The Lippmann–Schwinger Formula and One Dimensional Models with Dirac Delta Interactions

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Abstract We show how a proper use of the Lippmann–Schwinger equation simplifies the calculations to obtain scattering states for one dimensional systems perturbed by N Dirac delta equations. Here, we consider two situations. In the former, attractive Dirac deltas perturbed the free one dimensional Schrödinger Hamiltonian. We obtain explicit expressions for scattering and Gamow states. For completeness, we show that the method to obtain bound states use comparable formulas, although not based on the Lippmann–Schwinger equation. Then, the attractive N deltas perturbed the one dimensional Salpeter equation. We also obtain explicit expressions for the scattering wave functions. Here, we need regularisation techniques that we implement via heat kernel regularisation.

Keywords Scattering states · Schrödinger and Salpeter one dimensional Hamiltonians · Contact perturbations · Gamow wave functions · Lippmann–Schwinger equation

1 Introduction

One of the more used tools in order to understand quantum mechanics are the solvable models, in particular those which are one dimensional due to their simplicity [1–4]. The more often studied among these models is the free particle Schrödinger Hamiltonian decorated with Dirac delta interactions. Relativistic one

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dimensional approaches for the free particle Hamiltonian, such as those named after Salpeter or Dirac, have also been perturbed with contact interactions of delta type [5–7]. The purpose of the present article is to give a brief review of the recent work by the authors including the perturbation by N Dirac deltas of the one dimensional Schrödinger and Salpeter free Hamiltonians [6, 8, 9].

From the physics point of view, point potentials may represent interactions which are very localised in the space and strong and have a vast amount of applications for modelling real physical systems. A well-known model using Dirac delta potentials in non-relativistic quantum mechanics is the so-called Kronig–Penney model [10], and it is actually a reference model in describing the band gap structure of metals in solid state physics [11]. In addition, Dirac delta interactions in one or more dimensions serve as simple pedagogical toy models for the understanding of several quantum non-trivial concepts [12–19].

From the mathematical point of view, contact potentials are the result of the theory of self-adjoint extensions of symmetric operators with equal deficiency indices. In general, there are two methods to obtain these extensions. One is by defining some matching conditions at the nodes (points that support the contact potentials). Other uses the construction of the resolvent operator and often requires a renormalisation due to possible divergences in the construction of the resolvent of the self-adjoint extension. Still a third method relies on a theorem of von Neumann that characterises all self-adjoint extensions of a symmetric operator with equal deficiency indices, although this one has been less used.

We also want to show how the Lippmann–Schwinger formula is useful for this purpose as a simplifying computational tool. Here, we shall use the simplest form of this equation which acquires mathematical sense on Gelfand triplets. The Lippmann–Schwinger formula gives an equation satisfied by the incoming and outgoing plane waves after a scattering process due to a potential V. It has the following form:

$$|k^{\pm}\rangle = |k\rangle - R_0(E_k \pm i0)\, V\, |k^{\pm}\rangle\,, \tag{1}$$

where $|k^{\pm}\rangle$ refers to the full scattered incoming $(+)$ and outgoing $(-)$ plane waves, $|k\rangle$ is the free plane wave, V the potential and $R_0(E_k \pm i0)$ is the free resolvent, also called the Green operator. Since it is a function of the complex variable z, $R(z)$, and has a branch cut at the spectrum of the free Hamiltonian (usually $\mathbb{R}^+ \equiv [0, \infty)$), we denote by $R_0(E_k \pm i0)$ the upper and lower limits of $R(z)$ as the imaginary part of z goes to zero. Here, $E_k = (\hbar^2 k^2)/2m$.

This paper contains three more sections. In Sect. 2, we briefly discuss the consequences of adding N Dirac delta perturbations to the one dimensional free Schrödinger Hamiltonian. In Sect. 3, we do the same with the one dimensional Salpeter Hamiltonian. The analysis of bound states is particularly relevant in both cases. We finish our discussion with the concluding remarks.

2 One Dimensional Schrödinger Hamiltonian with N Dirac Delta Interactions

The objective of this section is to study the one dimensional Schrödinger Hamiltonian $H_0 = \frac{p^2}{2m}$ perturbed by N Dirac deltas located at some points in the real axis. This study includes the search for bound states, scattering coefficients and resonances provided they exist. As is well known, this perturbed Hamiltonian has the form

$$H := \frac{p^2}{2m} - \sum_{i=1}^{N} \lambda_i \, \delta(x - a_i) \,, \qquad V := -\sum_{i=1}^{N} \lambda_i \, \delta(x - a_i) \,, \tag{2}$$

where λ_i and $i = 1, 2, \ldots, N$, $i = 1, 2, \ldots, N$ are *positive* real numbers. The a_i show the points supporting the deltas and are called *nodes*. Each of the $-\lambda_i$, with $\lambda_i > 0$, is the intensity of the delta located at a_i for all value of i. These coefficients are chosen to be negative if we want to have bound states. The Schrödinger equation produced by (2) is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(x)}{dx^2} - \sum_{i=1}^{N} \lambda_i \, \delta(x - a_i) \, \psi(x) = E\psi(x) \,. \tag{3}$$

It is interesting to rewrite the interaction V in such a way that the calculations with the aid of the Lippmann–Schwinger equation become easy. For simplicity, let us assume that we have only one first. Then, the potential is $V = \lambda \, \delta(x - a)$ and the wave function is $\psi(x) = \langle x|\psi\rangle$ [20–24]. In this notation, $(V\psi)(x) = \langle x|V\psi\rangle$ and $\langle x|a\rangle = \delta(x - a)$. Thus,

$$(V\psi)(x) = \lambda \, \delta(x - a) \, \psi(a) \,. \tag{4}$$

Next, we note that the potential can be written as $V = \lambda \, |a\rangle\langle a|$, since then,

$$\langle x|V\psi\rangle = \lambda \, \langle x|a\rangle\langle a|\psi\rangle = \lambda \, \delta(x - a) \, \psi(a) = (V\psi)(x) \,. \tag{5}$$

The generalisation of the expression for the potential V in the case of having N nodes is the following:

$$V = -\sum_{i=1}^{N} \lambda_i \, |a_i\rangle\langle a_i| \,. \tag{6}$$

This is the desired expression. Let us clarify the vectors $|x\rangle$ for any real number x are the generalised eigenvalues of the position (multiplication) operator in one dimension with eigenvalue x. As is well known, these vectors do not belong to the

Hilbert space on which the multiplication operator acts, but instead to an extension of it endowed with a weak topology. We do not want to enter in these kind of details here, see [21–24]. Vectors $|a_i\rangle$ are precisely of this type with $x = a_i$.

The first objective is the search for scattering states. We are introducing the procedure in the sequel, although we shall skip some steps in order to reach the final result as straightforward as possible. Details may be found in [8, 9]. Let us use (6) in the Lippmann–Schwinger equation (1) and multiply the result from the left by the bra $\langle x|$. We have

$$\langle x|k^{\pm}\rangle = \langle x|k\rangle + \sum_{j=1}^{N} \lambda_j \, \langle x|G_0(E_k \pm i0)|a_j\rangle\langle a_j|k^{\pm}\rangle \, . \tag{7}$$

For convenience, we shall use the notation $G_0(x, y; E_k \pm i0) := \langle x|G_0(E_k \pm i0)|y\rangle$ in the sequel. Also, we recall that $\langle x|k\rangle$ is the free plane wave and $\psi_k^{\pm}(x) := \langle x|k^{\pm}\rangle$ the perturbed plane wave in the coordinate representation. In consequence, (7) can be written as (Henceforth we shall consider the sign plus in (7) only, for simplicity. Similar results would be obtained with the other choice.)

$$\psi_k^{+}(x) = e^{ikx} + \sum_{j=1}^{N} \lambda_j \, G_0(x, a_j; E_k + i0) \, \psi_k^{+}(a_j) \, , \tag{8}$$

The goal is now to obtain the explicit form of $\psi^{+}(x)$, for which we have to find the explicit form of the terms under the sum in (8). First, let as choose as values of x in (8) the $\{a_j\}$. We obtain the following linear system of N equations for N indeterminates:

$$\begin{aligned} e^{ika_i} &= \psi^{+}(a_i) \, [1 - \lambda_i \, G_0(a_i, a_i; E_k + i0)] \\ &\quad - \sum_{j \neq i}^{N} \lambda_j \, G_0(a_i, a_j; E_k + i0) \, \psi^{+}(a_j) \, , \qquad i = 1, 2, \ldots, N \, . \end{aligned} \tag{9}$$

This system can be rewritten in matrix form. If $\Phi \equiv \{\Phi_{ij}\}$ is the $N \times N$ matrix with matrix elements

$$\Phi_{ij}(E_k + i0) = \begin{cases} 1 - \lambda_i \, G_0(a_i, a_i; E_k + i0) \text{ if } i = j \, , \\ \lambda_j \, G_0(a_i, a_j; E_k + i0) \quad \text{if } i \neq j \, . \end{cases} \tag{10}$$

Then, Eqs. (9) take the form,

$$\sum_{j=1}^{N} \Phi_{ij}(E_k + i0) \, \psi_k^{+}(a_j) = e^{ika_j} \, , \qquad j = 1, 2, \ldots, N \, , \tag{11}$$

with solution,

$$\psi_k^+(a_j) = \sum_{j=1}^{N} \left[\Phi^{-1}(E_k + i0)\right]_{ij} e^{ika_j}\,, \tag{12}$$

where Φ^{-1} is the inverse of the matrix Φ. In consequence, the final form of (8) is

$$\psi_k^+(x) = e^{ikx} + \sum_{j=1}^{N} \lambda_j\, G_0\left(x, a_j; E_k + i0\right) \left[\Phi^{-1}(E_k + i0)\right]_{ij} e^{ika_j}\,. \tag{13}$$

Then, we have to find the Green function $G_0(x, a_j; E_k + i0)$. We do not intend to describe the procedure here, which is explained in detail in [9]. Once we have obtained this Green function, using (10), we finally get all matrix elements of Φ. The final results are

$$G_0\left(x, a_j; E_k + i0\right) = \frac{im}{\hbar^2 k}\, e^{k|x-a_i|} \tag{14}$$

and

$$\Phi_{ij}(E_k + i0) = \begin{cases} 1 - \dfrac{im\lambda_i}{\hbar^2 k} & \text{if} \quad i = j\,, \\ -\sqrt{\lambda_i\,\lambda_j}\, \dfrac{im}{\hbar^2 k}\, e^{ik|a_i - a_j|} & \text{if} \quad i \neq j\,. \end{cases} \tag{15}$$

Then, we have determined all the perturbed plane waves $\psi_k^+(x)$. For $\psi_k^-(x)$, we follow a similar procedure. Always recall that $E_k = (\hbar^2 k^2)/2m$.

2.1 Search for Bound States

So far, we have found the scattering states corresponding to the total (or perturbed) Hamiltonian, for which we have used the Lippmann–Schwinger equation as main tool. Next, we search for the possible existence of bound states, where the search could be carried out with similar tools to those used in the precedent discussion.

We proceed as follows: Let us use the simplified notation $|f_i\rangle := \sqrt{\lambda_i}\,|a_i\rangle$, so that the total Hamiltonian (2) may be written as

$$H = \frac{p^2}{2m} - \sum_{i=1}^{N} |f_i\rangle\langle f_i|\,. \tag{16}$$

The corresponding Schrödinger equation reads

$$\left\langle x \left| \frac{p^2}{2m} \right| \psi \right\rangle - \sum_{i=1}^{N} \langle x|f_i\rangle \langle f_i|\psi\rangle = E \langle x|\psi\rangle \,. \tag{17}$$

Bound states correspond to solutions of (17) with negative E and square integrable wave function $\psi(x) \equiv \langle x|\psi\rangle$.

Next, insert the completeness relation $1 = \frac{1}{2\pi\hbar}\int |p\rangle\langle p|\, dp$ in front of $|\psi\rangle$ and $|f_i\rangle$. Define $\widetilde{\psi}(p) := \langle p|\psi\rangle$, which is indeed the Fourier transform of $\langle x|\psi\rangle$, and write $\phi(a_i) := \langle f_i|\psi\rangle = \sqrt{\lambda_i}\,\langle a_i|\psi\rangle = \sqrt{\lambda_i}\,\psi(a_i)$. Recall that $\langle x|p\rangle = e^{\frac{i}{\hbar}px}$. Then, (17) becomes

$$\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{\frac{i}{\hbar}px}\, \widetilde{\psi}(p) \left(\frac{p^2}{2m} - E\right) = \sum_{i=1}^{N} \sqrt{\lambda_i} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, e^{\frac{i}{\hbar}p(x-a_i)}\, \phi(a_i) \,. \tag{18}$$

From (18) and the properties of the Fourier transform, we have that

$$\widetilde{\psi}(p) = \sum_{i=1}^{N} \sqrt{\lambda_i}\, \frac{e^{-\frac{i}{\hbar}pa_i}}{\dfrac{p^2}{2m} - E}\, \phi(a_i) \,. \tag{19}$$

But $\widetilde{\psi}(p)$ is the Fourier transform of the solution $\psi(x)$ of the Schrödinger equation (17). Let us use this idea to conclude that (take $x = a_i$)

$$\psi(a_i) = \sum_{i=1}^{N} \sqrt{\lambda_i} \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar}\, \frac{e^{-\frac{i}{\hbar}pa_i}}{\dfrac{p^2}{2m} - E}\, \phi(a_i) \,. \tag{20}$$

Multiply both sides in (20) by $\sqrt{\lambda_i}$ and recalling that $\phi(a_i) = \sqrt{\lambda_i}\,\psi(a_i)$, we arrive to an equation of the form:

$$\sum_{j=1}^{N} \Phi_{ij}(E)\, \phi(a_j) = 0 \,. \tag{21}$$

Find details in [8]. It is beyond a mere coincidence that the matrix elements $\Phi \equiv \{\Phi_{ij}(E)\}$ are identical to those of (15) with the replacement $k = \sqrt{2m|E|}$, so that [8]

$$\Phi_{ij}(E) = \begin{cases} 1 - \dfrac{m\lambda_i}{\hbar\sqrt{2m|E|}} & \text{if} \quad i = j \,, \\[2ex] -\dfrac{m\sqrt{\lambda_i\lambda_j}}{\hbar\sqrt{2m|E|}}\, \exp\left(-\sqrt{2m|E|}\,|a_i - a_j|/\hbar\right) & \text{if} \quad i \neq j \,. \end{cases} \tag{22}$$

Since Eq. (21) has come directly from (17), it is a necessary condition for the existence of solutions of (17) with the desired properties. This equation has non-

trivial solutions $\{\phi(a_j)\}$ if and only if $\det \Phi(E) = 0$. Therefore, the bound states energies are solutions of the transcendental equation $\det \Phi(E) = 0$.[1]

For a systematic calculation of the bound states, let us consider the following eigenvalue problem:

$$\Phi(E)A(E) = \omega(E)A(E)\,, \tag{23}$$

where $\omega(E)$ are the eigenvalues of the $N \times N$ matrix $\Phi(E)$ and $A(E)$ their corresponding eigenvectors. Equations (21) and (23) coincide if and only if $\omega(E) = 0$ and then, the bound states energies have to be the solutions of the transcendental equation $\omega(E) = 0$ its eigenvectors being those with components equal to $\phi(a_j)$. If we assume no degeneracy, the wave function corresponding to the energy value E_i with eigenvector $A(E) \equiv (\phi(a_1), \ldots, \phi(a_N))$ takes the form (19) with $E = E_i$. In the coordinate representation, the wave function is just its Fourier transform. For further comments, see [6, 8].

2.2 Resonances and Gamow States

The Lippmann–Schwinger equation is also useful for the construction of Gamow states, which are vector states for resonances. In a resonant scattering process [25] produced by a Hamiltonian pair, say $\{H_0, H\}$, where H_0 is a *free* Hamiltonian and $H = H_0 + V$, where V is the interaction, the Gamow vectors, $\psi^{\pm}$, for a resonance with energy E_R and inverse of the mean life given by Γ are two eigenvectors of H with respective eigenvalues $E_R \pm \Gamma/2$, i.e., $H\psi^{\pm} = (E_R \mp \Gamma/2)\,\psi^{\pm}$ [25]. This property shows that the Gamow vector ψ^{+} decays exponentially as $t \longmapsto \infty$ (and ψ^{-} decays exponentially as $t \longmapsto -\infty$, they are time reversal of each other). This situation produces two problems, one from the point of view of physics and the other from the point of view of mathematics.

Although exponential decay for simple quantum unstable systems has been detected for essentially for all values of time, deviations for these exponential law have been detected for very short or very large times [26, 27]. Since these deviations certainly occur under these conditions only, they are very difficult to be detected. For most values of time, exponential decay serves as an excellent approximation. This is why Gamow vectors are useful as good approximations of decaying states.

A self-adjoint operator on Hilbert space, as is the case of the Hamiltonian H, cannot have complex eigenvalues with corresponding eigenvectors in this Hilbert space. Thus, Gamow vectors are well-defined objects on some extensions of Hilbert spaces called rigged Hilbert spaces [25, 28–30].

[1] As a matter of fact, this also follows because $\Phi(E)$ appears in the denominator of the resolvent of the total Hamiltonian H.

Let us briefly sketch the use of (1) to obtain an explicit expression of the Gamow vectors as eigenvectors of H with eigenvalue $E_R \pm \Gamma/2$. Details may be found in [9, 31]. If we multiply Eq. (1) to the right by the bra $\langle\psi|$, we obtain a complex function on the variable k. With adequate choices of the space of bras, this results on meromorphic functions of complex variable defined at least on a half plane [29, 30]. Let us assume that this is the case and omit the bra $\langle\psi|$. Then, if we define k_R as

$$z_R := E_R - \Gamma/2 = \frac{k_R^2\,\hbar^2}{2m}\,, \tag{24}$$

we may consider the analytic extension of (1) to the value of k given by k_R,

$$|k_R^+\rangle = |k_R\rangle - G_0(z_R)\,V\,|k_R^+\rangle\,. \tag{25}$$

It is important to remark that z_R is a pole of the Green function corresponding to the total Hamiltonian H, but not of the free Hamiltonian H_0, just by the characterisation of resonances using the resolvent [32]. Then, $G_0(z_R)$ is well defined and so is $|k_R^+\rangle$, which has the property [9, 31, 33]

$$H|k_R^+\rangle = z_R\,|k_R^+\rangle\,. \tag{26}$$

Thus, $|k_R^+\rangle$ is one of the Gamow vectors with resonance pole z_R (the other can be obtained exactly in the same way, just replacing z_R by its complex conjugate z_R^* and taking the minus sign in (1). This Gamow vector in the coordinate representation is $\psi_R^+(x) := \langle x|k_R^+\rangle$, so that

$$(H\psi_R^+)(x) = \langle x|H|k_R^+\rangle = z_R\,\langle x|k_R^+\rangle = z_R\,\psi_R^+(x)\,. \tag{27}$$

Now, let us go back to the N Dirac deltas interaction and, consequently, take in (25) the form of the potential given by $V = -\sum_{i=1}^N \lambda_i\,|a_i\rangle\langle a_i|$. Multiply the result of this operation to the right by the bra $\langle x|$ and divide k_R into real and imaginary parts, $k_R = k_r - ik_I$. We have that $\langle x|k_R\rangle = e^{ik_R x} = e^{ik_r x}\,e^{-ik_I x}$ and

$$\begin{aligned}\psi_k^+(x) = \langle x|k_R^+\rangle &= \langle x|k_R\rangle + \sum_{i=1}^N \lambda_i\,\langle x|G_0(z_R)|a_i\rangle\langle a_i|k_R^+\rangle \\ &= e^{ik_r x}\,e^{k_I x} + \sum_{i=1}^N \lambda_i\,G_0(x,a_i;z_R)\,\psi_R^+(a_i) = e^{ik_r x}\,e^{k_I x} \\ &+ \sum_{i=1}^N \lambda_i \sum_{j=1}^N \frac{im\sqrt{\lambda_i\,\lambda_j}}{\hbar^2(k_r - ik_I)}\,\left[e^{i(k_r-ik_I)\,|x-a_i|}\,\Phi^{-1}(z_R)\right]_{ij}\,e^{i(k_r-ik_I)a_j}\,.\end{aligned} \tag{28}$$

A similar result can be obtained for the Gamow wave function $\psi^-(x)$. In principle, both Gamow functions will be equally suitable to play the role of wave function for the resonance state. The only technical difference is that one represents the time reversal of the other [30]. Observe that $\psi_k^+(x) \longmapsto \infty$ as $x \longmapsto \infty$. Gamow wave functions cannot be normalised in the usual sense of square integrable normalisation, but in sharp contrast with the plane waves (Dirac kets) which are not normalisable although bounded, Gamow functions show an exponential growing at the spatial infinite. This behaviour has been often called the exponential catastrophe. This is not such a problem with a proper interpretation of the Gamow wave function in terms of generalised functions in a suitable rigged Hilbert space. Still, this exponential behaviour creates some particular problems such as the difficulties arisen in order to fix a proper definition of averages of observables in Gamow states [34, 35].

3 One Dimensional Salpeter Hamiltonian with N Deltas

The one dimensional Salpeter Hamiltonian decorated with N Dirac deltas has the following form ($c = 1$):

$$H := \sqrt{p^2 + m^2} - \sum_{i=1}^{N} \lambda_i \, \delta(x - a_i) \,, \qquad H_0 := \sqrt{p^2 + m^2} \,. \tag{29}$$

Here, H_0 is the free Salpeter Hamiltonian. The definition of a self-adjoint version for H in (29) is not as simple as is in the Schrödinger case, where it is sufficient to impose correct matching conditions at the nodes. This self-adjoint version is usually determined by a proper choice of the resolvent operator of H, which should be obtained from the resolvent operator of H_0 by the Krein formula. However, this procedure leads to divergences in our case, so that a regularisation procedure is in order here [5, 6]. We have chosen heat kernel regularisation for several reasons discussed in [6]. Let us sketch briefly the procedure. First of all, we write the Hamiltonian H as in (29) as

$$H = \sqrt{p^2 + m^2} - \sum_{i=1}^{N} \lambda_i \, |a_i\rangle\langle a_i| \,, \tag{30}$$

exactly as we did for the cases studied in the previous section. The next step is to write an ϵ-regularised version of (30) as

$$H_\epsilon = \sqrt{p^2 + m^2} - \sum_{i=1}^{N} \lambda_i(\epsilon) \, |a_i^\epsilon\rangle\langle a_i^\epsilon| \,, \tag{31}$$

where the new kets $|a_i^\epsilon\rangle$ are defined in such a way that $\langle x|a_i^\epsilon\rangle := K_{\epsilon/2}(x, a_i)$, where the function $K_t(x, y)$ is the so-called heat kernel, which is the fundamental solution of the heat equation of the form:

$$\sqrt{p^2 + m^2}\, K_t(x, y) = -\frac{\partial\, K_t(x, y)}{\partial\, t}\,, \tag{32}$$

and the weights $\lambda(\epsilon)$ are also chosen as functions of the parameter ϵ, such that $\lim_{\epsilon\to 0^+} \lambda_i(\epsilon) \longmapsto \lambda_i, i = 1, 2, \ldots, N$. The interest of this choice for $\langle x|a_i^\epsilon\rangle$ comes after the limiting property $\langle x|a_i^\epsilon\rangle \longrightarrow \langle x|a_i\rangle = \delta(x - a_i)$ as $\epsilon \longmapsto 0^+$.

Now, we go back to the Lippmann–Schwinger equation (1), where in the present case $E_k = \sqrt{p^2 + m^2}$ and V is as in (31). This gives

$$|k^\pm(\epsilon)\rangle = |k\rangle + \sum_{j=1}^{N} \lambda_j(\epsilon)\, R_0(E_k \pm i0)\, |a_j^\epsilon\rangle\langle a_j^\epsilon|k^\pm\rangle\,. \tag{33}$$

Let us choose the plus sign in (33) and use for brevity the following notation: $|f_i^\epsilon\rangle := \sqrt{\lambda_i(\epsilon)}\, |a_i^\epsilon\rangle$. Then, we choose one subindex i and isolate the corresponding term in (33):

$$\begin{aligned} |k^+(\epsilon)\rangle = |k\rangle &+ R_0(E_k + i0)\, |f_i^\epsilon\rangle\langle f_i^\epsilon|k^+(\epsilon)\rangle \\ &+ \sum_{j\neq i}^{N} R_0(E_k + i0)\, |f_j^\epsilon\rangle\langle f_j^\epsilon|k^+(\epsilon)\rangle\,, \end{aligned} \tag{34}$$

before multiplying (34) to the left by the ket $\langle f_i^\epsilon|$. This gives

$$\begin{aligned} &\left[1 - \left\langle f_i^\epsilon\right| R_0(E_k + i0)\, \left|f_i^\epsilon\right\rangle\right] \left\langle f_i^\epsilon|k^+(\epsilon)\right\rangle \\ &- \sum_{i\neq j}^{N} \left[\left\langle f_i^\epsilon\right| R_0(E_k + i0)\, \left|f_i^\epsilon\right\rangle\right] \left\langle f_i^\epsilon|k^+(\epsilon)\right\rangle = \left\langle f_i^\epsilon|k\right\rangle\,, \end{aligned} \tag{35}$$

expression valid for $i = 1, 2, \ldots, N$. This may be written in the matrix form as

$$\sum_{j=1}^{N} T_{ij}(\epsilon, E_k + i0)\, \langle f_j^\epsilon|k^+(\epsilon)\rangle = \langle f_i^\epsilon|k\rangle\,, \qquad j = 1, 2, \ldots, N\,, \tag{36}$$

with

$$T_{ij}(\epsilon, E_k + i0) = \begin{cases} 1 - \langle f_i^\epsilon|\, R_0(E_k + i0)\, |f_i^\epsilon\rangle & \text{if} \quad i = j\,, \\ -\langle f_i^\epsilon|\, R_0(E_k + i0)\, |f_j^\epsilon\rangle & \text{if} \quad i \neq j\,. \end{cases} \tag{37}$$

Therefore, we may write the solution of (36) as

$$\left\langle f_i^{\epsilon}|k^{+}(\epsilon)\right\rangle = \sum_{j=1}^{N}\left[T^{-1}\left(\epsilon, E_k + i0\right)\right]_{ij}\left\langle f_j^{\epsilon}|k\right\rangle . \tag{38}$$

We use (38) in (35) and, then, multiply the result to the left by the bra $\langle x|$. This gives

$$\begin{aligned}\psi^{+}(\epsilon, x) &:= \langle x|k^{+}(\epsilon)\rangle \\ &= \langle x|k\rangle + \sum_{i,j=1}^{N}\langle x|\, R_0(E_k + i0)\,|f_i^{\epsilon}\rangle\,[T^{-1}(\epsilon, E_k + i0)]_{ij}\,\langle f_j^{\epsilon}|k\rangle \\ &= e^{ikx} + \sum_{i,j=1}^{N}\langle x|\, R_0(E_k + i0)\,|a_i^{\epsilon}\rangle\,[\Phi^{-1}(\epsilon, E_k + i0)]_{ij}\,\langle a_j^{\epsilon}|k\rangle\,, \end{aligned} \tag{39}$$

with

$$\Phi_{ij}(\epsilon, E_k + i0) = \begin{cases} \frac{1}{\lambda_i(\epsilon)} - \langle a_i^{\epsilon}|\, R_0(E_k + i0)\,|a_i^{\epsilon}\rangle & \text{if} \quad i = j\,, \\ -\langle a_i^{\epsilon}|\, R_0(E_k + i0)\,|a_j^{\epsilon}\rangle & \text{if} \quad i \neq j\,. \end{cases} \tag{40}$$

The next step is to take the limit $\epsilon \longmapsto 0$, for which we need a determination of the functions $\lambda_i(\epsilon)$ for all values of $i = 1, 2, \ldots, N$. This has been motivated and determined in Section II in [6] and is

$$\frac{1}{\lambda_i(\epsilon)} = \frac{1}{\lambda_i(M_i)} + \int_0^{\infty} dt\; K_{t+\epsilon}(a_i, a_i)\, e^{tM_i}\,, \tag{41}$$

where $K_t(x, y)$ is the heat kernel and M_i is an unphysical renormalisation scale that is chosen to be the energy of the bound state E_B^i corresponding to the bound state of the i-th delta [6]. This gives in the limit $\epsilon \longmapsto 0$,

$$\psi_k^{+}(x) = e^{ikx} + \sum_{i,j=1}^{N}\langle x|\, R_0(E_k + i0)\,|a_i\rangle\,[\Phi^{-1}(E_k + i0)]_{ij}\, e^{ika_j}\,. \tag{42}$$

Here,

$$\langle x|\, R_0(E_k + i0)\,|a_i\rangle = \frac{i\sqrt{k^2 + m^2}}{k}\, e^{ik|x - a_i|} + \frac{1}{\pi}\int_m^{\infty} d\mu\, e^{-\mu|x - a_i|}\,\frac{\sqrt{\mu^2 - m^2}}{\mu^2 + k^2}$$

and

$$\Phi_{ij}(E_k + i0) =$$
$$= \begin{cases} -\frac{1}{\lambda(E_\lambda, E_B^i)} - \frac{iE_k}{\sqrt{E_k^2 - m^2}} & \text{if } i = j\,, \\ -\frac{iE_k}{\sqrt{E_k^2-m^2}}\, e^{i\sqrt{E_k^2-m^2}\,|x-a_j|} - \frac{1}{\pi}\int_m^\infty d\mu\, e^{-\mu|x-a_i|}\, \frac{\sqrt{\mu^2-m^2}}{\mu^2+E_k^2-m^2} & \text{if } i \neq j\,, \end{cases}$$

where

$$\frac{1}{\lambda\left(E_\lambda, E_B^i\right)}$$
$$= \frac{-E_k}{\pi\sqrt{E_k^2-m^2}}\,\text{arctanh}\left(\frac{\sqrt{E_k^2-m^2}}{E_k}\right) - \frac{E_B^i}{\pi\sqrt{m^2-\left(E_B^i\right)^2}}\left(\frac{\pi}{2} + \arcsin\frac{E_B^i}{m}\right),$$

where E_B^i has been defined before and $\mu := \min_i E_B^i$. The conclusion is that the Lippmann–Schwinger equation gives in a rather straightforward manner the exact form of the scattering states in a rather cumbersome situation as the one discussed along the present section. Explicit expressions for transmission and reflection coefficients can be also derived from the above expressions.

4 Concluding Remarks

The Lippmann–Schwinger equation is a useful tool that permits to obtain explicit forms for the scattering states produced by some potential. When this potential is a finite set of Dirac delta interactions, one may find explicit expressions for these scattering states. We have shown that this is the case when perturbing the free Schrödinger one dimensional and the Salpeter Hamiltonians with N attractive deltas. In the first case, we have also shown that the Lippmann–Schwinger equation gives explicit expressions for Gamow wave functions which are the wave function for the purely exponential decay part of resonance states. The discussion on the search for bound states for the Schrödinger case includes similar methods.

The one dimensional Salpeter Hamiltonian with N attractive deltas is much more complicated as it requires of a regularisation procedure that we implement with the use of the heat kernel for the pseudo-differential operator $\sqrt{-d^2/dx^2 + m^2}$. In this case, we also obtain the exact form of the scattering states.

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