Coherent Gamow states for the hyperbolic Pöschl-Teller potential.

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Abstract

We have defined a pair of families of coherent states using creation and annihilation operators relating the Gamow states, vector states for non-relativistic quantum resonances. We have used an explicit one dimensional model, for which these operators may be explicitly written: the one dimensional Pöschl-Teller Hamiltonian. We have shown that this model may serve as an example for the pseudo-boson formalism.

1 Introduction

Since the discovering of quantum coherent states as states with minimal dispersion, a lot of work has been done in this direction as they are extremely useful in quantum optics. Among the immense amount of literature on the subject let us mention [1, 2, 3, 4, 5]. The definition of coherent states as eigenvectors of an annihilation operator for all complex eigenvalues was given for instance in [5].

One may wonder whether coherent states may be constructed with Gamow states, which are state vectors for quantum unstable states [6, 7]. The purpose of this note is to show that this is possible, at least in concrete models, for which Gamow states are related by creation-annihilation operators. An explicit construction has been given in [17] for the infinite set of resonances that appear in the one dimensional Pöschl-Teller model. After this construction, the consideration of coherent states becomes straightforward.

The one dimensional Pöschl-Teller potential has been considered in numerous publications. Let us mention only that this type of potentials have been used for a manageable approximation of other potentials. See, for instance [8].

As is well know, quantum resonances appear as pairs of poles of the analytic continuation of the S-matrix [10, 11]. If the S-matrix is given in the momentum representation, each pair of poles symmetrically distributed with respect to the imaginary axis in the third and fourth quadrant of the complex plane, represent the same resonance. Under certain conditions [17], the one-dimensional Pöschl-Teller potential has an infinite number of these pairs of poles.

If we go to the energy representation, each pair of resonance poles are located in the analytic continuation through the positive real axis of the S-matrix [10, 11] and are complex conjugate of each other. The real part corresponds to the resonance energy and the imaginary part is related with the inverse of the mean life. Then, the Gamow states are eigenstates of the total Hamiltonian with complex eigenvalues that coincide with the resonance poles in the energy representation. Since the Hamiltonian is self-adjoint, these Gamow states are not vectors on the Hilbert space on which the Hamiltonian acts, so that we need to extend the Hamiltonian to a larger space in the context of the rigged Hilbert space formalism [12, 13, 14]. Poles of the S matrix corresponding to a given pair and their corresponding Gamow states are related via the time reversal operator [15].

In order to construct the coherent states, we go back to the momentum representation and take the series of Gamow states corresponding to the poles in the third and forth quadrant independently. Each one of these series define a family of coherent states which may be treated independently of the other. Coherent states are given by series which weakly converges under a topology given by the rigged Hilbert space.

We have also shown that the two families of Gamow states behave as pseudo-bosons in the sense given by Bagarello in [19]. We have constructed respective topologies on the spaces spanned by both families of Gamow states and shown that the corresponding ladder operators are continuous under these topologies.

Concerning recent literature on coherent states, we would like to mention the book of Combescure and Didier [20], or a book with contributions on the subject [21].

Then, we construct the coherent states in Section 2 and give the link with the pseudo-boson formalism in Section 3. We have added two Appendices. In the former, we define the spaces on which Gamow vectors act. In the second one, we have shown that our coherent Gamow states satisfy two equivalent resolutions of the identity.

2 Resonances and their Gamow functions

Let us consider a one dimensional Hamiltonian with an hyperbolic Pöschl-Teller potential, which has the following form:

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - \frac{\hbar^2}{2m}\frac{\alpha^2\,\lambda(\lambda-1)}{\cosh^2\alpha x}\,, \tag{1}$$

where α is a fixed constant that we shall fix as $\alpha = 1$ and λ a parameter. The properties of H depend of the value taken by λ, which could be complex. In fact, if one chooses $\lambda = 1/2 + i\ell$ with $\ell > 0$, the potential is a real repulsive barrier. This Hamiltonian is self adjoint[1].

Writing $k^2 := 2mE/\hbar^2$, the Schrödinger equation for (1) takes the following form:

[1]In fact, $H_0 = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}$ is self adjoint when we choose its domain the Sobolev space of square integrable absolutely continuous functions, $f(x)$, on the real line with first absolutely continuous derivative and square integrable second derivative:

$$\int_{-\infty}^{\infty}\{|f(x)|^2 + |f''(x)|^2\}\,dx < \infty\,.$$

Since the hyperbolic Pöschl-Teller potential is bounded, the Kato Rellich theorem [22] guarantees the self-adjointness of the total Hamiltonian.

$$U''(x) + \left[k^2 + \frac{\lambda(\lambda-1)}{\cosh^2 x}\right] U(x) = 0\,. \tag{2}$$

Equation (2) has been solved in [17]. Its general solution is given by

$$U(x) = A(1+\tanh x)^{ik/2}(1-\tanh x)^{-ik/2}\,{}_2F_1\left(\lambda, 1-\lambda; ik+1; \frac{1+\tanh x}{2}\right)$$

$$+B\,2^{ik}\,(1+\tanh x)^{-ik/2}(1-\tanh x)^{-ik/2}\,{}_2F_1\left(\lambda-ik, 1-\lambda-ik; 1-ik; \frac{1+\tanh x}{2}\right)\,, \tag{3}$$

where A and B are arbitrary constants and ${}_2F_1()$ is the second kind Kummer function, solution of the hypergeometric differential equation. In order to obtain scattering data, we have to obtain the asymptotic forms of (3). Thus for $x \longmapsto -\infty$, this asymptotic form is simple:

$$U^-(x) = A\,e^{ikx} + B\,e^{-ikx}\,. \tag{4}$$

On the other hand, for $x \longmapsto x$, the situation is more complicated although it has still the form

$$U^+ = A'\,e^{ikx} + B'\,e^{-ikx}\,, \tag{5}$$

where A' and B' have been obtained in [17] in terms of A and B and Gamma functions depending on k and λ. The S-matrix is defined by its components $\{S_{ij}\}$ as

$$\begin{pmatrix} B \\ A' \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} A \\ B' \end{pmatrix}\,. \tag{6}$$

Related with the S-matrix is the transfer matrix $T \equiv \{T_{ij}\}$

$$\begin{pmatrix} A' \\ B' \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}\,. \tag{7}$$

The relation between the S-matrix and the transfer matrix T is given by

$$S = \frac{1}{T_{22}} \begin{pmatrix} -T_{21} & 1 \\ T_{11}\,T_{22} - T_{21}\,T_{12} & T_{12} \end{pmatrix} \tag{8}$$

Resonances are characterized by the solutions on k of the so called purely outgoing boundary conditions, which select the outgoing asymptotic wave function and ignores the incoming wave function. This choice implies that $A = B' = 0$ and $A' \neq 0 \neq B$. Since $B' = T_{21}\,A + T_{22}\,B$, purely outgoing boundary conditions imply that $T_{22}(k) = 0$ and this gives the poles of the analytic continuation on k of the S-matrix. The transmission coefficient is

$$T = \frac{1}{|T_{22}|^2}\,, \tag{9}$$

and therefore, the poles of the transmission coefficient coincide with the poles of the S-matrix.

The choice $\lambda = 1/2 + i\ell$ with $\ell > 0$ gives the following expression for the transmission coefficient:

$$T(k) = \frac{\sinh^2(\pi k)}{\cosh^2(\pi k) + \sinh^2(\pi \ell)}\,. \tag{10}$$

Observe that the transmission coefficient is a function of k (the same is true for the reflection coefficient) and that ℓ in (10) is just a constant parameter.

There are two infinite series of poles for the analytic continuation of T in terms of k, which are given by

$$k_1(n) = \ell - i\left(n + \frac{1}{2}\right)\,, \qquad k_2(n) = -\ell - i\left(n + \frac{1}{2}\right)\,, \qquad n = 0, 1, 2, \dots\,. \tag{11}$$

These poles are located on the lower half plane of the k-complex plane symmetrically located with respect to the imaginary axis. They are resonance poles and each resonance is given by one pair of symmetrically located poles. In the energy representation, they correspond to pairs of conjugate poles for the analytic continuation of $S(k)$ through the positive semi-axis (second sheet in the language of Riemann surfaces). These are

$$z_R(n) = \frac{\hbar^2}{2m}\,k_1(n)^2 = E_R - i\,\frac{\Gamma}{2}\,, \qquad z_R^*(n) = \frac{\hbar^2}{2m}\,k_2(n)^2 = E_R + i\,\frac{\Gamma}{2} \tag{12}$$

with

$$E_R = \frac{\hbar^2}{2m}\,(\ell^2 - \gamma_n^2)\,, \qquad \Gamma = \frac{\hbar^2}{2m}\,4\ell\gamma_n > 0\,, \qquad \gamma_n = n + \frac{1}{2}\,. \tag{13}$$

The Gamow states are vector states for the purely exponential decaying part of a resonance. They have been defined as eigenvectors φ^D of the total Hamiltonian $H = H_0 + V$ with eigenvalue z_R, so that $H\varphi^D = z_R\varphi^D$. Then, their time evolution is a purely decaying exponential. Taking into account (12), from $H\varphi^D = z_R\varphi^D$, one formally gets

$$e^{-itH}\,\varphi^D = e^{-itE_R}\,e^{-t\Gamma/2}\,\varphi^D\,, \tag{14}$$

expression which decays exponentially for $t > 0$. Since resonance poles appear in pairs, for each resonance there is a second Gamow vector, φ^G, which is an eigenvector of the Hamiltonian with eigenvector z_R^*, so that

$$e^{-itH}\,\varphi^G = e^{-itE_R}\,e^{t\Gamma/2}\,\varphi^G\,, \tag{15}$$

so that φ^G decays exponentially for $t \longmapsto -\infty$. Vectors φ^D and φ^G are time reversal of each other and represent exactly the same resonance.

One may argue that the Hamiltonian being a self-adjoint operator, it may not have complex eigenvalues. This is true in a Hilbert space. However, it is well known that Gamow functions, or wave functions representing Gamow vectors, are exponentially growing functions of the coordinates and, therefore, cannot be square integrable. The solution is well known, which is to extend the Hilbert space to a rigged Hilbert space, where the extension of H, e^{-itH} and Gamow states are well defined and relations such as (14) and (15) are well defined. See [12, 14].

Let us assume that φ_n^D is the Gamow vector corresponding to the $(n+1)$-th resonance pole $k_1(n)$, $n = 0, 1, 2, \dots$ which are located in the forth quadrant of the k-complex plane. We may construct ladder operators such that

$$B_n^- \,\varphi_n^D = \varphi_{n-1}^D\,, \qquad B_n^+ \,\varphi_{n-1}^D = \varphi_n^D\,. \tag{16}$$

These operators were determined in [17] and have the form

$$B_n^- = -\cosh x\,\partial_x + (i\ell + n + \frac{1}{2})\sinh x\,, \tag{17}$$

$$B_n^+ = \cosh x\,\partial_x + (i\ell + n + \frac{1}{2})\,\sinh x\,. \tag{18}$$

Resonances in the series $k_2(n)$ are located in the third quadrant of the complex k-plane. For these resonance poles, there exist creation and annihilation operators C_n^+ and C_n^- such that

$$C_n^- \,\varphi_n^G = \varphi_{n-1}^G\,, \qquad C_n^+ \,\varphi_{n-1}^G = \varphi_n^G\,. \tag{19}$$

These are,

$$C_n^- = -\cosh x\,\partial_x + (-i\ell + n + \frac{1}{2})\sinh x\,, \tag{20}$$

$$C_n^+ = \cosh x\,\partial_x + (-i\ell + n + \frac{1}{2})\,\sinh x\,. \tag{21}$$

The construction of the Gamow functions follows the standard procedure. We first obtain the Gamow functions corresponding to the value $n = 0$ as

$$B_0^- \,\varphi_0^D = 0\,, \qquad C_0^- \,\varphi_0^G = 0\,, \tag{22}$$

which gives simple differential equations giving the explicit form of $\varphi_0^D(x)$ and $\varphi_0^G(x)$. This are save for an arbitrary multiplicative constant:

$$\varphi_0^D(x) = (\cosh x)^{i\ell+1/2}\,, \qquad \varphi_0^G(x) = (\cosh x)^{-i\ell+1/2}\,. \tag{23}$$

Then, using the operators B_n^+ and C_n^+, we may construct all the Gamow functions of the series $\varphi_n^D(x)$ and $\varphi_n^G(x)$. They have the form,

$$\varphi_n^D(x) = P_n(\sinh x)\,\varphi_0^D(x)\,, \qquad \varphi_n^G(x) = Q_n(\sinh x)\,\varphi_0^G(x)\,, \tag{24}$$

where $P_n(\sinh x)$ and $Q_n(\sinh x)$ are polynomials of degree n on $\sinh x$. The proof of this statement is obvious. We may also show that

$$H\,\varphi_n^D(x) = \frac{\hbar^2}{2m}\,[k_1(n)]^2\,\varphi_n^D(x)\,, \qquad H\,\varphi_n^G(x) = \frac{\hbar^2}{2m}\,[k_2(n)]^2\,\varphi_n^G(x)\,, \tag{25}$$

which shows that $\varphi_n^D(x)$ and $\varphi_n^G(x)$, $n = 0, 1, 2, \dots$ are the Gamow functions corresponding to the resonance poles $k_1(n)$ and $k_2(n)$, respectively.

Following the procedure given in [17], the operators $B_n^\pm$ and $C_n^\pm$ defined in (17-18) and 20-21) yield to index-free operators defined on the linear spaces spanned by the functions $\varphi_n^D(x)$ and $\varphi_n^G(x)$, respectively, as

$$B^-\varphi_n^D := \sqrt{n}\,B_n^-\varphi_n^D\,, \quad B^+\varphi_n^D := \sqrt{n+1}\,B_n^+\varphi_n^D\,, \quad C^-\varphi_n^G := \sqrt{n}\,C_n^-\varphi_n^G\,, \quad C^+\varphi_n^G := \sqrt{n+1}\,C_n^+\varphi_n^G\,. \tag{26}$$

We may introduce the coefficients $\sqrt{n}$ and $\sqrt{n+1}$, since no normalization are required for the Gamow functions, as they grow exponentially at the infinity. On the other hand, these coefficients will be important for the weak convergence of the series defining the coherent states.

We may construct two distinct series of coherent states using one sequence of resonance poles or the other. Therefore, we have two collections of coherent states, defined by means of the equations

$$B^-|z^D\rangle = z\,|z^D\rangle\,, \qquad C^-|z^G\rangle = z\,|z^G\rangle\,, \quad \forall\, z \in \mathbb{C}\,. \tag{27}$$

The construction is straightforward, we write

$$|z^D\rangle = \sum_{n=0}^\infty c_n\,\varphi_n^D\,, \tag{28}$$

and use (27)

$$B^-\,|z^D\rangle = \sum_{n=1}^\infty c_n\,\sqrt{n}\,\varphi_{n-1}^D = z\sum_{n=0}^\infty c_n\,\varphi_n^D\,. \tag{29}$$

Then,

$$c_n = \frac{c_0}{\sqrt{n!}}\,z^n\,. \tag{30}$$

We may choose $c_0 = 1$. This gives the explicit form for $|z^D\rangle$ for any $z \in \mathbb{C}$. The same procedure gives $|z^G\rangle$. This derivation is trivial, although we need to show that the series defining $|z^D\rangle$ and $|z^G\rangle$ weakly converge on some space of test functions. Let us consider the spaces $\Phi_\pm$ defined as in [14]. The action of the functional $|\varphi_n^D\rangle \in \Phi_+^\times$ on any $\phi_+ \in \Phi_+$, $\langle\phi_+|\varphi_n^D\rangle$, is the value of the corresponding Hardy function (see [14]) on the lower half plane of the complex plane at the point $\frac{\hbar^2}{2m}\,k_1(n)^2$.

If $\phi_+(z)$ is a Hardy function on the lower half plane, it behaves as $|\phi_+(z)| \approx |z|^{-1/2}$ for large values of $|z|$ [18]. The modulus of the resonance poles goes to infinite as $n \longmapsto \infty$. This implies that the limit

$$\lim_{n\to\infty} |\phi_+(z_R(n))| = 0\,, \tag{31}$$

so that the sequence $|\phi_+(z_R(n))|$ is bounded for all values of n by a positive constant K.

Then, if we apply $|z^D\rangle$ to a test vector $\phi_+ \in \Phi_+$, it gives:

$$\langle \phi_+|z^D\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, \langle \phi_+|\varphi_n^D\rangle\,, \tag{32}$$

so that,

$$|\langle \phi_+|z^D\rangle| = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, |\langle \phi_+|\varphi^D\rangle| \le \sum_{n=0}^{\infty} \frac{K}{\sqrt{n!}} < \infty\,. \tag{33}$$

Therefore, $|z^D\rangle$ is well defined for any $z \in \mathbb{C}$ as the series (28) weakly converges. Same is true with $|z^G\rangle$.

In Appendix 2, we show that these coherent Gamow states satisfy two equivalent resolutions of the identity.

3 Gamow states as Pseudo-Bosons

Let us consider the set of decaying Gamow vectors $\{\varphi_n^D\}_{n=0}^{\infty}$, in brief $\{\varphi_n^D\}$, and let $\mathcal{D}^D$ the complex linear space generated by $\{\varphi_n^D\}$. For any $\psi \in \mathcal{D}^D$, it is clear that

$$B^-\,B^+\psi - B^+\,B^-\,\psi = \psi \Longleftrightarrow [B^-, B^+] = 1\,. \tag{34}$$

The set of growing Gamow vectors $\{\varphi_n^G\}$ spans the complex linear space $\mathcal{D}^G$. The idea is to consider $\mathcal{D}^G$ as the dual space of $\mathcal{D}^D$. This is done by defining for all $n, m = 0, 1, 2, \dots$ the duality relation

$$\langle \varphi_n^D|\varphi_m^G\rangle = \delta_{n,m}\,, \tag{35}$$

and extend it by bilinearity to all $\mathcal{D}^D$ and all $\mathcal{D}^G$. Note that C^- is the formal adjoint of B^+ and C^+ is the formal adjoint of B^-, so that for any $\psi \in \mathcal{D}^D$ and any $\phi \in \mathcal{D}^G$, we have

$$\langle B^-\psi|\phi\rangle = \langle \psi|C^+\phi\rangle\,, \qquad \langle B^+\psi|\phi\rangle = \langle \psi|C^-\phi\rangle\,. \tag{36}$$

This is a slight generalization of the formalism for pseudo-bosons introduced by Bagarello [19]. In fact, if we identify $a \equiv B^-$, $b \equiv B^+$, $\varphi_0 \equiv \varphi_0^D$, $\Psi_0 \equiv \varphi_0^G$, so that $a^\dagger \equiv C^+$, $b^\dagger \equiv C^-$, we have the following expressions: The vectors

$$\varphi_n := \frac{1}{\sqrt{n!}}\, b^n\, \varphi_0\,, \qquad \Psi_n := \frac{1}{\sqrt{n!}}\, {a^\dagger}^n\, \Psi_0\,, \tag{37}$$

generate $\mathcal{D}^D$ and $\mathcal{D}^G$, respectively, as they are nothing else that φ_n^D and φ_n^G, respectively. In addition, one obviously has

$$b\varphi_n = \sqrt{n+1}\,\varphi_{n+1}\,, \qquad n \geq 0\,; \quad a\varphi_0 = 0\,; \quad a\varphi_n = \sqrt{n}\,\varphi_{n-1}\,, \qquad n \geq 1\,, \tag{38}$$

and

$$a^\dagger\,\Psi_n = \sqrt{n+1}\,\Psi_{n+1}\,, \qquad n \geq 0\,; \quad b^\dagger\,\Psi_0 = 0\,; \quad b^\dagger\,\Psi_n = \sqrt{n}\,\Psi_{n-1}\,, \qquad n \geq 1\,. \tag{39}$$

This shows the analogy of our formalism with that in [19]. Nevertheless, there are some minor differences that should not affect the essential idea. First of all, φ_n^D and φ_n^G do not belong to the same Hilbert space. In fact, there is no Hilbert space here, as the wave functions for φ_n^D and φ_n^G are not normalizable. It is certain that we may introduce by hand a scalar product. However, this is not very natural, although we always may define a "regularized" scalar product, for instance using Zeldovich regularization [23]. It seems more natural to introduce in the linear space spanned by φ_n^D and φ_n^G a Krein pseudometrics, such that along (35), one has $\langle\varphi_n^D|\varphi_m^D\rangle = \langle\varphi_n^G|\varphi_m^G\rangle$ for all values of m and n [24, 25], see also [26]. In any case, the pair $(z^D\rangle, |z^G\rangle)$, $z \in \mathbb{C}$ may be looked as bi-coherent states in the sense of [19].

From the mathematical point of view, it would be interesting to endow the spaces $\mathcal{D}^D$ and $\mathcal{D}^G$ with respective locally convex topologies such that the operators $b = B^+$ and $a = B^-$ are continuous on $\mathcal{D}^D$ and $a^\dagger = C^+$ and $b^\dagger = C^-$ are continuous on $\mathcal{D}^G$. Let us focus our discussion on the space $\mathcal{D}^D$. The construction on $\mathcal{D}^G$ is completely analogous.

So far, we have assumed that $\mathcal{D}^D$ is the space of linear combinations of decaying Gamow vectors. Let us extend it to the space of vectors $\varphi = \sum_{n=0}^\infty a_n\,\varphi_n$ with

$$||\varphi||_p^2 := \sum_{n=0}^\infty (n+1)^{2p}\,|a_n|^2 < \infty\,, \qquad p = 0, 1, 2, \dots\,, \tag{40}$$

and endow this extended $\mathcal{D}^D$, that we still call it $\mathcal{D}^D$ for simplicity, with the locally convex topology spanned by the norms $||-||_p$, $p = 0, 1, 2, \dots$.

A linear operator A on $\mathcal{D}^D$ is continuous if and only if for any $p = 0, 1, 2, \dots$, there exists a positive number C_p and a finite family of norms $\{||-||_{p_i}\}$, which may be different for each value of p, such that for any $\varphi \in \mathcal{D}^D$ one has [27]

$$||A\varphi||_p \leq C_p\{||\varphi||_{p_1} + \cdots + ||\varphi||_{p_k}\}\,. \tag{41}$$

Then, to prove the continuity of $b = B^+$ on $\mathcal{D}^D$, we write for any $p = 0, 1, 2, \dots$:

$$||b\varphi||_p^2 = \sum_{n=0}^\infty (n+1)^{2p+1}\,|a_n|^2 \leq \sum_{n=0}^\infty (n+1)^{2p+2}\,|a_n|^2 = ||\varphi||_{p+1}^2\,. \tag{42}$$

Also, the continuity of $a = B^-$ is easily proven (note that $n + 2 \leq 2(n+1)$):

$$||a\varphi||_p^2 = \sum_{n=1}^{\infty} n\,(n+1)^{2p}\,|a_{n-1}|^2 = \sum_{n=0}^{\infty} (n+1)(n+2)^{2p}\,|a_n|^2$$

$$\leq 2^{2p} \sum_{n=0}^{\infty} (n+1)^2\,(n+1)^{2p}\,|a_n|^2 \leq 2^{2p} ||\varphi||_{p+1}^2\,. \tag{43}$$

The continuity on $\mathcal{D}^D$ of all elements of the algebra spanned by $b = B^+$ and $a = B^-$ is obvious. Same treatment is valid for $C^\pm$ on an analogous locally convex extension of $\mathcal{D}^G$. It is noteworthy that these topological versions of $\mathcal{D}^D$ and $\mathcal{D}^G$ are homeomorphic to the Schwartz space S, see [27], and therefore, these spaces are isomorphic to S.

A further comment: The commutation relations $[B^-, B^+] = 1 = [C^+, C^-]$ show that $B^\pm$ and $C^\pm$ separately behave like the creation and annihilation operators of the harmonic oscillator. At this point, we should mention that there are up to three types of hyperbolic Pöshl-Teller potentials, depending on the values of the parameter λ in (1). For values of λ in the ranges $\lambda > 0$ and $1/2 \leq \lambda < 1$, one may construct ladder operators connecting bound and antibound states in such a way that these operators satisfy the commutation relations of the generators of the Lie algebra $su(2)$ [17]. This is not the case here, where we have chosen $\lambda = i\ell + 1/2$, the only Pöschl-Teller potential that shows scattering resonances.

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4 Appendix 1. The spaces $\Phi_\pm$.

The idea of the construction of the spaces $\Phi_\pm$ goes as follows: The Hamiltonian H in (1) is self-adjoint as an unbounded operator on $L^2(\mathbb{R})$, with simple absolutely continuous spectrum given by $\mathbb{R}^+ \equiv [0, \infty)$. According to the spectral theorem [27], there exists a unitary operator $U : L^2(\mathbb{R}) \longmapsto L^2(\mathbb{R}^+)$ such that UHU^{-1} is the multiplication operator on $L^2(\mathbb{R}^+)$.

A Hardy function, $f(z)$, on the lower half plane is a complex analytic function on $\mathbb{C}^- := \{z \in \mathbb{C}\ ;\ \operatorname{Im} z < 0\}$, where $\mathbb{C}$ is the field of complex numbers, such that

$$\sup_{y>0} \int_{-\infty}^{\infty} |f(x - iy)|^2\,dx < \infty\,. \tag{44}$$

As a consequence, the boundary values of $f(z)$ exists for $z = x$ real and the resulting function is defined a.e. and is square integrable:

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx < \infty\,. \tag{45}$$

Hardy functions on the lower half plane form a closed subspace of $L^2(\mathbb{R})$ that we denote as $\mathcal{H}^2_-$. Similarly, we may define Hardy functions on the upper half plane. These functions form a closed subspace of $L^2(\mathbb{R})$ that we denote as $\mathcal{H}^2_-$. This is an immediate consequence of the *Paley-Wiener* theorem [28], which states that the spaces of Hardy functions are Fourier transforms of square integrable functions on a half line. If $\mathcal{F}$ represents the Fourier transform, the Paley Wiener theorem establish that

$$\mathcal{H}^2_\pm \equiv \mathcal{F}[L^2(\mathbb{R}^\mp)] \tag{46}$$

with $\mathbb{R}^- \equiv (-\infty, 0]$. In consequence,

$$L^2(\mathbb{R}) = \mathcal{H}^2_+ \oplus \mathcal{H}^2_- \,, \tag{47}$$

where $\oplus$ denotes orthogonal direct sum.

Another important property is that Hardy functions are uniquely determined by their boundary values on the positive semi-axis $\mathbb{R}^+$. Then, if we denote by $\mathcal{H}^2_\pm|_{\mathbb{R}^+}$ the spaces of restrictions of Hardy functions on the positive semi-axis, a result by van Winter [29] shows that $\mathcal{H}^2_\pm|_{\mathbb{R}^+}$ are dense in $L^2(\mathbb{R}^+)$ with the topology derived from the norm in $L^2(\mathbb{R}^+)$.

By S we denote the Schwartz space, which is the linear space of all complex functions, $f(x)$, on the line $\mathbb{R}$ such that: i.) Any $f(x) \in S$ admits derivatives of all orders and at all points; ii) Any $f(x) \in S$ as well as all its derivatives go to zero at the infinity faster than the inverse of any polynomial. The Schwartz space S is usually endowed with a topology given by an infinite countable set of norms, which may be defined in various equivalent ways [27]. For example, this family of norms

$$||f||_{m,n} := \sup_{x\in\mathbb{R}} \left| x^m \, \frac{d^n}{dx^n} \, f(x) \right| \,, \tag{48}$$

gives the mentioned topology, which is metrizable, complete (so that S is a Fréchet space) and nuclear.

Let us consider the spaces $S \cap \mathcal{H}^2_\pm$ with the topology inherited from the Schwartz space S, and, then, the space $[S \cap \mathcal{H}^2_\pm]_{\mathbb{R}^+}$ of its restrictions to the positive semi-axis. Since Hardy functions are fully determined by their boundary values on $\mathbb{R}^+$, we may consider the canonical injection $S \cap \mathcal{H}^2_\pm \longmapsto [S \cap \mathcal{H}^2_\pm]_{\mathbb{R}^+}$ that assigns each function in $S \cap \mathcal{H}^2_\pm$ to its restriction to $\mathbb{R}^+$. The space $[S \cap \mathcal{H}^2_\pm]_{\mathbb{R}^+}$ is dense in $L^2(\mathbb{R}^+)$ with the norm topology of the latter [13, 14]. On $[S \cap \mathcal{H}^2_\pm]_{\mathbb{R}^+}$, we have the metrizable topology transported by this canonical injection.

Now we define,

$$\Phi_\mp := U^{-1}[S \cap \mathcal{H}^2_\pm]_{\mathbb{R}^+} \,, \tag{49}$$

where U is the unitary mapping defined at the beginning of the present Appendix. Then, let us transport the topology from $S \cap \mathcal{H}^2_\pm$ to $\Phi_\mp$ by U^{-1}.

A mapping $F_\mp$ from $\Phi_\mp$ to the field of complex numbers $\mathbb{C}$ is said to be *antilinear* if for any pair of vectors $\varphi_\mp$ and $\psi_\mp$ and any pair of complex numbers α and β, we have that

$$\langle \alpha\varphi_\mp + \beta\psi_\mp | F_\mp \rangle = \alpha^* \langle \varphi_\mp | F_\mp \rangle + \beta^* \langle \psi_\mp | F_\mp \rangle \,, \tag{50}$$

where $\langle\psi_\mp|F_\mp\rangle$ represents the action of the functional $F_\mp$ on the vector $\psi_\mp$, respectively and the star denotes complex conjugation. Let us consider the vector spaces, $\Phi^\times_\mp$, spanned by the *continuous* antilinear functionals on $\Phi_\pm$. These spaces are usually endowed with the *weak* topology induced by the dual pair $(\Phi_\pm,\Phi^\times_\pm)$, [30]. Then, we obtain a couple of triplets or *rigged Hilbert spaces* [31], which are

$$\Phi_\pm \subset L^2(\mathbb{R}) \subset \Phi^\times_\pm\,. \tag{51}$$

After the above discussion, we see that to any vector $\phi_\pm \in \Phi_\pm$, it corresponds a Hardy function $\phi_\mp(z)$ uniquely determined. Then, the Gamow vectors $\varphi^D_n \in \Phi^\times_+$ and $\varphi^G_n \in \Phi^\times_-$ are defined by their action on each $\phi_+ \in \Phi_+$ and $\phi_- \in \Phi_-$, respectively, as [13, 14]

$$\langle\phi_+|\varphi^D_n\rangle := \phi^*_+(z_R(n))\,, \qquad \langle\phi_-|\varphi^G_n\rangle := \phi^*_-(z^*_R(n))\,. \tag{52}$$

Note that *both* spaces $\Phi_\pm$ are included in *both* anti-duals $\Phi^\times_\pm$. Therefore, there exists four canonical injections, which are all continuous. In the next Appendix, we are considering only two, the canonical injections $I_\pm : \Phi_\pm \longmapsto \Phi^\times_\mp$. Observe the change of signs.

Some additional properties of Gamow vectors could be found in the literature, particularly in [13, 14] and references quoted therein.

4.1 Appendix 2. Resolutions of the identity

These coherent states satisfy a couple of resolutions of the identity in a sense to be clarified here. A spectral decomposition of the Hamiltonian in terms of the Gamow vectors has the form [32]

$$H = \sum_n z_R(n)\,|\varphi^D_n\rangle\langle\psi^G_n| + \text{background}\,. \tag{53}$$

The background term gives the deviations of the exponential decay law for very small (Zeno effect) and very large (Khalfin effect) values of time, which are very difficult to be observed [33, 34]. It seems reasonable to neglect this term when we study most of effects produced by quantum resonances, outside these ranges of time. Once we have drop out the background term, the meaningful part of the spectral decomposition of the Hamiltonian is given by the sum in (53). This is a linear continuous mapping from Φ_- into $\Phi^\times_+$ as defined in Appendix I. Correspondingly, a decomposition of the identity is also a continuous mapping (in fact the canonical injection) from Φ_- into $\Phi^\times_+$ given by

$$I_- = \sum_{n=0}^\infty |\varphi^D_n\rangle\langle\psi^G_n|\,. \tag{54}$$

Next, let us consider

$$\int_{\mathbb{C}} |z^D\rangle\langle z^G|\,\frac{1}{\pi}\,e^{-|z|^2}\,dz\,, \tag{55}$$

with $dz = dx\,dy$, $z = x + iy$. Since $|z^D\rangle \in \Phi_+^\times$ and $|z^G\rangle \in \Phi_-^\times$, (55) makes sense if we take arbitrary $\varphi^+ \in \Phi_+$ and $\psi^- \in \Phi_-$ and write

$$\int_{\mathbb{C}} \langle\varphi^+|z^D\rangle\langle z^G|\psi^-\rangle\, \frac{1}{\pi}\, e^{-|z|^2}\, dz\,. \tag{56}$$

Let us calculate the brackets in (56) using (52) and (30):

$$\langle\varphi^+|z^D\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, \langle\varphi^+|\varphi_n^G\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\, \varphi^+(z_R(n))\,, \tag{57}$$

and

$$\langle z^G|\psi^-\rangle = \sum_{m=0}^{\infty} \frac{(z^*)^m}{\sqrt{m!}}\, \langle\varphi_n^G|\psi^-\rangle = \sum_{m=0}^{\infty} \frac{(z^*)^m}{\sqrt{m!}}\, \psi^-(z_R(m))^*\,. \tag{58}$$

Due to the fact that Hardy functions behave at the infinity as $|z|^{-1/2}$, it results that $|\langle\varphi^+|z^D\rangle\langle z^G|\psi^-\rangle|$ is bounded by $e^{2|z|}$, which is integrable with respect to the measure $e^{-|z|^2}\,dz$ on the complex plane. Therefore, we may use the Lebesgue theorem in (56), which gives

$$\int_{\mathbb{C}} \langle\varphi^+|z^D\rangle\langle z^G|\psi^-\rangle\, \frac{1}{\pi}\, e^{-|z|^2}\, dz = \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}\,\sqrt{m!}}\, \langle\varphi^+|\varphi_n^G\rangle\langle\varphi_n^G|\psi^-\rangle\, \frac{1}{\pi} \int_{\mathbb{C}} z^n\,(z^*)^m\, e^{-|z|^2}\, dz$$

$$= \sum_{n,m=0}^{\infty} \frac{1}{\sqrt{n!}\,\sqrt{m!}}\, \langle\varphi^+|\varphi_n^G\rangle\langle\varphi_n^G|\psi^-\rangle\, n!\,\delta_{n,m} = \sum_{n=0}^{\infty} \langle\varphi^+|\varphi_n^G\rangle\langle\varphi_n^G|\psi^-\rangle\,. \tag{59}$$

Then, if we omit the arbitrary φ^- and ψ^-, we finally obtain that

$$\int_{\mathbb{C}} |z^D\rangle\langle z^G|\, \frac{1}{\pi}\, e^{-|z|^2}\, dz = \sum_{n=0}^{\infty} |\varphi_n^D\rangle\langle\psi_n^G| = I_-\,, \tag{60}$$

where the I_- is the canonical injection from Φ_- into $\Phi_+^\times$. We have proven a sort of resolution of the identity using our Gamow coherent states. Analogously,

$$\int_{\mathbb{C}} |z^G\rangle\langle z^D|\, \frac{1}{\pi}\, e^{-|z|^2}\, dz = \sum_{n=0}^{\infty} |\varphi_n^G\rangle\langle\psi_n^D| = I_+\,. \tag{61}$$

Thus, we have indeed two resolutions of the identity, which look like formal adjoint of each other.

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