Zernike functions, rigged Hilbert spaces, and potential applications

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ABSTRACT

We revise the symmetries of the Zernike polynomials that determine the Lie algebra $su(1, 1) \oplus su(1, 1)$. We show how they induce discrete as well as continuous bases that coexist in the framework of rigged Hilbert spaces. We also discuss some other interesting properties of Zernike polynomials and Zernike functions. One of the areas of interest of Zernike functions has been their applications in optics. Here, we suggest that operators on the spaces of Zernike functions may play a role in optical image processing.

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I. INTRODUCTION

The presently known Zernike polynomials were introduced by Zernike in 1934[1] due to their possible applications in optics. Nowadays, they are the main ingredient in the construction of the Zernike functions, which are an orthonormal basis for the Hilbert space of square integrable functions on the unit disk. There is a wide bibliography on these functions and their mathematical properties.[2–12] Recent studies underline their role in the analysis of integrable and superintegrable systems as well as the determination of separable coordinates.[13–16] It is interesting to remark that Zernike polynomials are the analogs of the spherical harmonics for the disk.

In optical image processing, adaptive optics serves to clean signals,[17] where an auxiliary photoreceptor measures the wavefront deformations introduced by the medium and acts on the instrument to induce the opposite effect. Adaptive optics removes indeed the spurious phases in the complex function $f(r, \theta)$ that represents the optical signal on the disk, allowing us to obtain a cleaned image $|f(r, \theta)|^2$. This function may not be considered as the final result of the process, but an intermediate step, which may be further elaborated by means of an action called *soft adaptive optics*.[18] While "hard adaptive optics" acts on the perturbations of the phase introduced solely by the medium, the soft elaboration of the numerical image $|f(r, \theta)|^2$ operates on a set of pixels, so as to obtain another set of pixels. This is independent from the cause of the distortion, i.e., wind in atmosphere, diffraction in lens, defects of the apparatus, etc., and the particular instrument of measure. Hard adaptive optics can only be applied to the cleaning of images, although this soft approach converts images into images and can be employed everywhere. Such a transformation of images can play a role in, for instance, laser physics, microscopic images, radioastronomy, or in general instrumental improvement.

One of the objectives of the present paper is the construction of a theory of operators acting in the space of images defined on the disk. These operators transform images into images. To this end, it is desirable to have an algebra of operators with properties of continuity over some appropriate space. These operators are unbounded with a dense common domain in the space of square integrable functions on the unit disk, $L^2(\mathcal{D}, r\, dr\, d\theta)$ [$\equiv L^2(\mathcal{D})$], where the Zernike functions form an orthonormal basis. Ladder operators should be included in the algebra of unbounded operators. With this purpose, we need to endow the dense subspace supporting the algebra of operators with a topology stronger than the topology inherited from the Hilbert space $L^2(\mathcal{D})$. This leads to the concept of rigged Hilbert space (RHS).

A rigged Hilbert space, also called Gelfand triplet, is a set of three spaces $\Phi \subset \mathcal{H} \subset \Phi^\times$, where $\mathcal{H}$ is a Hilbert space, Φ a dense subspace of $\mathcal{H}$ endowed with a topology finer (with more open sets) than the topology that Φ has inherited from $\mathcal{H}$, and $\Phi^\times$ the dual space of Φ. We do not want to discuss properties and other applications of RHS here, since there is a vast available bibliography on the subject.[19–25] The space Φ is the common domain of the operators in the algebra, which with the topology on Φ becomes continuous and can be continuously extended into $\Phi^\times$.

In addition, RHS is the proper framework where discrete (complete orthonormal sets in separable Hilbert spaces) and continuous (widely used in quantum mechanics) bases coexist. This is one of the great advantages of RHS, which will also play a role in our discussion.

In previous works,[18,26–33] we have shown the closed relation existing between the bases of special functions, Lie algebra representations, and RHS. This is also the purpose of the present article, where we shall establish the relations existing between Zernike functions, unitary irreducible representations of $su(1,1) \oplus su(1,1)$, the universal enveloping algebra $UEA[su(1,1) \oplus su(1,1)]$, and our particular choice of RHS.

This paper is organized as follows: In Sec. II, we discuss some relevant properties of Zernike functions, while we leave for Sec. III a discussion of the RHS implementation for our purposes. In Sec. IV, we introduce the algebras of continuous operators that will be used in Sec. V to implement the procedure for soft adaptive optics. Finally, in Appendixes A and B, we present some interesting properties of the Zernike polynomials that we have used along this paper. In Appendix C, we show a new topology for the space of Zernike functions. This topology is obtained from a family of norms different from the norms used in Secs. III and IV, but we obtain similar results to those obtained with the original topology.

II. ZERNIKE FUNCTIONS: DISCRETE AND CONTINUOUS BASES ON THE UNIT DISK

Zernike functions $\mathcal{Z}_n^m(r,\theta)$ on the closed unit circle,

$$\mathcal{D} = \{(r,\theta), \ 0 \le r \le 1, \ \theta \in [0, 2\pi)\}, \tag{1}$$

are expressions of the form

$$\mathcal{Z}_n^m(r,\theta) := R_n^m(r)\, e^{im\theta}, \quad n \in \mathbb{N}, \ m \in \mathbb{Z} \tag{2}$$

such that

$$n \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad |m| \le n, \quad \frac{n - |m|}{2} \in \mathbb{N}, \tag{3}$$

where $R_n^m(r)$ are real polynomial, called Zernike radial polynomials, solutions of the following differential equation:

$$\left[(1 - r^2) \frac{d^2}{dr^2} - \left(3r - \frac{1}{r} \right) \frac{d}{dr} + n(n+2) - \frac{m^2}{r^2} \right] R_n^m(r) = 0, \tag{4}$$

with $R_n^m(1) = 1$. Note that in the differential Eq. (4), the label m appears as m^2. This shows that the Zernike polynomials have the symmetry $R_n^m(r) = R_n^{-m}(r)$. Also, it can be seen in the last relation of (3) that n and m have the same parity [i.e., $n \equiv m \pmod 2$], in other words n and m ought to be even or odd at the same time. An explicit formula for the Zernike polynomials is

$$R_n^m(r) = \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \binom{n-k}{k} \binom{n-2k}{\frac{n-m}{2} - k} r^{n-2k}. \tag{5}$$

Moreover, they are related with the Jacobi polynomials $J_n^{(\alpha,\beta)}(x)$ as follows:[2,5]

$$R_n^m(r) = (-1)^{(n-m)/2}\, r^m\, J_{(n-m)/2}^{(m,0)}(1 - 2r^2).$$

Zernike polynomials satisfy some important properties. First of all, for each fixed value of m, the polynomials $R_n^m(r)$ fulfill the following orthogonality condition:

$$\int_0^1 R_n^m(r)\, R_{n'}^m(r)\, r\, dr = \frac{\delta_{n,n'}}{2(n+1)}, \tag{6}$$

along a completeness relation of the type

$$\sum_{\substack{n=|m| \\ n \equiv m \,(\mathrm{mod}\ 2)}}^{\infty} R_n^m(r)\, R_n^m(r')\, (n+1) = \frac{\delta(r - r')}{2r}. \tag{7}$$

The Zernike polynomials may be extended to any $r \in [-1, 1]$. In Appendix A, we discuss the main facts relative to this extension and the enlarged symmetries. This generalization gives a relation between Zernike and Legendre polynomials.

A. W-Zernike functions

We redefine the Zernike functions (2) in a slightly different way by introducing a numerical factor and by changing the parameterization. We call "W-Zernike functions" and denote $W_{u,v}(r, \theta)$ to this new class of functions.

Let us introduce the parameters u and v, defined as follows:[6]

$$u := \frac{n+m}{2}, \qquad v := \frac{n-m}{2}. \tag{8}$$

The parameters u and v are positive integer numbers and are independent of each other. Hence, the W-Zernike functions, $W_{u,v}(r, \theta)$ (with $u, v = 0, 1, 2, \ldots$), are functions on the closed unit circle $\mathcal{D}$ (1) defined by

$$W_{u,v}(r, \theta) := \sqrt{\frac{u+v+1}{\pi}}\, \mathcal{Z}_{u+v}^{u-v}(r, \theta)$$

$$= \sqrt{\frac{u+v+1}{\pi}}\, R_{u+v}^{|u-v|}(r)\, e^{i(u-v)\theta}. \tag{9}$$

Note that when u and v have the same (different) parity, $R_{u+v}^{|u-v|}(r)$ are polynomials of even (odd) degree.

The W-Zernike functions have the following properties:

1. They are square integrable in $L^2(\mathcal{D}, r\,dr\,d\theta) \equiv L^2(\mathcal{D})$ because $R_n^m(r)$ are polynomials.
2. They satisfy the following identities:

$$W_{v,u}(r, \theta) = W_{u,v}(r, \theta)^* = W_{u,v}(r, -\theta), \tag{10}$$

where the star denotes complex conjugation. This symmetry property holds from $R_n^m(r) = R_n^{-m}(r)$ and (9).

3. Orthonormality in $L^2(\mathcal{D})$:

$$\langle W_{u',v'}, W_{u,v} \rangle = \int_0^{2\pi} d\theta \int_0^1 dr\, r\, W_{u',v'}(r, \theta)^*\, W_{u,v}(r, \theta) = \delta_{u,u'}\, \delta_{v,v'}. \tag{11}$$

4. The following completeness relation holds:

$$\sum_{u,v=0}^{\infty} W_{u,v}(r, \theta)\, W_{u,v}^*(r', \theta') = \frac{1}{2r}\, \delta(r-r')\, \delta(\theta-\theta'). \tag{12}$$

5. From the property of the Zernike polynomials $|R_n^m(r)| \le 1$, $0 \le r \le 1$, we find an upper bound for the Zernike functions,

$$|W_{u,v}(r, \theta)| \le \sqrt{\frac{u+v+1}{\pi}}, \quad \forall (r, \theta) \in \mathcal{D}. \tag{13}$$

B. Discrete and continuous bases on the unit disk

The above properties show that the set of Zernike functions $\{W_{u,v}(r, \theta)\}_{u,v \in \mathbb{N}}$ is an orthonormal basis in $L^2(\mathcal{D})$. Hence, for any function $f(r, \theta)$ of $L^2(\mathcal{D})$, we have, in the sense of convergence on the Hilbert space $L^2(\mathcal{D})$, that

$$f(r, \theta) = \sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r, \theta), \tag{14}$$

where $f_{u,v}$ are complex numbers given by

$$f_{u,v} = \int_0^{2\pi} d\theta \int_0^1 dr\, r\, W_{u,v}^*(r, \theta) f(r, \theta). \tag{15}$$

Moreover, from (11), (12), and (14) we obtain that

$$\langle f, f \rangle = \int_0^{2\pi} d\theta \int_0^1 dr\, r f^*(r, \theta) f(r, \theta)$$

$$= \sum_{u,v=0}^{\infty} |f_{u,v}|^2 < \infty.$$

In adaptive optics, one always chooses $f(r, \theta)$ real so that $f_{u,v} = f_{v,u}^*$.

As is customary in quantum mechanics, let us introduce the generalized continuous basis $\{|r, \theta\rangle\}_{(r,\theta)\in\mathcal{D}}$ whose elements have the following properties:

$$\langle r, \theta | r', \theta' \rangle = \frac{1}{r} \delta(r - r') \delta(\theta - \theta'),$$

$$\mathbb{I} = \int_0^{2\pi} d\theta \int_0^1 dr\, r |r, \theta\rangle\langle r, \theta|,$$

$$(16)$$

where $\mathbb{I}$ is the identity operator. Next, we define the kets $|u, v\rangle$ with the help of the Zernike functions $W_{u,v}(r, \theta)$

$$|u, v\rangle := \int_0^{2\pi} d\theta \int_0^1 r\, dr\, |r, \theta\rangle\, W_{u,v}(r, \theta), \qquad (17)$$

with $u, v = 0, 1, 2, \ldots$, which have the following properties as one can check from the orthogonality relations (11) and (16):

$$\langle u, v | u', v' \rangle = \delta_{u,u'} \delta_{v,v'},$$

$$\sum_{u,v=0}^{\infty} |u, v\rangle\langle u, v| = \mathcal{I}.$$

$$(18)$$

Due to the fact that $\{|u, v\rangle\}_{u,v\in\mathbb{N}}$ is a discrete basis and $\{|r, \theta\rangle\}_{(r,\theta)\in\mathcal{D}}$ a continuous basis, we make a distinction between the identities $\mathbb{I}$ and $\mathcal{I}$, which, in principle, should be different. From expression (17), and taking into account the first relation of (16), we obtain

$$\langle r, \theta | u, v \rangle = W_{u,v}(r, \theta) \qquad (19)$$

so that the set of vectors $\{|u, v\rangle\}$ forms an orthonormal basis on a Hilbert space, $\mathcal{H}$, unitarily equivalent to $L^2(\mathcal{D})$ and the Zernike functions are the transition elements between both bases. On this Hilbert space $\mathcal{H}$, the identity operator is $\mathcal{I}$ (18).

Taking into account relations (14), (15), and (19), we have for any $f(r, \theta) \in L^2(\mathcal{D})$ (14) that

$$|f\rangle := \sum_{u,v=0}^{\infty} f_{u,v} |u, v\rangle \in \mathcal{H}.$$

Then, the space of all vectors

$$|f\rangle = \sum_{u,v=0}^{\infty} f_{u,v} |u, v\rangle \qquad (20)$$

such that $\sum_{u,v=0}^{\infty} |f_{u,v}|^2 < \infty$ completes an abstract Hilbert space, $\mathcal{H}$, and the mapping

$$\mathcal{U} : \mathcal{H} \longmapsto L^2(\mathcal{D})$$
$$|f\rangle \longmapsto \mathcal{U}|f\rangle = \langle r, \theta | f \rangle = f(r, \theta) \qquad (21)$$

is unitary. The vectors $|f\rangle \in \mathcal{H}$ admit two representations in terms of the continuous basis and the discrete basis, respectively,

$$|f\rangle = \int_0^{2\pi} d\theta \int_0^1 dr\, r |r, \theta\rangle\langle r, \theta | f \rangle = \int_0^{2\pi} d\theta \int_0^1 dr\, r |r, \theta\rangle f(r, \theta),$$

$$|f\rangle = \sum_{u,v=0}^{\infty} |u, v\rangle\langle u, v | f \rangle = \sum_{u,v=0}^{\infty} |u, v\rangle f_{u,v}.$$

Although the first of these two relations gives the unitary mapping $\mathcal{U}$, as a matter of fact (21) is strictly valid on a dense subspace of $\mathcal{H}$. The second is just the span of $|f\rangle \in \mathcal{H}$ with respect to the basis $\{|u, v\rangle\}$. These two spans provide of two expressions for the scalar product of any two vectors $|g\rangle, |f\rangle \in \mathcal{H}$ as well as the norm of any vector $|f\rangle \in \mathcal{H}$ as

$$\langle g|f\rangle = \int_0^{2\pi} d\theta \int_0^1 dr\, r\, g(r,\theta)^* f(r,\theta) = \sum_{u,v=0}^{\infty} g_{u,v}^* f_{u,v},$$

$$\|f\|^2 := \langle f|f\rangle = \int_0^{2\pi} d\theta \int_0^1 dr\, r\, |f(r,\theta)|^2 = \sum_{u,v=0}^{\infty} |f_{u,v}|^2.$$

Note that, according to (21), we have for $|f\rangle \in \mathcal{H}$ that

$$\langle r,\theta|f\rangle = f(r,\theta) = (\mathcal{U}f)(r,\theta). \tag{22}$$

As already noted, the first identity in (22) is not valid for any $|f\rangle \in \mathcal{H}$, but only for those on a dense subspace of $\mathcal{H}$. We shall clarify this point later.

III. A PROPOSAL FOR RIGGED HILBERT SPACES

To begin with, let us consider the set $\Phi_1 \subset \mathcal{H}$ of vectors $|f\rangle$ (20) such that

$$\big\||f\rangle\big\|_p^2 := \sum_{u,v=0}^{\infty} |f_{u,v}|^2 (u+v+1)^{2p} < \infty \tag{23}$$

for any $p = 0, 1, 2, \ldots$. This is a countable normed subspace, and hence metrizable of $\mathcal{H}$. Its norms ($\| - \|_p$, $p = 0, 1, 2, \ldots$) are given by (23). Then, consider the subspace $\Psi_1 := \mathcal{U}\Phi_1$ of $L^2(\mathcal{D})$. Hence, Ψ_1 is the set of $f(r,\theta) \in L^2(\mathcal{D})$ (14) such that

$$\sum_{u,v=0}^{\infty} |f_{u,v}|^2 (u+v+1)^{2p} < \infty, \quad \forall p \in N.$$

It is obvious that $\Psi_1 = \mathcal{U}\Phi_1$ has the metrizable structure transported from Φ_1 by $\mathcal{U}$.

Next, we consider a subspace Ψ of Ψ_1 with the following additional conditions:

(i) The series (14), i.e.,

$$f(r,\theta) = \sum_{u,v=0}^{\infty} f_{u,v}\, W_{u,v}(r,\theta),$$

converges pointwise almost elsewhere in $\mathcal{D}$. Note that in general, L^2 convergence does not imply pointwise convergence.

(ii) If $f(r,\theta) \in \Psi$, then, $r\, e^{i\theta} f(r,\theta) \in \Psi$.

Condition (i) is satisfied by all finite linear combinations of Zernike functions $W_{u,v}(r,\theta)$.

In order to prove condition (ii), let us start by defining the operator P

$$Pf(r,\theta) := r\, e^{i\theta} f(r,\theta). \tag{24}$$

Then, P transforms the Zernike functions into linear combinations of (only two) Zernike functions as (see Property 2 in Appendix B for the proof)

$$P\, W_{u,v}(r,\theta) = \alpha_u^v\, W_{u+1,v}(r,\theta) + \beta_u^v\, W_{u,v-1}(r,\theta), \tag{25}$$

where the coefficients α_u^v and β_u^v are given by

$$\begin{aligned}
\alpha_u^v &= \frac{u+1}{\sqrt{(u+v+1)(u+v+2)}}, \\
\beta_u^v &= \frac{v}{\sqrt{(u+v)(u+v+1)}}.
\end{aligned} \tag{26}$$

Note that $0 \le \alpha_u^v \le 1$ and $0 \le \beta_u^v \le 1$.

The consequence is that, all finite linear combinations of Zernike functions satisfy condition (ii). Since finite linear combinations of elements in an orthonormal basis form a dense subspace of the Hilbert space, we must conclude that Ψ is dense in $L^2(\mathcal{D})$. Then, $\Phi := \mathcal{U}^{-1}\Psi$, which is dense in $\mathcal{H}$.

We have the following sequence of spaces:

$$\Phi \subset \Phi_1 \subset \mathcal{H} \subset \Phi_1^\times \subset \Phi^\times, \tag{27}$$

where $\Phi^\times$ is the antidual space of Φ. We denote the action of $F \in \Phi^\times$ on any $|f\rangle \in \Phi$ as $\langle F|f\rangle$, and this notation will be kept for the action of $F \in \Phi_1^\times$ on $|f\rangle \in \Phi_1$. Note that Φ should not necessarily be a closed subspace of Φ_1 (we do not have any proof thereof), and it may well happen that $\Phi_1^\times = \Phi^\times$. In any case, this is not relevant in our discussion. The space Φ will always be endowed with the topology inherited from that of Φ_1, i.e., the metrizable topology given by the countable set of norms (23).

Along to the spaces (27), we have their representations which are their images by $\mathcal{U}$. Note that if we have $\Phi \subset \mathcal{H} \subset \Phi^\times$, we may extend $\mathcal{U}$ to $\Phi^\times$ by using the *duality formula*

$$\langle \mathcal{U}F|(\mathcal{U}|f\rangle)\rangle := \langle F|f\rangle, \tag{28}$$

valid for any $|f\rangle \in \Phi$ and any $F \in \Phi^\times$. This defines $\mathcal{U}$ on $\Phi^\times$ (we have denoted the extension also by $\mathcal{U}$), and the same formula is valid to define $\mathcal{U}$ on $\Phi_1^\times$. Since $\Psi \equiv \mathcal{U}\Phi$ and $\Psi_1 \equiv \mathcal{U}\Phi_1$ and the respective topologies are those transported by $\mathcal{U}$, which is one-to-one and onto in both cases, it results that $\Psi^\times \equiv \mathcal{U}\Phi^\times$ and $\Psi_1^\times \equiv \mathcal{U}\Phi_1^\times$. Equivalently to the chain of spaces (27), we have another sequence

$$\Psi \subset \Psi_1 \subset L^2(\mathcal{D}) \subset \Psi_1^\times \subset \Psi^\times. \tag{29}$$

This chain of spaces given may be looked as a representation of (27) by means of the mapping $\mathcal{U}$. In other words, we have the diagram

$$\begin{array}{ccccccccc} \Phi & \subset & \Phi_1 & \subset & \mathcal{H} & & \subset & \Phi_1^\times & \subset & \Phi^\times \\ \mathcal{U}\downarrow & & \mathcal{U}\downarrow & & \mathcal{U}\downarrow & & \mathcal{U}\downarrow & & \mathcal{U}\downarrow \\ \Psi & \subset & \Psi_1 & \subset & L^2(\mathcal{D}) & \subset & \Psi_1^\times & \subset & \Psi^\times \end{array}.$$

While vectors in Φ, Φ_1, and $\mathcal{H}$ are abstract objects, vectors in Ψ, Ψ_1, and $L^2(\mathcal{D})$ are square integrable functions on the unit circle.

Before proceeding with our discussion, let us recall an important result concerning continuity of linear mappings on countably normed spaces as those under our consideration. Assume that the topology of an infinite dimensional vector space Φ is given by the countable set of norms $\{\|-\|_n\}_{n\in\mathbb{N}}$ on Φ. Then,[34]

1. A linear functional $F : \Phi \longmapsto \mathbb{C}$, where $\mathbb{C}$ is the field of complex numbers, is continuous if and only if there exists a constant $K > 0$ and a finite collection of norms $\{\|-\|_{n_1}, \|-\|_{n_2}, \ldots, \|-\|_{n_k}\}$ such that for all $f \in \Phi$ we have that

$$|F(f)| \le K \left(\|f\|_{n_1} + \|f\|_{n_2} + \cdots + \|f\|_{n_k}\right). \tag{30}$$

2. A linear mapping $A: \Phi \longmapsto \Phi$ is continuous if and only if for any norm $\|-\|_n$ there exists a positive constant $K_n > 0$ and $k(n)$ other norms [K_n and $k(n)$ will depend in general on n] such that for any $f \in \Phi$, we have

$$\|Af\|_n \le K_n \left(\|f\|_1 + \cdots + \|f\|_{k(n)}\right) \tag{31}$$

for any $n = 1, 2, \ldots$.

Let us go back to our general discussion. For $r \in [0, 1]$ and $\theta \in [0, 2\pi)$ fixed, let us define the functional $\langle r, \theta|$ on Φ as

$$\langle r, \theta|f\rangle := \mathcal{U}|f\rangle = f(r, \theta). \tag{32}$$

Then, we have the following:

Proposition 1. The functional $\langle r, \theta|$ is continuous on Φ for all $r \in [0, 1]$ and $\theta \in [0, 2\pi)$.

Proof. The functional is obviously well defined. Taking into account the upper-bound for the Zernike functions (13), and by our hypothesis, the series (14) converges pointwise almost elsewhere. We have that

$$|\langle r, \theta|f\rangle| = |f(r, \theta)| \le \sum_{u,v=0}^{\infty} |f_{u,v}| = \sum_{u,v=0}^{\infty} \frac{|f_{u,v}|\,(u+v+1)}{(u+v+1)}$$

$$\le \sqrt{\sum_{u,v=0}^{\infty} |f_{u,v}|^2 \,(u+v+1)^2} \sqrt{\sum_{u,v=0}^{\infty} \frac{1}{(u+v+1)^2}} = K \|f\|_1, \tag{33}$$

with $\|\,|f\rangle\|_1$ as given in (23) for $p = 1$ and K is the second square root in the second row of (33). This shows the continuity of $\langle r, \theta|$ on Φ. ∎

On the other hand, since Φ is metrizable, this implies that $\langle r, \theta|$ could be continuously extended to the closure of Φ, if it were not closed in Φ_1.

Note that in particular, $\langle r, \theta | u, v \rangle = W_{u,v}(r, \theta)$ (19) holds and is well defined now.

For the operator P (24), we can prove that for any $r \in [0, 1]$ and any $\theta \in [0, 2\pi)$

$$P|r, \theta\rangle = r\, e^{-i\theta}|r, \theta\rangle. \qquad (34)$$

Effectively, using the duality formula (with the same symbol P for either P on $L^2(\mathcal{D})$ and $U^{-1}PU$ on $\mathcal{H}$), the definition (32), and that $\langle r, \theta | f \rangle = \langle f | r, \theta \rangle^*$, we get that

$$\langle f | P | r, \theta \rangle = \langle Pf | r, \theta \rangle = \langle r, \theta | Pf \rangle^* = [r\, e^{i\theta} f(r, \theta)]^* = r\, e^{-i\theta}\langle f | r, \theta \rangle. \qquad (35)$$

Omitting the arbitrary $|f\rangle \in \Phi$, we prove the result (34). Note that P is bounded on both Hilbert spaces, but that we do not have any conclusion about the continuity or about P on Φ.

IV. ALGEBRAS OF CONTINUOUS OPERATORS ON Φ_1

Now, the idea is to show that Φ_1 serves as support of a Lie algebra and its generators are continuous and essentially self-adjoint.

A. Operators on Φ_1 and $L^2(\mathcal{D})$

To begin with, let us define the operators U and V on Φ_1,

$$U|u, v\rangle := u\,|u, v\rangle, \quad V|u, v\rangle := v\,|u, v\rangle. \qquad (36)$$

Then, one may define for $f \in \Phi_1$ [see (20) and (23)]

$$U|f\rangle = \sum_{u,v=0}^{\infty} u f_{u,v}\,|u, v\rangle,$$

$$V|f\rangle = \sum_{u,v=0}^{\infty} v f_{u,v}\,|u, v\rangle. \qquad (37)$$

Proposition 2. The operators U and V are continuous and satisfy the property $(X \pm iI)\Phi_1 = \Phi_1$, where X is either U or V.

Proof. Take for instance,

$$\|U|f\rangle\|_p^2 = \sum_{u,v=0}^{\infty} |f_{u,v}|^2 u^2 (u + v + 1)^{2p} \le \sum_{u,v=0}^{\infty} |f_{u,v}|^2 (u + v + 1)^{2p+2} = \||f\rangle\|_{p+1}^2,$$

which proves the continuity of U. In order to show that $(U \pm iI)\Phi_1 = \Phi_1$ let us define the vectors

$$|g_{\pm}\rangle := \sum_{u,v=0}^{\infty} \frac{f_{u,v}}{u \pm i}\,|u, v\rangle. \qquad (38)$$

Observe that

$$\||g_{\pm}\rangle\|_p^2 = \sum_{u,v=0}^{\infty} \frac{|f_{u,v}|^2}{u^2 + 1}\,(u + v + 1)^{2p} \le \sum_{u,v=0}^{\infty} |f_{u,v}|^2 (u + v + 1)^{2p} = \||f\rangle\|_p^2.$$

Hence, $|g_{\pm}\rangle \in \Phi_1$. Thus, $(U \pm iI)\Phi_1 = \Phi_1$, which proves our claim.

The proof for the identical results referred to V is similar. ∎

Let us define on appropriate dense subspaces of $L^2(\mathcal{D})$ the following operators:

1. $d/d\mathcal{R}$: derivation with respect to the radial variable r.
2. $\mathcal{R}$: multiplication by the radial variable r.

3. $e^{\pm i\Theta}$: multiplication by $e^{\pm i\theta}$, where θ is the angular variable.

Correspondingly, we have the following operators on dense domains in $\mathcal{H}$:

4. $D_R := U\left(d/d\mathcal{R}\right)U^{-1}$.
5. $R = U\,\mathcal{R}\,U^{-1}$.
6. $e^{\pm i\Theta} = U\,e^{\pm i\tilde{\Theta}}\,U^{-1}$.

Thus, we have the following formal operators that are symmetries of the Zernike functions (see Refs. 18 and 29 and Appendix A) on $\mathcal{H}$:

$$A_\pm = \frac{e^{\pm i\Theta}}{2}\left[\mp(1-R^2)D_R + R(U+V+1\pm 1) + \frac{1}{R}(U-V)\right]\sqrt{\frac{U+V+1\pm 1}{U+V+1}},$$

$$B_+ = \frac{e^{\mp i\Theta}}{2}\left[\mp(1-R^2)D_R + R(U+V+1\pm 1) - \frac{1}{R}(U-V)\right]\sqrt{\frac{U+V+1\pm 1}{U+V+1}}.$$

The operators $\mathcal{A}_\pm := UA_\pm U^{-1}$ and $\mathcal{B}_\pm := UB\pm U^{-1}$ with dense domain on $L^2(\mathcal{D})$ have the following properties:

$$\begin{aligned}
\mathcal{A}_+\,W_{u,v}(r,\theta) &= (u+1)\,W_{u+1,v}(r,\theta),\\
\mathcal{A}_-\,W_{u,v}(r,\theta) &= u\,W_{u-1,v}(r,\theta),\\
\mathcal{B}_+\,W_{u,v}(r,\theta) &= (v+1)\,W_{u,v+1}(r,\theta),\\
\mathcal{B}_-\,W_{u,v}(r,\theta) &= v\,W_{u,v-1}(r,\theta).
\end{aligned}\qquad(39)$$

Therefore,

$$\begin{aligned}
A_+\,|u,v\rangle &= (u+1)\,|u+1,v\rangle,\\
A_-\,|u,v\rangle &= u\,|u-1,v\rangle,\\
B_+\,|u,v\rangle &= (v+1)\,|u,v+1\rangle,\\
B_-\,|u,v\rangle &= v\,|u,v-1\rangle.
\end{aligned}$$

Proposition 3. *The operators $A_\pm$ and $B_\pm$ are continuous. Furthermore, $A_\pm$ are formal adjoints of each other and the same is true for $B_\pm$.*

Proof. It is quite similar to the prove showing same properties for U and V. It is sufficient to give it for one case, say A_+. Take $|f\rangle \in \Phi_1$ as in (20) and (23). Then,

$$A_+\,|f\rangle = \sum_{u,v=0}^{\infty} f_{u,v}(u+1)\,|u+1,v\rangle \qquad (40)$$

so that

$$\|A_+\,|f\rangle\|_p^2 = \sum_{u=1,v=0}^{\infty} |f_{u,v}|^2\,(u+1)^2\,(u+v+1)^{2p} \le \sum_{u,v=0}^{\infty} |f_{u,v}|^2\,(u+v+1)^{2(p+1)} = \||f\rangle\|_{p+1}^2,$$

which proves the continuity on Φ_1. ∎

B. The Lie algebra *su*(1, 1) ⊕ *su*(1, 1)

On Φ_1, we have the following commutation relations:

$$[U, A_\pm] = \pm A_\pm, \qquad [V, B_\pm] = \pm B_\pm. \qquad (41)$$

Then, let us define

$$A_3 := U + \frac{1}{2}, \qquad B_3 := V + \frac{1}{2} \qquad (42)$$

so that we have the following commutation relations:

$$[A_+, A_-] = -2A_3, \qquad [A_3, A_\pm] = \pm A_\pm,$$
$$[B_+, B_-] = -2B_3, \qquad [B_3, B_\pm] = \pm B_\pm, \tag{43}$$

showing that the operators $A_\pm$, A_3 on one side and $B_\pm$, B_3 on the other close $su(1, 1)$ Lie algebras with Casimir invariants, respectively,

$$\mathcal{C}_A = A_3^2 - \frac{1}{2}\{A_+, A_-\} \quad \Rightarrow \quad \mathcal{C}_A |u, v\rangle = -\frac{1}{4}|u, v\rangle,$$

$$\mathcal{C}_B = B_3^2 - \frac{1}{2}\{B_+, B_-\} \quad \Rightarrow \quad \mathcal{C}_B |u, v\rangle = -\frac{1}{4}|u, v\rangle,$$

where $\{X, Y\}$ denotes the anticommutator of the operators X and Y, i.e., $\{X, Y\} := XY + YX$. We may easily check that all A-operators commute with all B-operators, i.e.,

$$[A_i, B_j] = 0, \qquad i, j = +, -, 3. \tag{44}$$

Thus with the A and B-operators, we have obtained a realization of the six dimensional Lie algebra $su(1, 1) \oplus su(1, 1)$ recovering previous results by Ref. 9.

We may compare this result with the Casimir invariants for the discrete principal series of unitary irreducible representations for the group $SU(1, 1)$, which is given by $\mathcal{C} = j(j-1)\mathbb{I}$, with $j = 1/2, 1, 3/2, \ldots$.[35] Here, $j(j-1) = -1/4$, and therefore, $j = 1/2$. The space supporting this representation is usually denoted as $D_{1/2}^+$ so that the space spanned by the Zernike functions must be isomorphic to the space $D_{1/2}^+ \otimes D_{1/2}^+$, which supports an irreducible unitary representation of the group $SU(1, 1) \otimes SU(1, 1)$. Its corresponding Lie algebra is spanned by six operators $\{A_\pm, B_\pm, A_3, B_3\}$ that act on the basis, $|a, b\rangle$, of $D_{1/2}^+ \otimes D_{1/2}^+$ as

$$A_\pm |a, b\rangle = \left(a \pm \frac{1}{2}\right)|a \pm 1, b\rangle,$$

$$A_3 |a, b\rangle = a |a, b\rangle,$$

$$B_\pm |a, b\rangle = \left(b \pm \frac{1}{2}\right)|a, b \pm 1\rangle, \tag{45}$$

$$B_3 |a, b\rangle = b |a, b\rangle.$$

There is an immediate relation between $|a, b\rangle$ and $|u, v\rangle$ and is given by $a = u + 1/2$ and $b = v + 1/2$.

C. The universal enveloping algebra of $su(1, 1) \oplus su(1, 1)$

Now, let us call UEA[$su(1, 1) \oplus su(1, 1)$] to the universal enveloping algebra of $su(1, 1) \oplus su(1, 1)$. This is the vector space spanned by the ordered monomials of the form $A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3}$, where α_i and β_j, $i, j = 1, 2, 3$, are either zero or natural numbers (see the Poincaré-Birkoff-Witt theorem[36]). If we denote by $\overline{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\overline{\beta} = (\beta_1, \beta_2, \beta_3)$, any operator $O \in$ UEA[$su(1, 1) \oplus su(1, 1)$] has the following form:

$$O = \sum_{\overline{\alpha}, \overline{\beta}} O_{\overline{\alpha}, \overline{\beta}}$$
$$= \sum_{\overline{\alpha}, \overline{\beta}} c_{\overline{\alpha}, \overline{\beta}} A_+^{\alpha_1} A_3^{\alpha_2} A_-^{\alpha_3} B_+^{\beta_1} B_3^{\beta_2} B_-^{\beta_3}, \tag{46}$$

where $c_{\overline{\alpha}, \overline{\beta}}$ are complex numbers.

The unitary mapping $\mathcal{U} : \mathcal{H} \longmapsto L^2(\mathcal{D})$ transforms this abstract representation into a differential representation of the algebra $su(1, 1) \oplus su(1, 1)$ supported on $L^2(\mathcal{D})$. In fact, Zernike functions satisfy the same relation of the Zernike radial polynomials $R_n^m(r)$. Indeed, (4) can be rewritten as

$$\frac{d^2}{dr^2} W_{u,v}(r, \theta) = \frac{1}{1 - r^2}\left[\left(3r - \frac{1}{r}\right)\frac{d}{dr} - (u + v)(u + v + 2) + \frac{(u - v)^2}{r^2}\right] W_{u,v}(r, \theta)$$

so that for any linear combination $f(r, \theta)$ of the Zernike functions $W_{u,v}(r, \theta)$, we have that

$$D_R^2 f(r, \theta) = \frac{1}{1 - R^2}\left[\left(3R - \frac{1}{R}\right)D_R - (U + V)(U + V + 2) + \frac{1}{R^2}(U - V)^2\right] f(r, \theta).$$

This equation gives us a formal relation between D_R^2 and D_R. This is quite interesting, since this allows us to write any operator of the form (46) as a first order differential operator. As an example, we see that A_+^2 can be written as

$$A_+^2 = \frac{e^{2i\Theta}}{4} \left[h(U, V, R) D_R + k(U, V, R) \right],$$

where $h(U, V, R)$ and $k(U, V, R)$ are given functions of the operators U, V, and R, and the use of the Zernike equation allows us to show a linear dependence of A_+^2 on D_R. This result has an interesting consequence related with the fact that each element of the six dimensional group $SU(1,1) \otimes SU(1,1)$ can be written as a direct product $g(\mathbf{a}, \mathbf{b}) = g_A(\mathbf{a}) * g_B(\mathbf{b})$ with

$$\begin{aligned} g_A(\mathbf{a}) &= e^{i(a_1(A_+ + A_-) + ia_2(A_+ - A_-) + a_3 A_3)}, \\ g_B(\mathbf{b}) &= e^{i(b_1(B_+ + B_-) + ib_2(B_+ - B_-) + a_3 B_3)}, \end{aligned} \tag{47}$$

where $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. This shows that if $g \in SU(1,1) \otimes SU(1,1)$, then, $g \in UEA[su(1,1) \oplus su(1,1)]$ and, therefore, each $g(\mathbf{a}, \mathbf{b})$ as above may be written as a differential operator of first order in D_R. In conclusion, we have the following result:

$$g(\mathbf{a}, \mathbf{b}) = h_{\mathbf{a}, \mathbf{b}}(U, V, R, \Theta) D_R + k_{\mathbf{a}, \mathbf{b}}(U, V, R, \Theta),$$

where $h_{\mathbf{a}, \mathbf{b}}(U, V, R, \Theta)$ and $k_{\mathbf{a}, \mathbf{b}}(U, V, R, \Theta)$ are functions on the given arguments. For practical purposes, one truncates the series that yield to the exponentials (47) so as to obtain a simpler although sufficient approximation.

V. POTENTIAL APPLICATIONS: SOFT ADAPTIVE OPTICS

As Zernike functions have played a role in optical image processing, we have considered interesting to add a short section on possible applications which may even open a way for future research. This is an operator formalism on the space of Zernike functions intended to be applicable to optical image processing or adaptive optics.[17] This is an algebraic procedure that we call *soft adaptive optics*.[18] As a tool for image processing, we consider that soft adaptive optics, as any other manipulator of images, can improve other widely used methods. We believe that it could be an interesting tool to enhance the quality of images. In principle, it may offer some advantages as it is not a complicated procedure and the original image is saved so that it may undergo further manipulations.

Thus in what follows, we qualitatively sketch the applications to soft adaptive optics to the previous formalism. An elaborated example would have been quite interesting to illustrate the method. However, we have realized that the construction of such an example is far from trivial and could be the subject of another article. In any case, it goes beyond the scope of the present paper. Nevertheless, we add some figures at the end taken from numerical experiments. This is given in Appendix D.

Let us consider a real function $f(r, \theta) \in L^2(\mathcal{D})$. The images on the unit disk $\mathcal{D}$ are described by $|f(r, \theta)|^2$. Obviously, $|f(r, \theta)| \in L^2(\mathcal{D})$, so that, (14)

$$\left| f(r, \theta) \right| = \sum_{u, v = 0}^{\infty} \mathfrak{f}_{u, v} \, W_{u, v}(r, \theta).$$

Relation (15) allows us to obtain the components $\mathfrak{f}_{u, v}$ in terms of the basis $\{W_{u, v}(r, \theta)\}$ as

$$\mathfrak{f}_{u, v} = \int_0^{2\pi} d\theta \int_0^1 dr \, r \, W_{u, v}^*(r, \theta) \, |f(r, \theta)|.$$

Note that the properties of Zernike functions, in particular (10), show that $\mathfrak{f}_{u, v} = \mathfrak{f}_{v, u}^*$. In practical numerical calculations, we need truncation of the series spanning the function $|f(r, \theta)|$ in terms of the coefficients $\mathfrak{f}_{u, v}$ and the Zernike functions so that

$$\left| f(r, \theta) \right| \approx \sum_{u, v = 0}^{u_M, v_M} \mathfrak{f}_{u, v} \, W_{u, v}(r, \theta), \tag{48}$$

where u_M and v_M denote the maximum values of u and v in the sum (14), respectively, and are related to the digitalization of the image.

Then, any operator $\mathcal{O}$ performing a transformation from the initial image $|f(r, \theta)|$ to a final image $|g(r, \theta)|$, i.e.,

$$\mathcal{O} : \left| f(r, \theta) \right| \longrightarrow \left| g(r, \theta) \right|,$$

has the form of the sum (46), where we have to replace the operators A and B acting on kets $|a, b\rangle$ by $\mathcal{A}$ and $\mathcal{B}$ acting on the Zernike functions. We have to take into account that the term in $\overline{\alpha}, \overline{\beta}$ obeys the following approximate identity:

$$\mathcal{O}_{\overline{\alpha}, \overline{\beta}} \left| f(r, \theta) \right| = \sum_{u, v = 0}^{u_M, v_M} \mathfrak{f}_{u, v} \, c_{\overline{\alpha}, \overline{\beta}} \, \mathcal{A}_+^{\alpha_1} \, \mathcal{A}_3^{\alpha_2} \, \mathcal{A}_-^{\alpha_3} \, \mathcal{B}_+^{\beta_1} \, \mathcal{B}_3^{\beta_2} \, \mathcal{B}_-^{\beta_3} \, W_{u, v}(r, \theta). \tag{49}$$

In order to calculate the terms in this sum (49), we need to use the identities (39) and (42). This gives a result of the following form:

$$\mathcal{A}_+^{\alpha_1} \, \mathcal{A}_3^{\alpha_2} \, \mathcal{A}_-^{\alpha_3} \, \mathcal{B}_+^{\beta_1} \, \mathcal{B}_3^{\beta_2} \, \mathcal{B}_-^{\beta_3} \, W_{u, v}(r, \theta) = g_{u + \alpha_1 - \alpha_3, v + \beta_1 - \beta_3} \, W_{u + \alpha_1 - \alpha_3, v + \beta_1 - \beta_3}(r, \theta), \tag{50}$$

where the coefficients $g_{k,l}$ are given by

$$g_{u+\alpha_1-\alpha_3,v+\beta_1-\beta_3} = (u-\alpha_3+1)^{(\alpha_1)}\,(u-\alpha_3+1/2)^{\alpha_2}\,(u)_{\alpha_3}\,(v-\beta_3+1)^{(\beta_1)}\,(v-\beta_3+1/2)^{\beta_2}\,(v)_{\beta_3}, \tag{51}$$

with

$$(x)_n := x(x-1)(x-2)\cdots(x-n+1) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)}$$

the falling factorial and

$$x^{(n)} := x(x+1)(x+2)\cdots(x+n-1) = \frac{\Gamma(x+n)}{\Gamma(x)}$$

the raising factorial or Pochhammer symbol. This obviously gives

$$\mathcal{O}_{\overline{\alpha},\overline{\beta}}\left|f(r,\theta)\right| = \sum_{u,v=0}^{u_M,v_M} \mathfrak{f}_{u,v}\,c_{\overline{\alpha},\overline{\beta}}\,g_{u+\alpha_1-\alpha_3,v+\beta_1-\beta_3}W_{u+\alpha_1-\alpha_3,v+\beta_1-\beta_3}(r,\theta),$$

a result that provides the final expression of the object image as

$$\left|g(r,\theta)\right| = \mathcal{O}\left|f(r,\theta)\right|.$$

In Appendix D, we add an illustration of the proposed procedure.

VI. CONCLUSIONS

We have revisited some properties of the Zernike polynomials and their connections with the symmetry group $SU(1,1)\otimes SU(1,1)$. The introduction of the W-Zernike functions $W_{u,v}(r,\theta)$ in (9) is important in order to find in a very natural way the ladder operators that close the Lie algebra $su(1,1)\oplus su(1,1)$.

The Zernike functions can be seen as transition matrices between continuous and discrete bases on the unit disk $\mathcal{D}$, $\{|r,\theta\rangle\}_{(r,\theta)\in\mathcal{D}}$, and $\{|u,v\rangle\}_{u,v\in\mathbb{N}}$, respectively.

It is well known that discrete and continuous bases are quite often used in quantum physics although the continuous bases do not exist in the Hilbert space, which is often considered as the usual framework of quantum mechanics. This is why the formalism of rigged Hilbert spaces is needed, where both types of bases acquire full meaning. In this context, a continuous basis is a set of functionals over a space of test vectors. Some formal and useful relations between both kinds of bases are presented.

The elements of the Lie algebra $su(1,1)\oplus su(1,1)$ are unbounded as operators on Hilbert spaces. However, as operators on rigged Hilbert space all these unbounded operators become continuous. The same happens with the elements of the UEA$[su(1,1)\oplus su(1,1)]$ and, obviously, with the elements of the group $SU(1,1)\otimes SU(1,1)$.

These properties are interesting in possible applications to soft adaptive optics, where the original image can be transformed to a new one by means of continuous operators of UEA$[su(1,1)\oplus su(1,1)]$.

ACKNOWLEDGMENTS

APPENDIX A: ZERNIKE POLYNOMIALS FOR $|r| \leq 1$

The Zernike polynomials $R_n^m(r)$ can be enlarged for negative values of r, i.e., $r \in [-1,1]$. Hence,

$$R_n^m(-r) = (-1)^n R_n^m(r). \tag{A1}$$

In this case, the new orthogonality and completeness relations are now

$$\int_{-1}^{1} R_n^m(r)\,(n+1)\,R_{n'}^m(r)\,|r|\,dr = \delta_{n,n'},$$

$$\sum_{n=|m|}^{\infty} R_n^m(r)\,R_n^m(r')\,(n+1) = \delta(r^2 - r'^2)$$

$$= \frac{1}{2|r|}\left(\delta(r+r') + \delta(r-r')\right). \tag{A2}$$

The symmetries of the Zernike radial polynomials determine the Lie group $SU(1, 1) \otimes SU(1, 1)$.[9,18] Its Lie infinitesimal generators in the representation $R_{u+v}^{|u-v|}(r)$ have the following explicit form valid for $|r| \leq 1$:

$$\mathcal{A}_{\pm} := \frac{1}{2}\Big[\mp (1 - r^2)D_r + r(U + V + 1 \pm 1) + \frac{1}{r}(U - V)\Big],$$

$$\mathcal{B}_{\pm} := \frac{1}{2}\Big[\mp (1 - r^2)D_r + r(U + V + 1 \pm 1) - \frac{1}{r}(U - V)\Big].$$

Their action on $R_{u+v}^{|u-v|}(r)$ is

$$\mathcal{A}_{\pm}R_{u+v}^{|u-v|}(r) = \Big(u + \frac{1}{2} \pm \frac{1}{2}\Big) R_{(u\pm1)+v}^{|(u\pm1)-v|}(r),$$

$$\mathcal{B}_{\pm}R_{u+v}^{|u-v|}(r) = \Big(v + \frac{1}{2} \pm \frac{1}{2}\Big) R_{u+(v\pm1)}^{|u-(v\pm1)|}(r).$$

APPENDIX B: PROOF OF RELATION (25)

Let us consider the Zernike polynomial $R_n^m(r)$, and since the symmetry property $R_n^m(r) = R_n^{-m}(r)$ we can consider $m \geq 0$ without loss of generality. This is a polynomial for which monomials r^k are either even or odd. In the first case, n and m are both even, while in the second case both are odd. The first term is proportional to r^n and the last one to r^m so that a typical Zernike polynomial has the form

$$R_n^m(r) = a_n r^n + a_{n-2} r^{n-2} + \cdots + a_m r^m,$$

where the a_i are real numbers. Multiplying $R_n^m(r)$ by r, we have

$$r R_n^m(r) = a_n r^{n+1} + a_{n-2} r^{n-1} + \cdots + a_m r^{m+1}.$$

Since the Zernike polynomials $R_{n+1}^{m+1}(r), R_{n-1}^{m+1}(r)$, and $R_{n-3}^{m+1}(r), \ldots, R_{n-m+1}^{m+1}(r)$ are linearly independent polynomials of degree $n + 1, n - 1, \ldots,$ $m + 1$, respectively, we have that

$$a_n r^{n+1} + a_{n-2} r^{n-1} + \cdots + a_m r^{m+1} = b_{n+1} R_{n+1}^{m+1}(r) + b_{n-1} R_{n-1}^{m+1}(r) + \cdots + b_{m+1} R_{m+1}^{m+1}(r),$$

where the coefficients b_k are real numbers. In conclusion,

$$r R_n^m(r) = b_{n+1} R_{n+1}^{m+1}(r) + b_{n-1} R_{n-1}^{m+1}(r) + \cdots + b_{m+1} R_{m+1}^{m+1}(r).$$

However, we can refine prove the previous relation between Zernike polynomials. So, we establish the following:

Property 1. Any Zernike polynomial $R_n^m(r)$ such that $n \geq 1$ verifies the relation

$$r R_n^m(r) = a_n^m R_{n+1}^{m+1}(r) + b_n^m R_{n-1}^{m+1}(r), \tag{B1}$$

where

$$a_n^m = \frac{n + m + 2}{2(n + 1)}, \qquad b_n^m = \frac{n - m}{2(n + 1)}.$$

Proof. Effectively, let us start with this explicit formula of the Zernike polynomials R_n^m (5). Now from (5), the lhs of (B1) can be written as

$$r R_n^m(r) = \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \binom{n - k}{k} \binom{n - 2k}{\frac{n-m}{2} - k} r^{n+1-2k}. \tag{B2}$$

Also, the rhs of (B1) is equal to

$$a \sum_{k=0}^{\frac{n-m}{2}} (-1)^k \binom{n + 1 - k}{k} \binom{n + 1 - 2k}{\frac{n-m}{2} - k} r^{n-2k} + b \sum_{k=0}^{\frac{n-m-2}{2}} (-1)^k \binom{n - 1 - k}{k} \binom{n - 1 - 2k}{\frac{n-m-2}{2} - k} r^{n-1-2k}$$

so that we can rewrite as

$$\sum_{k=0}^{\frac{n-m}{2}} (-1)^k \left[a \binom{n+1-k}{k} \binom{n+1-2k}{\frac{n-m}{2}-k} - b\left(1-\delta_{0,k}\right) \binom{n-k}{k-1} \binom{n+1-2k}{\frac{n-m}{2}-k} \right] r^{n+1-2k}. \tag{B3}$$

From (B2) and (B3), we obtain the following relations for the coefficients of the powers of r^{n+1-2k} for $k = 0, 1, \ldots, (n-m)/2$:

$$\binom{n-k}{k} \binom{n-2k}{\frac{n-m}{2}-k} = a \binom{n+1-k}{k} \binom{n+1-2k}{\frac{n-m}{2}-k} - b\left(1-\delta_{0,k}\right) \binom{n-k}{k-1} \binom{n+1-2k}{\frac{n-m}{2}-k}.$$

From $k = 0$, the previous expression becomes

$$\binom{n}{\frac{n-m}{2}} = a \binom{n+1}{\frac{n-m}{2}}.$$

So, from the definition of the binomial coefficients we obtain that

$$a = \frac{n+m+2}{2(n+1)}, \tag{B4}$$

and for $k \geq 1$, we get

$$\binom{n-k}{k} \binom{n-2k}{\frac{n-m}{2}-k} = a \binom{n+1-k}{k} \binom{n+1-2k}{\frac{n-m}{2}-k} - b \binom{n-k}{k-1} \binom{n+1-2k}{\frac{n-m}{2}-k}$$

that developing the binomial coefficients in terms of factorial, we get that

$$1 = a \frac{2(n+1-k)}{n+m+2-2k} - b \frac{2k}{n+m+2-2k}.$$

Now from (B4) we get that

$$b = \frac{n-m}{2(n+1)},$$

which is independent of k.   ∎

As a corollary, we have the following property of the Zernike functions $W_{u,v}(r, \theta)$:

Property 2. Any Zernike function $W_{u,v}(r, \theta)$ such that $n \geq 1$ verifies the following relation:

$$r\,e^{i\theta}\,W_{u,v}(r,\theta) = \alpha_u^v\,W_{u+1,v}(r,\theta) + \beta_u^v\,W_{u,v-1}(r,\theta),$$

where

$$\alpha_u^v = \frac{u+1}{\sqrt{(u+v+1)(u+v+2)}}, \quad \beta_u^v = \frac{v}{\sqrt{(u+v)(u+v+1)}}.$$

Proof. The proof is trivial taking into account the definition (9) of the $W_{u,v}(r, \theta)$ as well as the relations (8) between the parameters (u, v) and (n, m).   ∎

After the definition of the space Φ, this proves the stability of Φ under the action of P. Note that we have not proved the continuity of P on Φ, neither some topological properties of Φ with respect to the topology inherited from Φ_1. This is not strictly necessary for our purposes.

APPENDIX C: ANOTHER TOPOLOGY FOR THE SPACE OF ZERNIKE FUNCTIONS

Along with the space Ψ_1 of functions $f(r, \theta)$ of $L^2(\mathcal{D})$ (14) such that

$$\sum_{u,v=0}^{\infty} |f_{u,v}|^2 (u + v + 1)^{2p} < \infty, \quad \forall p \in N,$$

we consider another one that we denote here as Ψ. This is the space of functions $f(r, \theta) \in L^2(\mathcal{D})$ (14) verifying

$$\sum_{u,v}^{\infty} |f_{u,v}| (u + v + 1)^q < \infty, \quad \forall q \in \mathbb{N}.$$

We endow Ψ with the set of norms $\|-\|_{1,q}$

$$\|f(r, \theta)\|_{1,q} := \sum_{u,v}^{\infty} |f_{u,v}| (u + v + 1)^q,$$

with $q = 0, 1, 2, \ldots$ so that Ψ has the structure of countably normed space and, hence, metrizable.

This space has the following properties:

1. It is dense in $L^2(\mathcal{D})$, since it contains all the basis elements $W_{u,v}(r, \theta)$.
2. The series

$$f(r, \theta) = \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r, \theta) \tag{C1}$$

converges absolutely and uniformly, and hence pointwise.

The proof is the following: since the functions $W_{u,v}(r, \theta)$ have the upper bound (13), i.e., $|W_{u,v}(r, \theta)| \le \sqrt{(u + v + 1)(\pi)}$, then,

$$\sum_{u,v=0}^{\infty} |f_{u,v}| \cdot |W_{u,v}(r, \theta)|$$

$$\le \sum_{u,v=0}^{\infty} |f_{u,v}| \sqrt{\frac{u + v + 1}{\pi}} \le \frac{1}{\sqrt{\pi}} \sum_{u,v=0}^{\infty} |f_{u,v}| (u + v + 1) < \infty.$$

Then, the Weierstrass M-Theorem guarantees the absolute and uniform convergence of the series.

3. Observe that for all absolutely convergent series $\sum_n a_n$, we have that

$$\sqrt{\sum_n |a_n|^2} \le \sum_n |a_n|.$$

This shows that if $\|-\|_r$ is the norm defined in (23), we have that

$$\|f\|_r = \sqrt{\sum_{u,v}^{\infty} |f_{u,v}|^2 (u + v + 1)^{2r}} \le \sum_{u,v}^{\infty} |f_{u,v}| (u + v + 1)^r =: p_r(f),$$

which shows that $\Psi \subset \Psi_1$ and also that the canonical injection

$$\Psi \xrightarrow{\ i\ } \Psi_1$$
$$f \longrightarrow i(f) = f$$

is continuous. This implies that the canonical injection $i : \Psi \longmapsto L^2(\mathcal{D})$ is also continuous so that $\Psi \subset L^2(\mathcal{D}) \subset \Psi^\times$ is a rigged Hilbert space.

4. The operators $\mathcal{U}, \mathcal{V}, \mathcal{A}_\pm, \mathcal{B}_\pm$ are continuous on Ψ.

The proof is straightforward. It is also important to show that the operator P, as defined in (25) and (26) is invariant and continuous on Φ. The proof is very simple. For any $f(r, \theta) \in \Psi$ as in (C1), we have

$$P \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r,\theta) = \sum_{u,v=0}^{\infty} \left(\alpha_{u-1}^{v} f_{u-1,v} + \beta_{u}^{v+1} f_{u,v+1} \right) W_{u,v}(r,\theta),$$

where we take $f_{-1,v} = 0$. Since $0 \le \alpha_{u}^{v} \le 1$ and $0 \le \beta_{u}^{v} \le 1$, we get that

$$\left\| P \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r,\theta) \right\|_{1,r} = \sum_{u,v=0}^{\infty} \left| \alpha_{u-1}^{v} f_{u-1,v} + \beta_{u}^{v+1} f_{u,v+1} \right| (u+v+1)^{r}$$

$$\le \sum_{u,v=0}^{\infty} \left| f_{u-1,v} \right| (u+v+1)^{r} + \sum_{u,v=0}^{\infty} \left| f_{u,v+1} \right| (u+v+1)^{r}.$$

The first term of the last inequality in the previous expression since $f_{-1,0} = 0$ can be rewritten as

$$\sum_{u,v=0}^{\infty} \left| f_{u-1,v} \right| (u+v+1)^{r} = \sum_{u,v=0}^{\infty} \left| f_{u,v} \right| (u+v+2)^{r}$$

$$\le 2^{r} \sum_{u,v=0}^{\infty} \left| f_{u,v} \right| (u+v+1)^{r} = 2^{r} \left\| \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r,\theta) \right\|_{1,r},$$

while the second one gives

$$\sum_{u,v=0}^{\infty} \left| f_{u,v+1} \right| (u+v+1)^{r} \le \sum_{u,v=0}^{\infty} \left| f_{u,v} \right| (u+v)^{r}$$

$$\le \sum_{u,v=0}^{\infty} \left| f_{u,v} \right| (u+v+1)^{r} = \left\| \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r,\theta) \right\|_{1,r}.$$

This shows that

$$\left\| P \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r,\theta) \right\|_{1,r} \le (2^{r}+1) \left\| \sum_{u,v=0}^{\infty} f_{u,v} W_{u,v}(r,\theta) \right\|_{1,r},$$

which proves our claim. From (33), we have that

$$\left| \langle r, \theta | f \rangle \right| = \left| f(r,\theta) \right| \le \sum_{u,v=0}^{\infty} \left| f_{u,v} \right| = \| f(r,\theta) \|_{1,0}$$

so that $\langle r, \theta |$ is a continuous mapping on Ψ.

APPENDIX D: $\mathcal{Z}$-ZERNIKE FUNCTIONS VS W-ZERNIKE FUNCTIONS

In optics, the use of the first $\mathcal{Z}$-Zernike functions which are defined as follows is very common:

$$\mathcal{Z}_{n}^{m}(r,\theta) := R_{n}^{m}(r) \cos(m\theta),$$
$$\mathcal{Z}_{n}^{-m}(r,\theta) := R_{n}^{m}(r) \sin(m\theta), \tag{D1}$$

(a) $\mathcal{Z}_3^{-1}(r,\theta)$

(b) $\operatorname{Im} W_{2,1}(r,\theta)$

(c) $\mathcal{Z}$

(d) W

(e) $\mathcal{Z}_{24}^{16}$

(f) $\operatorname{Re} W_{20,4}$
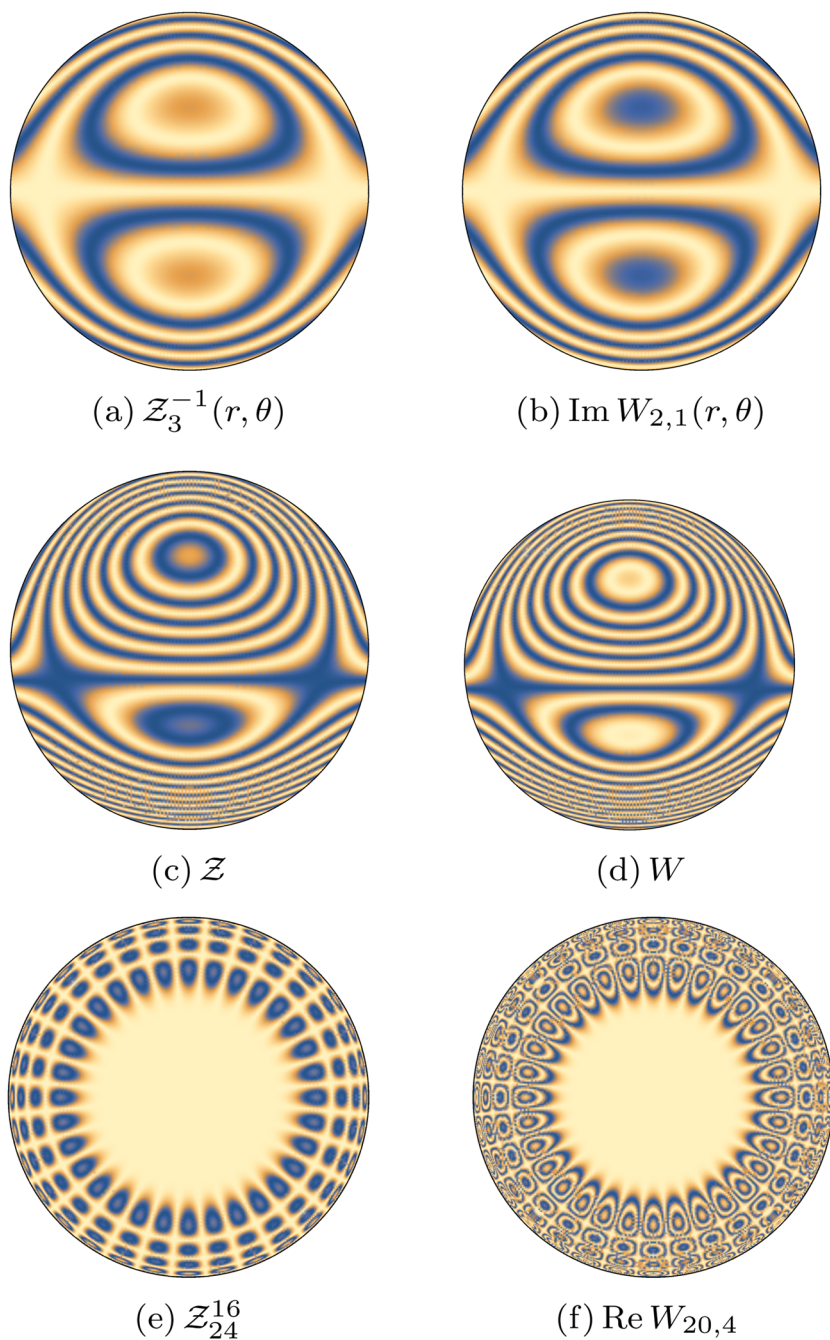
FIG. 1. $\mathcal{Z}$-Zernike functions vs W-Zernike functions, (c) $\mathcal{Z} = 1.25\,\mathcal{Z}_2^2 + 2.5\,\mathcal{Z}_3^{-1}$, and (d) $W = 1.25\,\operatorname{Re} W_{2,0} + 2.5\,\operatorname{Im} W_{2,1}$.

with $n,\ m \in \mathbb{N}$. The other conditions verified by m and n are displayed in expression (3). Both kinds of $\mathcal{Z}$-Zernike functions (D1) are included by the use of $e^{im\theta}$, as we have done in (2). On the other hand, the W-Zernike functions (9) contain a scale factor and they are well adapted to show the underlying symmetry of the Zernike functions that in the representation given by the $\mathcal{Z}$-Zernike functions is more difficult to see. Rewriting (9),
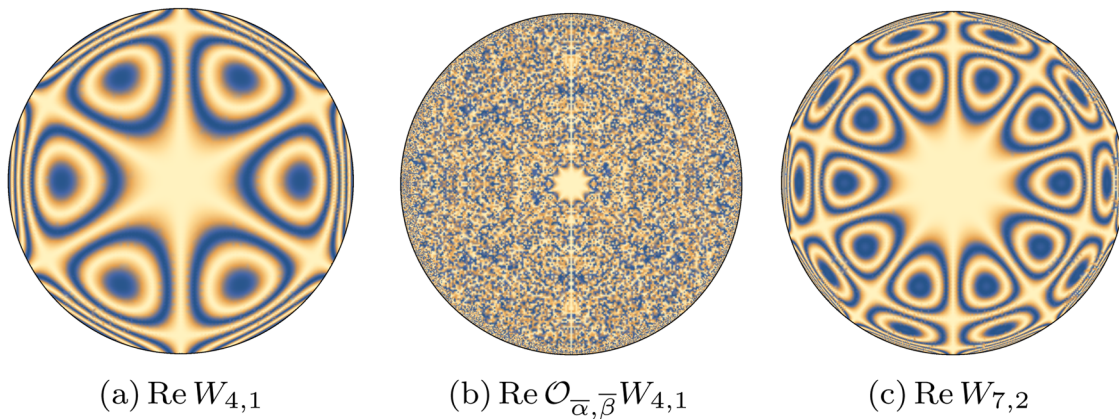
(a) $\mathrm{Re}\, W_{4,1}$ (b) $\mathrm{Re}\, \mathcal{O}_{\overline{\alpha},\overline{\beta}} W_{4,1}$ (c) $\mathrm{Re}\, W_{7,2}$

FIG. 2. (a) W-Zernike function; (b) its transformation under $\mathcal{O}_{\overline{\alpha},\overline{\beta}}$ with $\overline{\alpha} = (3,0,0)$ and $\overline{\beta} = (1,0,0)$.

$$W_{u,v}(r,\theta) := \sqrt{\frac{u+v+1}{\pi}} \, \mathcal{Z}_{u+v}^{u-v}(r,\theta)$$

$$= \sqrt{\frac{u+v+1}{\pi}} \, R_{u+v}^{|u-v|}(r) \, e^{i(u-v)\theta},$$

we easily can find the relation between the first $\mathcal{Z}$-Zernike functions vs W-Zernike functions. Thus,

$$W_0^0(r,\theta) = \sqrt{\frac{1}{\pi}} \, \mathcal{Z}_0^0(r,\theta), \quad W_1^0(r,\theta) = \sqrt{\frac{2}{\pi}} \, \mathcal{Z}_1^1(r,\theta), \quad W_1^1(r,\theta) = \sqrt{\frac{3}{\pi}} \, \mathcal{Z}_2^0(r,\theta),$$

$$W_2^0(r,\theta) = \sqrt{\frac{3}{\pi}} \, \mathcal{Z}_2^2(r,\theta), \quad W_2^1(r,\theta) = \sqrt{\frac{4}{\pi}} \, \mathcal{Z}_3^1(r,\theta), \quad W_2^2(r,\theta) = \sqrt{\frac{5}{\pi}} \, \mathcal{Z}_4^0(r,\theta),$$

$$W_3^0(r,\theta) = \sqrt{\frac{4}{\pi}} \, \mathcal{Z}_3^3(r,\theta), \quad W_3^1(r,\theta) = \sqrt{\frac{5}{\pi}} \, \mathcal{Z}_4^2(r,\theta), \quad W_3^2(r,\theta) = \sqrt{\frac{6}{\pi}} \, \mathcal{Z}_5^1(r,\theta),$$

$$W_3^3(r,\theta) = \sqrt{\frac{7}{\pi}} \, \mathcal{Z}_6^0(r,\theta), \quad W_4^1(r,\theta) = \sqrt{\frac{6}{\pi}} \, \mathcal{Z}_5^3(r,\theta), \quad W_4^2(r,\theta) = \sqrt{\frac{7}{\pi}} \, \mathcal{Z}_6^2(r,\theta).$$

In Fig. 1, we display some $\mathcal{Z}$-Zernike functions and their W-Zernike counterparts. From the definition (9), we see that the $W_{u,v}(r,\theta)$ and its counterpart $\mathcal{Z}_{u+v}^{u-v}(r,\theta)$ differs in the factor $\sqrt{(u+v+1)/\pi}$, whose influence is displayed in Fig. 1. For higher values of u and v the differences are more marked, as denoted Figs. 1(c) and 1(d).

In Fig. 2, we display $W_{4,1}(r,\theta)$, its transformation under the action of the operator $\mathcal{O}_{\overline{\alpha},\overline{\beta}}$ (49), where $\overline{\alpha} = (3,0,0)$ and $\overline{\beta} = (1,0,0)$, i.e.,

$$\mathcal{O}_{\overline{\alpha},\overline{\beta}} W_{4,1}(r,\theta) = \mathcal{A}_+^3 \, \mathcal{A}_3^0 \, \mathcal{A}_-^0 \, \mathcal{B}_+^1 \, \mathcal{B}_3^0 \, \mathcal{B}_-^0 \, W_{4,1}(r,\theta) = 420 \, W_{7,2}(r,\theta),$$

where the coefficient is computed according to formula (51). The figures have been plotted using the program Mathematica of Wolfram Research, Inc.

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