Trends in Supersymmetric Quantum Mechanics

David J. Fernández C

Abstract Along the years, supersymmetric quantum mechanics (SUSY QM) has been used for studying solvable quantum potentials. It is the simplest method to build Hamiltonians with prescribed spectra in the spectral design. The key is to pair two Hamiltonians through a finite order differential operator. Some related subjects can be simply analyzed, as the algebras ruling both Hamiltonians and the associated coherent states. The technique has been applied also to periodic potentials, where the spectra consist of allowed and forbidden energy bands. In addition, a link with non-linear second-order differential equations, and the possibility of generating some solutions, can be explored. Recent applications concern the study of Dirac electrons in graphene placed either in electric or magnetic fields, and the analysis of optical systems whose relevant equations are the same as those of SUSY QM. These issues will be reviewed briefly in this paper, trying to identify the most important subjects explored currently in the literature.

Keywords Supersymmetric quantum mechanics · Coherent states · Painlevé equations · Painlevé transcendents · Polynomial Heisenberg algebras · Factorization method · Exact solutions · Spectral design · Graphene

1 Introduction

The birth of supersymmetric quantum mechanics (SUSY QM) in 1981, as a toy model to illustrate the properties that systems involving both bosons and fermions have, was a breakthrough in the study of solvable quantum mechanical models [1]. One of the reasons is that SUSY QM is tightly related to other approaches used

Dedicated to my dear friend and colleague Véronique Hussin.

D. J. Fernández C (✉)
Departamento de Física, Cinvestav, Ciudad de México, Mexico
e-mail: david@fis.cinvestav.mx

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in the past to address this kind of systems, e.g., the factorization method, Darboux transformation, and intertwining technique [2–27].

On the other hand, it is well known that the factorization method was introduced by Dirac in 1935, to derive algebraically the spectrum of the harmonic oscillator [28]. The next important advance was done by Schrödinger in 1940, who realized that the procedure can be also applied to the Coulomb potential [29, 30]. Later on, Infeld and his collaborators push forward the technique [31, 32], supplying a general classification scheme including most of the exactly solvable Schrödinger Hamiltonians known up to that time [2]. As a consequence, the idea that the factorization method was essentially exhausted started to spread among the scientific community.

However, in 1984 Mielnik proved that this belief was wrong, by generalizing simply the Infeld–Hull factorization method when he was seeking the most general first-order differential operators which factorize the harmonic oscillator Hamiltonian in a certain given order [33]. The key point of his approach was that if the ordering of the generalized factorization operators is interchanged, then a new Hamiltonian is obtained which is intertwined with the oscillator one.

It is worth to stress that Mielnik's work represented the next breakthrough in the development of the factorization method, since it opened the way to look for new solvable quantum potentials. In particular, this generalization was immediately applied to the Coulomb problem [34]. Meanwhile, Andrianov's group [35, 36] and Nieto [37] identified the links of the factorization method with Darboux transformation and supersymmetric quantum mechanics, respectively. In addition, Sukumar indicated the way to apply Mielnik's approach to arbitrary potentials and factorization energies [38, 39], setting up the general framework where the factorization method would develop for the next decade [33, 34, 40–64].

Let us mention that up to the year 1993 the factorization operators, which at the same time are intertwining operators in this case, were first-order differential ones. A natural generalization, pursued by Andrianov and collaborators [65, 66], consists in taking the intertwining operators of order greater than one. This proposal was important, since it helped to circumvent the restriction of the first-order method, that only the energy of the initial ground state can be modified. Moreover, it made clear that the key of the generalization is the analysis of the intertwining relation rather than the factorized expressions. Let us note also that in 1995 Bagrov and Samsonov explored the same technique in a different but complementary way [67].

Our group got back to the subject in 1997 [68–72], although some works related with the method had been done previously [73]. In particular, several physically interesting potentials were addressed through this technique, as the standard harmonic oscillator [33, 69, 70], the radial oscillator, and Coulomb potentials [34, 73, 74], among others [75–77]. In addition, the coherent states associated to the SUSY partners of the harmonic oscillator were explored [78–81], and similar works dealing with more general one-dimensional Hamiltonians were done [82, 83]. Another important contribution has to do with the determination of the general systems ruled by polynomial Heisenberg algebras and the study of particular realizations based on the SUSY partners of the oscillator [80, 84–87]. The complex SUSY transformations involving either real or complex factorization

"energies" were implemented as well [88–92]. In addition, the analysis of the confluent algorithm, the degenerate case in which all the factorization energies tend to a single one, was also elaborated [74, 93–101]. The SUSY techniques for exactly solvable periodic potentials, as the Lamé and associated Lamé potentials, have been explored as well[102–109].

Some other groups have addressed the same subjects through different viewpoints, e.g., the N-fold supersymmetry by Tanaka and collaborators [110–114], the hidden non-linear supersymmetry by Plyushchay et al. [115–119], among others.

Especially important is the connection of SUSY QM with non-linear second-order ordinary differential equations, as KdV and Painlevé IV and V equations, as well as the possibility of designing algorithms to generate some of their solutions [81, 84–86, 90, 92, 120–135].

Another relevant subject related to SUSY QM is the so-called exceptional orthogonal polynomials (EOP) [136–150]. In fact, it seems that most of these new polynomials appear quite naturally when the seed solutions which are employed reduce to polynomial solutions of the initial stationary Schrödinger equation [144].

Recently, the SUSY methods started to be used also in the study of Dirac electrons in graphene and some of its allotropes, when external electric or magnetic fields are applied [151–161]. It is worth to mention as well some systems in optics, since there is a well-known correspondence between Schrödinger equation and Maxwell equations in the paraxial approximation, which makes that the SUSY methods can be applied directly in some areas of optics [162–169].

As we can see, the number of physical systems which are related with supersymmetric quantum mechanics is large enough to justify the writing of a new review paper, in which we will present the recent advances in the subject. If the reader is looking for books and previous review papers addressing SUSY QM from an inductive viewpoint, we recommend Refs. [5–27].

2 Supersymmetric Quantum Mechanics

In this section we shall present axiomatically the supersymmetric quantum mechanics, as a tool for generating solvable potentials $\widetilde{V}(x)$ departing from a given initial one $V(x)$.

The supersymmetry algebra with two generators introduced by Witten in 1981 [1]

$$[Q_i, H_{\rm ss}] = 0, \quad \{Q_i, Q_j\} = \delta_{ij} H_{\rm ss}, \quad i, j = 1, 2, \tag{1}$$

when realized in the following way:

$$Q_1 = \frac{Q^+ + Q}{\sqrt{2}}, \qquad Q_2 = \frac{Q^+ - Q}{i\sqrt{2}}, \tag{2}$$

$$Q = \begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & B^+ \\ 0 & 0 \end{pmatrix}, \tag{3}$$

$$H_{\rm ss} = \{Q, Q^+\} = \begin{pmatrix} B^+B & 0 \\ 0 & BB^+ \end{pmatrix} \tag{4}$$

is called supersymmetric quantum mechanics, where $H_{\rm ss}$ is the supersymmetric Hamiltonian, while Q_1, Q_2 are the supercharges. The kth order differential operators B, B^+ intertwine two Schrödinger Hamiltonians

$$\widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \widetilde{V}(x), \qquad H = -\frac{1}{2}\frac{d^2}{dx^2} + V(x), \tag{5}$$

in the way

$$\widetilde{H}B^+ = B^+H, \qquad HB = B\widetilde{H}. \tag{6}$$

There is a natural link with the *factorization method*, since the following relations are fulfilled:

$$B^+B = \prod_{j=1}^{k}(\widetilde{H} - \epsilon_j), \qquad BB^+ = \prod_{j=1}^{k}(H - \epsilon_j), \tag{7}$$

where ϵ_j, $j = 1, \ldots, k$ are k *factorization energies* associated to k *seed solutions* required to implement the intertwining (see Eqs. (5) and (6) and Sects. 2.1 and 2.2). Taking into account these expressions, it turns out that the supersymmetric Hamiltonian $H_{\rm ss}$ is a polynomial of degree kth in the diagonal matrix operator $H_{\rm p}$ which involves the two Schrödinger Hamiltonians H and $\widetilde{H}$ as follows:

$$H_{\rm ss} = \prod_{j=1}^{k}(H_{\rm p} - \epsilon_j), \qquad H_{\rm p} = \begin{pmatrix} \widetilde{H} & 0 \\ 0 & H \end{pmatrix}. \tag{8}$$

In particular, if $k = 1$ the standard (first-order) supersymmetric quantum mechanics is recovered, for which $H_{\rm ss}$ is a first degree polynomial in $H_{\rm p}$, $H_{\rm ss} = H_{\rm p} - \epsilon_1$. For $k > 1$, however, we will arrive to the so-called higher-order supersymmetric quantum mechanics, in which $H_{\rm ss}$ is a polynomial of degree greater than one in $H_{\rm p}$ (see, for example, [23]).

2.1 Standard SUSY Transformations

Let us suppose now that we select k solutions u_j of the initial stationary Schrödinger equation for k *different* factorization energies ϵ_j, $j = 1, \ldots, k$,

$$Hu_j = \epsilon_j u_j, \tag{9}$$

which are called *seed solutions*. From them we implement the intertwining transformation of Eq. (6), leading to a new potential $\widetilde{V}(x)$ which is expressed in terms of the initial potential and the seed solutions as follows:

$$\widetilde{V}(x) = V(x) - [\log W(u_1, \ldots, u_k)]'', \tag{10}$$

where $W(u_1, \ldots, u_k)$ denotes the Wronskian of u_j, $j = 1, \ldots, k$. The eigenfunctions $\widetilde{\psi}_n$ and eigenvalues E_n of $\widetilde{H}$ are obtained from the corresponding ones of H, ψ_n, and E_n, as follows:

$$\widetilde{\psi}_n = \frac{B^+\psi_n}{\sqrt{(E_n - \epsilon_1)\cdots(E_n - \epsilon_k)}} \propto \frac{W(u_1, \ldots, u_k, \psi_n)}{W(u_1, \ldots, u_k)}. \tag{11}$$

Moreover, $\widetilde{H}$ could have additional eigenfunctions $\widetilde{\psi}_{\epsilon_j}$ for some of the factorization energies ϵ_j (at most k, depending on either they fulfill or not the required boundary conditions) which are given by:

$$\widetilde{\psi}_{\epsilon_j} \propto \frac{W(u_1, \ldots, u_{j-1}, u_{j+1}, \ldots, u_k)}{W(u_1, \ldots, u_k)}. \tag{12}$$

We can conclude that, given the initial potential $V(x)$, its eigenfunctions ψ_n, eigenvalues E_n, and the k chosen seed solutions u_j, $j = 1, \ldots, k$, it is possible to generate algorithmically its kth order SUSY partner potential $\widetilde{V}(x)$ as well as the associated eigenfunctions and eigenvalues through expressions (10)–(12).

It is important to stress that the seed solutions must be carefully chosen in order that the new potential will not have singularities additional to those of the initial potential $V(x)$. When this happens, we say that the transformation is *non-singular*. If the initial potential is real, and we require the same for the final potential, then there are some criteria for choosing the real seed solutions u_j according to their number of nodes, which also depend on the values taken by the associated factorization energies ϵ_j (see, for example, [23]). Although non-exhaustive, let us report next a list of some important criteria, which will make the final potential $\widetilde{V}(x)$ to be real and without any extra singularity with respect to $V(x)$.

– If $k = 1$ (first-order SUSY QM), the factorization energy ϵ_1 must belong to the infinite energy gap $\epsilon_1 < E_0$ in order that u_1 could be nodeless inside the x-domain of the problem, where E_0 is the ground state energy of H. Moreover, since in this ϵ_1-domain the seed solution u_1 could have either one node or none,

then we additionally require to identify the right nodeless solution. With these conditions, the transformation will be non-singular and the spectrum of the new Hamiltonian $\widetilde{H}$ will have an extra level ϵ_1 with respect to H (creation of a new level). Note that also it is possible to select the seed solution with a node at one of the edges of the x-domain; thus, the SUSY transformation will be still non-singular but the factorization energy ϵ_1 will not belong to the spectrum of $\widetilde{H}$ (isospectral transformation).

- If $k = 1$, $\epsilon_1 = E_0$, and $u_1 = \psi_0$ (the seed solution is the ground state, which has one node at each edge of the x-domain), then the SUSY transformation will be non-singular and the spectrum of the new Hamiltonian will not have the level E_0 (deletion of one level).
- If $k = 2$ (standard second-order SUSY QM), first of all both ϵ_1 and ϵ_2 must belong to the same energy gap, either to the infinite one below E_0 or to a finite gap defined by two neighbor energy levels (E_m, E_{m+1}). Let us order the two factorization energies in the way $\epsilon_2 < \epsilon_1$. In order that the Wronskian of u_1 and u_2 would be nodeless, the seed solution u_2 associated to the lower factorization energy ϵ_2 should have one extra node with respect to the solution u_1 associated to the higher factorization energy ϵ_1 [23]. In particular, in the infinite gap u_2 should have one node and u_1 should be nodeless. On the other hand, when both factorization energies are in the finite gap (E_m, E_{m+1}) the seed solutions u_2 and u_1 should have $m + 2$ and $m + 1$ nodes, respectively. In both cases the spectrum of the new Hamiltonian will contain two extra eigenvalues ϵ_1, ϵ_2 (creation of two levels). Moreover, the seed solutions can be chosen such that the transformation is still non-singular but either ϵ_1, ϵ_2 or both will not belong to the spectrum of $\widetilde{H}$ (either creation of one new level or isospectral transformation).
- If $k = 2$, $\epsilon_2 = E_m$, $u_2 = \psi_m$, $\epsilon_1 = E_{m+1}$, $u_1 = \psi_{m+1}$, then the SUSY transformation will be non-singular and the spectrum of the new Hamiltonian will not have the two levels E_m, E_{m+1} (deletion of two levels).
- If $k > 2$, the corresponding non-singular SUSY transformation can be expressed as the product of a certain number of first and second-order SUSY transformations, each one having to be consistent with any of the previous criteria to be non-singular.

2.2 Confluent SUSY Transformations

An important degenerate case of the SUSY transformation for $k \geq 2$ appears when all the factorization energies ϵ_j, $j = 1, \ldots, k$ tend to a fixed single value ϵ_1 [74, 93–101, 170–172]. Let us note that the expression for the new potential of Eq. (10) is still valid, but the seed solutions have to be changed if non-trivial modifications in the new potential are going to appear. In fact, the seed solutions u_j, $j = 1, \ldots, k$ instead of being just normal eigenfunctions of H should generate a Jordan chain of

generalized eigenfunctions for H and ϵ_1 as follows:

$$(H-\epsilon_1)u_1 = 0, \tag{13}$$

$$(H-\epsilon_1)u_2 = u_1, \tag{14}$$

$$\vdots$$

$$(H-\epsilon_1)u_k = u_{k-1}. \tag{15}$$

First let us assume that the seed solution u_1 satisfying Eq. (13) is given, then we need to find the general solution for u_j, $j = 2, \ldots, k$ (precisely in that order!) in terms of u_1. There are two methods essentially different to determine such a general solution: the first one is known as integral method, in which through the technique of variation of parameters one simplifies each inhomogeneous equation in the chain and when integrating the resulting equation every solution u_j is found. In fact, by applying this procedure the solution to the inhomogeneous equations

$$(H-\epsilon_1)u_j = u_{j-1}, \quad j = 2, \ldots, k, \tag{16}$$

is given by

$$u_j(x) = -2\,u_1(x)\,v_j(x), \tag{17}$$

$$v_j(x) = v_j(x_0) + \int_{x_0}^{x} \frac{w_j(y)}{u_1^2(y)}dy, \tag{18}$$

$$w_j(x) = w_j(x_0) + \int_{x_0}^{x} u_1(z)\,u_{j-1}(z)dz, \tag{19}$$

where x_0 is a point in the initial domain of the problem. Thus, Eq. (19) with $j = 2$ determines w_2, by inserting then this result in Eq. (18) with $j = 2$ we find v_2 which in turn fixes u_2 through Eq. (17) [74]. By using then this expression for u_2 it is found w_3 through Eq. (19) and then v_3 and u_3 by means of Eqs. (18) and (17), respectively [95]. We continue this process to find at the end the expression for u_k, and then we insert all the u_j, $j = 1, \ldots, k$ in Eq. (10) in order to obtain the new potential [170].

An alternative is the so-called differential method, in which one identifies in a clever way (through parametric differentiation with respect to the factorization energy ϵ_1) one particular solution for each inhomogeneous equation of the chain [96, 100]. It is straightforward then to find the general solution for each u_j, $j = 2, \ldots, k$. Instead of supplying the resulting formulas for arbitrary $k > 1$, let us derive the results just for the simplest case with $k = 2$.

2.2.1 Confluent Second-Order SUSY QM

For $k = 2$ we just need to solve the following system of equations:

$$(H - \epsilon_1)u_1 = 0, \tag{20}$$

$$(H - \epsilon_1)u_2 = u_1. \tag{21}$$

The result for the integral method in this case is achieved by making $k = 2$ in Eqs. (17)–(19), which leads to [74]:

$$u_2(x) = -2\,u_1(x)\,v_2(x), \tag{22}$$

$$v_2(x) = v_2(x_0) + \int_{x_0}^{x} \frac{w_2(y)}{u_1^2(y)}dy, \tag{23}$$

$$w_2(x) = w_2(x_0) + \int_{x_0}^{x} u_1^2(y)dy. \tag{24}$$

Thus we obtain:

$$W(u_1, u_2) = -2\,w_2(x). \tag{25}$$

Up to a constant factor, this is the well-known formula generated for the first time in [94], which will induce non-trivial modifications in the new potential $\widetilde{V}(x)$ (see Eq. (10)).

Let us solve now the system of Eqs. (20)–(21) through the differential method [96]. If we derive Eq. (20) with respect to ϵ_1, assuming that the Hamiltonian H does not depend explicitly on ϵ_1, we obtain a particular solution of the inhomogeneous Eq. (21), namely

$$(H - \epsilon_1)\frac{\partial u_1}{\partial \epsilon_1} = u_1. \tag{26}$$

Thus, the general solution for u_2 we were looking for becomes:

$$u_2(x) = c_2\,u_1 + d_2\,u_1 \int_{x_0}^{x} \frac{dy}{u_1^2(y)} + \frac{\partial u_1}{\partial \epsilon_1}. \tag{27}$$

Hence:

$$W(u_1, u_2) = d_2 + W\left(u_1, \frac{\partial u_1}{\partial \epsilon_1}\right). \tag{28}$$

Let us note that both methods have advantages and disadvantages, as compared with each other. For instance, in the integral method often it is hard to find explicit analytic solutions for the involved integrals, then in such cases we can try to use

the differential method. However, for numerical calculation of the new potential it is simple and straightforward to use the integral formulas. On the other hand, there are not many potentials for which we can calculate in a simple way the corresponding derivative with respect to the factorization energy. At the end both methods turn out to be complementary to each other. A final remark has to be done: the family of new potentials generated through both algorithms (the integral and differential one) is the same, but if we want to generate a specific member of the family through both methods we need to be sure that we are using the same pair of seed solutions u_1, u_2. In practice, given u_1, u_2, with u_2 generated, for example, through the integral method (which means that we have fixed the constants $v_2(x_0)$ and $w_2(x_0)$ of Eqs. (23), (24)) we have to look for the appropriate coefficients c_2 and d_2 of Eq. (27) in order to guarantee that the same seed solution u_2 is going to be used for the differential algorithm (see the discussion in [99]).

As in the non-confluent SUSY approach, once again we have to choose carefully the seed solution u_1 in order that the new potential will not have extra singularities with respect to $V(x)$. In the case of the second-order confluent algorithm, the way of selecting such a seed solution is the following [94]:

- In the first place u_1 must vanish at one of the two edges of the x-domain. If this happens, then there will be some domain of the parameter $w_2(x_0)$ for which the key function w_2 of Eq. (24) will not have any node.
- The above requirement can be satisfied, in principle, by seed solutions u_1 associated to any real factorization energy; thus, we can create an energy level at any place on the energy axis.
- In particular, any eigenfunction of H satisfies the conditions to produce non-singular confluent second-order SUSY transformations, and the corresponding energy eigenvalue can be also kept in the spectrum of the new Hamiltonian (isospectral transformations).
- When an eigenfunction of H is used, a zero for w_2 could appear at one of the edges of the x domain. In such a case, the SUSY transformation stays non-singular, but the corresponding eigenvalue will disappear from the spectrum of $\widetilde{H}$ (deletion of one level).

3 SUSY QM and Exactly Solvable Potentials

The methods discussed previously can be used to generate, from an exactly solvable potential, plenty of new exactly solvable Hamiltonians with spectra quite similar to the initial one. In this section we will employ the harmonic oscillator to illustrate the technique. Although in this case the spectrum consists of an infinite number of non-degenerate discrete energy levels, the method works as well for Hamiltonians with mixed spectrum (discrete and continuous) or even when there is just a continuous one (see, e.g., [173]). This is what happens for periodic potentials [102–109], where the spectrum consists of allowed energy bands separated by forbidden gaps.

Moreover, the technique has been applied also to a very special system whose spectrum is the full real line, with each level being doubly degenerate: the so-called repulsive oscillator [128].

3.1 Harmonic Oscillator

The harmonic oscillator potential is given by:

$$V(x) = \frac{x^2}{2}. \tag{29}$$

In order to apply the SUSY methods, it is required to find the general solution $u(x)$ of the stationary Schrödinger equation for an arbitrary factorization energy ϵ:

$$-\frac{1}{2}u''(x) + \frac{x^2}{2}u(x) = \epsilon\, u(x). \tag{30}$$

Up to a constant factor, the general solution to this equation is a linear combination (characterized by the parameter ν) of an even and odd linearly independent solutions, given by [80]:

$$\begin{aligned} u(x) &= e^{-\frac{x^2}{2}}\left[{}_1F_1\left(\frac{1-2\epsilon}{4}, \frac{1}{2}; x^2\right) + 2\nu\frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\, x\, {}_1F_1\left(\frac{3-2\epsilon}{4}, \frac{3}{2}; x^2\right)\right] \\ &= e^{\frac{x^2}{2}}\left[{}_1F_1\left(\frac{1+2\epsilon}{4}, \frac{1}{2}; -x^2\right) + 2\nu\frac{\Gamma(\frac{3-2\epsilon}{4})}{\Gamma(\frac{1-2\epsilon}{4})}\, x\, {}_1F_1\left(\frac{3+2\epsilon}{4}, \frac{3}{2}; -x^2\right)\right]. \end{aligned} \tag{31}$$

In order to produce non-singular SUSY transformations we need to know the number of nodes that u has, according to the position of the parameter ϵ on the energy axis. Let us note first of all that, if ϵ is any real number, u will have an even number of nodes for $|\nu| < 1$, while this number will be odd for $|\nu| > 1$. This implies that, when ϵ is in the infinite energy gap $\epsilon < E_0$, this solution will have one node for $|\nu| > 1$ and it will be nodeless for $|\nu| < 1$. On the other hand, if $E_m < \epsilon < E_{m+1}$ with m even, then u will have $m+2$ nodes for $|\nu| < 1$ and it will have $m+1$ nodes for $|\nu| > 1$, while for odd m it will have $m+2$ and $m+1$ nodes for $|\nu| > 1$ and $|\nu| < 1$, respectively.

Now, although the SUSY methods can supply an infinity of new exactly solvable potentials, their expressions become in general too long to be explicitly reported. The simplest formulas appear when the factorization energies become either some of the eigenvalues $E_n = n + \frac{1}{2}, n = 0, 1, \ldots$ of H or some other special values, defined by the sequence $\mathcal{E}_m = -(m + \frac{1}{2}), m = 0, 1, \ldots$ In both cases it is possible to reduce the Schrödinger solution u to the product of one exponential factor $e^{\pm x^2/2}$

times a Hermite polynomial, either of a real variable when one of the E_n is taken or of an imaginary one when any of the $\mathcal{E}_m$ is chosen [80]. We supply next some explicit expressions for exactly solvable potentials, generated through the SUSY methods for such special values of the factorization energies. Let us note that we have sticked strictly to the criteria pointed out at Sect. 1 for producing non-singular SUSY transformations on the full real line. It is just for the first-order transformation that we have employed one general solution to show explicitly the simplest family of exactly solvable potential generated through SUSY QM.

3.1.1 First-Order SUSY Partners of the Oscillator

For $k = 1, \epsilon_1 = -\frac{1}{2}, |\nu_1| < 1$ it is obtained (see also [33]):

$$\widetilde{V}(x) = \frac{x^2}{2} - \left(\frac{2\,\nu_1\, e^{-x^2}}{\sqrt{\pi}\,[1 + \nu_1\, \mathrm{erf}(x)]} \right)' - 1, \tag{32}$$

where $\mathrm{erf}(x)$ is the error function.

For $k = 1, \epsilon_1 = -\frac{5}{2}, \nu_1 = 0$ we get:

$$\widetilde{V}(x) = \frac{x^2}{2} - \left(\frac{4x}{2x^2 + 1} \right)' - 1. \tag{33}$$

For $k = 1, \epsilon_1 = -\frac{9}{2}, \nu_1 = 0$ it is obtained:

$$\widetilde{V}(x) = \frac{x^2}{2} - \left[\frac{8x(2x^2 + 3)}{4x^4 + 12x^2 + 3} \right]' - 1. \tag{34}$$

Let us note that in all these three cases the spectrum of the new Hamiltonian $\widetilde{H}$, besides having the eigenvalues of H, will contain also a new energy level at ϵ_1.

3.1.2 Second-Order SUSY Partners of the Oscillator

For $k = 2, \epsilon_1 = -\frac{5}{2}, \nu_1 = 0, \epsilon_2 = -\frac{7}{2}, \nu_2 \to \infty$ it is obtained:

$$\widetilde{V}(x) = \frac{x^2}{2} - \left(\frac{16x^3}{4x^4 + 3} \right)' - 2. \tag{35}$$

For $k = 2, \epsilon_1 = -\frac{9}{2}, \nu_1 = 0, \epsilon_2 = -\frac{11}{2}, \nu_2 \to \infty$ we get:

$$\widetilde{V}(x) = \frac{x^2}{2} - \left[\frac{32x^3(4x^4 + 12x^2 + 15)}{16x^8 + 64x^6 + 120x^4 + 45} \right]' - 2. \tag{36}$$

For $k=2$, $\epsilon_1=-\frac{5}{2}$, $\nu_1=0$, $\epsilon_2=-\frac{11}{2}$, $\nu_2\to\infty$ it is obtained:

$$\widetilde{V}(x)=\frac{x^2}{2}-\left[\frac{4x(12x^4+20x^2+5)}{8x^6+20x^4+10x^2+5}\right]'-2. \tag{37}$$

Once again, in all these cases the spectrum of the new Hamiltonian $\widetilde{H}$ will have two new levels at ϵ_1, ϵ_2, besides the eigenvalues E_n of H.

On the other hand, when deleting two neighbor energy levels of H in order to create $\widetilde{H}$ we could obtain again some of the potentials reported above, up to an energy shift to align the corresponding energy levels (see, e.g., [174]). For instance, if we delete the first and second excited states of H we recover the potential given in Eq. (33), if we delete the second and third excited states we get again the potential in Eq. (35). Let us generate now a new potential by deleting the third and fourth excited states, which leads to:

$$\widetilde{V}(x)=\frac{x^2}{2}-\left[\frac{12x(4x^4-4x^2+3)}{8x^6-12x^4+18x^2+9}\right]'+2. \tag{38}$$

Note that the corresponding Hamiltonian $\widetilde{H}$ will not have the levels $E_3=7/2$, $E_4=9/2$.

In order to present some potentials obtained through the confluent second-order SUSY QM, let us use once again the eigenstates of H. If the ground state is taken to implement the transformation, it is generated the same family of potentials of Eq. (32). However, if the first excited state is employed, the following one-parameter family of potentials isospectral to the oscillator is gotten (see Eqs. (10), (24), (25)):

$$\widetilde{V}(x)=\frac{x^2}{2}-\left[\frac{4x^2}{\sqrt{\pi}(2b_2+1)e^{x^2}+\sqrt{\pi}e^{x^2}\mathrm{erf}(x)-2x}\right]', \tag{39}$$

where $b_2\equiv w_2(-\infty)$. For $b_2>0$ the new Hamiltonian $\widetilde{H}$ is isospectral to H. However, if $b_2=0$ the level E_1 will disappear from the spectrum of $\widetilde{H}$.

Let us note that if a general eigenfunction $\psi_n(x)$ of H is used to perform the confluent second-order transformation, an explicit expression for the key function $w_2(x)$ has been obtained, which will induce non-trivial modifications in the new potential [94].

4 Algebraic Structures of H, $\widetilde{H}$, and Coherent States

In this section we are going to analyze the kind of algebra that the SUSY partner Hamiltonian $\widetilde{H}$ will inherit from the initial one H. We are going to suppose that H has an algebraic structure general enough to include the most important one-dimensional Hamiltonians appearing currently in the literature, as the harmonic oscillator [82].

4.1 Algebraic Structure of H

Let us suppose that the initial Schrödinger Hamiltonian H has an infinite discrete spectrum whose non-degenerate energy levels E_n, $n = 0, 1, \ldots$ are ordered as usual, $E_n < E_{n+1}$. Moreover, there is an explicit functional dependence between the eigenvalues E_n and the index n, i.e., $E_n = E(n)$, where $E(n)$ is well defined on the non-negative integers. For example, for the harmonic oscillator it turns out that $E(n) = n + \frac{1}{2}$. In this section we will use Dirac notation, so that the eigenstates and eigenvalues satisfy:

$$H|\psi_n\rangle = E_n|\psi_n\rangle, \quad n = 0, 1, \ldots \tag{40}$$

The number operator N is now introduced as

$$N|\psi_n\rangle = n|\psi_n\rangle. \tag{41}$$

It can be defined now a pair of ladder operators of the system through

$$a^-|\psi_n\rangle = r(n)|\psi_{n-1}\rangle, \tag{42}$$

$$a^+|\psi_n\rangle = r^*(n+1)|\psi_{n+1}\rangle, \tag{43}$$

$$r(n) = e^{i\tau(E_n - E_{n-1})}\sqrt{E_n - E_0}, \quad \tau \in \mathbb{R}, \tag{44}$$

where $r^*(n)$ denotes the complex conjugate of $r(n)$. Thus, the *intrinsic algebra* of the system is defined by:

$$[N, a^\pm] = \pm a^\pm, \tag{45}$$

$$a^+a^- = E(N) - E_0, \tag{46}$$

$$a^-a^+ = E(N+1) - E_0, \tag{47}$$

$$[a^-, a^+] = E(N+1) - E(N) \equiv f(N), \tag{48}$$

$$[H, a^\pm] = \pm f(N - 1/2 \mp 1/2)a^\pm. \tag{49}$$

Let us note that, depending on the key function $E(n)$ associated to the initial Hamiltonian, the system could be ruled by a Lie algebra, in case that $E(n)$ is either linear or quadratic in n. However, it could be also ruled by non-Lie algebras, when $E(n)$ has a more involved dependence with n.

Once we have characterized the algebra for the initial Hamiltonian, it is possible to analyze the corresponding structure for its SUSY partner Hamiltonians $\widetilde{H}$.

4.2 Algebraic Structure of $\widetilde{H}$

The most important properties of $\widetilde{H}$ come from its connection with the initial Hamiltonian H through the intertwining operators (see Eq. (6)). In fact, from these expressions it is simple to identify the *natural* ladder operators for $\widetilde{H}$ as follows [33, 78, 80, 82]:

$$\widetilde{a}^{\pm} = B^{+} a^{\pm} B. \tag{50}$$

Its action on the eigenstates of $\widetilde{H}$ can be straightforwardly calculated, leading to:

$$\widetilde{a}^{\pm} |\widetilde{\psi}_{\epsilon_j}\rangle = 0, \tag{51}$$

$$\widetilde{a}^{-} |\widetilde{\psi}_n\rangle = \widetilde{r}(n) |\widetilde{\psi}_{n-1}\rangle, \tag{52}$$

$$\widetilde{a}^{+} |\widetilde{\psi}_n\rangle = \widetilde{r}^{*}(n+1)\, |\widetilde{\psi}_{n+1}\rangle, \tag{53}$$

$$\widetilde{r}(n) = \left[\prod_{i=1}^{k} [E(n) - \epsilon_i][E(n-1) - \epsilon_i]\right]^{\frac{1}{2}} r(n). \tag{54}$$

In order to simplify the discussion, from now on we will assume that none of the ϵ_j, $j = 1, \ldots, k$ coincide with some eigenvalue of H, and that k new energy levels are created for $\widetilde{H}$ at ϵ_j, $j = 1, \ldots, k$. It is important as well to define the number operator $\widetilde{N}$ for the system ruled by $\widetilde{H}$, through its action on the corresponding energy eigenstates:

$$\widetilde{N} |\widetilde{\psi}_{\epsilon_j}\rangle = 0, \tag{55}$$

$$\widetilde{N} |\widetilde{\psi}_n\rangle = n |\widetilde{\psi}_n\rangle. \tag{56}$$

The *natural algebra* of the system is now defined by:

$$[\widetilde{N}, \widetilde{a}^{\pm}] = \pm \widetilde{a}^{\pm}, \tag{57}$$

$$[\widetilde{a}^{-}, \widetilde{a}^{+}] = \left[\widetilde{r}^{*}(\widetilde{N}+1)\,\widetilde{r}(\widetilde{N}+1) - \widetilde{r}^{*}(\widetilde{N})\,\widetilde{r}(\widetilde{N})\right] \sum_{n=0}^{\infty} |\widetilde{\psi}_n\rangle \langle \widetilde{\psi}_n|, \tag{58}$$

where $\widetilde{r}(n)$ is given by Eqs. (54), (44).

4.3 Coherent States of H and $\widetilde{H}$

We have just identified the annihilation and creation operators for the SUSY partner Hamiltonians H and $\widetilde{H}$. The coherent states for such systems can be looked for as

eigenstates of the annihilation operator with complex eigenvalues z, namely:

$$a^-|z,\tau\rangle = z|z,\tau\rangle, \tag{59}$$

$$\widetilde{a}^-|\widetilde{z,\tau}\rangle = z|\widetilde{z,\tau}\rangle. \tag{60}$$

If we expand the coherent states in the basis of energy eigenstates, substitute them in Eqs. (59), (60) to obtain a recurrence relation for the coefficients of the expansion, and express such coefficients in terms of the first one and normalize them, we arrive at the following expressions:

$$|z,\tau\rangle = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\tau(E_m-E_0)}\frac{z^m}{\sqrt{\rho_m}}|\psi_m\rangle, \tag{61}$$

$$\rho_m = \begin{cases} 1 & \text{if } m=0 \\ (E_m-E_0)\cdots(E_1-E_0) & \text{if } m>0 \end{cases} \tag{62}$$

and

$$|\widetilde{z,\tau}\rangle = \left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\widetilde{\rho}_m}\right)^{-\frac{1}{2}}\sum_{m=0}^{\infty}e^{-i\tau(E_m-E_0)}\frac{z^m}{\sqrt{\widetilde{\rho}_m}}|\widetilde{\psi}_m\rangle, \tag{63}$$

$$\widetilde{\rho}_m = \begin{cases} 1 & \text{if } m=0 \\ \rho_m\prod_{i=1}^{k}(E_m-\epsilon_i)(E_{m-1}-\epsilon_i)^2\ldots(E_1-\epsilon_i)^2(E_0-\epsilon_i) & \text{if } m>0. \end{cases} \tag{64}$$

It is important to ensure that our coherent states fulfill a completeness relation, in order that an arbitrary state can be decomposed in terms of them. In our case the two completeness relations are:

$$\int |z,\tau\rangle\langle z,\tau|\,d\mu(z) = 1, \tag{65}$$

$$d\mu(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\rho_m}\right)\rho(|z|^2)\,d^2z, \tag{66}$$

and

$$\sum_{i=1}^{k}|\widetilde{\psi}_{\epsilon_i}\rangle\langle\widetilde{\psi}_{\epsilon_i}| + \int |\widetilde{z,\tau}\rangle\langle\widetilde{z,\tau}|\,d\widetilde{\mu}(z) = 1, \tag{67}$$

$$d\widetilde{\mu}(z) = \frac{1}{\pi}\left(\sum_{m=0}^{\infty}\frac{|z|^{2m}}{\widetilde{\rho}_m}\right)\widetilde{\rho}(|z|^2)\,d^2z. \tag{68}$$

They will be fulfilled if we would find two measure functions $\rho(y)$ and $\widetilde{\rho}(y)$ solving the following moment problems [78, 80, 175–177]:

$$\int_0^\infty y^m \rho(y)\, dy = \rho_m, \tag{69}$$

$$\int_0^\infty y^m \widetilde{\rho}(y)\, dy = \widetilde{\rho}_m, \qquad m = 0, 1, \ldots. \tag{70}$$

The fact that two coherent states of a given family in general are not orthogonal is contained in the so-called reproducing kernel, which turns out to be:

$$\langle z_1, \tau | z_2, \tau \rangle = \left(\sum_{m=0}^{\infty} \frac{|z_1|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \left(\sum_{m=0}^{\infty} \frac{|z_2|^{2m}}{\rho_m} \right)^{-\frac{1}{2}} \left(\sum_{m=0}^{\infty} \frac{(\bar{z}_1 z_2)^m}{\rho_m} \right), \tag{71}$$

$$\langle \widetilde{z_1, \tau} | \widetilde{z_2, \tau} \rangle = \left(\sum_{m=0}^{\infty} \frac{|z_1|^{2m}}{\widetilde{\rho}_m} \right)^{-\frac{1}{2}} \left(\sum_{m=0}^{\infty} \frac{|z_2|^{2m}}{\widetilde{\rho}_m} \right)^{-\frac{1}{2}} \left(\sum_{m=0}^{\infty} \frac{(\bar{z}_1 z_2)^m}{\widetilde{\rho}_m} \right). \tag{72}$$

Concerning dynamics, the coherent states evolve as follows:

$$U(t)|z, \tau\rangle = \exp(-itH)|z, \tau\rangle = e^{-itE_0}|z, \tau + t\rangle, \tag{73}$$

$$\widetilde{U}(t)|\widetilde{z, \tau}\rangle = \exp(-it\widetilde{H})|\widetilde{z, \tau}\rangle = e^{-itE_0}|\widetilde{z, \tau + t}\rangle. \tag{74}$$

Let us note that, while the eigenvalue $z = 0$ of a^- is non-degenerate (if $z = 0$ is made in Eq. (61) the ground state of H is achieved), for $\widetilde{a}^-$ this eigenvalue is $(k+1)$th degenerate, since all states $\widetilde{\psi}_{\epsilon_i}$, $i = 1, \ldots, k$ are annihilated by $\widetilde{a}^-$ and for $z = 0$ Eq. (63) reduces to the eigenstate $|\widetilde{\psi}_0\rangle$ of $\widetilde{H}$ associated to E_0.

4.4 Example: Harmonic Oscillator

The simplest system available to illustrate the previous treatment is the harmonic oscillator. In this case there is a linear relation between the number operator and the Hamiltonian H, $H = E(N) = N + 1/2$. In addition, the function characterizing the action of $a^\pm$ onto the eigenstates of H becomes:

$$r(n) = \sqrt{E_n - E_0} = \sqrt{n}, \tag{75}$$

where, since the phase factors of Eq. (44) are independent of n, we have fixed them by taking $\tau = 0$. The function characterizing the commutator between the annihilation and creation operators is now (see Eq. (48)):

$$f(N) = E(N+1) - E(N) = 1. \tag{76}$$

Thus, the commutation relations for the intrinsic algebra of the oscillator become:

$$[N, a^{\pm}] = \pm a^{\pm}, \tag{77}$$

$$[a^{-}, a^{+}] = 1, \tag{78}$$

which is the well-known Heisenberg–Weyl algebra.

On the other hand, for the SUSY partner Hamiltonian $\widetilde{H}$ we have that:

$$\widetilde{r}(n) = \left[\prod_{i=1}^{k} (E_n - \epsilon_i - 1)\,(E_n - \epsilon_i)\right]^{\frac{1}{2}} r(n). \tag{79}$$

If we insert this expression in Eq. (58) it is obtained a polynomial Heisenberg algebra, since in this case the commutator of $\widetilde{a}^{-}$ and $\widetilde{a}^{+}$ is a polynomial of degree $2k$ either in $\widetilde{H}$ or in $\widetilde{N}$.

Concerning coherent states, in the first place the coefficients ρ_m and $\widetilde{\rho}_m$, which are also the moments arising in Eqs. (69), (70), become:

$$\rho_m = m!, \tag{80}$$

$$\widetilde{\rho}_m = m! \prod_{i=1}^{k} \left(\frac{1}{2} - \epsilon_i\right)_m \left(\frac{3}{2} - \epsilon_i\right)_m, \tag{81}$$

where $(c)_m = \Gamma(c+m)/\Gamma(c)$ is a Pochhammer's symbol. It is straightforward to find now the explicit expressions for the coherent states:

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} |\psi_m\rangle, \tag{82}$$

$$|\widetilde{z}\rangle = \sum_{m=0}^{\infty} \frac{z^m |\widetilde{\psi}_m\rangle}{\sqrt{{}_0F_{2k}\left(\frac{1}{2}-\epsilon_1, \frac{3}{2}-\epsilon_1, \ldots, \frac{1}{2}-\epsilon_k, \frac{3}{2}-\epsilon_k; |z|^2\right) m! \prod_{i=1}^{k} \left(\frac{1}{2}-\epsilon_i\right)_m \left(\frac{3}{2}-\epsilon_i\right)_m}}. \tag{83}$$

The solutions to the moment problems of Eqs. (69), (70) are given by:

$$\rho(y) = \exp(-y), \tag{84}$$

$$\widetilde{\rho}(y) = \frac{G^{2k+1\ 0}_{0\ 2k+1}\left(y|0, -\epsilon_1 - \frac{1}{2}, \ldots, -\epsilon_k - \frac{1}{2}, \frac{1}{2} - \epsilon_1, \ldots, \frac{1}{2} - \epsilon_k\right)}{\prod_{i=1}^{k} \Gamma\left(\frac{1}{2} - \epsilon_i\right) \Gamma\left(\frac{3}{2} - \epsilon_i\right)}, \tag{85}$$

where G is a Meijer G-function [80]. The reproducing kernel in both cases turns out to be:

$$\langle z_1|z_2\rangle = \exp\left[-\tfrac{1}{2}(|z_1|^2+|z_2|^2-2z_1^* z_2)\right], \tag{86}$$

$$\langle \widetilde{z_1}|\widetilde{z_2}\rangle = \frac{{}_0F_{2k}\left(\frac{1}{2}-\epsilon_1,\frac{3}{2}-\epsilon_1,\ldots,\frac{1}{2}-\epsilon_k,\frac{3}{2}-\epsilon_k;z_1^* z_2\right)}{\sqrt{{}_0F_{2k}\left(\frac{1}{2}-\epsilon_1,\frac{3}{2}-\epsilon_1,\ldots,\frac{1}{2}-\epsilon_k,\frac{3}{2}-\epsilon_k;|z_1|^2\right){}_0F_{2k}\left(\frac{1}{2}-\epsilon_1,\frac{3}{2}-\epsilon_1,\ldots,\frac{1}{2}-\epsilon_k,\frac{3}{2}-\epsilon_k;|z_2|^2\right)}}. \tag{87}$$

As we can see, the coherent states for the initial Hamiltonian H are the standard ones, which minimize the Heisenberg uncertainty relation, namely $(\Delta X)(\Delta P) = 1/2$. It would be important to know if the coherent states associated to $\widetilde{H}$ have also this property. However, the calculation of $(\Delta X)(\Delta P)$ for general SUSY transformations, with arbitrary factorization energies and associated constants ϵ_j, ν_j, $j = 1,\ldots,k$ involved in the Schrödinger solution of Eq. (31), is difficult. Such an uncertainty can be analytically calculated in the harmonic oscillator limit for an arbitrary k. In particular, for $k = 1$, $\epsilon_1 = -\frac{1}{2}$, $\nu_1 = 0$ it is obtained [78] ($r = |z|$):

$$(\Delta X)(\Delta P) = \sqrt{\left\{\tfrac{3}{2}-[\mathrm{Re}(z)]^2\xi_1(r)\right\}\left\{\tfrac{3}{2}-[\mathrm{Im}(z)]^2\xi_1(r)\right\}}, \tag{88}$$

$$\xi_1(r) = 2\left[\frac{{}_0F_2(2,2;r^2)}{{}_0F_2(1,2;r^2)}\right]^2 - \left[\frac{{}_0F_2(2,3;r^2)}{{}_0F_2(1,2;r^2)}\right], \tag{89}$$

while for $k = 2$, $(\epsilon_1,\epsilon_2) = (-\frac{1}{2},-\frac{3}{2})$, $(\nu_1,\nu_2) = (0,\infty)$ we arrive at [80]:

$$(\Delta X)(\Delta P) = \sqrt{\left\{\tfrac{5}{2}-[\mathrm{Re}(z)]^2\xi_2(r)\right\}\left\{\tfrac{5}{2}-[\mathrm{Im}(z)]^2\xi_2(r)\right\}}, \tag{90}$$

$$\xi_2(r) = \frac{1}{2}\left[\frac{{}_0F_4(2,2,3,3;r^2)}{{}_0F_4(1,2,2,3;r^2)}\right]^2 - \frac{1}{6}\left[\frac{{}_0F_4(2,3,3,4;r^2)}{{}_0F_4(1,2,2,3;r^2)}\right]. \tag{91}$$

Plots of the Heisenberg uncertainty relations of Eqs. (88) and (90) as functions of z are shown in Figs. 1 and 2, respectively. It is seen that these coherent states are no longer minimum uncertainty states. However, for $k = 1$ there are some directions in the complex plane for which the minimum value $(\Delta X)(\Delta P) = 1/2$ is achieved when $|z| \to \infty$ (see Fig. 1).

5 SUSY QM and Painlevé Equations

In a general context, the polynomial Heisenberg algebras (PHA) of degree m are deformations of the Heisenberg–Weyl algebra for which the commutators of the Hamiltonian H (of form given in Eq. (5)) with $(m+1)$th order differential ladder operators $L^{\pm}$ are standard, while the commutator between L^- and L^+ is a

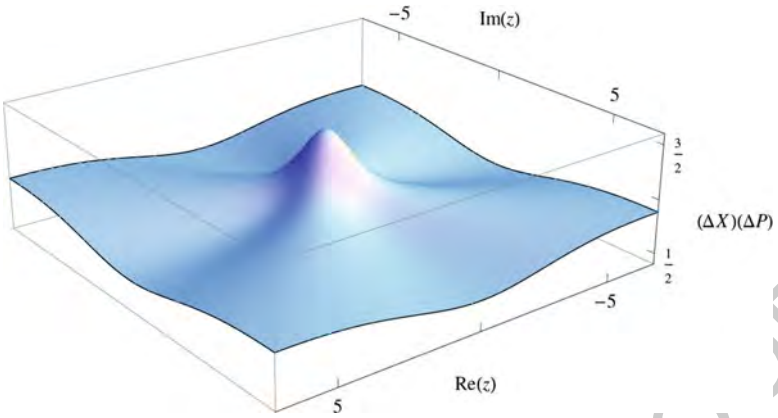

Fig. 1 Uncertainty relation $(\Delta X)(\Delta P)$ for the coherent states $|\widetilde{z}\rangle$ with $k = 1$ in the harmonic oscillator limit, when $\epsilon_1 = -\frac{1}{2}$, $\nu_1 = 0$

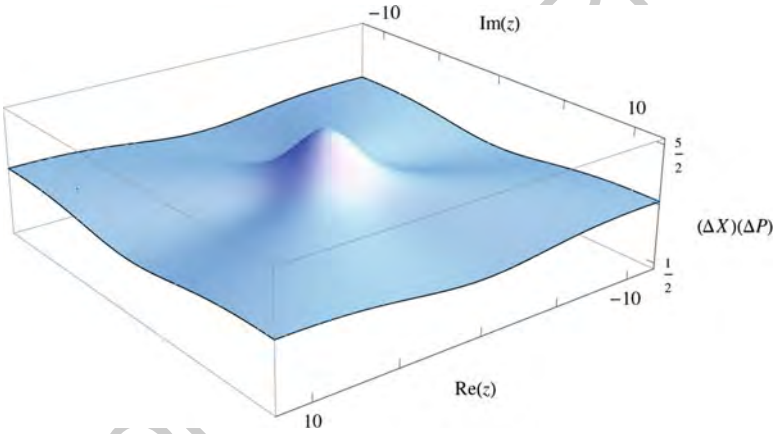

Fig. 2 Uncertainty relation $(\Delta X)(\Delta P)$ for the coherent states $|\widetilde{z}\rangle$ with $k = 2$ in the harmonic oscillator limit, when $(\epsilon_1, \epsilon_2) = (-\frac{1}{2}, -\frac{3}{2})$, $(\nu_1, \nu_2) = (0, \infty)$

polynomial of degree mth in H [85], i.e., 431

$$[H, L^{\pm}] = \pm L^{\pm}, \tag{92}$$

$$[L^{-}, L^{+}] = q_{m+1}(H+1) - q_{m+1}(H) = p_m(H), \tag{93}$$

$$L^{+}L^{-} = q_{m+1}(H) = \prod_{j=1}^{m+1}(H - \mathcal{E}_j), \tag{94}$$

$$L^{-}L^{+} = q_{m+1}(H+1) = \prod_{j=1}^{m+1}(H - \mathcal{E}_j + 1). \tag{95}$$

Systems ruled by PHA of degree m have $m+1$ extremal states $\psi_{\mathcal{E}_j}$, $j=1,\ldots,m+1$, which are annihilated by L^- and are formal eigenstates of H associated to $\mathcal{E}_j$.

Previously it was shown that the SUSY partner Hamiltonians of the harmonic oscillator are ruled by PHA of degree $2k$, with their natural ladder operators being of order $2k+1$ (see Eq. (50)). Hence, the first-order SUSY partners of the harmonic oscillator are ruled by second-degree polynomial Heisenberg algebras generated by third-order ladder operators, and so on. Thus, through SUSY QM plenty of particular realizations of such algebras can be supplied. However, it would be important to identify the general Hamiltonians H, of form given in Eq. (5), which have $(m+1)$th order differential ladder operators. This question has been addressed recurrently in the past, and nowadays there are some definite answers: if $m=0$ the general potential having first-order ladder operators is the harmonic oscillator, while for $m=1$ (second-order ladder operators) it is the radial oscillator. On the other hand, for $m=2$ ($m=3$) the general potential with third-order (fourth-order) ladder operators is expressed in terms of a function which satisfies the Painlevé IV (V) equation [85].

This connection suggests the possibility of going in the inverse direction, so if we could identify a Hamiltonian with third-order (fourth-order) ladder operators, perhaps we could use some information (the extremal state expressions and associated factorization energies $\mathcal{E}_j$) to generate solutions to the Painlevé IV (V) equation (also called Painlevé IV (V) transcendents). This is in fact what happens; thus, the game reduces to find Hamiltonians with third-order (fourth-order) ladder operators for generating Painlevé IV (V) transcendents through the extremal states of the system [85, 90, 129, 132].

Let us present next these statements as two algorithms to generate solutions for such non-linear second-order ordinary differential equations.

5.1 Generation of Painlevé IV Transcendents

Let us suppose that we have identified a Hamiltonian of the form given in Eq. (5), which has third-order differential ladder operators $L^{\pm}$ satisfying Eqs. (92)–(95) with $m=2$, as well as its three extremal states $\psi_{\mathcal{E}_j}$ and associated factorization energies $\mathcal{E}_j$, $j=1,2,3$. Thus, a solution to the Painlevé IV (PIV) equation

$$g''=\frac{g'^2}{2g}+\frac{3}{2}g^3+4xg^2+2(x^2-\alpha)g+\frac{\beta}{g} \tag{96}$$

is given by

$$g(x)=-x-\{\ln[\psi_{\mathcal{E}_3}(x)]\}', \tag{97}$$

where the parameters α, β of the PIV equation are related with $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{E}_3$ in the way

$$\alpha = \mathcal{E}_1 + \mathcal{E}_2 - 2\mathcal{E}_3 - 1, \qquad \beta = -2(\mathcal{E}_1 - \mathcal{E}_2)^2. \tag{98}$$

Let us note that, if the indices assigned to the extremal states are permuted cyclically, we will obtain three PIV transcendents, one for each extremal state when it is labeled as $\psi_{\mathcal{E}_3}$.

Summarizing, our task has been reduced to identify systems ruled by second-degree PHA and the corresponding extremal states [85, 90]. The harmonic oscillator supplies several such possibilities, for instance, the two operator pairs $\{a^3, (a^+)^3\}$, $\{a^+a^2, (a^+)^2a\}$ are third-order ladder operators satisfying Eqs. (92)–(95) (the level spacing has to be adjusted in the first case), and it is simple to identify the corresponding extremal states. On the other hand, the first-order SUSY partners of the oscillator also have natural third-order ladder operators, and well-identified extremal states. For the SUSY partners of the oscillator with $k \geq 2$ the natural ladder operators are not of third order (they are in general of order $2k + 1$). However, it is possible to induce a reduction process, by choosing connected seed solutions $u_{j+1} = au_j$, $\epsilon_{j+1} = \epsilon_j - 1$, $j = 1, \ldots, k-1$ instead of general seed solutions, so that the $(2k + 1)$th order ladder operators reduce to third-order ones.

Some examples of real PIV transcendents associated to real PIV parameters α, β, which are generated through this algorithm, are presented next.

5.1.1 Harmonic Oscillator

If we take the ladder operators $L^- = a^3$, $L^+ = (a^+)^3$ for the harmonic oscillator Hamiltonian we get the PIV transcendents reported in Table 1 [178]. Note that in order that the level spacing induced by this pair of ladder operators coincides with the standard one ($\Delta E = 1$) of Eqs. (92)–(95), we need to change variables $y = \sqrt{3}x$ and scale the factorization energies (dividing by 3). Remember also that $\psi_j(x)$ are the eigenfunctions of the harmonic oscillator associated to the first three energy levels $E_j = j + 1/2$, $j = 0, 1, 2$.

Table 1 PIV transcendents generated from the harmonic oscillator Hamiltonian with $L^- = a^3$, $L^+ = (a^+)^3$

$\psi_{\mathcal{E}_3}$	$\psi_0(x)$	$\psi_1(x)$	$\psi_2(x)$
$\mathcal{E}_3$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{5}{2}$
$g(y)$	$-\frac{2y}{3}$	$-\frac{2y}{3} - \frac{1}{y}$	$-\frac{2y}{3} - \frac{4y}{2y^2-3}$
α	0	-1	-2
β	$-\frac{2}{9}$	$-\frac{8}{9}$	$-\frac{2}{9}$

Table 2 PIV transcendents generated from the first-order SUSY partner Hamiltonian $\widetilde{H}$ with $L^- = B^+aB,\ L^+ = B^+a^+B$

$\psi_{\mathcal{E}_3}$	$\frac{1}{u_1}$	$B^+\psi_0$	$B^+a^+u_1$	t4.1
$\mathcal{E}_3$	$-\frac{5}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	t4.2
$g(x)$	$\frac{4x}{1+2x^2}$	$-\frac{4x^4+3}{4x^5+8x^3+3x}$	$\frac{8x^5+6x}{1-4x^4}$	t4.3
α	3	-6	0	t4.4
β	-8	-2	-18	t4.5

Table 3 PIV transcendents generated from the second-order SUSY partner Hamiltonian $\widetilde{H}$ and the third-order ladder operators obtained by reducing $L^- = B^+aB,\ L^+ = B^+a^+B$

$\psi_{\mathcal{E}_3}$	$\frac{u_1}{W(u_1,u_2)}$	$B^+\psi_0$	$B^+a^+u_1$	t7.1
$\mathcal{E}_3$	$-\frac{7}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	t7.2
$g(x)$	$\frac{4x(4x^4+4x^2-3)}{8x^6+4x^4+6x^2+3}$	$-\frac{4x(16x^8+72x^2+27)}{32x^{10}+48x^8+96x^6+54x^2-27}$	$\frac{-16x^8+32x^6-48x^4+9}{x(2x^2-3)(4x^4+3)}$	t7.3
α	5	-7	-1	t7.4
β	-8	-8	-32	t7.5

5.1.2 First-Order SUSY Partner of the Harmonic Oscillator

For $\epsilon_1 = -\frac{5}{2},\ \nu_1 = 0$, and the third-order ladder operators $L^- = B^+aB,\ L^+ = B^+a^+B$ of $\widetilde{H}$, we get the PIV transcendents reported in Table 2. The seed solution employed is $u_1(x) = e^{\frac{x^2}{2}}(1+2x^2)$.

5.1.3 Second-Order SUSY Partner of the Harmonic Oscillator

For $\epsilon_1 = -\frac{5}{2},\ \nu_1 = 0$, and the third-order ladder operators of $\widetilde{H}$ obtained from the reduction of the fifth-order ones $L^- = B^+aB,\ L^+ = B^+a^+B$, we get the PIV transcendents reported in Table 3. Once again, the seed solution u_1 employed is $u_1(x) = e^{\frac{x^2}{2}}(1+2x^2)$ and $u_2 = au_1$.

5.2 Generation of Painlevé V Transcendents

Let us suppose now that the Hamiltonian H we have identified has fourth-order ladder operators and satisfy Eqs. (92)–(95) with $m = 3$. We know also its four extremal states $\psi_{\mathcal{E}_j}$ and associated factorization energies $\mathcal{E}_j,\ j = 1, 2, 3, 4$. Thus, one solution to the Painlevé V (PV) equation

$$w'' = \left(\frac{1}{2w} - \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha\, w + \frac{\beta}{w}\right) + \gamma\,\frac{w}{z} + \delta\,\frac{w(w+1)}{w-1} \quad (99)$$

is given by

$$w(z) = 1 + \frac{\sqrt{z}}{g(\sqrt{z})}, \tag{100}$$

$$g(x) = -x - \frac{d}{dx}\left\{\ln\left[W(\psi_{\mathcal{E}_3}(x), \psi_{\mathcal{E}_4}(x))\right]\right\}, \tag{101}$$

where the prime in Eq. (99) means derivative with respect to z, and the PV parameters $\alpha,\ \beta,\ \gamma,\ \delta$ are related with $\mathcal{E}_1,\ \mathcal{E}_2,\ \mathcal{E}_3,\ \mathcal{E}_4$ through

$$\alpha = \frac{(\mathcal{E}_1-\mathcal{E}_2)^2}{2}, \quad \beta = -\frac{(\mathcal{E}_3-\mathcal{E}_4)^2}{2}, \quad \gamma = \frac{\mathcal{E}_1+\mathcal{E}_2}{2} - \frac{\mathcal{E}_3+\mathcal{E}_4+1}{2}, \quad \delta = -\frac{1}{8}. \tag{102}$$

Note that if the indices of the extremal states are permuted, we will obtain at the end six PV transcendents (in principle different), one for each pair of extremal states when they are labeled as $\psi_{\mathcal{E}_3},\ \psi_{\mathcal{E}_4}$ [132].

Once again, now we require just to identify systems ruled by third degree PHA and their four extremal states. The harmonic oscillator also supplies some possibilities, the simplest one through the fourth order ladder operators $\{L^- = a^4,\ L^+ = (a^+)^4\}$, which satisfy Eqs. (92)–(95) if we change variables and adjust the levels spacing, with the extremal states being the eigenstates associated to the four lowest energy levels of the oscillator. Another system closely related to PV equation is the radial oscillator, for which its ladder operators $b^{\pm}$ are of second order [132]. Thus, the second powers of such operators are also fourth order ladder operators that will give place to PV transcendents. Concerning SUSY partners, those of the radial oscillator give place to PHA of degree $2k+1$, with natural ladder operators of order $2k+2$. Thus, the first-order SUSY partners of the radial oscillator have natural fourth-order ladder operators and well-identified extremal states. For $k \geq 2$, it is possible to produce again a reduction process, by connecting the seed solutions in the way $u_{j+1} = b^- u_j,\ \epsilon_{j+1} = \epsilon_j - 1,\ j = 1, \ldots, k-1$, so that the $(2k+2)$th order natural ladder operators reduce to fourth-order ones [132]. Remember that the first-order SUSY partners of the harmonic oscillator also have fourth-order ladder operators, given by $L^- = B^+ a^2 B,\ L^+ = B^+ (a^+)^2 B$, but we will have to change variables and adjust the level spacing to stick to the standard convention $\Delta E = 1$.

Some examples of real PV transcendents associated to real parameters $\alpha,\ \beta,\ \gamma,\ \delta$, generated through this algorithm, are now presented.

5.2.1 Harmonic Oscillator

If we take $L^- = a^4, L^+ = (a^+)^4$ as ladder operators, we generate the PV transcendents reported in Table 4. Note that here $z = 4x^2$ and $\psi_j(x),\ j = 0, 1, 2, 3$ are the eigenfunctions for the four lowest eigenvalues of the harmonic oscillator. We

Table 4 PV transcendents generated from the harmonic oscillator Hamiltonian and $L^- = a^4$, $L^+ = (a^+)^4$

Permutation	α	β	γ	$w(z)$	
1234	$\frac{1}{32}$	$-\frac{1}{32}$	0	-1	t9.2
4231	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{2-z}{z+2}$	t9.3
1432	$\frac{1}{32}$	$-\frac{9}{32}$	$-\frac{1}{2}$	$\frac{6-z}{z+2}$	t9.4
3241	$\frac{9}{32}$	$-\frac{1}{32}$	$-\frac{1}{2}$	$\frac{2-z}{z+6}$	t9.5
3142	$\frac{1}{8}$	$-\frac{1}{8}$	$-\frac{3}{4}$	$\frac{6-z}{z+6}$	t9.6
3412	$\frac{1}{32}$	$-\frac{1}{32}$	-1	$-\frac{(z-6)(z-2)}{(z+2)(z+6)}$	t9.7

Table 5 PV transcendents generated from the first-order SUSY partner Hamiltonian $\widetilde{H}$ of the oscillator and $L^- = B^+a^2B$, $L^+ = B^+(a^+)^2B$

Permutation	α	β	γ	$w(z)$	
1234	$\frac{1}{8}$	$-\frac{1}{2}$	$\frac{3}{4}$	$-\frac{2}{z-1}$	t11.2
4231	$\frac{1}{2}$	$-\frac{9}{8}$	$\frac{1}{4}$	$\frac{z+3}{2}$	t11.3
1432	$\frac{1}{8}$	-2	$-\frac{1}{4}$	$\frac{z^2+2z-1}{z-1}$	t11.4
3241	2	$-\frac{1}{8}$	$-\frac{3}{4}$	$\frac{z+3}{z^2+2z+3}$	t11.5
3142	$\frac{9}{8}$	$-\frac{1}{2}$	$-\frac{5}{4}$	$\frac{2(z^2+2z-1)}{z^3+z^2+z-3}$	t11.6
3412	$\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{7}{4}$	$-\frac{z^3+5z^2+5z-3}{2(z^2+2z+3)}$	t11.7

initially order the extremal states as

$$\psi_{\mathcal{E}_1}(x) = \psi_2(x), \qquad \mathcal{E}_1 = \tfrac{5}{2}, \tag{103}$$

$$\psi_{\mathcal{E}_2}(x) = \psi_3(x), \qquad \mathcal{E}_2 = \tfrac{7}{2}, \tag{104}$$

$$\psi_{\mathcal{E}_3}(x) = \psi_0(x), \qquad \mathcal{E}_3 = \tfrac{1}{2}, \tag{105}$$

$$\psi_{\mathcal{E}_4}(x) = \psi_1(x), \qquad \mathcal{E}_4 = \tfrac{3}{2}, \tag{106}$$

and this permutation will be denoted as 1234. We do not include the parameter δ in this table since it is constant ($\delta = -\frac{1}{8}$).

5.2.2 First-Order SUSY Partner of the Harmonic Oscillator

For $\epsilon_1 = -\frac{5}{2}$, $\nu_1 = 0$, and the fourth-order ladder operators $L^- = B^+a^2B$, $L^+ = B^+(a^+)^2B$ of $\widetilde{H}$, we will get the PV transcendents reported in Table 5, where $z = 2x^2$. The seed solution employed is $u_1(x) = e^{\frac{x^2}{2}}(1 + 2x^2)$. The initial order for the extremal states, denoted as 1234 in the table, is

$$\psi_{\mathcal{E}_1}(x) = \frac{W(u_1, \psi_0)}{u_1}, \qquad \mathcal{E}_1 = \frac{1}{2}, \tag{107}$$

$$\psi_{\mathcal{E}_2}(x) = \frac{W(u_1, \psi_1)}{u_1}, \qquad \mathcal{E}_2 = \frac{3}{2}, \tag{108}$$

$$\psi_{\mathcal{E}_3}(x) = \frac{1}{u_1}, \qquad \mathcal{E}_3 = -\frac{5}{2}, \tag{109}$$

$$\psi_{\mathcal{E}_4}(x) = B^+(a^+)^2 u_1, \qquad \mathcal{E}_4 = -\frac{1}{2}. \tag{110}$$

We conclude this section by stating that an infinity of PIV and PV transcendents can be derived through the techniques described here. It is an open question to determine if any exact solution to such equations that exists in the literature can be derived through these methods. However, the algorithms are so simple and direct that we felt it was the right time to try to make them known to a wider and diversified community, not just to people working on solutions to non-linear differential equations.

6 Recent Applications of SUSY QM

Some recent interesting applications of SUSY QM are worth of some discussion. We would like to mention in the first place the motion of electrons in graphene, a single layer of carbon atoms arranged in a hexagonal honeycomb lattice. Since close to the Dirac points in the Brillouin zone there is a gapless linear dispersion relation, obtained in the low energy regime through a tight binding model, one ends up with an electron description in terms of the massless Dirac–Weyl equation, with Fermi velocity $v_F \approx c/300$ instead of the speed of light c. If the graphene layer is subject to external magnetic fields orthogonal to its surface (the $x - y$ plane), the Dirac–Weyl equation reads:

$$\mathbf{H}\Psi(x, y) = v_F \boldsymbol{\sigma} \cdot \left[\mathbf{p} + \frac{e\mathbf{A}}{c}\right] \Psi(x, y) = E\Psi(x, y), \tag{111}$$

where $v_F \sim 8 \times 10^5$ m/s is the Fermi velocity, $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ are the Pauli matrices, $\mathbf{p} = -i\hbar(\partial_x, \partial_y)^T$ is the momentum operator in the $x - y$ plane, $-e$ is the electron charge, and $\mathbf{A}$ is the vector potential leading to the magnetic field through $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$. For magnetic fields which change just along x-direction, $\mathbf{B} = \mathcal{B}(x)\hat{e}_z$, in the Landau gauge we have that $\mathbf{A} = \mathcal{A}(x)\hat{e}_y$, $\mathcal{B}(x) = \mathcal{A}'(x)$. Since there is a translational invariance along y axis, we can propose

$$\Psi(x, y) = e^{iky} \begin{bmatrix} \psi^+(x) \\ i\psi^-(x) \end{bmatrix}, \tag{112}$$

where k is the wave number in the y direction and $\psi^{\pm}(x)$ describe the electron amplitudes on two adjacent sites in the unit cell of graphene. Thus we arrive to:

$$\left(\pm\frac{d}{dx} + \frac{e}{c\hbar}\mathcal{A} + k\right) \psi^{\mp}(x) = \frac{E}{\hbar v_F}\psi^{\pm}(x). \tag{113}$$

By decoupling these set of equations it is obtained:

$$H^{\pm}\psi^{\pm}(x) = \mathcal{E}\psi^{\pm}(x), \qquad \mathcal{E} = \frac{E^2}{\hbar^2 v_F^2}, \tag{114}$$

$$H^{\pm} = -\frac{d^2}{dx^2} + V^{\pm} = -\frac{d^2}{dx^2} + \left(\frac{e\mathcal{A}}{c\hbar} + k\right)^2 \pm \frac{e}{c\hbar}\frac{d\mathcal{A}}{dx}. \tag{115}$$

Let us note that these expressions are characteristic of the first-order SUSY QM. In fact, through the identification[1]:

$$B^{\pm} = \mp\frac{d}{dx} + \mathcal{W}(x), \tag{116}$$

where

$$\mathcal{W}(x) = \frac{e\mathcal{A}(x)}{c\hbar} + k \tag{117}$$

is the superpotential, it turns out that

$$B^{\mp}\psi^{\mp}(x) = \sqrt{\mathcal{E}}\psi^{\pm}(x). \tag{118}$$

The SUSY partner Hamiltonians $H^{\pm}$ thus satisfy:

$$H^{\pm} = B^{\mp}B^{\pm}, \quad V^{\pm}(x) = \mathcal{W}^2 \pm \mathcal{W}', \tag{119}$$

$$H^{\pm}B^{\mp} = B^{\mp}H^{\mp}. \tag{120}$$

By comparing these expressions with the formalism of Sect. 2, one realizes that $H^{\pm}$ can be identified with any of the two SUSY partner Hamiltonians H and $\widetilde{H}$ (up to a constant factor), depending on which one will be taken as the departure Hamiltonian. Moreover, by deriving the superpotential with respect to x it is obtained:

$$\mathcal{B}(x) = \frac{c\hbar}{e}\frac{d\mathcal{W}}{dx}. \tag{121}$$

This formula suggests a method to proceed further: the magnetic field $\mathcal{B}(x)$ has to be chosen cleverly, in order to arrive to a pair of exactly solvable potentials $V^{\pm}$. In particular, it has been chosen in several different ways but taking care that $V^{\pm}$ are shape invariant potentials [151]. An important case of this type appears for constant homogeneous magnetic fields: in such a situation both $V^{\pm}$ become

[1] We choose here a notation consistent with Sect. 2. Please do not confuse the intertwining operators of Eq. (116) with the magnetic field **B**, its magnitude $\mathcal{B}(x)$, or any of its components.

harmonic oscillator potentials. It is worth to mention also that the shape invariance condition has been generalized, thus supplying a method for generating magnetic fields which are deformed with respect to the chosen initial one, but leading once again to an exactly solvable problem [157].

Let us note that the SUSY methods have been applied also to other carbon allotropes, as the carbon nanotubes, and it has been successfully implemented when electrostatic fields are applied, with or without static magnetic fields. In addition, the coherent state methods have been started to be applied recently to graphene subject to static homogeneous magnetic fields [179]. As can be seen, the SUSY methods applied to Dirac materials is a very active field which surely will continue its development in the near future [151–161].

At this point, it is worth to mention also the applications of SUSY QM to optical system, since there is a well-known correspondence between Schrödinger equation and Maxwell equations in the paraxial approximation. Thus, it seems natural to think that many techniques successfully used to deal with quantum mechanical problems can be directly applied to optical systems in an appropriate approximation. In a way, we are dealing with the optical analogues of quantum phenomena, which have been realized, for example, in waveguide arrays, optimization of quantum cascade lasers, among others. In particular, the optical analogues of SUSY QM is an emergent field which could supply a lot of interesting physical information [162–169].

7 Conclusions

It has been shown that supersymmetric quantum mechanics is a simple powerful tool for generating potentials with known spectra departing from a given initial solvable one. Since the spectrum of the new Hamiltonian differs slightly from the initial one, the method can be used to implement the spectral design in quantum mechanics.

In this direction, let us note that here we have discussed real SUSY transformations, by employing just real seed solutions which will produce at the end real SUSY partner potentials $\tilde{V}(x)$. However, most of these formulas can be used without any change for implementing complex SUSY transformations. If we would introduce this procedure gradually, in the first place we could use complex seed solutions associated to real factorization energies in order to generate complex potentials with real spectrum [90, 180]. This offers immediately new possibilities of spectral design which were not available for real SUSY transformations, for example, through a complex first-order SUSY transformation with real factorization energies a new energy level can be created at any position on the real energy axis. In a second step of this approach, one can use complex seed solutions associated to complex factorization energies for an initial potential which is real [88], thus generating new levels at arbitrary positions in the complex energy plane. The third step for making complex the SUSY transformation is to apply the method to initial potentials which are complex from the very beginning [92]. In all these steps we will get at the

end new potentials which are complex, but the spectrum will depend on the initial potential as well as of the kind of seed solutions employed.

We want to finish this paper by noting that the factorization method and intertwining techniques have been also applied with success to some discrete versions of the stationary Schrödinger equation [181–186]. The connections that could be established between such problems and well-known finite difference equations [187, 188] could contribute to the effort of classifying the known solutions and generate new ones, as it has happened in the continuous case for more than 80 years.

As it was pointed out previously, one of our aims when writing this article was to make a short review of the most recent advances of SUSY QM, either on purely theoretical or applied directions. We hope to have succeeded; perhaps the reader will find interesting and/or useful the ideas here presented.

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