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Second harmonic Hamiltonian: Algebraic and Schrödinger approaches

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ABSTRACT

We study in detail the behavior of the energy spectrum for the second harmonic generation (SHG) and a family of corresponding quasi-exactly solvable Schrödinger potentials labeled by a real parameter b. The eigenvalues of this system are obtained by the polynomial deformation of the Lie algebra $sl(2, R)$ representation space. We have found the bi-confluent Heun equation (BHE) corresponding to this system in a differential realization approach, by making use of the symmetries. By means of a b-transformation from this second-order equation to a Schrödinger one, we have found a family of quasi-exactly solvable potentials. For each invariant n-dimensional subspace of the second harmonic generation, there are either n potentials, each with one known solution, or one potential with n-known solutions. Well-known potentials like a sextic oscillator or that of a quantum dot appear among them.

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1. Introduction

The second harmonic generation (SHG) is a nonlinear optical process in which photons with the frequency of ω interact with a nonlinear medium and generate photons with twice this frequency [1]. This process was firstly shown when a ruby laser beam passed through the crystal-line quartz and the final beam was twice intense [2]. From classical point of view, the general solution of this system was obtained based on elliptic integrals [3]. The quantum mechanical treatment of this nonlinear optical process is made in Refs. [4–7]. Some authors have considered some non-classical features for this Hamiltonian such as anti-bunching that happened for coherent light incident [8,9], squeezing [10,11], revival-collapse effect in mean photon number of the field modes [12,13], quantum correlation [14,15].

The dynamics of this Hamiltonian can be dealt with in different ways. A semi-classical JWKB analysis is performed for a one dimensional Schrödinger equation and asymptotic formulas are obtained when the input number of photons is large enough [16]. The asymptotic behavior of eigenvalues for large number of input photons and the equivalency between this nonlinear system and one-dimensional Schrödinger with a sextic polynomial potential are shown in Ref. [17]. The exact analytical expression for eigenvalues and eigenstates of this Hamiltonian has been obtained based on the unknown parameter λ [18].

The other well-known approach which has been extensively applied in quantum optical system is the polynomial deformation of Lie algebras. These algebras were introduced for the first time by Higgs [19] and Sklyanin [20]. The motivation to apply this method is the existence of nonlinear symmetries in multi-photon Hamiltonian that can not be described by Lie algebras [21]. Furthermore, the general expression for energy spectra of some nonlinear quantum model has been obtained by polynomial deformation of Lie algebra $su(2)$ in [22]. In [23] the effective Hamiltonian for a nonlinear optical system is found in terms of diagonal operator of polynomial algebra in far-off resonant limit. The SHG system has been considered by the polynomial Lie algebra method in [24] where the authors found an algorithm to achieve eigenvalues. But this algorithmic form could only be applied in some dimensions of a subspace of the system while they had to use the $su(2)$ quasi-classical approximation in other dimensions. These two methods displayed consistence for some parameters of model but the partial distinctions between them require more enhancements in the approximation method.

The main objective of this paper is to show the general behavior of the energy spectrum for this system by an algebraic approach and to obtain other families of potentials in addition to the sextic potential for this nonlinear optical system within the differential realization approach. In particular we want to illustrate the role of symmetries.

The organization of this paper is as follows. In Section 2 we will use the polynomial deformation of Lie algebra $sl(2, R)$ and its representations to obtain the characteristic equation. By means of a decomposition of the tridiagonal matrix corresponding to the

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Hamiltonian, we can discuss the energy spectrum. We can see that unlike Ref. [16] in which the energy spectra is obtained in the form of convergent power series by a semi-classical approximation, our results can be expressed in an exact manner. In Section 3, the second order partial differential equation of the SHG Hamiltonian will be given through Bargmann-Fock variables and derive an ordinary differential equation, the bi-confluent Heun equation (BHE), by separation in a new set of variables. A further transformation which is characterized by a real label b enables us to pass the form of the BHE into a quasi-solvable Schrödinger equation. The resulting potentials will be parameterized by eigenvalues of the system E and by the transformation label b. Some examples are displayed in Section 4. Finally, we will outline the main conclusions at the end of the paper.

2. Algebraic approach

The Hamiltonian of the second harmonic generation system with perfect resonance takes the form (in units $\hbar$),

$$H_{SHG} = \omega a_1^+ a_1 + 2\omega a_2^+ a_2 + g(a_1^2 a_2^+ + (a_1^+)^2 a_2), \quad (2.1)$$

where $a_1^+(a_1)$ and $a_2^+(a_2)$ are the creation (annihilation) operators of the input photons with the frequency ω and the output photon with frequency 2ω, respectively. The parameter g is the coupling constant multiplying a non-diagonal term that describes the interaction between the two coupled bosons. Therefore, the Hamiltonian (2.1) can be rewritten as

$$H_{SHG} = H_0 + g H_I, \quad (2.2)$$

showing two basic terms. The role of H_0 is that of an initial non-interacting Hamiltonian including both type of photons. A set of symmetries (only three of them are functionally independent) of H_0 is

$$N_1 = a_1^+ a_1, \quad N_2 = a_2^+ a_2, \quad J_+ = \left(a_1^+\right)^2 a_2, \quad J_- = a_1^2 a_2^+. \quad (2.3)$$

So, as a two-dimensional Hamiltonian, H_0 is superintegrable and solvable [25]. The eigenspaces $\mathcal{W}_n$ of H_0 with eigenvalue n are easy to characterize and all of them are finite dimensional (later we will deal with this point). On the other hand, the interaction term of H_I is made of the symmetries J_+ and J_- of H_0. This is a key point, since it implies that each eigenspace $\mathcal{W}_n$ of H_0 will also be invariant under H_I. This means that the problem of the diagonalization of the Hamiltonian H_{SHG} can restrict to the family of the finite dimensional eigenspaces of H_0.

Once we have introduced the interacting term, the total Hamiltonian H_{SHG} has only one independent additional symmetry given by $H_0 = \omega(N_1 + 2N_2)$. Therefore the initial system H_0 was superintegrable, while the total Hamiltonian H_{SHG} becomes (only) completely integrable [25,26].

In this section, we will write the Hamiltonian in the terms of generators defined by

$$J_+ = \left(a_1^+\right)^2 a_2, \quad J_- = a_1^2 a_2^+, \quad J_0 = \frac{a_1^+ a_1 - a_2^+ a_2}{3},$$
$$N = a_1^+ a_1 + 2a_2^+ a_2 \quad (2.4)$$

as follows:

$$H_{SHG} = \omega N + g(J_+ + J_-) \quad (2.5)$$

The commutators of these generators take the following form

$$[J_+, J_-] = 12J_0^2 - \frac{N^2}{3} - N, \quad [J_0, J_\pm] = \pm J_\pm,$$
$$[N, J_\pm] = [N, J_0] = 0 \quad (2.6)$$

This means that the generators $J_\pm$ are lowering/raising with respect to the diagonal operator J_0, while N is a central generator commuting with all the others. The above set of commutators describes a polynomial deformation of the Lie algebra $sl(2,\mathbb{R})$. The finite dimensional representation space for these operators in the space spanned by the basis $|J, M\rangle$, $M = -J, ..., J$ is defined by [27]:

$$J_0|J, M\rangle = (M + c)|J, M\rangle \quad (2.7)$$
$$J_\pm|J, M\rangle = [(J \mp M)(J \pm M + 1)(12c + 4M \pm 2)]^{\frac{1}{2}}$$
$$\times |J, M \pm 1\rangle, \quad (2.8)$$

where the parameter c is a real number, J is half integer and the dimension of this space is $2J + 1$. From the commutator J_+ and J_- and the constraint for finite dimensional space, we have the following relation for c:

$$c^2 = \frac{-J(J+1)}{3} + \frac{1}{12}\left(\frac{n^2}{3} + n\right), \quad (2.9)$$

where $n = n_1 + 2n_2$ is the positive integer value of the generator N. For the reality of c in (2.9), we have:

$$J \leqslant -\frac{1}{2} + \frac{1}{2}\left(\frac{n^2}{3} + n\right)^{\frac{1}{2}}. \quad (2.10)$$

The eigenvalue equation of Hamiltonian (2.1) for a state $|\psi_k\rangle = \sum_{M=-J}^{J} C_M^K |J, M\rangle$ in an invariant subspace will be

$$n\omega C_M^k + g C_{M-1}^k [(J+M)(J-M+1)(12c+4M-2)]^{\frac{1}{2}} \quad (2.11)$$
$$+ g C_{M+1}^k [(J-M)(J+M+1)(12c+4M+2)]^{\frac{1}{2}} = E_k C_M^k.$$

In matrix notation, the above relation can be written as

$$\mathbf{AC} = \mathbf{0}, \quad (2.12)$$

i.e., the product of a tridiagonal symmetric matrix denoted $\mathbf{A}$ of dimensions $\hat{n} \times \hat{n}$ by a column matrix $\mathbf{C}$ of the coefficients C_M^K. The elements of main diagonal are $n\omega - E_K$ and the elements of the other two diagonals are the coefficients of $C_{M\pm1}^K$ which will be called a_i, $i = 1, \dots, \hat{n} - 1$. Hence, the eigenvalues E_K will be found from the determinant of this tridiagonal matrix.

We can face this problem under different points of view. One of them is that used in Ref. [28]. In this paper, this type of matrix is decomposed to two matrices. One of them is a lower triangular matrix where all elements of the main diagonal are equal to 1 and the other one is an upper triangular matrix. The determinant of tridiagonal matrix will be product of the determinant of these matrices which gives the product of the elements of main diagonal of the second one. In this way, we have the following relation for the determinant of A:

$$\det(A) = (-\lambda)\left(-\lambda - \frac{a_1^2}{-\lambda}\right)\left(-\lambda - \frac{a_2^2}{-\lambda - \frac{a_1^2}{-\lambda}}\right)\cdots$$
$$\left(-\lambda - \frac{a_{\hat{n}}^2}{-\lambda - \frac{a_{\hat{n}-1}^2}{\frac{\cdots}{-\lambda - \frac{a_1^2}{-\lambda}}}}\right) \quad (2.13)$$

where $-\lambda = n\omega - E_K$. We can also write this relation as follows,

$$\det(A) = (-\lambda)\left(\delta - \frac{\gamma_1}{\delta}\right)\left(\delta - \frac{\gamma_2}{\delta - \frac{\gamma_1}{\delta}}\right)\cdots\left(\delta - \frac{\gamma_{\hat{n}}}{\delta - \frac{\gamma_{\hat{n}-1}}{\frac{\cdots}{\delta - \frac{\gamma_1^2}{\delta}}}}\right) \tag{2.14}$$

with $\delta = -\lambda$ and $\gamma_i = a_i^2$. Next, we can see the recurrence relation for the determinant by using the following definition:

$$d_0 = 1, \tag{2.15}$$

$$d_1 = \delta,$$

$$d_2 = \delta\, d_1 - \gamma_1,$$

$$d_3 = \delta\, d_2 - \gamma_2\, d_1,$$

$$d_4 = \delta\, d_3 - \gamma_3\, d_2,$$

$$\vdots = \vdots$$

$$d_{\hat{n}} = \delta\, d_{\hat{n}-1} - \gamma_{\hat{n}-1}\, d_{\hat{n}-2}.$$

In this way, the $\det(A)$ will be

$$\det(A) = d_1 \times \left(\frac{d_2}{d_1}\right) \times \left(\frac{d_3}{d_2}\right) \times \left(\frac{d_4}{d_3}\right) \times \cdots \times \left(\frac{d_{\tilde{n}-1}}{d_{\hat{n}-2}}\right)$$
$$\times \left(\frac{d_{\hat{n}}}{d_{\hat{n}-1}}\right), \tag{2.16}$$

then, the condition $\det(A) = 0$ can be expressed as

$$d_{\hat{n}} = \delta\, d_{\hat{n}-1} - \gamma_{\hat{n}-1}\, d_{\hat{n}-2} = 0. \tag{2.17}$$

By means of Eq. (2.17), we can discuss the eigenvalues of the system. The above recurrence relation for Hamiltonian (2.1) was obtained in [18] but there it was assumed that the form of eigenvalues of this system as $E = \omega N + g\lambda$ with the unknown parameter λ and then it was found the recurrence relation for λ. In our approach, all elements in (2.17) are known and they are related to parameters of the system.

2.1. The finite dimensional subspace $\mathcal{W}_n$

In order to discuss about the eigenvalues of this system, we have to characterize the eigenspace $\mathcal{W}_n$ of H_0 with eigenvalue n. For this aim, we change our basis from $|J, M\rangle$ to the basis of two-mode Fock space $|n_1, n_2\rangle$ with two constrains $n = n_1 + 2n_2$ and $M + c = \frac{n_1 - n_2}{3}$. The subspaces $\mathcal{W}_n$ have the two following features:

– The subspaces $\mathcal{W}_n$ are finite–dimensional i.e., for each $N = n$ photon number, a basis will be formed by the following $\left[\frac{n}{2}\right] + 1$ vectors,

$$|n, 0\rangle, |n - 2, 1\rangle, |n - 4, 2\rangle, \cdots, \left|n - 2\left[\frac{n}{2}\right], \left[\frac{n}{2}\right]\right\rangle. \tag{2.18}$$

So, the dimension of this subspace for the odd and even values $n = 2k$ and $n = 2k + 1$ is equal to $\hat{n} = k + 1$.

– The subspaces $\mathcal{W}_n$ are invariant under H and its restriction to these invariant subspaces result in the finite dimensional matrix representation for H denoted as H_n. In this way, a portion of the spectrum of H can be obtained by diagonalizing the matrix H_n. According to the definition in [29] we can use the term "quasi-exactly solvable" for the second harmonic generation system.

Now we have two situations for this system:

a) Even dimension $\hat{n} = 2\tilde{k}$, $\tilde{k} \in \mathbb{N}$.
   In this case, the determinant $d_{\hat{n}}$ has the general form

$$d_{2\tilde{k}} = \delta^{2\tilde{k}} + b_1\delta^{2\tilde{k}-2} + b_2\delta^{2\tilde{k}-4} + \cdots + b_n\delta^2 + b_0 = 0$$
$$\implies \delta = \pm\sqrt{\xi_i}, \tag{2.19}$$

   where ξ_i is a combination of b_i's $i = (1, 2, ..., n)$ and b_i's are used for different combination of a_i's. When the dimension of the system is even, the behavior of the eigenvalues will take the form $E = E_0 \pm \sqrt{\xi_i}$. Also, let us mention that when the number of photons n is 18 this means that the degree of the polynomial equation (2.19) in terms of δ^2 is 5. Based on Abel-Ruffini theorem, we can not find radical expressions for the roots of any polynomial equation with degree more than five [30], hence for $n \geq 18$ we may not have algebraic solutions.

b) Odd dimension $\hat{n} = 2\tilde{k} + 1$, $\tilde{k} \in \mathbb{N}$.
   In this case $d_{\hat{n}}$ will have the form

$$d_{2\tilde{k}+1} = \delta^{2\tilde{k}+1} + b_1\delta^{2\tilde{k}-1} + b_2\delta^{2\tilde{k}-3} + \cdots + b_n\delta = 0$$
$$\implies \delta = 0, \quad \delta = \pm\sqrt{\xi_i}. \tag{2.20}$$

   So, when the dimension of the system is odd, it is seen that one of the roots for equation (2.20) will be zero $\delta = 0$, and its corresponding eigenvalue will be $E_k = E_0 = n\omega$.

As an example, we can see this behavior in two cases $n = 2$ and $n = 4$.

1) $n = 2$
   The two-dimensional Fock subspace $\mathcal{W}_2$ has the basis $\{|2, 0\rangle, |0, 1\rangle\}$. The Hamiltonian restricted to this subspace in this basis has the matrix form

$$H_2 = \begin{pmatrix} 2\omega & g\sqrt{2} \\ g\sqrt{2} & 2\omega \end{pmatrix}. \tag{2.21}$$

   The eigenvalue equation will be

$$(2\omega - E_k)^2 - 2g^2 = 0, \tag{2.22}$$

   whose eigenvalues E_k are

$$E_k = 2\omega \pm \sqrt{2}g. \tag{2.23}$$

2) $n = 4$
   The subspace $\mathcal{W}_4$ is generated by the vectors $|4, 0\rangle$, $|2, 1\rangle$, $|0, 2\rangle$. The matrix representation in this subspace will be

$$H_4 = \begin{pmatrix} 4\omega & 2g\sqrt{3} & 0 \\ 2g\sqrt{3} & 4\omega & 2g \\ 0 & 2g & 4\omega \end{pmatrix}. \tag{2.24}$$

   From the following eigenvalue equation

$$(4\omega - E_k)^3 - 16g^2(4\omega - E_k) = 0. \tag{2.25}$$

   The eigenvalues from this equation will be $E_1 = 4\omega + 4g$, $E_2 = 4\omega - 4g$, $E_3 = 4\omega$. As we can see, one of the eigenvalues (E_3) is in the form $n\omega$.

3. The associated Schrödinger equations

3.1. Differential realizations of the second harmonic generation model

The Hamiltonian (2.1) in the Bargmann-Fock representation by using following expression of creation and annihilation operators

$$a_1 = \partial_{z_1}, \quad a_1^+ = z_1, \qquad a_2 = \partial_{z_2}, \quad a_2^+ = z_2, \tag{3.1}$$

adopts the differential form

$$H_{SHG} = g z_2 \, \partial_{z_1}^2 + g z_1^2 \, \partial_{z_2} + 2 z_2 \, \partial_{z_2} + z_1 \, \partial_{z_1}, \qquad \omega = 1. \tag{3.2}$$

As we mentioned in the previous section, this system has a set of two independent and commuting symmetry operators, including the Hamiltonian H_{SHG},

$$\{ H_{SHG}, \ H_0 = \omega \, (N_1 + 2N_2) \}. \tag{3.3}$$

The symmetry H_0 will be a first order differential operator in the Bargmann-Fock representation

$$H_0 = z_1 \, \partial_{z_1} + 2 z_2 \, \partial_{z_2}, \quad \omega = 1. \tag{3.4}$$

In this realization, the Fock states are represented by the monomials

$$|n_1, n_2\rangle = \frac{(z_1)^{n_1} (z_2)^{n_2}}{(n_1! n_2!)^{1/2}}. \tag{3.5}$$

The subspace $\mathcal{W}_n$ will be generated by the above monomials such that $n_1 + 2n_2 = n = E_0$ (we suppose $\omega = 1$). The symmetry operator H_0 will help us to make a transformation from variables (z_1, z_2) to (s, t) so that the Hamiltonian H can be made separable. We can find this transformation by imposing that H_0 depends only on one variable. After straightforward calculations, we arrive at the following (non unique) simple solution:

$$s = z_2, \quad t = \frac{z_1^2}{z_2}. \tag{3.6}$$

We can express the partial derivatives $\frac{\partial}{\partial z_1}$ and $\frac{\partial}{\partial z_2}$ in terms of new variables as follows,

$$\frac{\partial}{\partial z_1} = 2 \sqrt{\frac{t}{s}} \frac{\partial}{\partial t}, \tag{3.7}$$

$$\frac{\partial}{\partial z_2} = \frac{\partial}{\partial s} - \frac{t}{s} \frac{\partial}{\partial t}. \tag{3.8}$$

The Hamiltonians H_{SHG} and H_0 in the new variables will have the following expression

$$H_{SHG} = 4gt \frac{\partial^2}{\partial t^2} + g(2 - t^2) \frac{\partial}{\partial t} + gts \frac{\partial}{\partial s} + 2s \frac{\partial}{\partial s}, \tag{3.9}$$

$$H_0 = 2s \frac{\partial}{\partial s}. \tag{3.10}$$

The simultaneous eigenfunctions for H_{SHG} and H_0 represented as $\Psi(s, t)$ satisfy,

$$H_{SHG} \Psi(s, t) = E \Psi(s, t)$$
$$H_0 \Psi(s, t) = E_0 \Psi(s, t) \implies \Psi(s, t) = s^{\frac{E_0}{2}} \psi(t), \tag{3.11}$$

where $E_0 = n$ is determined by the eigenspace $\mathcal{W}_n$ of H_0 in which we are restricting the Hamiltonian H. Replacing this separated eigenfunctions in the eigenvalue equation for H_{SHG} in (3.9), we get a second-order reduced differential equation for $\psi(t)$:

$$4gt \frac{d^2\psi}{dt^2} + g(2 - t^2) \frac{d\psi}{dt} + (E_0 - E + \frac{gtE_0}{2})\psi = 0. \tag{3.12}$$

This equation can be related to the bi-confluent Heun equation (BHE) because it has two singularities, at the origin and at the infinity, the first one being regular and the second one irregular [31]. The standard bi-confluent Heun equation has the following form

$$x \left(\frac{d^2 y}{dx^2}\right) + \left(1 + \alpha + \beta x - 2x^2\right)\left(\frac{dy}{dx}\right)$$
$$+ \left(-\frac{\delta + (1 + \alpha)\beta}{2} + (\gamma - \alpha - 2)x\right) y = 0. \tag{3.13}$$

In order to get this form we make a new change of variable $t = m\rho$, so we can rewrite the eigenvalue equation (3.12) as follows

$$\rho \left(\frac{d^2\psi}{d\rho^2}\right) + \left(\frac{1}{2} - \frac{m^2\rho^2}{4}\right)\left(\frac{d\psi}{d\rho}\right)$$
$$+ \left(\frac{(E_0 - E)m}{4g} + \frac{E_0 m^2 \rho}{8}\right) \psi(\rho) = 0. \tag{3.14}$$

Then, if we choose $m^2 = 8$ we will have that (3.14) becomes:

$$\rho \frac{d^2\psi}{d\rho^2} + (\frac{1}{2} - 2\rho^2)\frac{d\psi}{d\rho} + \left(\frac{m(E_0 - E)}{4g} + E_0\rho\right) \psi = 0, \tag{3.15}$$

which is exactly in the form of (3.13) with the following identification of parameters:

$$\alpha = -\frac{1}{2}, \quad \beta = 0, \quad \gamma = E_0 + \frac{3}{2}, \quad \delta = -\frac{m(E - E_0)}{2g}. \tag{3.16}$$

If we apply power series (Frobenius) method to Eq. (3.15) with $\psi(\rho)$ in the form $\sum_{i=0}^{\infty} \tilde{b}_i(\rho)^{i+\kappa}$, we will find two sets of solutions for $\kappa = 0$ and $\kappa = 1/2$ in following form:

$$\psi(\rho) = \sum_{i=0}^{\infty} \tilde{b}_i \rho^i \qquad \text{for} \quad \kappa = 0, \tag{3.17}$$

$$\psi(\rho) = \sum_{i=0}^{\infty} \tilde{b}_i \rho^{i+1/2} \quad \text{for} \quad \kappa = 1/2. \tag{3.18}$$

Then, the wavefunctions Ψ in the old variables z_1, z_2 will be

$$\Psi(z_1, z_2) = \sum_{i=0}^{\infty} \tilde{b}_i (z_1)^{2i} (z_2)^{\frac{E_0}{2} - i} \qquad \text{for} \quad \kappa = 0, \tag{3.19}$$

$$\Psi(z_1, z_2) = \sum_{i=0}^{\infty} \tilde{b}_i (z_1)^{2i+1} (z_2)^{\frac{E_0}{2} - i - \frac{1}{2}} \quad \text{for} \quad \kappa = 1/2. \tag{3.20}$$

From equation (3.5), we can see that the power of z_2 has to be integer. So we consider $E_0 = 2k$ in (3.19) and $E_0 = 2k + 1$ in (3.20). Then we have the following relation

$$\Psi(z_1, z_2) = \sum_{i=0}^{\infty} \tilde{b}_i (z_1)^{2i} (z_2)^{k-i} \qquad \text{for} \quad E_0 = 2k, \tag{3.21}$$

$$\Psi(z_1, z_2) = \sum_{i=0}^{\infty} \tilde{b}_i (z_1)^{2i+1} (z_2)^{k-i} \quad \text{for} \quad E_0 = 2k + 1. \tag{3.22}$$

As we see, the function $\Psi(z_1, z_2)$ will have a singularity for $i > k$ in both cases. Then, in order $\Psi(z_1, z_2)$ to be an analytical and polynomial solution, the power of z_2 must be greater or equal than zero and the upper bound in (3.21) and (3.22) power series has to be k. If we write $\Psi(z_1, z_2)$ in new variables (s, ρ), the function $\Psi(s, \rho)$ will be

$$\Psi(s, \rho) = (s)^k \sum_{i=0}^{k} \tilde{b}_i m^i \rho^i, \qquad\qquad E_0 = 2k, \tag{3.23}$$

$$\Psi(s, \rho) = (s)^{k+\frac{1}{2}} \sum_{i=0}^{k} \tilde{b}_i m^{i+\frac{1}{2}} \rho^{i+1/2}, \quad E_0 = 2k + 1, \tag{3.24}$$

with

$$\psi(\rho) = \sum_{i=0}^{k} \tilde{b}_i(m\rho)^i, \qquad E_0 = 2k, \tag{3.25}$$

$$\psi(\rho) = \sum_{i=0}^{k} \tilde{b}_i(m\rho)^{i+\frac{1}{2}}, \quad E_0 = 2k+1. \tag{3.26}$$

3.2. QES Schrödinger potentials

Every second-order second order equation of the form (3.15) can be transformed into a Schrödinger equation by means of a suitable change of variables and a gauge transformation [29]. We consider the case $m = 2\sqrt{2}$ (we will discuss the other sign later) and use the similarity transformation and the change of variables given by

$$\psi(\rho) = \exp[-W(x)]\,\chi(x), \qquad x = x(\rho). \tag{3.27}$$

Then, if the function $W(x)$ and the variable $x(\rho)$ satisfy the following equation:

$$\ddot{x} - 2(\dot{x})^2 \frac{dW}{dx} + \dot{x}\left(\frac{1}{2\rho} - 2\rho\right) = 0, \tag{3.28}$$

the function $\chi(\rho)$ will satisfy an Schrödinger equation (without the first order derivative) of the form

$$-\chi'' + V(x)\chi = \lambda\chi, \tag{3.29}$$

where $\dot{x} = \dfrac{dx}{d\rho}$ and the expression for the potential will be

$$V(x) = \frac{\ddot{x}}{\dot{x}^2}\frac{dW}{dx} - \left(\frac{dW}{dx}\right)^2 + \frac{d^2W}{dx^2} + \frac{1}{\dot{x}}\frac{dW}{dx}\left[\frac{1}{2\rho} - 2\rho\right] - \frac{a_0 + \rho E_0}{\rho\,\dot{x}^2}, \tag{3.30}$$

where a_0 is

$$a_0 = \frac{\sqrt{2}(E_0 - E)}{2g}. \tag{3.31}$$

We can find different forms for the function $W(x)$ and the potential $V(x)$ based on the new variable $x(\rho)$. A simple choice of this function is just a power $x(\rho) = \rho^b$, with $b \in \mathbb{R}^+$. In this case we get the following expressions for $V(x)$ and $\chi(x)$:

$$V_b(x) = \left(-\frac{1}{4} + \frac{1}{16b^2}\right)x^{-2} - \left(\frac{a_0}{b^2}\right)(x)^{-2+\frac{1}{b}} - \left(\frac{3+2E_0}{2b^2}\right)(x)^{-2+\frac{2}{b}} + \frac{1}{b^2}(x)^{-2+\frac{4}{b}}, \tag{3.32}$$

$$\chi_b(x) = x^{-(1-2b)/(4b)}\exp\left[-\frac{x^{\frac{2}{b}}}{2}\right]\psi(\rho(x)). \tag{3.33}$$

The wavefunction χ in (3.33) will be physical square integrable for $b \geqslant \dfrac{1}{2}$, assuming that $\psi(\rho)$ is a polynomial. We can get different realizations of this potential for the following values of b: 1/2, 1, and 2. In this way we obtain some simplified expressions by the vanishing of one of the powers of x in (3.32).

1) $b = 1/2$.
   For this value of b, the potential and wavefunction respectively will be

$$V_{1/2}(x) = 4x^6 - x^2(6 + 4E_0) - 4a_0, \tag{3.34}$$

$$\chi_{\frac{1}{2}}(x) = \exp\left[-\frac{x^4}{2}\right]\psi(\rho(x)). \tag{3.35}$$

According to the definition of a_0 in (3.31), we can replace

$$a_0 = \frac{\sqrt{2}E_0}{2g} - \frac{\sqrt{2}}{2g}E, \tag{3.36}$$

and redefine

$$\tilde{V}_{1/2}(x) = 4x^6 - x^2(6 + 4E_0) - \frac{2\sqrt{2}E_0}{g}. \tag{3.37}$$

Then, the resulting Schrödinger equation is:

$$-\chi_{1/2}''(x) + \tilde{V}_{1/2}(x)\chi_{1/2}(x) = -\frac{2\sqrt{2}}{g}E\chi_{1/2}(x). \tag{3.38}$$

Therefore in this case we have a sextic potential depending on the parameter E_0 of the finite dimensional eigenspace while the energy λ for $\chi_{1/2}(x)$ correspond to the values of $(-2\sqrt{2}E)/g$ in this eigenspace. The potential $\tilde{V}_{1/2}(x)$ depends on x^2 and wave function $\chi_{1/2}(x)$ depends also on polynomials in x^2. This means that this potential is symmetric in x and the even and odd solutions can be obtained from (3.35) with (3.25) and (3.26) respectively for even and odd number of photons.

2) $b = 1$.
   In this case the potential function is

$$V_1(x) = -\frac{3}{2} - \frac{3}{16x^2} + x^2 - \frac{a_0}{x} - E_0. \tag{3.39}$$

We can discuss about this potential in two cases $a_0 = 0$ and $a_0 \neq 0$.
When the dimension of the eigenspace is odd, then according to the odd case we mentioned in Section 2.1, the parameter a_0 will be zero for one of the eigenvalues ($E_k = E_0$). So we can consider the potential $\tilde{V}_1(x)$ for this case as an oscillator potential, plus a centrifugal term, with energy E_0. The potential $\tilde{V}_1(x)$ is

$$\tilde{V}_1(x) = -\frac{3}{2} - \frac{3}{16x^2} + x^2. \tag{3.40}$$

In other cases ($a_0 \neq 0$), there will coexist an oscillator and centrifugal plus a Coulomb potential. This potential as

$$\tilde{\tilde{V}}_1(x) = -\frac{3}{2} - \frac{3}{16x^2} + x^2 - \frac{a_0}{x}, \tag{3.41}$$

can be obtained from a system of two interacting electrons in an external harmonic potential. Examples of this kind of system are given as quantum dots, and it is mentioned in many references, see for example [32–34].
The energy λ of the Schrödinger equation for both cases is E_0 and the corresponding wavefunction $\chi_1(x)$ is

$$\chi_1(x) = x^{\frac{1}{4}}\exp\left[-\frac{x^2}{2}\right]\psi(\rho(x)). \tag{3.42}$$

3) $b = 2$.
   In this case, the potential is

$$V_2(x) = \frac{1}{4} - \frac{15}{64x^2} - \frac{a_0}{4x^{3/2}} + \frac{-3 - 2E_0}{8x}. \tag{3.43}$$

In this case, the potential includes as intrinsic parameters both a_0 and E_0, which can not take the role of a non-vanishing energy, thus the energy of the corresponding Schrödinger equation is zero:
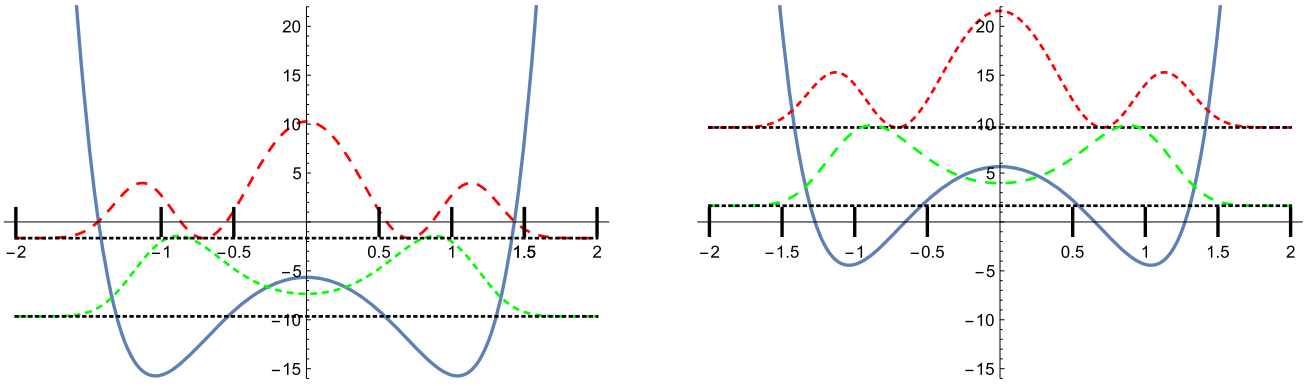
Fig. 1. The potential $\tilde{V}_{1/2}(x)$ (solid line) and probability density function $|\chi_p(x)|^2$ (dashed line) for $n = 2$ in cases $t = 2\sqrt{2}\rho$ placed at the energy levels (dotted lines) $\varepsilon_\pm = -2\sqrt{2}(2 \pm \sqrt{2})$ (left) and $t = -2\sqrt{2}\rho$ placed at the energy levels (dotted lines) $\varepsilon_\pm = 2\sqrt{2}(2 \pm \sqrt{2})$ (right).

$$-\chi_2''(x) + V_2(x)\chi_2(x) = 0. \tag{3.44}$$

The wavefunction $\chi_2(x)$ has the expression

$$\chi_2(x) = x^{\frac{3}{8}} \exp\left[-\frac{x}{2}\right] \psi(\rho(x)). \tag{3.45}$$

If we replace ρ by $-\rho$ in (3.15), we will have the same differential equation but for a negative value $m = -2\sqrt{2}$. Hence we encounter another set of potentials in which the relation between x and ρ are in the form $x = (-\rho)^b$ rather than $x = (\rho)^b$.

4. Two examples: the sextic and the quantum dot potentials

As we have seen in the last section, potentials $\tilde{V}_1(x)$, $\tilde{\tilde{V}}_1(x)$ and $\tilde{V}_{1/2}(x)$ represent well known physical system. In this section, we concentrate on the illustration of these potentials and corresponding probability density functions for (*i*) two-dimensional ($\mathcal{W}_2$) and (*ii*) three-dimensional ($\mathcal{W}_3$) subspaces. The examples of these two subspaces were discussed in Section 2. The potentials $\tilde{\tilde{V}}_1(x)$ for $n = 2$ and $\tilde{V}_{1/2}(x)$ for $n = 2$ and $n = 3$ are considered in two-dimensional subspace. In three-dimensional subspace, the potentials $\tilde{V}_1(x)$ and $\tilde{\tilde{V}}_1(x)$ are illustrated for $n = 4$ as well the even and odd solutions for the potential $\tilde{V}_{1/2}(x)$ are shown for $n = 4$ and $n = 5$ respectively.

4.1. Two-dimensional subspace $\mathcal{W}_2$

The bases and matrix representation for two-dimensional subspace of $n = 2$ was obtained as an example in Section 2. If we choose $\omega = 1$ and $g = 1$ for simplicity, the eigenvalues and eigenfunctions corresponding to the two by two matrix Hamiltonian are

$$|\psi_p\rangle = \gamma_{p,1}|2, 0\rangle + \gamma_{p,2}|0, 1\rangle, \quad p = 1, 2, \tag{4.1}$$

with

p	E_p	$\gamma_{p,1}$	$\gamma_{p,2}$
1	$2 + \sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
2	$2 - \sqrt{2}$	$1/\sqrt{2}$	$-1/\sqrt{2}$

We can write the eigenfunctions (4.1) in terms of the variables ρ, s (see (3.25)) as follows:

$$\Psi_p(\rho, s) = s\left(\gamma_{p,2} + 2\gamma_{p,1}\rho\right) = s\,\psi(\rho). \tag{4.2}$$

The wavefunctions are

$$\chi_p(x) = e^{W(x)}\left(\gamma_{p,2} + 2\gamma_{p,1}\rho\right). \tag{4.3}$$

When we choose $b = 1/2$, that is $x = \sqrt{\rho}$, we have the Schrödinger equation with potential $\tilde{V}_{1/2}(x)$ as (3.37) with the energy $\varepsilon_\pm = -2\sqrt{2}(2 \pm \sqrt{2})$. The potential $\tilde{V}_{1/2}(x)$ and the probability density function $|\chi_p(x)|^2$ are depicted in Fig. 1 (left). The corresponding potential and density $|\chi_p(x)|^2$ are shown for the case $t = -2\sqrt{2}\rho$ with the energy $\varepsilon_\pm = 2\sqrt{2}(2 \pm \sqrt{2})$ in Fig. 1 (right), in this case the new potential is the same as before except for the addition of a constant (see Fig. 1 (right)). As we can see, the potential $\tilde{V}_{1/2}(x)$ is symmetric in x and the solutions here obtained are even solutions.

If we want to show the odd solutions for the potential $\tilde{V}_{1/2}(x)$, we have to find the wavefunctions for $n = 3$. The bases of two-dimensional subspace are $\{|3, 0\rangle, |1, 1\rangle\}$ for this case. The corresponding two by two Hamiltonian in this subspace has the following eigenfunctions and eigenvalues

$$|\psi_p\rangle = \gamma_{p,1}|3, 0\rangle + \gamma_{p,2}|1, 1\rangle, \quad p = 1, 2, \tag{4.4}$$

with

p	E_p	$\gamma_{p,1}$	$\gamma_{p,2}$
1	$3 + \sqrt{6}$	$1/\sqrt{2}$	$1/\sqrt{2}$
2	$3 - \sqrt{6}$	$1/\sqrt{2}$	$-1/\sqrt{2}$

The eigenfunctions (4.4) can be written as a function with variables ρ, s (see (3.26)) as follows:

$$\Psi_p(\rho, s) = s^{\frac{3}{2}}\left(\gamma_{p,2}\,\rho^{\frac{1}{2}} + \frac{2}{\sqrt{3}}\gamma_{p,1}\,\rho^{\frac{3}{2}}\right) = s\,\psi(\rho). \tag{4.5}$$

The wavefunctions are

$$\chi_p(x) = e^{W(x)}\left(\gamma_{p,2}\,x + \frac{2}{\sqrt{3}}\gamma_{p,1}\,x^3\right). \tag{4.6}$$

The $\tilde{V}_{1/2}(x)$ and corresponding probability density functions $|\chi_p(x)|^2$ are depicted in Fig. 2 for $t = \pm 2\sqrt{2}\rho$ with corresponding energies.

For another case $b = 1$, that is, $x(\rho) = \rho$, we face potential $\tilde{\tilde{V}}_1(x)$. The corresponding potential and the probability density function $|\chi_p(x)|^2$ are shown for $t = -2\sqrt{2}\rho$ in Fig. 3 and $t = -2\sqrt{2}\rho$ in Fig. 4. In this case the potentials for different signs are simply interchanged.

4.2. Three-dimensional subspace $\mathcal{W}_3$

For $n = 4$, we encounter a three dimensional subspace as we mentioned in Section 2.1 where the Hamiltonian has a 3×3 matrix representation. The eigenvalues and eigenfunctions in this subspace are
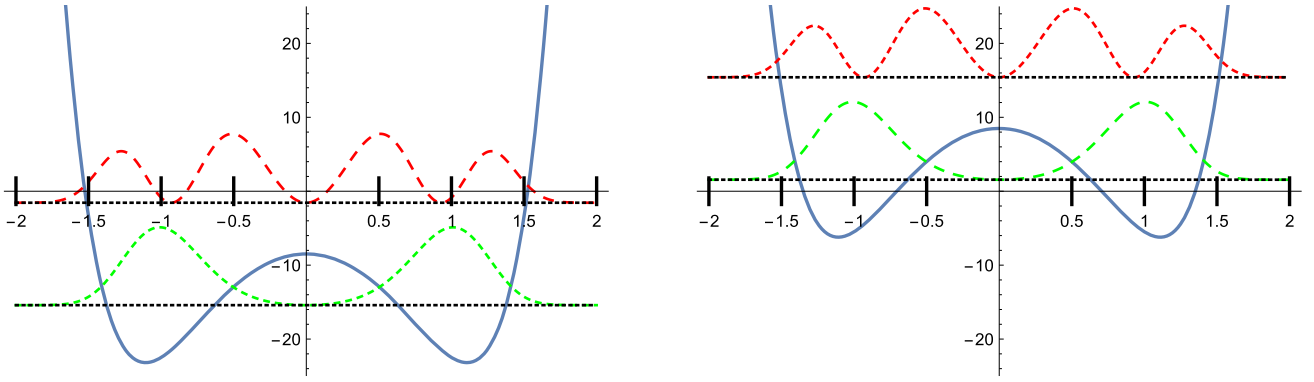
Fig. 2. The potential $\tilde{V}_{1/2}(x)$ (solid line) and probability density function $|\chi_p(x)|^2$ (dashed line) for $n=3$ in cases $t=2\sqrt{2}\rho$ placed at the energy levels (dotted lines) $\varepsilon_\pm = -2\sqrt{2}(2\pm\sqrt{2})$ (left) and $t=-2\sqrt{2}\rho$ placed at the energy levels (dotted lines) $\varepsilon_\pm = 2\sqrt{2}(2\pm\sqrt{2})$ (right).
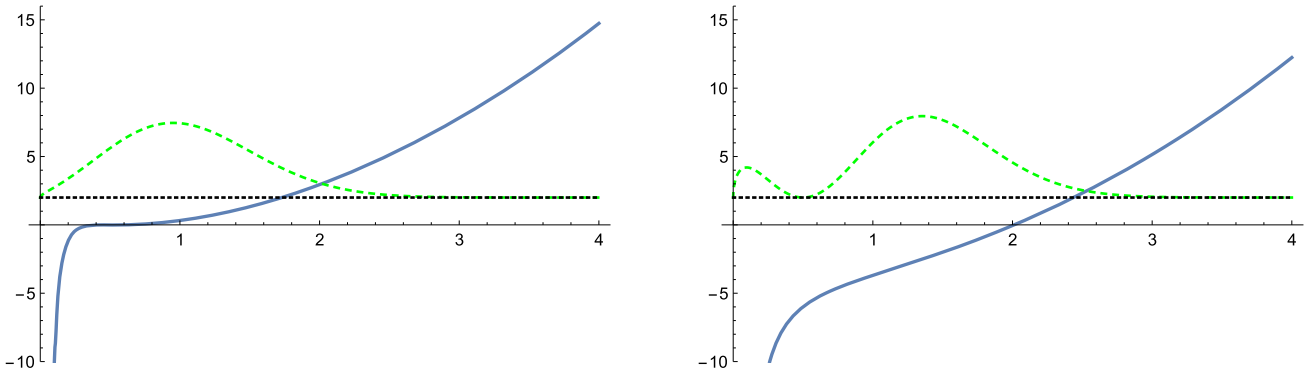

Fig. 3. The potential $\tilde{\tilde{V}}_1(x)$ (solid line) and probability density function $|\chi_p(x)|^2$ (dashed line) for $E_1 = 2+\sqrt{2}$ (left) and $E_2 = 2-\sqrt{2}$ (right) at the energy level $\varepsilon = 2$.
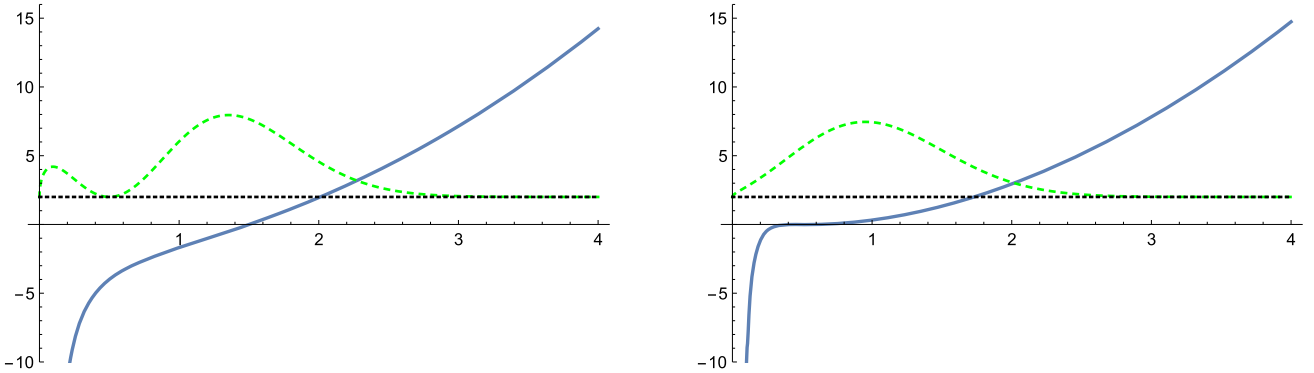

Fig. 4. The plots for case $t=-2\sqrt{2}\rho$ of the potential $\tilde{\tilde{V}}_1(x)$ (solid line) and probability density function $|\chi_p(x)|^2$ (dashed line) for $E_1 = 2+\sqrt{2}$ (left) and $E_2 = 2-\sqrt{2}$ (right) at the energy level $\varepsilon = 2$.

$$|\psi_p\rangle = \gamma_{p,1}|4,0\rangle + \gamma_{p,2}|2,1\rangle + \gamma_{p,3}|0,2\rangle, \quad p=1,2,3, \quad (4.7)$$

with

p	E_p	$\gamma_{p,1}$	$\gamma_{p,2}$	$\gamma_{p,3}$
1	8	$\sqrt{3}/(2\sqrt{2})$	$1/\sqrt{2}$	$1/(2\sqrt{2})$
2	0	$-\sqrt{3}/(2\sqrt{2})$	$1/\sqrt{2}$	$-1/(2\sqrt{2})$
3	4	$1/2$	0	$-\sqrt{3}/2$

We can write these eigenfunctions in terms of the variables ρ, s as follows:

$$\Psi_p(\rho,s) = s^2\left(\frac{1}{\sqrt{2}}\gamma_{p,3} + 2\gamma_{p,2}\,\rho + \frac{2\sqrt{6}}{3}\gamma_{p,1}\,\rho^2\right). \quad (4.8)$$

The wavefunctions are

$$\chi_p(x) = e^{W(x)}\left(\frac{1}{\sqrt{2}}\gamma_{p,3} + 2\gamma_{p,2}\,\rho + \frac{2\sqrt{6}}{3}\gamma_{p,1}\,\rho^2\right). \quad (4.9)$$

When we choose $b=1$, that is, $x(\rho) = \rho$, we find the potential $\tilde{V}_1(x)$ and $\tilde{\tilde{V}}_1(x)$. The Schrödinger equation for both of these potentials has energy E_0. These potentials and the probability density functions $|\chi_p(x)|^2$ are shown in Fig. 5.

On the other hand, when we choose $b=1/2$, that is $x=\sqrt{\rho}$, we have Schrödinger equation with potential $\tilde{V}_{\frac{1}{2}}(x)$ as shown in (3.37) with the energy $\varepsilon_p = -2\sqrt{2}(E_p)$. The potential $\tilde{V}_{\frac{1}{2}}(x)$ and the probability density function $|\chi_p(x)|^2$ can be seen in Fig. 6 (left). The same comments on the even eigenvalues and eigenfunctions apply in this example as in the previous $n=2$ and $b=1/2$ case.
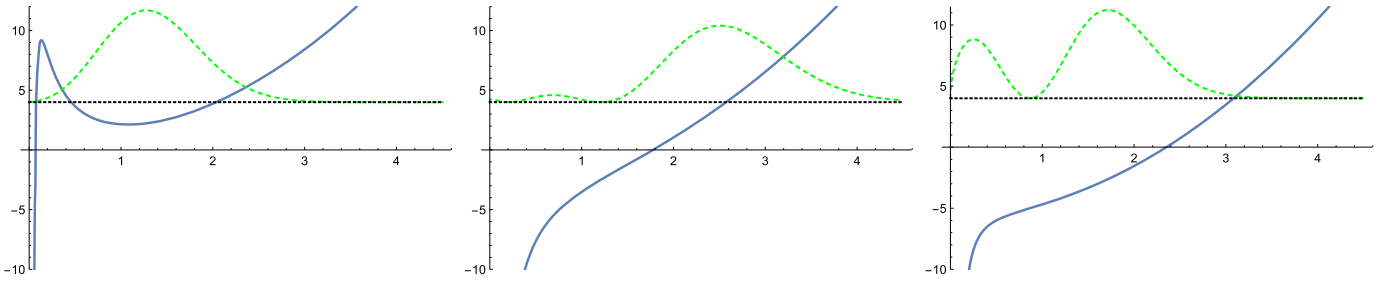
Fig. 5. The potentials (solid line) $\tilde{V}_1(x)$ for $E_1 = 8$ (left), $E_2 = 0$ (center), $\tilde{\tilde{V}}_1(x)$ for $E_3 = 4$ (right) and probability density functions $|\chi_p(x)|^2$ (dashed lines) at the energy level $\varepsilon = 4$ (dotted line).
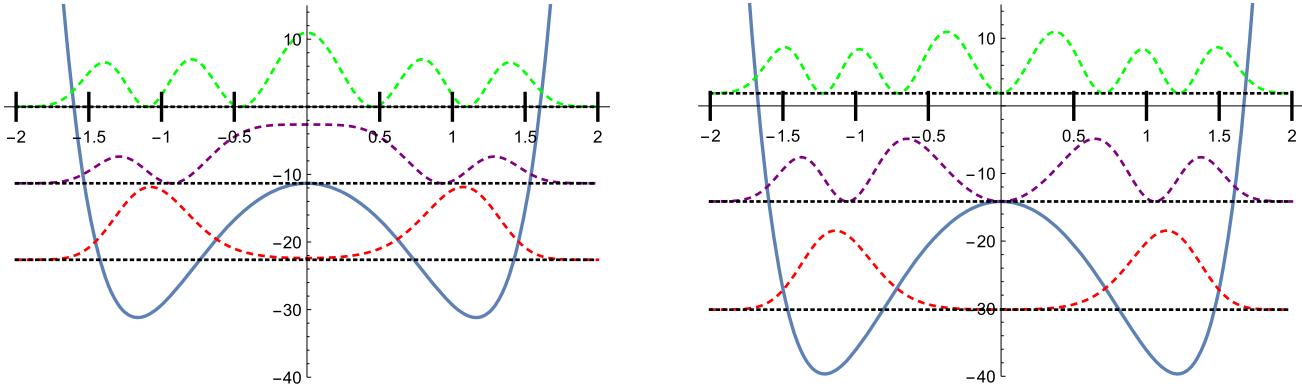

Fig. 6. The potential $\tilde{V}_{1/2}(x)$ (solid line) and probability density functions $|\chi_p(x)|^2$ (dashed lines) placed at the energy levels $\varepsilon_p = -2\sqrt{2}E_p$ (dotted lines) for $n = 4$ (left) and $n = 5$ (right).

The wavefunctions for $n = 5$ show the odd solutions for the potential $\tilde{V}_{\frac{1}{2}}(x)$ in this three-dimensional subspace. The eigenvalues and eigenfunctions in this subspace are

$$|\psi_p\rangle = \gamma_{p,1}|5,0\rangle + \gamma_{p,2}|3,1\rangle + \gamma_{p,3}|1,2\rangle, \quad p = 1,2,3, \quad (4.10)$$

with

p	E_p	$\gamma_{p,1}$	$\gamma_{p,2}$	$\gamma_{p,3}$
1	$5 + 4\sqrt{2}$	$\sqrt{5}/4$	$1/\sqrt{2}$	$\sqrt{3}/4$
2	$5 - 4\sqrt{2}$	$-\sqrt{5}/4$	$1/\sqrt{2}$	$-\sqrt{3}/4$
3	5	$\sqrt{3}/\sqrt{8}$	0	$-\sqrt{5}/\sqrt{8}$

After writing these eigenfunctions in terms of variables (ρ, s), the wavefunctions will be in the following form:

$$\chi_p(x) = e^{W(x)}\left(\frac{1}{\sqrt{2}}\gamma_{p,3}\,x + \frac{2}{\sqrt{3}}\gamma_{p,2}\,x^3 + \frac{2\sqrt{2}}{\sqrt{15}}\gamma_{p,1}\,x^5\right).$$

$$(4.11)$$

In Fig. 6, the potential $\tilde{V}_{1/2}(x)$ and probability density functions $|\chi_p(x)|^2$ are illustrated for $n = 4$ and $n = 5$.

5. Conclusions

In this work, in a first step we have studied the algebraic structure underlying the second harmonic generation Hamiltonian and later, as an application, we have developed some interesting differential realizations based on its symmetries. We began by writing the Hamiltonian of this system based on rising J_+ and lowering J_- the operators of the polynomial deformations of the Lie algebra $sl(2, R)$, using the finite dimensional space, with basis vectors $|J, M\rangle$, to calculate eigenvalue equation. Then we found a recurrence relation for the determinant of the matrix representation of the characteristic equation.

In deriving our results, we showed that we can predict the behavior of eigenvalues for the large number of photons. When the dimension of the subspace is even, the eigenvalues are in the form $E_0 \pm g\sqrt{\xi_i}$. Also one of the eigenvalues will be zero for odd-dimensional subspace. On the other hand, we found that there is no radical expression for $n \geq 18$ which was obtained in [18].

The natural differential realization of the SHG Hamiltonian is obtained by means of the Bargmann-Fock representation. The change of variables to the new set of variables by using symmetry enables us to find the bi-confluent Heun equation as differential form of this system. The transformation between bi-confluent Heun equation and Schrödinger equation has been considered in many Refs. [35–39]. We made the map from this second-order differential equation to Schrödinger equation by a change of variable in the form $x = \rho^b$ and we have different set of quasi-exactly solvable potentials for different values of b.

As examples, we provided some explicit expressions for potential and wavefunctions for two values of $b = 1$ and $b = 1/2$ in two and three dimensional subspaces. We consider $\tilde{V}_1(x)$ and $\tilde{\tilde{V}}_1(x)$, $\tilde{V}_{1/2}(x)$ for $n = 2, 3$ and $n = 4, 5$ which have two and three dimensional subspace respectively. The potential $\tilde{V}_{1/2}(x)$ is a sextic potential which was obtained for this optical system in Refs. [27,40], while the other potential $\tilde{V}_1(x)$ is used to describe quantum dots. In two dimensional subspace, we have only one sextic potential for two different eigenfunctions for $b = 1/2$. We have found even eigenfunctions that correspond to ground state and second excited state for $n = 2$ in Fig. 1 and odd solutions that correspond to first and third excited state for $n = 3$ in Fig. 2. On the other hand, for $b = 1$, we have one potential $\tilde{\tilde{V}}_1(x)$ for each eigenfunctions. These eigenfunctions correspond to ground and first excited state which both of them have energy $\varepsilon = 2$.

In second case, $n = 4$, we have similar considerations. For $b = 1/2$, one sextic potential corresponds to three eigenfunctions with negative energies. For $n = 4$, we have seen even eigenfunctions

correspond to ground, second and fourth excited states and odd eigenfunctions correspond to first, third and fifth excited states for $n = 5$ in Fig. 6. For $b = 1$, we have one potential for each of three eigenfunctions. These eigenfunctions correspond to ground, first and second excited states which all of them have energy $\varepsilon = 4$.

As a brief summary, we found a family of quasi-exactly solvable potentials, labeled by the parameter b, which are related to the second harmonic generation. The two special values $b = 1$ and $b = 1/2$ have been considered in detail. For $b = 1/2$ we have found one potential with the first n even eigenvalues. In the case $b = 1$ for each $n \in \mathbb{N}$, we have found one eigenfunction (starting from the first up to the n-th) of n different potentials.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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