}_{\kappa_1, \dots, \kappa_N}(N + 1)$ within the context of Klein’s consideration of most geometries as subgeometries of Projective Geometry and also to Cayley’s theory of projective metrics [38, 42, 43]. Nevertheless, the complete classification of these geometries was not given by Klein himself. The two-dimensional (2D) case was studied under the name of “quadratic geometries” by Poincaré, following a modern group theoretical procedure, and the classification for arbitrary dimension N was given by Sommerville in 1910 [40]. In the latter work, he showed that there are 3^N different geometries in dimension N , each corresponding to a different choice of the kind of measure of distance between points, lines, 2-planes, . . . which can be either elliptic, parabolic or hyperbolic. This result can be recovered by introducing N graded contraction parameters since a positive/zero/negative value of $\kappa_1, \kappa_2, \kappa_3, \dots$ corresponds, in this order, to a kind of measure of elliptic/parabolic/hyperbolic type between points, lines, 2-planes, etc. Furthermore, the CK groups allow for the construction of a set of symmetrical homogeneous spaces (as coset spaces), which are interpreted as the spaces of points, lines, 2-planes, . . . , each of them of constant curvature equal to $\kappa_1, \kappa_2, \kappa_3, \dots$ [8]. Nevertheless, in the literature only the ND space of points $\mathrm{SO}_{\kappa_1, \dots, \kappa_N}(N + 1)/\mathrm{SO}_{\kappa_2, \dots, \kappa_N}(N)$ is usually considered.

The aim of this paper is two-fold. On the one hand, we focus on the nine 2D CK geometries and study them as a set of *three* symmetrical homogeneous spaces: of points and of two kinds of lines. This enables us to describe the main properties of the CK geometries from a global approach and to explain several relations among them. On the other, we extend the notion of CK homogeneous spaces to Poisson homogeneous spaces, which can be considered as the semiclassical counterparts of CK noncommutative spaces which are invariant under quantum deformations of the CK groups [6, 7, 10, 31].

The structure of the paper is as follows. In the next Section we review the nine 2D CK geometries. The kinematical interpretation for six of them as Newtonian and Lorentzian spaces of constant curvature is summarized in Section 3. A set of new “dualities” for the CK algebras/spaces, which generalize the known ordinary duality of Projective Geometry that interchanges points with lines, is presented in Section 4. The metric structure and several sets of geodesic coordinates for the CK spaces are introduced in Section 5. Finally, we recall the basics on Poisson-Lie groups and quantum deformations in Section 6, which are further applied in the last Section in order to obtain new Poisson homogeneous spaces for the CK geometries.

2 The Nine Two-Dimensional Cayley–Klein Geometries

Let us consider the real Lie algebra $\mathfrak{so}(3)$ with generators $\{J_{01}, J_{02}, J_{12}\}$ fulfilling the commutation rules

$$[J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -J_{01}, \quad [J_{01}, J_{02}] = J_{12}$$

and with Casimir given by

$$\mathcal{C} = J_{01}^2 + J_{02}^2 + J_{12}^2.$$

In this basis $\mathfrak{so}(3)$ can be endowed with a $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ group of commuting involutive automorphisms generated by

$$\begin{aligned} \Theta_0(J_{01}, J_{02}, J_{12}) &= (-J_{01}, -J_{02}, J_{12}) \\ \Theta_{01}(J_{01}, J_{02}, J_{12}) &= (J_{01}, -J_{02}, -J_{12}) \end{aligned} \quad (1)$$

such that the remaining automorphisms are the composition $\Theta_0\Theta_{01}$ and the identity. By applying the graded contraction theory [35, 37], a particular solution of the set of $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ -graded contractions from $\mathfrak{so}(3)$ leads to a two-parametric family of Lie algebras, denoted as $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$, with commutators given by [22]

$$[J_{12}, J_{01}] = J_{02}, \quad [J_{12}, J_{02}] = -\kappa_2 J_{01}, \quad [J_{01}, J_{02}] = \kappa_1 J_{12} \quad (2)$$

where κ_1 and κ_2 are two real graded contraction parameters. The corresponding Casimir reads

$$\mathcal{C} = \kappa_2 J_{01}^2 + J_{02}^2 + \kappa_1 J_{12}^2. \quad (3)$$

Note that each parameter κ_i ($i = 1, 2$) can take any real value and it can be reduced to the values $\{+1, 0, -1\}$ through a rescaling of the Lie algebra generators. Hence the family $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ comprises nine specific Lie algebras (some of them isomorphic). In particular, $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ covers simple Lie algebras when both parameters $\kappa_i \neq 0$ (the initial $\mathfrak{so}(3)$ for positive values and $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}_2(\mathbb{R})$ otherwise), as well as non-simple ones when at least one $\kappa_i = 0$ (the inhomogeneous $\mathfrak{iso}(2)$, $\mathfrak{iso}(1, 1)$ and $\mathfrak{uiso}(1)$ where $\mathfrak{iso}(1) \equiv \mathbb{R}$). The relevant fact is that $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ contains all the Lie algebras of the motion groups of the 2D CK geometries [6, 18, 19, 23, 24, 25, 33, 34, 38, 42] and therefore $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ is called orthogonal CK algebra or quasi-simple orthogonal one [24].

Let us make the connection of $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ with the CK geometries more explicit. Each automorphism (1) gives rise to a Cartan decomposition in the form

$$\begin{aligned} \mathfrak{so}_{\kappa_1, \kappa_2}(3) &= \mathfrak{h}_0 \oplus \mathfrak{p}_0, & \mathfrak{h}_0 &= \text{span}\{J_{12}\}, & \mathfrak{p}_0 &= \text{span}\{J_{01}, J_{02}\} \\ \mathfrak{so}_{\kappa_1, \kappa_2}(3) &= \mathfrak{h}_{01} \oplus \mathfrak{p}_{01}, & \mathfrak{h}_{01} &= \text{span}\{J_{01}\}, & \mathfrak{p}_{01} &= \text{span}\{J_{02}, J_{12}\}. \end{aligned} \quad (4)$$

Usually, a 2D CK geometry is understood as the *set of points*, the “plane”, which corresponds to the symmetrical homogeneous space [17] coming from the first decomposition (4) and associated with the involution Θ_0 . In this way the CK homogeneous space of points is defined by the quotient of the CK Lie group $\text{SO}_{\kappa_1, \kappa_2}(3)$ by the Lie group H_0 corresponding to \mathfrak{h}_0 , that is, as the coset space

$$\mathbb{S}_{[\kappa_1], \kappa_2}^2 := \text{SO}_{\kappa_1, \kappa_2}(3)/H_0, \quad H_0 = \text{SO}_{\kappa_2}(2) = \langle J_{12} \rangle. \quad (5)$$

The space $\mathbb{S}_{[\kappa_1], \kappa_2}^2$ turns out to be of constant curvature equal to κ_1 and with signature of the metric given by $\text{diag}(+, \kappa_2)$, so determined by the second parameter κ_2 . Therefore the generator J_{12} leaves a point O invariant, the origin, generating rotations around O . The remaining generators J_{01} and J_{02} , that belong to the subspace \mathfrak{p}_0 , generate translations which move O in two basic directions.

However, we can also consider the *set of lines* as the symmetrical homogeneous space coming from the second decomposition (4) and associated to Θ_{01} , namely

$$\mathbb{S}_{\kappa_1, [\kappa_2]}^2 := \text{SO}_{\kappa_1, \kappa_2}(3)/H_{01}, \quad H_{01} = \text{SO}_{\kappa_1}(2) = \langle J_{01} \rangle. \quad (6)$$

The space $\mathbb{S}_{\kappa_1, [\kappa_2]}^2$ is also of constant curvature, now equal to κ_2 and with signature of the metric given by $\text{diag}(+, \kappa_1)$. In this case, J_{01} leaves a point of the space invariant (a line), while J_{02} and J_{12} move it, so the former behaves as a rotation and the latter as translations in $\mathbb{S}_{\kappa_1, [\kappa_2]}^2$.

Moreover, it is also possible to consider a second set of lines associated to the composition $\Theta_0\Theta_{01} \equiv \Theta_{02}$ as the one defined by the coset space

$$\text{SO}_{\kappa_1, \kappa_2}(3)/H_{02}, \quad H_{02} = \text{SO}_{\kappa_1\kappa_2}(2) = \langle J_{02} \rangle. \quad (7)$$

Hereafter we shall call (6) the space of first-kind lines and (7) the space of second-kind ones. By a CK geometry we will understand the set of the above three symmetrical homogeneous spaces. We display in Table 1 the nine 2D CK geometries along with their three isotropy subgroups.

Recall that, besides their curvature/signature role, the coefficients κ_i determine the *kind of measure of separation* between points and lines in the Klein's sense [42]:

- The kind of measure of distance between two points on a first-kind line is elliptical/parabolical/hyperbolic according to whether κ_1 is greater than/ equal to/lesser than zero.
- Likewise for two points on a second-kind line depending on the product $\kappa_1\kappa_2$.
- Likewise for the kind of measure of angle between two lines through a point according to κ_2 .

Hence in the first row of Table 1 with $\kappa_2 > 0$, one finds the three classical Riemannian geometries with elliptical kind of measure of angles. The second row with $\kappa_2 = 0$ shows the three semi-Riemannian geometries with parabolic kind of measure of angles. And the third row with $\kappa_2 < 0$ displays the pseudo-Riemannian geometries with hyperbolic kind of measure of angles. When Table 1 is read by columns, one sees the spaces of points (5) with positive/zero/negative curvature and with elliptical/parabolical/hyperbolic kind of measure of distance between two points on a first-kind line.

We remark that the use of the real parameters κ_i allows for dealing, simultaneously, with different real forms of Lie algebras, and that making zero a given κ_i parameter corresponds to an Inönü–Wigner contraction [26, 41]. In particular, each automorphism (1) determines a

Table 1: The nine 2D CK geometries with their specific Lie group $\text{SO}_{\kappa_1, \kappa_2}(3)$ and isotropy subgroups of a point H_0 (5), a first-kind line H_{01} (6) and a second-kind one H_{02} (7), according to the value of the pair (κ_1, κ_2) .

| | | |
|---|--|--|
| <ul style="list-style-type: none"> • Spherical (+, +): $\text{SO}(3)$ $H_0 = \text{SO}(2)$ $H_{01} = \text{SO}(2)$ $H_{02} = \text{SO}(2)$ | <ul style="list-style-type: none"> • Euclidean (0, +): $\text{ISO}(2)$ $H_0 = \text{SO}(2)$ $H_{01} = \mathbb{R}$ $H_{02} = \mathbb{R}$ | <ul style="list-style-type: none"> • Hyperbolic (-, +): $\text{SO}(2, 1)$ $H_0 = \text{SO}(2)$ $H_{01} = \text{SO}(1, 1)$ $H_{02} = \text{SO}(1, 1)$ |
| <ul style="list-style-type: none"> • Co-Euclidean (Oscillating NH) (+, 0): $\text{ISO}(2)$ $H_0 = \mathbb{R}$ $H_{01} = \text{SO}(2)$ $H_{02} = \mathbb{R}$ | <ul style="list-style-type: none"> • Galilean (0, 0): $\text{HISO}(1)$ $H_0 = \mathbb{R}$ $H_{01} = \mathbb{R}$ $H_{02} = \mathbb{R}$ | <ul style="list-style-type: none"> • Co-Minkowskian (Expanding NH) (-, 0): $\text{ISO}(1, 1)$ $H_0 = \mathbb{R}$ $H_{01} = \text{SO}(1, 1)$ $H_{02} = \mathbb{R}$ |
| <ul style="list-style-type: none"> • Co-Hyperbolic (Anti-de Sitter) (+, -): $\text{SO}(2, 1)$ $H_0 = \text{SO}(1, 1)$ $H_{01} = \text{SO}(2)$ $H_{02} = \text{SO}(1, 1)$ | <ul style="list-style-type: none"> • Minkowskian (0, -): $\text{ISO}(1, 1)$ $H_0 = \text{SO}(1, 1)$ $H_{01} = \mathbb{R}$ $H_{02} = \mathbb{R}$ | <ul style="list-style-type: none"> • Doubly Hyperbolic (De Sitter) (-, -): $\text{SO}(2, 1)$ $H_0 = \text{SO}(1, 1)$ $H_{01} = \text{SO}(1, 1)$ $H_{02} = \text{SO}(2)$ |

contraction which is obtained by keeping fixed the invariant generator and multiplying the two anti-invariant ones by a parameter ε , and next taking the limit $\varepsilon \rightarrow 0$, that is,

$$\begin{aligned}
 \Theta_0 : \quad J'_{01} &= \varepsilon J_{01}, & J'_{02} &= \varepsilon J_{02}, & J'_{12} &= J_{12}, & \varepsilon &\rightarrow 0 \\
 \Theta_{01} : \quad J'_{01} &= J_{01}, & J'_{02} &= \varepsilon J_{02}, & J'_{12} &= \varepsilon J_{12}, & \varepsilon &\rightarrow 0
 \end{aligned} \tag{8}$$

where J'_{ij} are the new generators. Thus the first contraction is a local contraction, around a point, and corresponds to set $\kappa_1 = 0$ (middle column in Table 1), while the second one is an axial contraction, around a first-kind line, corresponding to take $\kappa_2 = 0$ (middle row in Table 1).

It is also worth mentioning that the 2D CK geometries have been widely studied in terms of hypercomplex numbers [18, 19, 38, 42] instead of graded contraction parameters κ_i . For a detailed use of hypercomplex numbers applied to the geometries with isometry group isomorphic to $\text{SL}_2(\mathbb{R})$ together with a deep insight into their properties, including their contractions, see [27, 28] and references therein.

Explicitly, consider real coordinates (x, y) and a hypercomplex unit ι such that

$$\iota^2 \in \{-1, +1, 0\}. \tag{9}$$

The hypercomplex number z is defined as $z := x + \iota y$ with conjugate $\bar{z} \equiv x - \iota y$ so that

$$|z|^2 \equiv z\bar{z} = x^2 - \iota^2 y^2.$$

According to each specific hypercomplex unit (9) we find the following three algebra structures on \mathbb{R}^2 over \mathbb{R} :

- If $\iota^2 = -1$, then ι is an *elliptical* unit leading to the usual *complex numbers* such that $|z|^2 = z\bar{z} = x^2 + y^2$.
- When $\iota^2 = +1$, ι is a *hyperbolic* unit providing the so-called *split complex, double or Clifford numbers* with $|z|^2 = z\bar{z} = x^2 - y^2$.
- And if $\iota^2 = 0$, ι is a *parabolic* unit and z is known as a *dual or Study number*, which can be regarded as a *contracted* case since $|z|^2 = z\bar{z} = x^2$.

From this approach, it is necessary to consider two hypercomplex units ι_1 and ι_2 to describe the 2D CK geometries, whose different possibilities lead to the nine particular geometries (see e.g. [42]), enabling one to also deal with different real forms of Lie algebras. Since the real graded contraction parameters κ_i can be reduced to the standard values $\{+1, 0, -1\}$, it is obvious that these are somewhat related with the hypercomplex units $\iota_i \sim \sqrt{\kappa_i}$. Hence one can naively think that both procedures are related by a mere identification $\iota_i \equiv \sqrt{\kappa_i}$. Nevertheless, the main differences between both approaches arise in the pure contracted case corresponding to consider the parabolic or dual-Study unit with $\iota^2 = 0$ and to set $\kappa = 0$. This can clearly be appreciated by considering, for instance, the following contraction of exponentials of a Lie generator J :

$$\exp(\iota^2 J) \rightarrow 1, \quad \exp(\iota J) \rightarrow 1 + \iota J, \quad \exp(\kappa J) \rightarrow 1, \quad \exp(\sqrt{\kappa} J) \rightarrow 1.$$

We remark that these kind of exponentials often appear in quantum group theory [10, 31], so that these two approaches could give rise to different results (see e.g. [5] where this fact appears explicitly in the contraction of $\mathfrak{so}_q(3)$ and $\mathfrak{so}_q(3, 2)$). We stress that throughout the paper we will make use of the graded contraction approach, and a smooth and well-defined $\kappa \rightarrow 0$ limit of all the expressions will be always feasible.

3 Kinematical Cayley–Klein Spaces

The six CK groups with $\kappa_2 \leq 0$ are kinematical groups, that is, motion groups of (1+1)D spacetimes of constant curvature [23, 25], which are displayed in the second and third rows of Table 1 (NH means Newton–Hooke). These spacetimes are the main cases within the classification of (3+1)D kinematical Lie algebras formerly performed in [2] (see also [14, 15, 36] and references therein).

The geometrical-kinematical relationship is established under the following identification between the geometrical generators J_{ij} and the infinitesimal generators of time translations P_0 , space translations P_1 and boosts K :

$$J_{01} \equiv P_0, \quad J_{02} \equiv P_1, \quad J_{12} \equiv K. \quad (10)$$

Hence the graded contraction parameters κ_i inherit physical dimensions in such a manner that they are related to the cosmological constant Λ and the speed of light c , namely

$$\kappa_1 = -\Lambda, \quad \kappa_2 = -1/c^2. \quad (11)$$

Thus the commutation rules (2) and Casimir (3) can be rewritten as

$$[K, P_0] = P_1, \quad [K, P_1] = \frac{1}{c^2}P_0, \quad [P_0, P_1] = -\Lambda K$$

$$\mathcal{C} = -\frac{1}{c^2}P_0^2 + P_1^2 - \Lambda K^2.$$

The automorphisms (1) are identified with the parity operation $\mathcal{P} \equiv \Theta_{01}$ and time-reversal $\mathcal{T} \equiv \Theta_{02} = \Theta_0\Theta_{01}$, so that the composition $\mathcal{PT} \equiv \Theta_0$ [2]. The substitutions $\kappa_1 = 0$ and $\kappa_2 = 0$ correspond to the spacetime and speed-space contractions, respectively (see (8)).

The physical interpretation of the three homogeneous spaces within each of the six kinematical CK geometries is as follows:

- The space of points (5) is just the (1+1)D *spacetime* and its curvature κ_1 is related to the universe (time) radius τ by $\kappa_1 = \pm 1/\tau^2$. The metric has signature given by $\text{diag}(+, -1/c^2)$.
- The space of first-kind lines (6) corresponds to the 2D space of *time-like lines* with curvature $\kappa_2 = -1/c^2$. Its metric now has signature $\text{diag}(+, \kappa_1)$.
- The space of second-kind lines (7) is the 2D space of *space-like lines*.

As it is shown in Table 1, the three Lorentzian spacetimes of constant curvature $\kappa_1 = -\Lambda$ arise for $\kappa_2 < 0$: Anti-de Sitter ($\kappa_1 > 0$), Minkowski ($\kappa_1 = 0$ or $\tau \rightarrow \infty$), and de Sitter ($\kappa_1 < 0$). Their non-relativistic limit is provided by the contraction $\kappa_2 = 0$ ($c \rightarrow \infty$), leading, in this order, to the oscillating NH ($\kappa_1 > 0$), Galilei ($\kappa_1 = 0$) and expanding NH ($\kappa_1 < 0$).

Finally, we point out that besides the role of (κ_1, κ_2) as contraction parameters, these can also be regarded as *classical deformation* ones [8, 14, 15]. In particular, let us consider the Galilean geometry, which is the most contracted case with parameters $(0, 0)$. This means that both the spacetime and the space of time-like lines are flat. If a non-zero parameter $\kappa_2 = -1/c^2$ is introduced, then one arrives at the Minkowskian geometry $(0, -)$ with a curved (hyperbolic) space of time-like lines, but keeping a flat spacetime. Next, curvature on the spacetime can be introduced through $\kappa_1 = -\Lambda$ giving rise to the (anti-)de Sitter spacetimes $(\kappa_1, -)$. Likewise one can proceed through other directions in the deformation process. The sequence of classical deformations ends with the (anti-)de Sitter geometries since their motion groups are always semisimple Lie groups ($\text{SO}(2, 1)$ at this dimension) and no further curvature (or physical constant) can be added if a motion Lie group is required. However, the deformation sequence can still continue in some sense if quantum deformations of Lie algebras and groups are considered. In this way another deformation parameter, the ‘‘quantum’’ one $q = e^z$, is introduced and, in some cases, the latter can be interpreted as a second fundamental relativistic invariant (besides c) which is related to the Planck scale, and thus giving rise to the so-called *Doubly Special Relativity* theories (see [1, 16, 29] and references therein).

4 Generalized Dualities

As we have already commented and can be seen in Table 1, some of the CK geometries have isomorphic Lie algebras. According to (κ_1, κ_2) , we find that $\mathfrak{iso}(2)$ appears twice for $(+, 0)$ and $(0, +)$; $\mathfrak{iso}(1, 1)$ also twice for $(-, 0)$ and $(0, -)$; and $\mathfrak{so}(2, 1) \simeq \mathfrak{sl}_2(\mathbb{R})$ three times for $(-, +)$, $(+, -)$ and $(-, -)$. Differences among the corresponding geometries emerge when the three homogeneous spaces are taken into account altogether, which amounts to focus on the isotropy subgroups of a point H_0 , a first-kind line H_{01} and a second-kind one H_{02} .

In fact, there exists an “automorphism” for the whole family of CK geometries that we shall name *ordinary duality* \mathcal{D}_0 [23] which is defined by

$$\mathcal{D}_0 : (\tilde{J}_{01}, \tilde{J}_{02}, \tilde{J}_{12}) = (-J_{12}, -J_{02}, -J_{01}) \quad (12)$$

where \tilde{J}_{ij} are the transformed generators. If we compute the new commutation rules, we find that these are again (2) but now with transformed contraction parameters given by

$$(\tilde{\kappa}_1, \tilde{\kappa}_2) = (\kappa_2, \kappa_1).$$

This, in turn, means that \mathcal{D}_0 interchanges the spaces of points and first-kind lines, leaving the space of second-kind lines invariant, that is,

$$\mathbb{S}_{[\kappa_1], \kappa_2}^2 \leftrightarrow \mathbb{S}_{\kappa_1, [\kappa_2]}^2, \quad H_0 \leftrightarrow H_{01}. \quad (13)$$

Hence the Euclidean, Minkowskian and hyperbolic geometries are dual under \mathcal{D}_0 to the co-Euclidean, co-Minkowskian and co-hyperbolic ones, respectively, meanwhile the three remaining geometries (sphere, Galilean and doubly hyperbolic) remain invariant. Therefore the prefix “co-” refers to this geometrical property [42] which actually corresponds to the known duality in Projective Geometry.

Nevertheless, \mathcal{D}_0 does not explain other relationships within the CK geometries which should concern the space of second-kind lines and so explaining the connections among the three geometries coming from $\mathfrak{so}(2, 1)$. With this aim in mind, let us formulate the map (12) in terms of a permutation on the set \mathcal{S} of indices of the generators J_{ij} , that is, $\mathcal{S} = \{0, 1, 2\}$. Then \mathcal{D}_0 corresponds to the 2-cycle (02) and its action on the isotropy subgroups (13) is consistently obtained by identifying $H_0 \equiv H_{12}$. Therefore the number 3! of permutations on \mathcal{S} provides *six generalized dualities*, each of them being determined by a permutation element and eventually a coefficient κ_i which means that the duality cannot be applied on the geometries with $\kappa_i = 0$.

The resulting generalized dualities are displayed in Table 2. Notice that only the ordinary duality (and, obviously, the identity) have no κ_i -restriction and can be applied to the nine geometries. We schematically represent their action on the nine CK geometries in Fig. 1, where these are considered in the same order (rows and columns) as in Table 1. Note also that the six dualities are always well defined on the four geometries with simple Lie group (at the corners) and that the sphere is the only geometry which always remains invariant.

Now let us explain some of these new results. The duality $\mathcal{D}_2 = (12)\kappa_2$ interchanges both spaces of first- and second-kind lines, $H_{01} \leftrightarrow H_{02}$, keeping the space of points. The three Riemannian geometries with $\kappa_2 > 0$ remain invariant, showing the known fact that the sets

Table 2: Transformation of the generators J_{ij} and contraction parameters (κ_1, κ_2) for the generalized dualities on the CK algebras.

| | \mathcal{D}_0 (0 2) | \mathcal{D}_1 (0 1) κ_1 | \mathcal{D}_2 (1 2) κ_2 | $\mathcal{D}_0\mathcal{D}_1$ (0 2 1) κ_1 | $\mathcal{D}_0\mathcal{D}_2$ (0 1 2) κ_2 | \mathcal{D}_0^2 Id |
|--------------------|--------------------------|-------------------------------------|-------------------------------------|--|--|-------------------------|
| \tilde{J}_{01} | $-J_{12}$ | $-J_{01}$ | J_{02} | $-J_{02}$ | J_{12} | J_{01} |
| \tilde{J}_{02} | $-J_{02}$ | $\kappa_1 J_{12}$ | $\kappa_2 J_{01}$ | $-\kappa_1 J_{12}$ | $-\kappa_2 J_{01}$ | J_{02} |
| \tilde{J}_{12} | $-J_{01}$ | J_{02} | $-J_{12}$ | J_{01} | $-J_{02}$ | J_{12} |
| $\tilde{\kappa}_1$ | κ_2 | κ_1 | $\kappa_1\kappa_2$ | $\kappa_1\kappa_2$ | κ_2 | κ_1 |
| $\tilde{\kappa}_2$ | κ_1 | $\kappa_1\kappa_2$ | κ_2 | κ_1 | $\kappa_1\kappa_2$ | κ_2 |

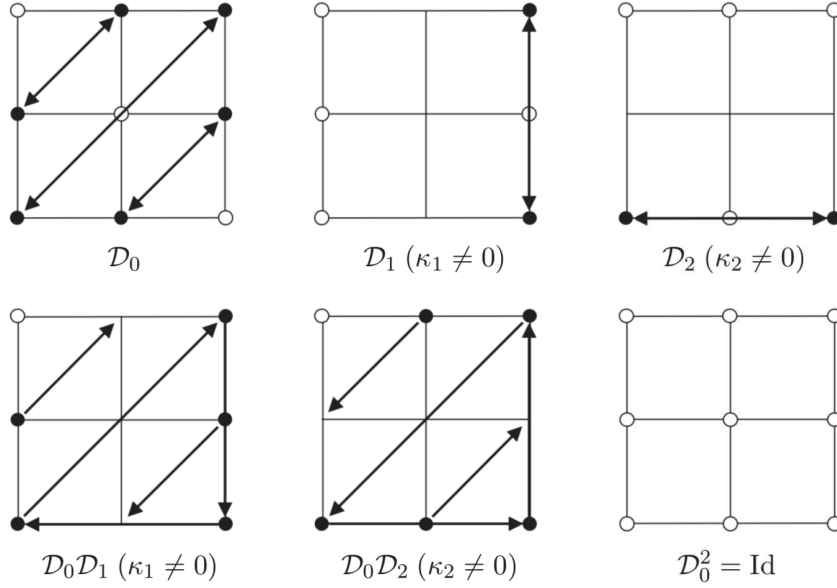


Figure 1: Generalized dualities on the nine 2D CK geometries displayed as in Table 1. White circles represent invariant geometries and black ones geometries which are transformed under the duality.

of first- and second-kind lines coincide (only in these cases the generators J_{01} and J_{02} are conjugated). On the three Newtonian geometries with $\kappa_2 = 0$, \mathcal{D}_2 is not defined, which reflects that time-like lines are just the “absolute-time” and cannot be related with the spatial lines (recall that the metric on the spacetime is degenerate). For the three Lorentzian geometries with $\kappa_2 < 0$, this duality relates anti-de Sitter with de Sitter, keeping Minkowskian geometry invariant. Notice that time-like lines are compact in anti-de Sitter ($H_{01} = \text{SO}(2)$) while space-like lines are non-compact ($H_{02} = \text{SO}(1, 1)$) and the converse is true in de Sitter space; by contrast, in the Minkowskian case $H_{01} \equiv H_{02} \equiv \mathbb{R}$.

The composition $\mathcal{D}_0\mathcal{D}_1 = (021)\kappa_1$, which cannot be applied to the three geometries of the second column, transforms simultaneously the three homogeneous spaces for each geometry providing the sequence

$$H_0 \rightarrow H_{02} \rightarrow H_{01} \rightarrow H_0.$$

Thus the co-Euclidean geometry arrives at the Euclidean one, but there is no reciprocity; e.g. $H_0 = \text{SO}(2)$ for the Euclidean geometry and $H_{02} = \mathbb{R}$ for the co-Euclidean one (see Table 1). Furthermore, this transformation can be regarded as a kind of “*triality*” for the three geometries associated with $\text{SO}(2,1)$ since

$$\text{Hyperbolic} \rightarrow \text{De Sitter} \rightarrow \text{Anti-de Sitter} \rightarrow \text{Hyperbolic}.$$

The two remaining dualities can be interpreted under a similar framework.

5 Vector Model and Geodesic Coordinates for the Space of Points

A faithful matrix representation of the CK algebra, $\rho : \mathfrak{so}_{\kappa_1, \kappa_2}(3) \rightarrow \text{End}(\mathbb{R}^3)$, is given by [23, 25]

$$\rho(J_{01}) = -\kappa_1 e_{01} + e_{10}, \quad \rho(J_{02}) = -\kappa_1 \kappa_2 e_{02} + e_{20}, \quad \rho(J_{12}) = -\kappa_2 e_{12} + e_{21} \quad (14)$$

where e_{ij} is the 3×3 matrix with a single non-zero entry 1 at row i and column j ($i, j = 0, 1, 2$). It can be checked that

$$\rho(J_{ij})^T \mathbb{I}_\kappa + \mathbb{I}_\kappa \rho(J_{ij}) = 0, \quad \mathbb{I}_\kappa = \text{diag}(1, \kappa_1, \kappa_1 \kappa_2). \quad (15)$$

Through matrix exponentiation of (14) we obtain the following matrix realization of the isotropy subgroups of the CK group $\text{SO}_{\kappa_1, \kappa_2}(3)$:

$$\begin{aligned} H_{01} = \text{SO}_{\kappa_1}(2) : \quad \exp(\alpha \rho(J_{01})) &= \begin{pmatrix} C_{\kappa_1}(\alpha) & -\kappa_1 S_{\kappa_1}(\alpha) & 0 \\ S_{\kappa_1}(\alpha) & C_{\kappa_1}(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ H_{02} = \text{SO}_{\kappa_1 \kappa_2}(2) : \quad \exp(\beta \rho(J_{02})) &= \begin{pmatrix} C_{\kappa_1 \kappa_2}(\beta) & 0 & -\kappa_1 \kappa_2 S_{\kappa_1 \kappa_2}(\beta) \\ 0 & 1 & 0 \\ S_{\kappa_1 \kappa_2}(\beta) & 0 & C_{\kappa_1 \kappa_2}(\beta) \end{pmatrix} \\ H_0 = \text{SO}_{\kappa_2}(2) : \quad \exp(\gamma \rho(J_{12})) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & C_{\kappa_2}(\gamma) & -\kappa_2 S_{\kappa_2}(\gamma) \\ 0 & S_{\kappa_2}(\gamma) & C_{\kappa_2}(\gamma) \end{pmatrix} \end{aligned} \quad (16)$$

where we have introduced the κ -dependent cosine and sine functions [6, 23]

$$C_\kappa(x) := \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l}}{(2l)!} = \begin{cases} \cos \sqrt{\kappa} x & \kappa > 0 \\ 1 & \kappa = 0 \\ \cosh \sqrt{-\kappa} x & \kappa < 0 \end{cases}$$

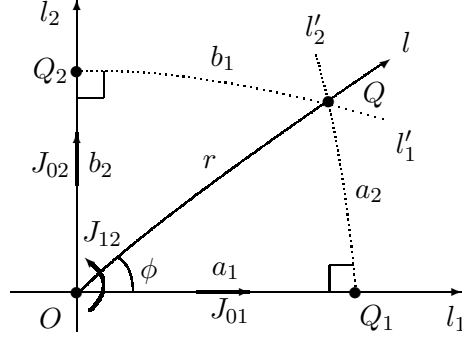


Figure 2: Infinitesimal generators $\{J_{01}, J_{02}, J_{12}\}$ of isometries and geodesic coordinates (a_1, a_2) , (b_1, b_2) and (r, ϕ) of a point $Q = (s_0, s_1, s_2)$ on the 2D CK space of points $\mathbb{S}_{[\kappa_1, \kappa_2]}^2$.

$$S_\kappa(x) := \sum_{l=0}^{\infty} (-\kappa)^l \frac{x^{2l+1}}{(2l+1)!} = \begin{cases} \frac{1}{\sqrt{\kappa}} \sin \sqrt{\kappa} x & \kappa > 0 \\ x & \kappa = 0 \\ \frac{1}{\sqrt{-\kappa}} \sinh \sqrt{-\kappa} x & \kappa < 0 \end{cases} .$$

The κ -tangent is defined as

$$T_\kappa(x) := \frac{S_\kappa(x)}{C_\kappa(x)} .$$

Hence these functions are just the circular and hyperbolic ones for $\kappa = \pm 1$, while under the contraction $\kappa = 0$ they reduce to the parabolic or Galilean functions: $C_0(x) = 1$ and $S_0(x) = T_0(x) = x$. Some relations for the above κ -functions are given by [23]

$$C_\kappa^2(x) + \kappa S_\kappa^2(x) = 1, \quad C_\kappa(2x) = C_\kappa^2(x) - \kappa S_\kappa^2(x), \quad S_\kappa(2x) = 2 S_\kappa(x) C_\kappa(x)$$

and their derivatives read

$$\frac{d}{dx} C_\kappa(x) = -\kappa S_\kappa(x), \quad \frac{d}{dx} S_\kappa(x) = C_\kappa(x), \quad \frac{d}{dx} T_\kappa(x) = \frac{1}{C_\kappa^2(x)} .$$

In what follows we present the metric and several sets of geodesic coordinates for the space of points $\mathbb{S}_{[\kappa_1, \kappa_2]}^2$ (5). We remark that, at this dimension, a similar procedure can be applied to the (dual) space of first-kind lines (6).

The matrix realization (16) allows us to consider the group action of $\text{SO}_{\kappa_1, \kappa_2}(3)$ on \mathbb{R}^3 as isometries of the bilinear form \mathbb{I}_κ (15); notice that $g^T \mathbb{I}_\kappa g = \mathbb{I}_\kappa$ for a 3×3 matrix $g \in \text{SO}_{\kappa_1, \kappa_2}(3)$. Then the subgroup $H_0 = \text{SO}_{\kappa_2}(2)$ (16) is the isotropy subgroup of the point $O := (1, 0, 0)$, which is thus taken as the *origin* in the space $\mathbb{S}_{[\kappa_1, \kappa_2]}^2$. Therefore, as commented in Section 2, the generator J_{12} is a rotation on this space, while J_{01} and J_{02} behave as translation generators moving O along two basic geodesics l_1 (of first-kind) and l_2 (of second-kind), which are orthogonal at O . This is schematically represented in Fig. 2.

Next, we consider coordinates $(s_0, s_1, s_2) \in \mathbb{R}^3$. The orbit of the origin $O = (1, 0, 0)$ is contained in the “ κ -sphere” determined by \mathbb{I}_κ (15):

$$\Sigma_\kappa : s_0^2 + \kappa_1 s_1^2 + \kappa_1 \kappa_2 s_2^2 = 1. \quad (17)$$

The connected component of Σ_κ can be identified with the space $\mathbb{S}_{[\kappa_1, \kappa_2]}^2$ and the action of $\text{SO}_{\kappa_1, \kappa_2}(3)$ is transitive on it. The coordinates (s_0, s_1, s_2) , satisfying the constraint (17) are called *ambient space* or *Weierstrass coordinates*, while $(s_1/s_0, s_2/s_0)$ are the usual *Beltrami coordinates* in Projective Geometry.

We define two metrics on $\mathbb{S}_{[\kappa_1, \kappa_2]}^2$: the *main metric* $d\sigma_{(1)}^2$, which comes from the flat ambient metric in \mathbb{R}^3 divided by the curvature κ_1 and restricted to (17), and a *subsidiary metric* $d\sigma_{(2)}^2$, proportional to the former one:

$$d\sigma_{(1)}^2 := \frac{1}{\kappa_1} (ds_0^2 + \kappa_1 ds_1^2 + \kappa_1 \kappa_2 ds_2^2) \Big|_{\Sigma_\kappa}, \quad d\sigma_{(2)}^2 := \frac{1}{\kappa_2} d\sigma_{(1)}^2. \quad (18)$$

On the Riemannian spaces ($\kappa_2 > 0$) both metrics are equivalent; on the Lorentzian spacetimes ($\kappa_2 < 0$) these correspond, in this order, to the time- and space-like metrics; but on the Newtonian spaces with $\kappa_2 = 0$ the main metric is degenerate with signature $(+, 0)$ and there exists an invariant foliation under the action of the CK group $\text{SO}_{\kappa_1, 0}(3)$ on $\mathbb{S}_{[\kappa_1], 0}^2$, in such a manner that the subsidiary metric is restricted to each leaf of the foliation.

The ambient coordinates (17) can be parametrized in terms of two intrinsic variables in different ways. In particular, let us introduce the so-called *geodesic parallel I* (a_1, a_2) , *geodesic parallel II* (b_1, b_2) and *geodesic polar* (r, ϕ) coordinates of a point $Q = (s_0, s_1, s_2) \in \mathbb{S}_{[\kappa_1, \kappa_2]}^2$ [25], which are defined through the following action of the one-parametric subgroups (16) on O :

$$\begin{aligned} (s_0, s_1, s_2)^T &= \exp(a_1 \rho(J_{01})) \exp(a_2 \rho(J_{02})) O^T \\ &= \exp(b_2 \rho(J_{02})) \exp(b_1 \rho(J_{01})) O^T \\ &= \exp(\phi \rho(J_{12})) \exp(r \rho(J_{01})) O^T. \end{aligned} \quad (19)$$

This yields

$$\begin{aligned} s_0 &= C_{\kappa_1}(a_1) C_{\kappa_1 \kappa_2}(a_2) = C_{\kappa_1}(b_1) C_{\kappa_1 \kappa_2}(b_2) = C_{\kappa_1}(r) \\ s_1 &= S_{\kappa_1}(a_1) C_{\kappa_1 \kappa_2}(a_2) = S_{\kappa_1}(b_1) = S_{\kappa_1}(r) C_{\kappa_2}(\phi) \\ s_2 &= S_{\kappa_1 \kappa_2}(a_2) = C_{\kappa_1}(b_1) S_{\kappa_1 \kappa_2}(b_2) = S_{\kappa_1}(r) S_{\kappa_2}(\phi). \end{aligned} \quad (20)$$

As shown in Fig. 1, r is the distance between the origin O and the point Q measured along the (first-kind) geodesic l that joins both points and ϕ is the angle of l with respect to l_1 . If Q_1 denotes the intersection point of l_1 with its orthogonal (second-kind) geodesic l'_2 through Q , then a_1 is the geodesic distance between O and Q_1 measured along l_1 and a_2 is the geodesic distance between Q_1 and Q measured along l'_2 . Similarly, for the geodesic parallel II coordinates (b_1, b_2) . Notice that $(a_1, a_2) \neq (b_1, b_2)$ when $\kappa_1 \neq 0$. On the flat cases with $\kappa_1 = 0$, the relations (20) show that $s_0 = 1$ and $(s_1, s_2) = (a_1, a_2) = (b_1, b_2)$ thus reducing them to Cartesian coordinates and (r, ϕ) to the polar ones. By introducing (20) in the main metric (18) we obtain

$$\begin{aligned} d\sigma_{(1)}^2 &= C_{\kappa_1 \kappa_2}^2(a_2) da_1^2 + \kappa_2 da_2^2 = db_1^2 + \kappa_2 C_{\kappa_1}^2(b_1) db_2^2 \\ &= dr^2 + \kappa_2 S_{\kappa_1}^2(r) d\phi^2. \end{aligned} \quad (21)$$

As expected, it can easily be checked that the Gaussian curvature is κ_1 and the signature is given by $\text{diag}(+, \kappa_2)$.

Let us now focus on the metric in geodesic parallel I coordinates (a_1, a_2) . When $\kappa_2 = 0$, the metric is degenerate and there appears an invariant foliation determined by $a_1 = a_1^0 = \text{constant}$, with subsidiary metric (18) $d\sigma_{(2)}^2 = da_2^2$ defined on each leaf $a_1 = a_1^0$. From a kinematical viewpoint (apply the identifications (10) and (11)), it turns out that (a_1, a_2) are just the time and space variables (x_0, x_1) ; e.g. in the Minkowskian spacetime we recover $d\sigma_{(1)}^2 = dx_0^2 - \frac{1}{c^2}dx_1^2$. In the Newtonian spaces with $\kappa_2 = 0$ ($c \rightarrow \infty$), the main metric is just $d\sigma_{(1)}^2 = dx_0^2$ which provides “absolute-time” x_0 , the leaves of the foliation are the “absolute-space” at $x_0 = x_0^0$, and $d\sigma_{(2)}^2 = dx_1^2$ is the spatial metric defined on each leaf.

Isometry vector fields for the CK generators, fulfilling (2), can be obtained from the 3×3 matrix representation (14). In ambient coordinates (s_0, s_1, s_2) , satisfying the constraint (17), these are given by

$$J_{01} = \kappa_1 s_1 \partial_{s_0} - s_0 \partial_{s_1}, \quad J_{02} = \kappa_1 \kappa_2 s_2 \partial_{s_0} - s_0 \partial_{s_2}, \quad J_{12} = \kappa_2 s_2 \partial_{s_1} - s_1 \partial_{s_2}.$$

They can be written in any geodesic coordinate system through (20). For instance, in terms of geodesic parallel I coordinates they turn out to be

$$\begin{aligned} J_{01} &= -\partial_{a_1} \\ J_{02} &= -\kappa_1 \kappa_2 S_{\kappa_1}(a_1) T_{\kappa_1 \kappa_2}(a_2) \partial_{a_1} - C_{\kappa_1}(a_1) \partial_{a_2} \\ J_{12} &= \kappa_2 C_{\kappa_1}(a_1) T_{\kappa_1 \kappa_2}(a_2) \partial_{a_1} - S_{\kappa_1}(a_1) \partial_{a_2}. \end{aligned} \tag{22}$$

Under such vector fields, the Casimir (3) gives rise to the Laplace–Beltrami operator on $\mathbb{S}_{[\kappa_1], \kappa_2}^2$, namely

$$\mathcal{C} = \frac{\kappa_2}{C_{\kappa_1 \kappa_2}^2(a_2)} \partial_{a_1}^2 + \partial_{a_2}^2 - \kappa_1 \kappa_2 T_{\kappa_1 \kappa_2}(a_2) \partial_{a_2}$$

which, in fact, provides the wave equation for the Lorentzian spaces [25] (set (11) and $(a_1, a_2) = (x_0, x_1)$); e.g. in the Minkowski case we have

$$\mathcal{C}\Phi(x_0, x_1) = 0 \quad \Rightarrow \quad \left(-\frac{1}{c^2} \partial_{x_0}^2 + \partial_{x_1}^2\right) \Phi(x_0, x_1) = 0.$$

We illustrate the above results by presenting in Table 3 the metric and vector fields in geodesic parallel I coordinates for each particular space of points.

Table 3: The nine CK spaces of points $\mathbf{S}_{[\kappa_1, \kappa_2]}^2$ (5) according to the “normalized” values of the contraction parameters $\kappa_i \in \{1, 0, -1\}$. For each space it is shown, in geodesic parallel I coordinates (a_1, a_2) (20), the metric (21) and the Lie vector fields of isometries (22). When $\kappa_2 = 0$, the second metric is defined on $a_1 = \text{constant}$.

| | | |
|---|---|--|
| <ul style="list-style-type: none"> • Spherical $\mathbb{S}_{[+],+}^2 = \text{SO}(3)/\text{SO}(2)$ $d\sigma_{(1)}^2 = \cos^2 a_2 da_1^2 + da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\sin a_1 \tan a_2 \partial_{a_1} - \cos a_1 \partial_{a_2}$ $J_{12} = \cos a_1 \tan a_2 \partial_{a_1} - \sin a_1 \partial_{a_2}$ | <ul style="list-style-type: none"> • Euclidean $\mathbb{S}_{[0],+}^2 = \text{ISO}(2)/\text{SO}(2)$ $d\sigma_{(1)}^2 = da_1^2 + da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\partial_{a_2}$ $J_{12} = a_2 \partial_{a_1} - a_1 \partial_{a_2}$ | <ul style="list-style-type: none"> • Hyperbolic $\mathbb{S}_{[-],+}^2 = \text{SO}(2, 1)/\text{SO}(2)$ $d\sigma_{(1)}^2 = \cosh^2 a_2 da_1^2 + da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = \sinh a_1 \tanh a_2 \partial_{a_1} - \cosh a_1 \partial_{a_2}$ $J_{12} = \cosh a_1 \tanh a_2 \partial_{a_1} - \sinh a_1 \partial_{a_2}$ |
| <ul style="list-style-type: none"> • Co-Euclidean/Oscillating NH $\mathbb{S}_{[+],0}^2 = \text{ISO}(2)/\mathbb{R}$ $d\sigma_{(1)}^2 = da_1^2, \quad d\sigma_{(2)}^2 = da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\cos a_1 \partial_{a_2}$ $J_{12} = -\sin a_1 \partial_{a_2}$ | <ul style="list-style-type: none"> • Galilean $\mathbb{S}_{[0],0}^2 = \text{ISO}(1)/\mathbb{R}$ $d\sigma_{(1)}^2 = da_1^2, \quad d\sigma_{(2)}^2 = da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\partial_{a_2}$ $J_{12} = -a_1 \partial_{a_2}$ | <ul style="list-style-type: none"> • Co-Minkowskian/Expanding NH $\mathbb{S}_{[-],0}^2 = \text{ISO}(1, 1)/\mathbb{R}$ $d\sigma_{(1)}^2 = da_1^2, \quad d\sigma_{(2)}^2 = da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\cosh a_1 \partial_{a_2}$ $J_{12} = -\sinh a_1 \partial_{a_2}$ |
| <ul style="list-style-type: none"> • Co-Hyperbolic/Anti-de Sitter $\mathbb{S}_{[+],-}^2 = \text{SO}(2, 1)/\text{SO}(1, 1)$ $d\sigma_{(1)}^2 = \cosh^2 a_2 da_1^2 - da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = \sin a_1 \tanh a_2 \partial_{a_1} - \cos a_1 \partial_{a_2}$ $J_{12} = -\cos a_1 \tanh a_2 \partial_{a_1} - \sin a_1 \partial_{a_2}$ | <ul style="list-style-type: none"> • Minkowskian $\mathbb{S}_{[0],-}^2 = \text{ISO}(1, 1)/\text{SO}(1, 1)$ $d\sigma_{(1)}^2 = da_1^2 - da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\partial_{a_2}$ $J_{12} = -a_2 \partial_{a_1} - a_1 \partial_{a_2}$ | <ul style="list-style-type: none"> • Doubly Hyperbolic/De Sitter $\mathbb{S}_{[-],-}^2 = \text{SO}(2, 1)/\text{SO}(1, 1)$ $d\sigma_{(1)}^2 = \cos^2 a_2 da_1^2 - da_2^2$ $J_{01} = -\partial_{a_1}$ $J_{02} = -\sinh a_1 \tan a_2 \partial_{a_1} - \cosh a_1 \partial_{a_2}$ $J_{12} = -\cosh a_1 \tan a_2 \partial_{a_1} - \sinh a_1 \partial_{a_2}$ |

6 Quantum Groups and Poisson Homogeneous Spaces

In this Section we introduce the basic background on quantum deformations and their connection with Poisson–Lie groups and Poisson homogeneous spaces.

Let us recall that *quantum groups* are quantizations of Poisson–Lie (PL) groups *i.e.*, quantizations of the Poisson–Hopf algebras of multiplicative Poisson structures on Lie groups [10, 12, 31]. PL structures (G, Π) on a simply connected Lie group G are in one-to-one correspondence with Lie bialgebra structures (\mathfrak{g}, δ) [11], where \mathfrak{g} is the Lie algebra of G and δ is the skewsymmetric cocommutator map $\delta : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$. The cocommutator δ must fulfil two conditions:

(i) δ is a 1-cocycle,

$$\delta([X, Y]) = [\delta(X), Y \otimes 1 + 1 \otimes Y] + [X \otimes 1 + 1 \otimes X, \delta(Y)], \quad \forall X, Y \in \mathfrak{g}.$$

(ii) The dual map $\delta^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie bracket on \mathfrak{g}^* .

Each quantum group G_z , with quantum deformation parameter $q = e^z$, can be associated with a PL group G , and this with a unique Lie bialgebra structure (\mathfrak{g}, δ) .

The dual version of quantum groups are *quantum algebras* $\mathcal{U}_z(\mathfrak{g})$, which are Hopf algebra deformations of universal enveloping algebras $\mathcal{U}(\mathfrak{g})$, and are constructed as formal power series in z and coefficients in $\mathcal{U}(\mathfrak{g})$. The Hopf algebra structure in $\mathcal{U}_z(\mathfrak{g})$ is provided by a coassociative coproduct map $\Delta_z : \mathcal{U}_z(\mathfrak{g}) \rightarrow \mathcal{U}_z(\mathfrak{g}) \otimes \mathcal{U}_z(\mathfrak{g})$, which is an algebra homomorphism, along with the counit ϵ and antipode γ mappings. If we write the coproduct as a formal power series in z , its first-order determines the cocommutator in the form

$$\Delta_z = \Delta_0 + z \Delta_1 + o[z^2], \quad \delta = z(\Delta_1 - \sigma \circ \Delta_1) \quad (23)$$

where $\Delta_0(X) = X \otimes 1 + 1 \otimes X$ and $\sigma(X \otimes Y) = Y \otimes X$. Hence, each quantum deformation turns out to be related to a unique Lie bialgebra structure (\mathfrak{g}, δ) . Explicitly, if we consider a basis for \mathfrak{g} where

$$[X_i, X_j] = c_{ij}^k X_k$$

any cocommutator δ will be of the form

$$\delta(X_i) = f_i^{jk} X_j \wedge X_k$$

being f_i^{jk} the structure tensor of the dual Lie algebra \mathfrak{g}^*

$$[\hat{\xi}^j, \hat{\xi}^k] = f_i^{jk} \hat{\xi}^i \quad (24)$$

where $\langle \hat{\xi}^j, X_k \rangle = \delta_k^j$. The cocycle condition for δ implies the following compatibility equations among the structure constants c_{ij}^k and f_i^{jk} :

$$f_k^{lm} c_{ij}^k = f_i^{lk} c_{kj}^m + f_i^{km} c_{kj}^l + f_j^{lk} c_{ik}^m + f_j^{km} c_{ik}^l.$$

The connection of these structures with *noncommutative spaces* arises when G is a group of isometries of a given space. Then X_i are the Lie algebra generators and $\hat{\xi}^j$ can be considered as a noncommutative counterpart of the local coordinates ξ^j on the group. For a quantum

algebra $\mathcal{U}_z(\mathfrak{g})$, the cocommutator δ is non-vanishing and the commutator (24) among the space coordinates associated to the translation generators of the group will be in general non-zero. This is the way in which noncommutative spaces are constructed from quantum groups. Higher-order contributions to the noncommutative space (24) can be obtained from higher-orders of the full quantum coproduct Δ_z .

In many cases the cocommutator δ is a coboundary one, which means that it is obtained through

$$\delta(X) = [X \otimes 1 + 1 \otimes X, r], \quad \forall X \in \mathfrak{g} \quad (25)$$

where $r = r^{ij} X_i \wedge X_j$ is an r -matrix, which is a solution of the modified classical Yang–Baxter equation

$$[X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X, [[r, r]]] = 0, \quad \forall X \in \mathfrak{g} \quad (26)$$

being $[[r, r]]$ the Schouten bracket defined as

$$[[r, r]] := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]$$

where $r_{12} = r^{ij} X_i \otimes X_j \otimes 1$, $r_{13} = r^{ij} X_i \otimes 1 \otimes X_j$, $r_{23} = r^{ij} 1 \otimes X_i \otimes X_j$. Recall that $[[r, r]] = 0$ is just the classical Yang–Baxter equation. When the PL group G is a coboundary one, its Poisson structure Π is given by the Sklyanin bracket [10]

$$\{f, g\} = r^{ij} (\nabla_i^L f \nabla_j^L g - \nabla_i^R f \nabla_j^R g), \quad f, g \in C^\infty(G) \quad (27)$$

where ∇_i^L, ∇_i^R are left- and right-invariant vector fields on G .

A *Poisson homogeneous space* (PHS) of a PL group (G, Π) is a Poisson manifold (M, π) endowed with a transitive group action $\triangleright : G \times M \rightarrow M$ which is a Poisson map with respect to the Poisson structure on M and the product $\Pi \times \pi$ of the Poisson structures on G and M [13]. In particular, let us consider a homogeneous space $M = G/H$ with isometry Lie group G and isotropy subgroup H . A PHS (M, π) is constructed by endowing G with the PL structure Π (27), and the space M with a Poisson bracket π that has to be compatible with the group action \triangleright . Since G may admit several PL structures Π (i.e., \mathfrak{g} may admit several Lie bialgebra structures (\mathfrak{g}, δ)), a given homogeneous space $M = G/H$ could lead to several non-equivalent PHSs.

Finally, it is known [13] that each PHS is in one-to-one correspondence with a Lagrangian subalgebra of the double Lie algebra of \mathfrak{g} associated with the cocommutator δ . This statement is equivalent to imposing the so-called *coisotropy condition* for the cocommutator δ with respect to the isotropy subalgebra \mathfrak{h} (see [9] and references therein)

$$\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{g}. \quad (28)$$

In the more restrictive case with $\delta(\mathfrak{h}) \subset \mathfrak{h} \wedge \mathfrak{h}$, the subgroup H have a sub-Lie bialgebra structure which implies that the PHS is constructed through an isotropy subgroup which is a Poisson subgroup with respect to Π .

7 Cayley–Klein Poisson Homogeneous Spaces of Points

Among the possible classical r -matrices for the CK algebra $\mathfrak{so}_{\kappa_1, \kappa_2}(3)$ (2), let us consider [3]

$$r = zJ_{12} \wedge J_{02} \quad (29)$$

which is a solution of the modified classical Yang–Baxter equation (26). For $\mathfrak{so}(3)$ this generates the usual Drinfel’d–Jimbo quantum deformation, and the corresponding cocommutator (25) reads

$$\delta(J_{01}) = 0, \quad \delta(J_{02}) = z\kappa_2 J_{01} \wedge J_{02}, \quad \delta(J_{12}) = z\kappa_2 J_{01} \wedge J_{12}. \quad (30)$$

These relations are the first-order in z (23) of the full coproduct for the quantum CK algebra $\mathcal{U}_z(\mathfrak{so}_{\kappa_1, \kappa_2}(3))$ which can be expressed as

$$\begin{aligned} \Delta_z(J_{01}) &= J_{01} \otimes 1 + 1 \otimes J_{01} \\ \Delta_z(J_{02}) &= J_{02} \otimes \exp(-\frac{z}{2}\kappa_2 J_{01}) + \exp(\frac{z}{2}\kappa_2 J_{01}) \otimes J_{02} \\ \Delta_z(J_{12}) &= J_{12} \otimes \exp(-\frac{z}{2}\kappa_2 J_{01}) + \exp(\frac{z}{2}\kappa_2 J_{01}) \otimes J_{12}. \end{aligned} \quad (31)$$

This is a homomorphism map for the deformed commutation rules given by

$$[J_{12}, J_{01}]_z = J_{02}, \quad [J_{12}, J_{02}]_z = -\frac{\sinh(z\kappa_2 J_{01})}{z}, \quad [J_{01}, J_{02}]_z = \kappa_1 J_{12}$$

which under the classical limit $z \rightarrow 0$ reduce to the Lie brackets (2).

Let us denote by \hat{x}^{ij} the quantum group coordinates dual to J_{ij} , that is $\langle \hat{x}^{ij}, J_{lm} \rangle = \delta_{lm}^{ij}$. By using (24) the cocommutator (30) gives rise to the commutation relations of the dual Lie algebra that we will denote $\mathfrak{so}_{\kappa_1, \kappa_2}^*(3)$, namely

$$[\hat{x}^{01}, \hat{x}^{02}] = z\kappa_2 \hat{x}^{02}, \quad [\hat{x}^{02}, \hat{x}^{12}] = 0, \quad [\hat{x}^{01}, \hat{x}^{12}] = z\kappa_2 \hat{x}^{12}. \quad (32)$$

As far as the isotropy subalgebras of a point $\mathfrak{h}_0 = \text{span}\{J_{12}\}$, a first-kind line $\mathfrak{h}_{01} = \text{span}\{J_{01}\}$ and a second-kind one $\mathfrak{h}_{02} = \text{span}\{J_{02}\}$ are concerned, the cocommutator (30) shows that the coisotropy condition (28) is fulfilled for all of them:

$$\delta(\mathfrak{h}_0) \subset \mathfrak{h}_0 \wedge \mathfrak{g}, \quad \delta(\mathfrak{h}_{01}) = 0, \quad \delta(\mathfrak{h}_{02}) \subset \mathfrak{h}_{02} \wedge \mathfrak{g}$$

for $\mathfrak{g} = \mathfrak{so}_{\kappa_1, \kappa_2}(3)$. Then the first commutator in (32) can be interpreted as the noncommutative counterpart of the space of points $\mathbb{S}_{[\kappa_1], \kappa_2}^2$ (5), at first-order in the deformation parameter and quantum coordinates, since it involves the coordinates dual to the translation generators in this space. Likewise, the second and third commutators in (32) correspond to the (first-order) noncommutative spaces associated with the spaces of first-kind (6) and second-kind lines (7), respectively.

Notice that the noncommutative space of first-kind lines is, in fact, commutative at first-order and that the quantum deformation $\mathcal{U}_z(\mathfrak{so}_{\kappa_1, \kappa_2}(3))$ is determined by the generator J_{01} which remains undeformed (31) (and it gives a Poisson subgroup). This is a consequence of the initial classical r -matrix (29) that we have considered. Therefore we shall say that this deformation is of first-kind (or time-like) type. Different results would come out if one starts

with $r = zJ_{12} \wedge J_{01}$, which would lead to a second-kind (or space-like) deformation, which is just the one fully worked out in [6, 7] (see also [20] for quantum deformations of CK algebras in terms hypercomplex units).

We remark that for the three Newtonian algebras with $\kappa_2 = 0$ the quantum deformation $\mathcal{U}_z(\mathfrak{so}_{\kappa_1, \kappa_2}(3))$ is the trivial one and that (32) provides commutative coordinates. By contrast, for the Lorentzian algebras with $\kappa_2 < 0$, the first relation (32) defines the well-known (noncommutative) kappa-Minkowski spacetime [30, 32]. Therefore, the *three* Lorentzian algebras share the same noncommutative spacetime but only at the first-order in z and the same happens for the Riemannian cases with $\kappa_2 > 0$.

Higher-order terms can be obtained either by computing the full quantum duality or by constructing the noncommutative space as the quantization of the corresponding PHS associated with the r -matrix that generates the deformation. Let us make this statement more explicit by constructing the full noncommutative space of points. Let $\{a_1, a_2, \xi\}$ be the local coordinates on the CK group $\text{SO}_{\kappa_1, \kappa_2}(3)$ associated, in this order, with the Lie generators $\{J_{01}, J_{02}, J_{12}\}$. Then we consider the following group element

$$g = \exp(a_1 \rho(J_{01})) \exp(a_2 \rho(J_{02})) \exp(\xi \rho(J_{12})) \quad (33)$$

written under the representation (14). We stress that the order chosen for the product of the matrices enables us to identify the local coordinates (a_1, a_2) with the geodesic parallel I type (19), so we keep the same notation. From (33), the left- and right-invariant vector fields on $\text{SO}_{\kappa_1, \kappa_2}(3)$ are found to be [7]

$$\begin{aligned} \nabla_{J_{01}}^L &= \frac{C_{\kappa_2}(\xi)}{C_{\kappa_1 \kappa_2}(a_2)} \partial_{a_1} + S_{\kappa_2}(\xi) \partial_{a_2} - \kappa_1 T_{\kappa_1 \kappa_2}(a_2) C_{\kappa_2}(\xi) \partial_\xi \\ \nabla_{J_{02}}^L &= -\kappa_2 \frac{S_{\kappa_2}(\xi)}{C_{\kappa_1 \kappa_2}(a_2)} \partial_{a_1} + C_{\kappa_2}(\xi) \partial_{a_2} + \kappa_1 \kappa_2 T_{\kappa_1 \kappa_2}(a_2) S_{\kappa_2}(\xi) \partial_\xi \\ \nabla_{J_{12}}^L &= \partial_\xi \\ \nabla_{J_{01}}^R &= \partial_{a_1} \\ \nabla_{J_{02}}^R &= \kappa_1 \kappa_2 S_{\kappa_1}(a_1) T_{\kappa_1 \kappa_2}(a_2) \partial_{a_1} + C_{\kappa_1}(a_1) \partial_{a_2} - \kappa_1 \frac{S_{\kappa_1}(a_1)}{C_{\kappa_1 \kappa_2}(a_2)} \partial_\xi \\ \nabla_{J_{12}}^R &= -\kappa_2 C_{\kappa_1}(a_1) T_{\kappa_1 \kappa_2}(a_2) \partial_{a_1} + S_{\kappa_1}(a_1) \partial_{a_2} + \frac{C_{\kappa_1}(a_1)}{C_{\kappa_1 \kappa_2}(a_2)} \partial_\xi. \end{aligned}$$

By computing the Sklyanin bracket (27) for the classical r -matrix (29), we obtain the Poisson structure Π on the CK group $\text{SO}_{\kappa_1, \kappa_2}(3)$

$$\begin{aligned} \{\xi, a_1\} &= -z \kappa_2 \frac{S_{\kappa_2}(\xi)}{C_{\kappa_1 \kappa_2}(a_2)}, & \{\xi, a_2\} &= z \frac{C_{\kappa_1 \kappa_2}(a_2) C_{\kappa_2}(\xi) - 1}{C_{\kappa_1 \kappa_2}(a_2)} \\ \{a_1, a_2\} &= z \kappa_2 T_{\kappa_1 \kappa_2}(a_2). \end{aligned} \quad (34)$$

The canonical projection of the Π -brackets to the homogeneous space with coordinates (a_1, a_2) gives rise to the PHS of points which is just the last π -Poisson bracket in (34). Since no ordering

Table 4: The Poisson structure (34) on the six CK groups $\text{SO}_{\kappa_1, \kappa_2}(\mathbb{3})$ with $\kappa_2 \neq 0$ and $\kappa_i \in \{1, 0, -1\}$. The Poisson bracket $\{a_1, a_2\}$ and $\{x_0, x_1\}$ defines the PHS of points for the Riemannian and Lorentzian cases, respectively.

| | | |
|--|--|---|
| <ul style="list-style-type: none"> • Spherical $\mathbb{S}_{[+],+}^2$ $\{\xi, a_1\} = -z \frac{\sin \xi}{\cos a_2}$ $\{\xi, a_2\} = z \frac{\cos a_2 \cos \xi - 1}{\cos a_2}$ $\{a_1, a_2\} = z \tan a_2$ | <ul style="list-style-type: none"> • Euclidean $\mathbb{S}_{[0],+}^2$ $\{\xi, a_1\} = -z \sin \xi$ $\{\xi, a_2\} = z(\cos \xi - 1)$ $\{a_1, a_2\} = z a_2$ | <ul style="list-style-type: none"> • Hyperbolic $\mathbb{S}_{[-],+}^2$ $\{\xi, a_1\} = -z \frac{\sin \xi}{\cosh a_2}$ $\{\xi, a_2\} = z \frac{\cosh a_2 \cos \xi - 1}{\cosh a_2}$ $\{a_1, a_2\} = z \tanh a_2$ |
| <ul style="list-style-type: none"> • Co-Hyperbolic Anti-de Sitter $\mathbb{S}_{[+],-}^2$ $\{\xi, x_0\} = z \frac{\sinh \xi}{\cosh x_1}$ $\{\xi, x_1\} = z \frac{\cosh x_1 \cosh \xi - 1}{\cosh x_1}$ $\{x_0, x_1\} = -z \tanh x_1$ | <ul style="list-style-type: none"> • Minkowskian $\mathbb{S}_{[0],-}^2$ $\{\xi, x_0\} = z \sinh \xi$ $\{\xi, x_1\} = z(\cosh \xi - 1)$ $\{x_0, x_1\} = -z x_1$ | <ul style="list-style-type: none"> • Doubly Hyperbolic De Sitter $\mathbb{S}_{[-],-}^2$ $\{\xi, x_0\} = z \frac{\sinh \xi}{\cos x_1}$ $\{\xi, x_1\} = z \frac{\cos x_1 \cosh \xi - 1}{\cos x_1}$ $\{x_0, x_1\} = -z \tan x_1$ |

ambiguities appear in the r.h.s. of the bracket $\{a_1, a_2\}$, this can directly be quantized yielding the noncommutative space of points

$$[\hat{a}_1, \hat{a}_2] = z\kappa_2 \text{T}_{\kappa_1 \kappa_2}(\hat{a}_2) \quad (35)$$

whose first-order is given by the first commutator in (32) under the identification $\hat{x}^{01} \equiv \hat{a}_1$ and $\hat{x}^{02} \equiv \hat{a}_2$. The noncommutative space (35) is, as expected, different from the one coming from the deformation of second-kind type determined by the classical r -matrix $r = zJ_{12} \wedge J_{01}$, namely [7]:

$$[\hat{a}_1, \hat{a}_2] = zS_{\kappa_1}(\hat{a}_1). \quad (36)$$

From a kinematical point of view, (36) can be interpreted as space-like noncommutative spacetimes. The noncommutative Minkowski spacetime ($\kappa_1 = 0$) corresponds to $[\hat{x}_0, \hat{x}_1] = z \hat{x}_0$, and it is different from kappa-Minkowski [30, 32] which is actually contained in (35) as $[\hat{x}_0, \hat{x}_1] = -z \frac{1}{c^2} \hat{x}_1$. The non-relativistic limit $\kappa_2 = 0$ of (36) now leads to non-trivial noncommutative Newtonian spacetimes.

To summarize, we present in Table 4 the Poisson brackets (34) for the six CK groups with $\kappa_2 \neq 0$, since for $\kappa_2 = 0$ all the Poisson brackets vanish. In the Lorentzian cases with $\kappa_2 < 0$ the geodesic parallel I coordinates (a_1, a_2) are written as the spacetime ones (x_0, x_1) and ξ is a rapidity. The PHSs of first- and second-kind lines can be constructed in a similar way, but this would require to consider the appropriate order for the exponentiation giving rise to the matrix group element g acting transitively on the chosen coordinates.

To end with, we stress that the classical picture of orthogonal CK algebras and (Poisson/noncommutative) spaces can be generalized to higher dimensions [3, 8, 21] and to other families of semisimple Lie algebras [21, 39]. Nevertheless, the quantum deformation scheme is rather involved, not only because to raise the dimension requires cumbersome computations,

but mainly due to the fact that in higher dimensions different possible non-equivalent deformations (and, therefore, PL structures and PHSs) can be considered. In this respect, see [4] and references therein for recent results on Lorentzian kinematical algebras.

Acknowledgements

This work was partially supported by Ministerio de Ciencia, Innovación y Universidades (Spain) under grant MTM2016-79639-P (AEI/FEDER, UE) and by the Action MP1405 QSPACE from the European Cooperation in Science and Technology (COST). I.G-S. acknowledges a predoctoral grant from Junta de Castilla y León (Spain) and the European Social Fund.

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