Jacobi Polynomials as $su(2,2)$ Unitary Irreducible Representation

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Abstract An infinite-dimensional irreducible representation of $su(2,2)$ is explicitly constructed in terms of ladder operators for the Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ and the Wigner d_j-matrices where the integer and half-integer spins $j := n + (\alpha + \beta)/2$ are considered together. The 15 generators of this irreducible representation are realized in terms of zero or first order differential operators and the algebraic and analytical structure of operators of physical interest discussed.

Keywords Jacobi polynomials · Lie algebras · Irreducible representations · Wigner matrices · Operators on special functions

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1 Introduction

The classification of the functions that can be defined “special,” where “special” means something more than “useful,” is an open problem [1].

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The actual main line of work for a possible unified theory of special functions is the Askey scheme that is based on the analytical theory of linear differential equations [2–4].

A possible scheme, different from the Askey one, seems to emerge in these last years by means of a generalization of the classical special functions, principally related to the introduction of d-orthogonal polynomials by means of difference equations, q-polynomials, and exceptional polynomials [5–15].

We follow here a point of view closely related to a field of mathematics seemingly quite far from special functions: Lie algebras. It is an idea first introduced by Wigner [16] and Talman [17] and later developed mainly by Miller [18] and Vilenkin and Klimyk [19–21].

However, our approach starts from well-established concepts, the "old style" orthogonal polynomials and looks for possible connections with the "old style" Lie group theory. Thus in this paper, as Jacobi polynomials have three parameters we simply attempt to relate them with a Lie algebra of rank three.

While other researches are focused on the general relations between special functions and Lie algebras we consider a further step connecting special functions and irreducible representations (IR) of Lie algebras. This restriction of the Lie counterpart that has quite more properties of the abstract algebra gives a lot of additional information on the special functions [22, 23].

Starting from the seminal work by Truesdell [24], where a sub-class of special functions was defined by means of a set of formal properties, we propose indeed a possible definition of a fundamental sub-class of special functions that we call "algebraic special functions" (ASF).

These ASF are related to the hypergeometric functions but they are constructed from the following algebraic assumptions:

1. A set of differential recurrence relations exists on these ASF that can be associated with a set of operators that span a Lie algebra.
2. These ASF support a characteristic IR of this algebra.
3. A vector space can be constructed on these ASF where the ladder operators have all the appropriate properties for realizing this IR of the associated Lie algebra.
4. The differential equations that define the ASF are related to the diagonal elements of the universal enveloping algebra (UEA) and, in particular, to the Casimir invariants of the whole algebra and subalgebras.

From these assumptions, we have that:

1. The exponential maps of the algebra define the associated group and allow to obtain from the ASF other different sets of functions. If the transformation is unitary, another algebraically equivalent basis of the space is thus obtained. When the transformations are not unitary, as in the case of coherent states, sets with different properties are found (like overcomplete sets).
2. The vector space of the operators acting on the L^2-space of functions is isomorphic to the UEA built on the algebra.

The starting point of our work has been the paradigmatic example of Hermite functions that are a basis on the Hilbert space of the square integrable functions defined on the configuration space $\mathbb{R}$. As it is well known from the algebraic discussion of the harmonic oscillator, besides the continuous basis $\{|x\rangle\}_{x\in\mathbb{R}}$ determined by the configuration space, a discrete basis $\{|n\rangle\}_{n\in\mathbb{N}}$—related to the Weyl–Heisenberg algebra $h(1)$—can be introduced such that Hermite functions are the transition matrix elements from one basis to the other.

In previous papers we have presented the direct connection between some special functions and specific IRs of Lie algebras in cases where the Lie structure was smaller [25–28].

In this paper we discuss in detail the symmetries of the Jacobi functions introduced in [29]. The fact that a $su(2, 2)$ symmetry exists inside the hypergeometric functions ${}_2F_1$ [30, 31] is, of course, the starting point of our discussion.

This is a further confirmation of the line introduced in [25–27] in terms of the Jacobi polynomials that satisfy the required conditions 1–4 and thus deserves an additional analysis to that presented in [29]. As shown there, Jacobi polynomials indeed can be associated with well-defined "algebraic Jacobi functions" (AJF) that satisfy the preceding assumptions.

The AJF support an IR of $su(2, 2)$ (a real form of A_3) a Lie algebra of rank 3 related to the three parameters, $\{n, \alpha, \beta\}$, of the Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ and, alternatively, to the three parameters $\{j, m, q\}$ of the AJF. These two triplets of parameters are indeed belonging to the Cartan subalgebra of $su(2, 2)$.

The procedure consists in starting from well-known orthogonality conditions of the Jacobi polynomials and defines the orthonormal AJF. The recurrence relations of the Jacobi polynomials are then rewritten by means of differential operators acting on the AJF as ladder operators, whose explicit action remembers the operators $J_\pm$ of the $su(2)$ representation. In this way we obtain twelve non-diagonal operators that together with three Cartan (diagonal) operators close the Lie algebra $su(2, 2)$ in a well-defined IR of AJF. All this analysis can also be transferred to the d_j-Wigner matrices [32].

From the Lie algebra point of view for both, AJF and Wigner d_j–matrices, the relevant algebraic chains are $su(2, 2) \supset su(2) \otimes su(2) \supset su(2)$ to consider together integer and half-integer spin j and $su(2, 2) \supset su(1, 1)$ to describe separately bosons and fermions.

The paper is organized as follows. Section 2 is devoted to recall the main properties of the AJF relevant for our discussion and their relations with the Wigner d_j-matrices. In Sect. 3 we study the symmetries of the AJF that keep invariant the principal parameter j changing only m and/or q. We thus construct the ladder operators that determine a $su(2) \oplus su(2)$ algebra and allow to build up the irreducible representations defined by the same Casimir invariant of both $su(2)$, i.e., $su_j(2) \otimes su_j(2)$. In Sect. 4 we construct four new sets of ladder operators that change the three parameters j, m, and q adding to all of them $\pm 1/2$. Each of these sets generates a $su(1, 1)$ algebra to which ∞-many IRs of $su(1, 1)$—supported by the AJF and the d_j-matrices—are associated. In Sect. 5 we show that the ladder operators, obtained in the previous sections, span all together a $su(2,2)$ algebra and that both AJF and Wigner d_j-matrices are a basis of the IR of $su(2,2)$ (that is characterized by the eigenvalue $-3/2$ of the quadratic Casimir of $su(2,2)$). Finally some conclusions and comments are included.

2 Algebraic Jacobi Functions and Their Structure

The Jacobi polynomial of degree $n \in \mathbb{N}$, $J_n^{(\alpha,\beta)}(x)$, is defined in terms of the hypergeometric functions ${}_2F_1$ [33–35] by

$$J_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} \;\; {}_2F_1\left[-n, 1+\alpha+\beta+n; \alpha+1; \frac{1-x}{2}\right], \qquad (1)$$

where $(a)_n := a\,(a+1)\cdots(a+n-1)$ is the Pochhammer symbol.

Now we include an x-depending factor related to the integration measure of the Jacobi polynomials and we define—alternatively to $\{n,\alpha,\beta\}$—three other parameters $\{j,m,q\}$:

$$j := n + \frac{\alpha+\beta}{2}\,, \qquad m := \frac{\alpha+\beta}{2}, \qquad q := \frac{\alpha-\beta}{2}\,,$$

such that

$$n = j - m, \qquad \alpha = m + q, \qquad \beta = m - q\;.$$

In order to obtain an algebra representation, as we will prove later, we have to impose the following restrictions for $\{j,m,q\}$:

$$j \geq |m|, \qquad j \geq |q|, \qquad 2j \in \mathbb{N}, \quad j-m \in \mathbb{N}, \quad j-q \in \mathbb{N}\,, \qquad (2)$$

thus $\{j,m,q\}$ are all together integers or half-integers. The conditions (2) rewritten in terms of the original parameters $\{n,\alpha,\beta\}$ exhibit that they are all integers satisfying

$$n \in \mathbb{N}, \qquad \alpha,\beta \in \mathbb{Z}, \qquad \alpha \geq -n, \qquad \beta \geq -n, \qquad \alpha+\beta \geq -n.$$

We thus define

$$\hat{\mathcal{J}}_j^{m,q}(x) := \sqrt{\frac{\Gamma(j+m+1)\,\Gamma(j-m+1)}{\Gamma(j+q+1)\,\Gamma(j-q+1)}}$$

$$\times \left(\frac{1-x}{2}\right)^{\frac{m+q}{2}} \left(\frac{1+x}{2}\right)^{\frac{m-q}{2}} \; J_{j-m}^{(m+q,m-q)}(x)\,. \qquad (3)$$

Note that usually the Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ are defined for $\alpha > -1$ and $\beta > -1$ ($\alpha,\ \beta \in \mathbb{R}$) in such a way that a unique weight function $w(x)$ allows their normalization. However (see also [36, p. 49]) we have to change such restrictions since the normalization inside the functions and their algebraic properties requires Eq. (2). So, in addition to integer or half-integer conditions, we have to restrict to $j \geq |m|$ in Eq. (3) ($\hat{\mathcal{J}}_j^{m,q}(x) = 0$ when $|q| > j \in \mathbb{N}/2$). This can be obtained assuming

$$\mathcal{J}_j^{m,q}(x) := \lim_{\varepsilon\to 0} \hat{\mathcal{J}}_{j+\varepsilon}^{m,q}(x)$$

indeed

$$\mathcal{J}_j^{m,q}(x) = \begin{cases} \hat{\mathcal{J}}_j^{m,q}(x)\ \forall\{j,m,q\} & \text{verifying all conditions (2)} \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

In conclusion, the basic objects of this paper that we call "algebraic Jacobi functions" (AJF) have the final form (4).

The AJF (4) reveal additional symmetries hidden inside the Jacobi polynomials. Indeed we have

$$\begin{aligned}
\mathcal{J}_j^{m,q}(x) &= \mathcal{J}_j^{q,m}(x), \\
\mathcal{J}_j^{m,q}(x) &= (-1)^{j-m}\,\mathcal{J}_j^{m,-q}(-x), \\
\mathcal{J}_j^{m,q}(x) &= (-1)^{j-q}\,\mathcal{J}_j^{-m,q}(-x), \\
\mathcal{J}_j^{m,q}(x) &= (-1)^{m+q}\,\mathcal{J}_j^{-m,-q}(x)\,.
\end{aligned} \tag{5}$$

The proof of these properties is straightforward. The first one can be proved taking into account the following property of the Jacobi polynomials for integer coefficients (n, α, β) [36]:

$$J_n^{\alpha,\beta}(x) = \frac{(n+\alpha)!\,(n+\beta)!}{n!\,(n+\alpha+\beta)!}\left(\frac{x+1}{2}\right)^{-\beta} J_{n+\beta}^{\alpha,-\beta}(x)\,,$$

while the second relation can be derived from the well-known symmetry of the Jacobi polynomials [33]

$$J_n^{(\alpha,\beta)}(x) = (-1)^n\,J_n^{(\beta,\alpha)}(-x), \tag{6}$$

and the last two properties can be proved using the first two ones.

The AJF for m and q fixed verify the orthonormality relation

$$\int_{-1}^{1} \mathcal{J}_j^{m,q}(x)\,(j+1/2)\,\mathcal{J}_{j'}^{m,q}(x)\,dx = \delta_{j\,j'} \tag{7}$$

as well as the completeness relation

$$\sum_{j=\sup(|m|,|q|)}^{\infty} \mathcal{J}_j^{m,q}(x)\;\; (j+1/2)\;\; \mathcal{J}_j^{m,q}(y) = \delta(x-y). \tag{8}$$

Both relations are similar to those of the Legendre polynomials [25] and the associated Legendre polynomials [26]: all are orthonormal up to the factor $j+1/2$. These relations allow us to state that $\{\mathcal{J}_j^{m,q}(x);\; m, q \text{ fixed}\}_{j=\sup(|m|,|q|)}^{\infty}$ is a basis in the space of square integrable functions defined in $\mathbb{E} = [-1, 1]$. Considering

$$\mathbb{E}\times\mathbb{Z}\times\mathbb{Z}/2 := \bigcup_{m-q\in\mathbb{Z}}\; \bigcup_{q\in\mathbb{Z}/2} \mathbb{E}_{m,q}\,,$$

where $\mathbb{E}_{m,q}$ is the configuration space $\mathbb{E} = [-1, 1]$ with m and q fixed and $\mathbb{Z}\times\mathbb{Z}/2$ is related to the set of pairs (m, q) with m and q both integer or half-integer, then $\{\mathcal{J}_j^{m,q}(x)\}$ is a basis of $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$ [29].

The Jacobi equation

$$E_n^{(\alpha,\beta)}\; J_n^{(\alpha,\beta)}(x) = 0\,,$$

where

$$E_n^{(\alpha,\beta)} \equiv (1-x^2)\frac{d^2}{dx^2} - ((\alpha+\beta+2)x + (\alpha-\beta))\,\frac{d}{dx} + n(n+\alpha+\beta+1)\,,$$

rewritten in terms of these new functions $\mathcal{J}_j^{m,q}(x)$ and of the new parameters $\{j, m, q\}$ becomes

$$\mathcal{E}_j^{m,q}\; \mathcal{J}_j^{m,q}(x) = 0\,, \tag{9}$$

with

$$\mathcal{E}_j^{m,q} \equiv -\left(1-x^2\right)\,\frac{d^2}{dx^2} + 2\,x\,\frac{d}{dx} + \frac{2\;m\;q\;x + m^2 + q^2}{1-x^2} - j(j+1)\,, \tag{10}$$

where the symmetry under the interchange between m and q is evident.

It is worth noticing that the AJFs (4), with the substitution $x = \cos\beta$ with $0 \le \beta \le \pi$, are essentially the Wigner d_j rotation matrices [32, 36]

$$d^j(\beta)^m_q = \sqrt{\frac{(j+m)!(j-m)!}{(j+q)!(j-q)!}}\left(\sin\frac{\beta}{2}\right)^{q-m}\left(\cos\frac{\beta}{2}\right)^{m+q} J_{j-m}^{(m-q,m+q)}(\cos\beta)$$

that verify the conditions (2). The explicit relation between them is

$$d^j(\beta)^m_q = \mathcal{J}_j^{m,-q}(\cos\beta). \tag{11}$$

Equation (5) are equivalent to the well-known relations among the $d^j(\beta)^m_q$, for instance,

$$d^j(\beta)^q_m = (-1)^{q-m}\, d^j(\beta)^m_q .$$

The starting point for finding the algebra representation of the AJF is now the construction of the rising/lowering differential applications [18] that change the labels $\{j, m, q\}$ of the AJF by 0 or 1/2. The fundamental limitation of the analytical approach [16–21] is that the indices are considered as parameters that, in iterated applications, must be introduced by hand. This problem has been solved in [25] where a consistent vector space framework was introduced to allow the iterated use of recurrence formulas by means of operators of which the parameters involved are eigenvalues.

Indeed—in order to realize the needed operator structure on the set $\{\mathcal{J}^{m,q}_j(x)\}$—we introduce not only the operators X and D_x of the configuration space :

$$X\, f(x) = x\, f(x), \qquad D_x\, f(x) = f'(x),$$

but also three other operators J, M, and Q such that

$$(J,\ M,\ Q)\ :\ \mathcal{J}^{m,q}_j(x) \to (j,\ m,\ q)\, \mathcal{J}^{m,q}_j(x), \tag{12}$$

that are diagonal on the AJF and, thus, belong—in the algebraic scheme—to the Cartan subalgebra.

3 Algebra Representations for $\varDelta j = 0$

We start from the differential–difference applications verified by the Jacobi functions (a complete list of which can be found in Refs. [33–35]). The procedure is laborious, so that, we only sketch the simplest case with $\varDelta j = 0$, related to $su(2)$ and well known for the d_j in terms of the angle [37].

Let us start from the operators that change the values of m only. The relations [33]

$$\begin{aligned} \tfrac{d}{dx} J^{(\alpha,\beta)}_n(x) &= \tfrac{1}{2}(n+\alpha+\beta+1)\, J^{(\alpha+1,\beta+1)}_{n-1}(x)\ , \\ \tfrac{d}{dx}\Big[(1-x)^\alpha(1+x)^\beta J^{(\alpha,\beta)}_n(x)\Big] &= -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1} J^{(\alpha-1,\beta-1)}_{n+1}(x) \end{aligned}$$

allow us to define the operators

$$A_\pm := \pm\sqrt{1-X^2}\, D_x + \frac{1}{\sqrt{1-X^2}}\,(XM+Q), \tag{13}$$

that act on the algebraic Jacobi functions $\mathcal{J}_j^{m,q}(x)$ as

$$A_\pm\ \mathcal{J}_j^{m,q}(x) = \sqrt{(j \mp m)\,(j \pm m + 1)}\ \ \mathcal{J}_j^{m\pm 1,\,q}(x). \tag{14}$$

The operators (13) are a generalization for $Q \neq 0$ of the operators $J_\pm$ introduced in [26] for the associated Legendre functions related to the AJF with $q = 0$. Indeed Eq. (14) that are independent from q coincide with Eqs. (2.11) and (2.12) of Ref. [26].

Defining now $A_3 := M$ and taking into account the action of the operators $A_\pm$ and A_3 on the AJFs, Eqs. (14) and (12), it is easy to check that $A_\pm$ and A_3 close a $su(2)$ algebra that commutes with J and Q, denoted in the following by $su_A(2)$:

$$[A_3, A_\pm] = \pm A_\pm \qquad [A_+, A_-] = 2A_3.$$

Thus, the AJFs $\{\mathcal{J}_j^{m,q}(x)\}$, with j and q fixed such that $2j \in \mathbb{N}$, $j - m \in \mathbb{N}$ and $-j \leq m \leq j$, support the $(2j+1)$-dimensional IR of the Lie algebra $su_A(2)$ independent from the value of q.

Similarly to [26], starting from the differential realization (13) of the $A_\pm$ operators, the Jacobi differential equation (9) is shown to be equivalent to the Casimir equation of $su_A(2)$

$$[\mathcal{C}_A - J(J+1)]\ \ \mathcal{J}_j^{m,q}(x) \equiv \left[A_3^2 + \frac{1}{2}\{A_+, A_-\} - J(J+1)\right] \mathcal{J}_j^{m,q}(x) = 0\ .$$

Indeed, this equation reproduces the operatorial form of (9), i.e., it gives

$$\mathcal{E}_J^{M,Q} \equiv -(1-X^2)D_x^2 + 2XD_x + \frac{1}{1-X^2}(2XMQ + M^2 + Q^2) - J(J+1)\,. \tag{15}$$

On the other hand, we can make use of the factorization method [38–40], relating second order differential equations to product of first order ladder operators in such a way that the application of the first operator modifies the values of the parameters of the second one. Taking into account this fact, iterated application of (13) gives the two equations

$$\begin{aligned} &[A_+\,A_- - (J+M)\,(J-M+1)]\ \ \mathcal{J}_j^{m,q}(x) = 0\ , \\ &[A_-\,A_+ - (J-M)\,(J+M+1)]\ \ \mathcal{J}_j^{m,q}(x) = 0\ , \end{aligned} \tag{16}$$

that reproduce again the operator form of the Jacobi equation (9). These are particular cases of a general property: the defining Jacobi equation can be recovered applying to $\mathcal{J}_j^{m,q}$ the Casimir operator of any involved algebra and subalgebra as well as any diagonal product of ladder operators.

Now, using the symmetry under the interchange of the labels m and q of the AJF (see first relation of (5)), we construct the algebra of operators that changes q

leaving j and m unchanged. From $A_\pm$ two new operators $B_\pm$ are thus defined

$$B_\pm := \pm\sqrt{1-X^2}\, D_x + \frac{1}{\sqrt{1-X^2}}\,(XQ+M)\,, \tag{17}$$

and their action on the AJF is

$$B_\pm\, \mathcal{J}_j^{m,q}(x) = \sqrt{(j\mp q)\,(j\pm q+1)}\;\; \mathcal{J}_l^{m,q\pm 1}(x). \tag{18}$$

Obviously also the operators $B_\pm$ and $B_3 := Q$ close a $su(2)$ algebra we denote $su_B(2)$

$$[B_3, B_\pm] = \pm B_\pm \qquad [B_+, B_-] = 2B_3,$$

and the AJFs $\{\mathcal{J}_j^{m,q}(x)\}$, with j and m fixed such that $2j \in \mathbb{N}$, $j-q \in \mathbb{N}$ and $-j \leq q \leq j$, close the $(2j+1)$-dimensional IR of the Lie algebra $su(2)_B$ independent from the value of m.

Again we can recover the Jacobi equation (9) from the Casimir, $\mathcal{C}_B$, of $su_B(2)$

$$[\mathcal{C}_B - J(J+1)]\, \mathcal{J}_j^{m,q}(x) = \left[B_3^2 + \frac{1}{2}\{B_+, B_-\} - J(J+1)\right] \mathcal{J}_j^{m,q}(x) = 0\,.$$

A more complex algebraic scheme appears in common applications of the operators $A_\pm$ and $B_\pm$. As the operators $\{A_\pm, A_3\}$ commute with $\{B_\pm, B_3\}$, the algebraic structure is the direct sum of the two Lie algebras

$$su_A(2) \oplus su_B(2).$$

A new symmetry of the AJFs emerges in the space of $\mathcal{J}_j^{m,q}(x)$ when only j is fixed. Both for $\{j, m, q\}$, integer or half-integer (see Eqs. (14), (18) and (12)) we have the IR of the algebra $su(2) \oplus su(2)$

$$su_j(2) \oplus su_j(2)\,.$$

So that the AJFs $\{\mathcal{J}_j^{m,q}(x)\}$ for fixed j and $-j \leq m \leq j$, $-j \leq q \leq j$ determine the IR with $\mathcal{C}_A = \mathcal{C}_B = j(j+1)$. From (13) and (17), taking into account that always the operators M and Q have been written at the right of X and D_x, it can be shown that $A_\pm^\dagger = A_\mp$, $B_\pm^\dagger = B_\mp$ and the representation would be unitary with a suitable inner product. In Fig. 1 the action of the operators $A_\pm$, $B_\pm$ on the parameters $\{j, m, q\}$ that label the AJFs corresponds to the plane $\Delta j = 0$.

In conclusion, $\{\mathcal{J}_j^{m,q}(x)\}$ with j fixed is the basis of an IR of $su(2) \oplus su(2)$ of dimension $(2j+1)^2$ symmetrical under the interchange of A with B.

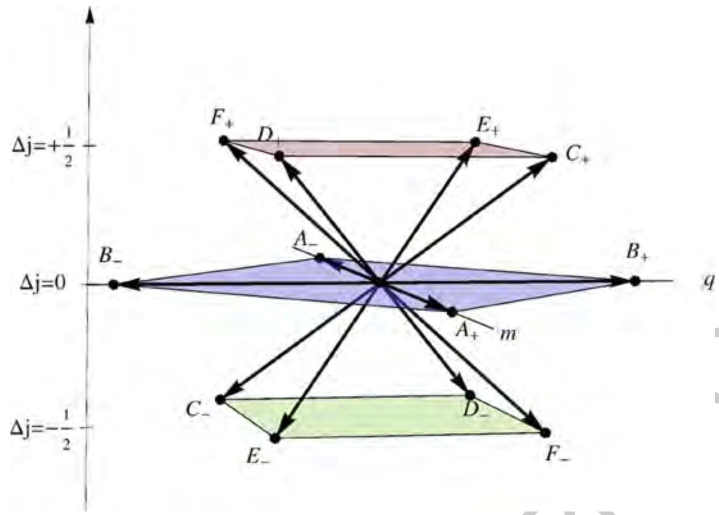

Fig. 1 Root diagram of $su(2,2)$. The coordinates displayed on the planes correspond to the pairs $\{m, q\}$, while the parameter Δj is represented in the vertical axis. The Cartan elements at the origin are not included

4 Other Ladder Operators Acting on AJF and $su(1,1)$ Representations

As we mentioned before there are many differential–difference relations between the Jacobi polynomials for different values of the parameters [33, 34]. Starting from them we construct a $su(2,2)$ representation supported by the AJF. The Lie algebra $su(2,2)$ has fifteen infinitesimal generators, where three of them are Cartan generators (for instance, J, M, and Q). As the four generators that commute with J (i.e., $A_\pm$ and $B_\pm$) have been introduced in the preceding paragraph, we have to construct eight non-diagonal operators more. They are

$$
\begin{aligned}
C_\pm &:= \pm\frac{(1+X)\sqrt{1-X}}{\sqrt{2}}\,D_x - \frac{1}{\sqrt{2(1-X)}}\Big(X\,(J+\tfrac{1}{2}\pm\tfrac{1}{2}) - (J+\tfrac{1}{2}\pm\tfrac{1}{2}+M+Q)\Big)\,,\\
D_\pm &:= \mp\frac{(1-X)\sqrt{1+X}}{\sqrt{2}}\,D_x + \frac{1}{\sqrt{2\,(1+X)}}\;\Big(X(J+\tfrac{1}{2}\pm\tfrac{1}{2}) + (J+\tfrac{1}{2}\pm\tfrac{1}{2}+M-Q)\Big)\,,\\
E_\pm &:= \mp\frac{(1-X)\sqrt{1+X}}{\sqrt{2}}\,D_x + \frac{1}{\sqrt{2\,(1+X)}}\Big(X(J+\tfrac{1}{2}\pm\tfrac{1}{2}) + (J+\tfrac{1}{2}\pm\tfrac{1}{2}-M+Q)\Big)\,,\\
F_\pm &:= \mp\frac{(1+X)\sqrt{1-X}}{\sqrt{2}}\,D_x + \frac{1}{\sqrt{2\,(1-X)}}\Big(X(J+\tfrac{1}{2}\pm\tfrac{1}{2}) - (J+\tfrac{1}{2}\pm\tfrac{1}{2}-M-Q)\Big)\,.
\end{aligned}
\tag{19}
$$

All these differential operators act on the space $\{\mathcal{J}_j^{m,q}\}$ for $\{j, m, q\}$ integer and half-integer such that $j \geq |m|, |q|$. The explicit form of their action is

$$\begin{aligned}
C_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{\left(j+m+\tfrac{1}{2}\pm\tfrac{1}{2}\right)\left(j+q+\tfrac{1}{2}\pm\tfrac{1}{2}\right)}\; \mathcal{J}_{j\pm 1/2}^{m\pm 1/2,\, q\pm 1/2}(x),\\
D_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{\left(j+m+\tfrac{1}{2}\pm\tfrac{1}{2}\right)\left(j-q+\tfrac{1}{2}\pm\tfrac{1}{2}\right)}\; \mathcal{J}_{j\pm 1/2}^{m\pm 1/2,\, q\mp 1/2}(x)\\
E_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{\left(j-m+\tfrac{1}{2}\pm\tfrac{1}{2}\right)\left(j+q+\tfrac{1}{2}\pm\tfrac{1}{2}\right)},\; \mathcal{J}_{j\pm 1/2}^{m\mp 1/2,\, q\pm 1/2}(x),\\
F_\pm\, \mathcal{J}_j^{m,q}(x) &= \sqrt{\left(j-m+\tfrac{1}{2}\pm\tfrac{1}{2}\right)\left(j-q+\tfrac{1}{2}\pm\tfrac{1}{2}\right)}\; \mathcal{J}_{j\pm 1/2}^{m\mp 1/2,\, q\mp 1/2}(x).
\end{aligned} \tag{20}$$

From (19) or (20) we have

$$C_\pm^\dagger = C_\mp, \qquad D_\pm^\dagger = D_\mp, \qquad E_\pm^\dagger = E_\mp, \qquad F_\pm^\dagger = F_\mp,$$

i.e., all these rising/lowering operators could have the hermiticity properties required by the representation to be unitary. The operators (19) change all parameters by $\pm 1/2$, so that in Fig. 1 they correspond to the planes $\Delta j = \pm 1/2$. In [29] also quadratic forms of operators (19) that change the parameters in $(\pm 1, 0)$ instead of $\pm 1/2$ have been considered.

From Eq. (19) it is easily stated that

$$\begin{aligned}
D_\pm(X, D_x, M, Q) &= C_\pm(-X, -D_x, M, -Q),\\
E_\pm(X, D_x, M, Q) &= C_\pm(-X, -D_x, -M, Q),\\
F_\pm(X, D_x, M, Q) &= -C_\pm(X, D_x, -M, -Q).
\end{aligned} \tag{21}$$

Thus, because of the Weyl symmetry of the roots, we limit ourselves to discuss the operators $C_\pm$. Taking thus into account their action on the Jacobi functions we get

$$[C_+, C_-] = -2C_3, \qquad [C_3, C_\pm] = \pm C_\pm \tag{22}$$

where

$$C_3 := J + \frac{1}{2}(M+Q) + \frac{1}{2}. \tag{23}$$

Hence $\{C_\pm, C_3\}$ close a $su(1,1)$ algebra we can denote $su_C(1,1)$.

As in the cases of the operators $A_\pm$ and $B_\pm$, we obtain the Jacobi differential equation from the Casimir $\mathcal{C}_C$ of $su_C(1,1)$, written in terms of (19) and (23),

$$\mathcal{C}_C\, \mathcal{J}_j^{m,q}(x) \equiv \left[C_3^2 - \frac{1}{2}\{C_+, C_-\}\right] \mathcal{J}_j^{m,q}(x) = \frac{1}{4}\left[(m+q)^2 - 1\right] \mathcal{J}_j^{m,q}(x).$$

Indeed

$$
\begin{aligned}
&\left[\mathcal{C}_C - \frac{1}{4}(M+Q)^2 + \frac{1}{4}\right] \mathcal{J}_j^{m,q}(x) \\
&\qquad \equiv \left[C_3^2 - \tfrac{1}{2}\{C_+, C_-\} - \tfrac{1}{4}\,(M+Q)^2 + 1/4\right] \mathcal{J}_j^{m,q}(x) = 0
\end{aligned} \tag{24}
$$

allows us to recover the Jacobi equation (9). Analogously the same result derives from eqs.

$$
\begin{aligned}
&[C_+\,C_- - (J+M)\,(J+Q)]\ \mathcal{J}_j^{m,q}(x) = 0, \\
&[C_-\,C_+ - (J+1+M)\,(J+1+Q)]\ \mathcal{J}_j^{m,q}(x) = 0,
\end{aligned} \tag{25}
$$

obtained by the factorization method.

From (24) we see that since $(m+q) = 0, \pm 1, \pm 2, \pm 3, \cdots$ the unitary IRs of $su(1,1)$ with $\mathcal{C}_C = (m+q)^2/4 - 1/4 = -1/4, 0, 3/4, 2, 15/4, \cdots$ are obtained. Hence, the set of AJF supports infinite unitary IRs of the discrete series of $su_C(1,1)$ [41].

Similar results can be found for the other ladder operators $D\pm$, $E\pm$, $F\pm$, up to an eventual multiplicative factor, with the substitutions (21) in all Eqs. (22)–(25).

5 The AJF Representation of $su(2,2)$

To obtain the root system of the simple Lie algebra A_3 (that has $su(2,2)$ as one of its real forms) we have only simply to add to Fig. 1 the three points in the origin corresponding to the elements J, M, and Q of the Cartan subalgebra.

The commutators of the generators $A_\pm$, $B_\pm$, $C_\pm$, $D_\pm$, $E_\pm$, $F_\pm$, J, M, Q are

$$
\begin{array}{llll}
[J, A_\pm] = 0, & [J, M] = 0, & [J, B_\pm] = 0, & [J, Q] = 0, \\[2ex]
[J, C_\pm] = \pm\frac{C_\pm}{2}, & [J, D_\pm] = \pm\frac{D_\pm}{2}, & [J, E_\pm] = \pm\frac{E_\pm}{2}, & [J, F_\pm] = \pm\frac{F_\pm}{2}, \\[2ex]
[M, B_\pm] = 0, & [M, Q] = 0, & & \\[2ex]
[M, C_\pm] = \pm\frac{C_\pm}{2}, & [M, D_\pm] = \pm\frac{D_\pm}{2}, & [M, E_\pm] = \mp\frac{E_\pm}{2}, & [M, F_\pm] = \mp\frac{F_\pm}{2}, \\[2ex]
[Q, A_\pm] = 0, & & & \\[2ex]
[Q, C_\pm] = \pm\frac{C_\pm}{2}, & [Q, D_\pm] = \mp\frac{D_\pm}{2}, & [Q, E_\pm] = \pm\frac{E_\pm}{2}, & [Q, F_\pm] = \mp\frac{F_\pm}{2}, \\[2ex]
[A_+, A_-] = 2A_3, & [A_3, A_\pm] = \pm A_\pm, & (A_3 = M), &
\end{array}
$$

$$
\begin{array}{llll}
[B_+, B_-] = 2B_3, & [B_3, B_\pm] = \pm B_\pm, & (B_3 = Q), & \\
[C_+, C_-] = -2C_3, & [C_3, C_\pm] = \pm C_\pm, & (C_3 = J + \frac{1}{2}(M+Q) + \frac{1}{2}), & \\
[D_+, D_-] = -2D_3, & [D_3, D_\pm] = \pm D_\pm, & (D_3 = J + \frac{1}{2}(M-Q) + \frac{1}{2}), & \\
[E_+, E_-] = -2E_3, & [E_3, E_\pm] = \pm E_\pm, & (E_3 = J + \frac{1}{2}(-M+Q) + \frac{1}{2}), & \\
[F_+, F_-] = -2F_3, & [F_3, F_\pm] = \pm F_\pm, & (F_3 = J - \frac{1}{2}(M+Q) + \frac{1}{2}), & \\
[A_\pm, B_\pm] = 0, & [A_\pm, B_\mp] = 0, & & \\
[A_\pm, C_\pm] = 0, & [A_\pm, C_\mp] = \pm E_\mp, & [A_\pm, D_\pm] = 0, & [A_\pm, D_\mp] = \mp F_\mp, \\
[A_\pm, E_\pm] = \pm C_\pm, & [A_\pm, E_\mp] = 0, & [A_\pm, F_\pm] = D_\pm, & [A_\pm, F_\mp] = 0, \\
[B_\pm, C_\pm] = 0, & [B_\pm, C_\mp] = \mp D_\mp, & [B_\pm, D_\pm] = \pm C_\pm, & [B_\pm, D_\mp] = 0, \\
[B_\pm, E_\pm] = 0, & [B_\pm, E_\mp] = \mp F_\mp, & [B_\pm, F_\pm] = \pm E_\pm, & [B_\pm, F_\mp] = 0, \\
[C_\pm, D_\pm] = 0, & [C_\pm, D_\mp] = \mp B_\pm, & [C_\pm, E_\pm] = 0, & [C_\pm, E_\mp] = \mp A_\pm, \\
[C_\pm, F_\pm] = 0, & [C_\pm, F_\mp] = 0, & & \\
[D_\pm, E_\pm] = 0, & [D_\pm, E_\mp] = 0, & [D_\pm, F_\pm] = 0, & [D_\pm, F_\mp] = \mp A_\pm, \\
[E_\pm, F_\pm] = 0, & [E_\pm, F_\mp] = \mp B_\pm. & &
\end{array}
$$

The quadratic Casimir of $su(2,2)$ has the form

$$
\begin{aligned}
\mathcal{C}_{su(2,2)} &= \tfrac{1}{2}\left(\{A_+, A_-\} + \{B_+,\ B_-\} - \{C_+, C_-\} - \{D_+, D_-\} - \{E_+, E_-\}\right. \\
&\qquad \left. -\{F_+, F_-\}\right) + \tfrac{1}{2}\left(A_3^2 + B_3^2 + C_3^2 + D_3^2 + E_3^2 + F_3^2\right) \\
&= \tfrac{1}{2}\left(\{A_+, A_-\} + \{B_+,\ B_-\} - \{C_+, C_-\} - \{D_+, D_-\} - \{E_+, E_-\}\right. \\
&\qquad \left. -\{F_+, F_-\}\right) + 2J(J+1) + M^2 + Q^2 + \tfrac{1}{2}\,,
\end{aligned}
$$

that, applied on the $\{\mathcal{J}_j^{m,q}(x)\}$, gives

$$
\mathcal{C}_{su(2,2)}\ \mathcal{J}_j^{m,q}(x) = -\frac{3}{2}\,\mathcal{J}_j^{m,q}(x)\,. \tag{26}
$$

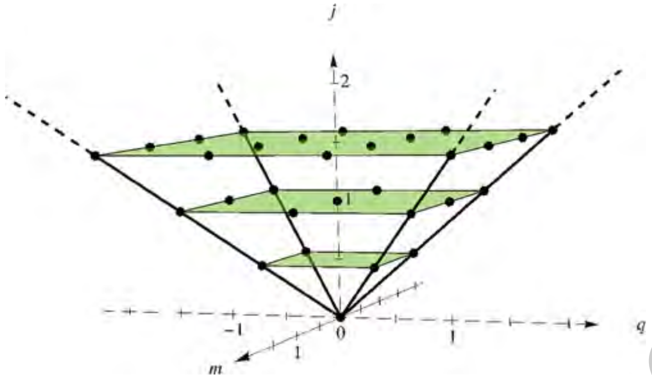

Fig. 2 IR of $su(2,2)$ supported by the AJF $\mathcal{J}_l^{m,q}(x)$ represented by the black points. The horizontal planes correspond to IR of $su_A(2) \oplus su_B(2)$

The relation (26) shows that the infinite-dimensional IR of $su(2,2)$ generated by $\{\mathcal{J}_j^{m,q}(x)\}$ contains all $j = 0, 1/2, 1, \ldots,$. From it and taking into account the differential realization of the operators involved, (12), (13), (17), and (19), we recover again the Jacobi equation (9) that, as in the previous sections, can be obtained also from the Casimir of any subalgebra of $su(2,2)$ as well as from any diagonal product of ladder operators.

In this IR of $su(2,2)$ the integer and half-integer values of $\{j, m, q\}$ are put all together (see Fig. 2). The symmetries of the AJF, where integer and half-integer values of $\{j, m, q\}$ belong to different IRs, have been considered in [29].

6 Resume and Conclusions

The Jacobi polynomials and the d_j-matrices look to be more general examples of the properties described in [25–29] for special functions. This suggests that the following properties could be assumed for a possible classification of the ASF, a relevant subset of generic special functions:

1. ASF are a basis of $L^2(\mathbb{F})$, the space of integrable functions defined on an appropriate space $\mathbb{F}$.
2. ASF are a basis of an IR of a Lie algebra $\mathcal{G}$.
3. All the diagonal elements of the UEA[$\mathcal{G}$] can be written in terms of the fundamental second order differential equation determined by the quadratic Casimir of $\mathcal{G}$.
4. All the non-diagonal elements of the UEA[$\mathcal{G}$] can be written as first order differential operators.
5. Every basis of $L^2(\mathbb{F})$ can be obtained applying an element of the Lie group G to the ASF.

6. Every operator acting on $L^2(\mathbb{F})$ belongs to UEA[$\mathcal{G}$].

Returning now to the particular case of the AJF the previous remarks become:

1. AJF are a basis of an IR of the Lie algebra $su(2, 2)$.
2. All the diagonal elements of the UEA[$su(2, 2)$] can be obtained from Eq. (9).
3. All the non-diagonal elements of the UEA[$su(2, 2)$] can be written as first order differential operators.
4. The set of AJF $\{\mathcal{J}_j^{m,q}(x)\}$ is a basis in $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$, where $\mathbb{E} = [-1, 1]$.
5. Every basis of $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$ can be obtained under the action of $SU(2, 2)$ on the set of AJF, i.e., it can be written as $\{g\, \mathcal{J}_j^{m,q}(x)\}$ where $g \in SU(2, 2)$.
6. Every operator acting on $L^2(\mathbb{E}, \mathbb{Z}, \mathbb{Z}/2)$ belongs to the UEA[$su(2, 2)$].

As a final point we recall the connection between the IR of $SU(2)$,

$$D_j(\alpha, \beta, \gamma)_m^{m'} = e^{-i\alpha m'}\, d_j(\beta)_m^{m'}\, e^{-i\gamma m},$$

where α, β, γ are the Euler angles [37], the Wigner d_j-matrices, and the Jacobi polynomials $P_{j-m'}^{m'-m,m'+m}$. This implies that all the results of this paper can be extended to $\{D_j(\alpha, \beta, \gamma)_m^{m'}\}$ that have similar properties of the $\{\mathcal{J}_j^{m,q}(x)\}$ and are a basis of the square integrable functions defined in the space $\{\alpha, \beta, \gamma\}$.

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