Classical ladder functions for Rosen-Morse and curved Kepler-Coulomb systems

L. Delisle-Doray[a], V. Hussin[b], Ş. Kuru[c], J. Negro[d]

[a]Département de Physique & Centre de Recherches Mathématiques, Université de Montréal, QC, H3C 3J7, Canada
[b]Département de Mathématiques et de Statistique & Centre de Recherches Mathématiques, Université de Montréal, QC, H3C 3J7, Canada
[c]Department of Physics, Faculty of Science, Ankara University, 06100 Ankara, Turkey
[d]Departamento de Física Teórica, Atómica y Óptica, Universidad de Valladolid, 47011 Valladolid, Spain

January 1, 2019

Abstract

Ladder functions in classical mechanics are defined in a similar way as ladder operators in the context of quantum mechanics. In the present paper, we develop a new method for obtaining ladder functions of one dimensional systems by means of a product of two 'factor functions'. We apply this method to the curved Kepler-Coulomb and Rosen-Morse II systems whose ladder functions were not found yet. The ladder functions here obtained are applied to get the motion of the system.

PACS:

KEYWORDS:

1 Introduction

In quantum mechanics the knowledge of symmetry operators $\hat{S}$ that commute with the Hamiltonian operator $\hat{H}$ of the system is very important. The set of symmetry operators closes, under the composition operation, a 'symmetry algebra'. This type of operators gives useful information on the system, in particular they explain the degeneracy of the energy levels and also they can supply the spectrum. A second type of interesting operators are called ladder operators $\hat{A}$. They do not commute with the Hamiltonian, but satisfy simple commutation rules of conformal type: $[\hat{H},\hat{A}]=\alpha(\hat{H})\hat{A}$, where $\alpha(\hat{H})$ designs a certain function of $\hat{H}$. When such operators exist, they connect eigenspaces of different energies; the paradigmatic case is given by the lowering and raising operators of the harmonic oscillator $a^{\pm}$. If we include both, symmetry and ladder operators, we get a 'spectrum generating algebra' [1, 2, 3].

In the frame of the Hamiltonian formalism of classical mechanics, the symmetries are generated by functions $S(x,p)$ of the canonical variables such that the Poisson bracket with the Hamiltonian function $H(x,p)$ vanishes: $\{H,S\}=0$. The surfaces defined by the values of such functions, $S(x,p)=s_0$, are made of trajectories of the system and any trajectory must be included in one of such surfaces. Although much less known, one can also define ladder functions $A(x,p)$ for classical systems as those satisfying the corresponding Poisson bracket with the Hamiltonian: $\{H,A\}=i\,\alpha(H)A$. In the classical context, the ladder functions supply information on the motion of the system.

The aim of this paper is to find the ladder functions of two families of one dimensional systems known as Rosen-Morse II (RMII) [4, 5] and curved Kepler-Coulomb (KC) [6, 7, 8, 9]. Ladder functions for other one dimensional systems have been computed in previous works [10, 11], but they were still missing for the RMII and curved KC systems, in particular. The ladder functions of these two systems are considerably more elaborate, and the purpose of this work is to introduce a procedure to get them. In this way, we want to complete the knowledge of this type of functions for the whole list of classical systems corresponding to the quantum systems solved by means of the factorization method as given by Infeld-Hull [4, 12].

The structure of the paper is as follows. In Section 2, we introduce the ladder functions and some of their basic properties. Next, in Section 3, we show our method to compute ladder functions as a product of two 'factor functions'. Then, in Section 4 we apply this method to the RMII and show that it includes the well known results for the Pöschl-Teller system as a particular case. Next, in Section 5 we deal with the curved KC system. Section 6 considers the limit from curved to flat KC systems and the case with zero angular momentum in order to get the formulas obtained previously by other methods. Some remarks will end the paper in the last section.

2 Basic theory

2.1 Definition of ladder functions

Let H be a one-dimensional Hamiltonian with canonical coordinates (x,p) defined as

$$H=p^2+V(x)\,. \tag{2.1}$$

We want to find two complex functions $A^{\pm}(x,p)$ defined on (some part of) the phase space (x,p), that together with the Hamiltonian $H(x,p)$ satisfy the following Poisson bracket algebra [10, 11, 13]:

$$\{H, A^{\pm}\} = \mp i\alpha(H)A^{\pm}, \tag{2.2}$$

$$\{A^{+}, A^{-}\} = i\beta(H), \tag{2.3}$$

where, in principle, $\alpha(H)$ and $\beta(H)$ are real functions which depend on H. We also may assume that $\alpha(H)$ be positive. Such an algebra is a classical analog of the generalized Heisenberg algebra (GHA), which is satisfied by ladder operators, with respect to commutators, in quantum mechanics [14, 15]. Since (2.3) can be deduced from (2.2), and since the ladder functions $A^{\pm}$ can in general be taken to be complex conjugate of each other, $A^{+} = (A^{-})^{*}$ (at least for bound states), only one of the equations (2.2) will be relevant. From (2.2) we can also deduce a type of factorization in the following form:

$$\delta(H) = A^{+}A^{-}, \tag{2.4}$$

where $\delta(H)$ is a certain function depending only on H. This is the reason why the ladder functions are also referred as the classical counterpart to the factorization method in quantum mechanics. We will note however that (2.4) is actually weaker than its quantum analog because of the commutativity of the product of phase-space functions. We will refer to it as the factorization condition, satisfied by a much richer variety of couples of "factor functions" than just the ladder functions. In fact, this condition will be quite useful to construct the ladder functions.

In this paper, we assume that the potential $V(x)$ has the form of a well, and we want to describe the bounded motion of the system in that well. Then, the motion with energy $H = E$ will be periodic between two turning points $x_{\pm}$ determined by two solutions of the equation $E = V(x)$ (where $p = 0$). The value of the physical frequency $\omega(H)$ (and its period $T(H)$) is given by

$$\omega(H) \equiv \frac{2\pi}{T(H)} = \frac{2\pi}{\int_{x_-}^{x_+} \frac{dx'}{\sqrt{H - V(x')}}}\,. \tag{2.5}$$

Now, the ladder functions $A^{\pm}$ of this system, satisfying (2.2), will determine the constants of motion $Q^{\pm}$ (depending explicitly on time) defined by [11]

$$Q^{\pm}(x,p,t) = A^{\pm}(x,p)e^{\mp i\alpha(H)t}\,. \tag{2.6}$$

This can be easily proved, since

$$\frac{dQ^{\pm}}{dt} = \{Q^{\pm}, H\} + \frac{\partial Q^{\pm}}{\partial t} = 0\,.$$

The constant complex values of $Q^{\pm}$ will be denoted by $qe^{\pm i\theta_0}$. Then, the equations

$$A^{\pm}(x,p)e^{\mp i\alpha(H)t} = qe^{\pm i\theta_0} \tag{2.7}$$

will lead to the motion $(x(t), p(t))$ of the system in the phase space in an algebraic way. As a consequence, the motion will have a frequency given by the exponent $\alpha(H)$, and therefore it should be equal to the physical frequency, $\alpha(H) = \omega(H)$ for 'fundamental' ladder functions $A^\pm$. Other ladder functions, such as $\tilde{A}^\pm = (A^\pm)^n$, $n = 1, 2, \dots$ will produce multiple frequencies $\alpha(H) = n\,\omega(H)$. The ladder operators (or functions) with multiple frequencies $n > 1$ have applications in some problems, for instance in order to find higher order symmetries of superintegrable systems [16, 17].

2.2 Ladder functions in the variables (x, H)

The object of this section is to change from the canonical variables (x, p) to another more practical set (x, H) and to develop an integral formula for a ladder function satisfying (2.2). This formula will not necessarily be useful to calculate an explicit (analytic) form for the ladder functions, but it will supply us with some information about its behaviour in the phase space.

If we introduce in the differential equation (2.2) the change of variables $(x, p) \to (x, H)$ with $p(x, H) = \pm\sqrt{H - V(x)}$ for each part of the half planes $p > 0$ and $p < 0$, then, using the chain rule, the Poisson bracket (2.2) for the functions $A^\pm(x, H)$ becomes:

$$\{H, A^\pm(x, H)\} = -2p\,\frac{\partial A^\pm(x, H)}{\partial x} = \mp i\alpha(H)A^\pm(x, H)\,. \tag{2.8}$$

Remark that this is not a change of canonical variables since (x, H) are not canonical. Notice also that eq. (2.8) is only valid for each half-plane taken separately and not for $p = 0$. This equation represents in fact two differential equations, one for each half-plane. To make this a bit clearer we introduce an index variable $\eta = \pm 1$ defined by $p = \eta\sqrt{H - V(x)}$. Then, eq. (2.8) is rewritten as:

$$i\alpha(H)A^\pm(x, H, \eta) = \pm 2\eta\sqrt{H - V(x)}\,\frac{\partial A^\pm(x, H, \eta)}{\partial x}\,, \tag{2.9}$$

and its integration yields the formula (for each H, $V(x) < H$):

$$A^\pm(x, H, \eta) = B(H, \eta)\exp\left(\pm i\eta\frac{\alpha(H)}{2}\int_{x_m}^{x}\frac{dx'}{\sqrt{H - V(x')}}\right), \tag{2.10}$$

where $B(H, \eta)$ is an integration constant, and the definite integral runs from any position $x_m \in (x_-, x_+)$ such that $V(x_m) < H$. Thus, formula (2.10) represents any algebraic ladder function A that satisfies (2.2) with respect to the function $\alpha(H)$ in these two regions.

The turning points $x_\pm$ are the ones for which $p = 0$, as mentioned above. In the coordinates (x, H) they are given by $x_\pm(H)$ with $H = V(x_\pm)$, so that the points $x_\pm$ also determine the value $H(x_\pm)$. The continuity of the ladder functions $A^\pm(x, p)$ implies that $A^\pm(x, H, \eta)$, defined by (2.10), satisfy:

$$\lim_{(x,H)\to(x_\pm, H(x_\pm))} A^\pm(x, H, +) = \lim_{(x,H)\to(x_\pm, H(x_\pm))} A^\pm(x, H, -)\,.$$

3 Construction of ladder functions as a product of factor functions

3.1 Contribution of factor functions

As it was mentioned in the introduction, we want to combine 'factor functions' to obtain a ladder function. To do this we need a way to keep track of the contribution of each factor function. Let us take two factor functions $f(x,H)$ and $g(x,H)$, such that they verify the factorization condition (2.4):

$$f^* f = \delta_f(H), \qquad g^* g = \delta_g(H)\,, \tag{3.1}$$

for some functions $\delta_f(H)$ and $\delta_g(H)$ of H. Now, notice that since the Poisson bracket satisfies the Leibniz's rule,

$$\{H, fg\} = f\{H,g\} + \{H,f\}g\,, \tag{3.2}$$

dividing by fg, we have

$$\frac{\{H,fg\}}{fg} = \frac{\{H,f\}}{f} + \frac{\{H,g\}}{g}. \tag{3.3}$$

It is thus useful to introduce the following notation:

$$\Lambda(f) = \frac{\{H,f\}}{f}\,. \tag{3.4}$$

This function $\Lambda(f)$ will be called the 'contribution' of f and eq. (3.3) will be reformulated as

$$\Lambda(fg) = \Lambda(f) + \Lambda(g). \tag{3.5}$$

The ladder bracket (2.2) is now equivalent to

$$\Lambda(A^{\pm}) = \mp i\alpha(H). \tag{3.6}$$

Hence, if after adding the contributions (3.5) of two factor functions, the sum is a function depending only on H, then the product fg will constitute a good ladder function (at least algebraically speaking).

The most useful ansatz for a factor function in the search of ladder functions, suggested by the form of other simpler cases [10], is

$$f(x,H) = a(x,H) + i\,b(x,H)p\,, \tag{3.7}$$

with $a(x,H)$ and $b(x,H)$ real functions depending on the variables x and H. For such a function, using $p^2 = H - V(x)$ and $p' = -\frac{V'(x)}{2p}$, its contribution is given by:

$$\Lambda(f(x,p)) = i\,\frac{2(H-V)(a'b - b'a) + V'b\,a}{\delta_f(H)}\,, \tag{3.8}$$

where the derivative is taken with respect to x maintaining H constant.

Another useful fact about Λ is that we can handle exponents more easily. Let us assume that we can define the exponent of a function f using a well chosen determination of the complex logarithm $f^{\rho(H)} = \exp[\rho(H)\log(f)]$, then it is easily seen from (3.2) and (3.4) that we will have:

$$\Lambda(f^{\rho(H)}) = \rho(H)\Lambda(f). \tag{3.9}$$

Some technicalities arising from the proper definition of exponent in the complex plane will have to be addressed if necessary.

3.2 Signature

In this subsection we will introduce a necessary and sufficient condition for a ladder function to be dynamically valid, i.e., $\alpha(H) = n\omega(H)$, $n = 1, 2, \ldots$ (under the standard assumption of continuous differentiability). We start by assuming that we have the functions $A^{\pm}(x, p)$ which satisfy (2.2) with a certain function $\alpha = \alpha(H)$ in a symmetric simply connected region R of the plane (including a segment of $p = 0$ as its axis of symmetry). We can then introduce two corresponding functions $A^{\pm}(x, H, \eta)$ of (x, H) in each region $p \geq 0$ and $0 \leq p$, using the index parameter $\eta = \pm 1$ defined in Sect. 2 ($p = \eta\sqrt{H - V(x)}$). Now, this means that for the two half-regions $p > 0$ and $p < 0$, the representation (2.10) is valid. Then, if $\alpha(H) = n\omega(H)$ with $\omega(H)$ defined by (2.5), we will have that

$$A^{\pm}(x_+, H, \eta) = (-1)^n A^{\pm}(x_-, H, \eta). \tag{3.10}$$

To see this, we just divide $A^{\pm}(x_+, H, \eta)$ by $A^{\pm}(x_-, H, \eta)$ and we expand the argument of the exponential using the fact that $\int_{x_-}^{x_m} + \int_{x_m}^{x_+} = \int_{x_-}^{x_+}$. Taking into account (2.5), this gives $e^{in\pi} = (-1)^n$. In the same way, it is easily shown that (3.10) implies $\alpha(H) = n\omega(H)$ (sufficiency). We call condition (3.10) antiperiodic if n is odd (this contains the privileged case $n = 1$, i.e. the 'fundamental' ladder functions) and periodic if n is even.

Now, we can define for any factor function $f(x, H)$ its signature $\Gamma(f)$ by

$$\Gamma(f) = \frac{f(x_+, H)}{f(x_-, H)}\,. \tag{3.11}$$

Thus, $\Gamma(f) = 1$ corresponds to the periodic case and $\Gamma(f) = -1$ to the antiperiodic case. The following simple property of the signature is really useful:

$$\Gamma(fg) = \Gamma(f)\Gamma(g). \tag{3.12}$$

If $A^{\pm} = fg$, this will allow us to calculate the signature of $A^{\pm}$ from that of each factor function.

The behaviour of the signature with respect to exponentiation is subtler and has to be examined more carefully. First, remark that in the case of a function of the form f given

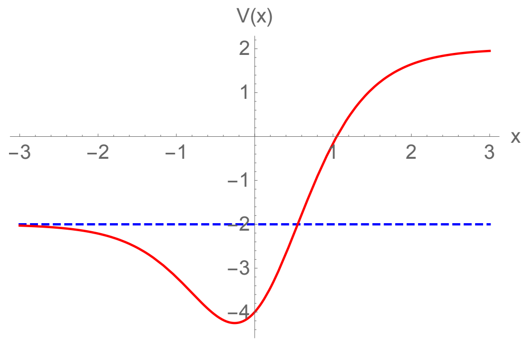

Figure 1: Plot of the Rosen-Morse II potential for $B = 2$, $C = 4$. The dashed line corresponds to $E = -B$.

by (3.7), the signature reduces to the calculation of the ratio:

$$\Gamma(f) = \frac{a(x_+, H)}{a(x_-, H)} . \tag{3.13}$$

Therefore, if both $a(x_+, H)$ and $a(x_-, H)$ are positive, we will simply have:

$$\Gamma(f^{\rho(H)}) = \Gamma(f)^{\rho(H)}. \tag{3.14}$$

Finally, the function f given by (3.7) that satisfies the factorization condition (2.4) with its complex conjugate, leads to

$$a^2 + b^2 p^2 = \delta_f(H). \tag{3.15}$$

From this and (3.13), it is easy to see that $\Gamma(f)^2 = 1$ so that $\Gamma(f) = \pm 1$ at $p = 0$. This will allow us to determine the signature of the function by simple analytic arguments.

4 The Rosen-Morse II system

In this section we consider the RMII system [4, 18] defined by the Hamiltonian

$$H = p^2 + B \tanh x - \frac{C}{\cosh^2 x} , \qquad x \in \mathbb{R} , \tag{4.1}$$

where B and C are real parameters. In order to have the shape of a well, we will assume that $C > 0$. We will look for ladder functions of this system in the region of the phase-space $H(x, p) < -|B|$, where the motion is bounded and periodic (see Fig. 1).

4.1 Ladder functions

We will look for the fundamental ladder functions satisfying (2.2) and such that $\alpha(H)$ is the frequency. The process is similar to the simplest cases as the harmonic oscillator, Pöschl-Teller, etc. but in this case it is a bit more complex. We will need two sets of factor functions that factorize the RMII Hamiltonian in the sense (2.4), such that they will be the basic ingredients of the fundamental ladder functions.

The first set of factor functions f_ϵ, where $\epsilon = \pm 1$, is given by

$$f_\epsilon = a_\epsilon + ib\,p, \tag{4.2}$$

where

$$b(x) = \cosh x, \quad a_\epsilon(x,H) = b(x)\left(\phi_\epsilon(H)\tanh x + \frac{B}{2\phi_\epsilon(H)}\right). \tag{4.3}$$

Then, the factorization condition (3.1) is fulfilled with

$$\delta_{f_\epsilon}(H) = C - \phi_\epsilon(H)^2, \tag{4.4}$$

where $\phi_\epsilon(H)$ is

$$\phi_\epsilon(H) = \frac{\sqrt{-H+B} + \epsilon\sqrt{-H-B}}{2}, \tag{4.5}$$

and

$$\phi_\epsilon(H)^2 = \frac{-H + \epsilon\sqrt{H^2 - B^2}}{2}. \tag{4.6}$$

The second set of factor functions g_ϵ is given by

$$g_\epsilon = c_\epsilon - id_\epsilon p, \tag{4.7}$$

where

$$d_\epsilon(x) = \frac{1}{\tanh x - \epsilon}, \quad c_\epsilon(x,H) = d_\epsilon(x)\left(\frac{(B+2\epsilon C)\tanh x + \epsilon B - 2(H+C)}{2\sqrt{-H+\epsilon B}}\right). \tag{4.8}$$

The factorization condition (3.1) is satisfied with the function

$$\delta_{g_\epsilon}(H) = \frac{B^2 + 4C(C+H)}{4(-H+\epsilon B)}. \tag{4.9}$$

One may be surprised by the fact that there are essentially two simple forms of ladder functions given by f_ϵ and g_ϵ. However, the origin of these solutions is easy to explain. The first factorization comes after writing the Hamiltonian (4.1) in the form

$$C = (B\tanh x - H)\cosh^2 x + p^2\cosh^2 x = (a_\epsilon + ibp)(a_\epsilon - ibp) + \phi_\epsilon(H)^2. \tag{4.10}$$

The second factorization comes if we rewrite (4.1) in terms of $\tanh x$,

$$H + 2C = p^2 + (B + \epsilon 2C)\tanh x + C(\tanh x - \epsilon)^2\,, \qquad x \in \mathbb{R}\,, \tag{4.11}$$

or

$$C = \frac{p^2}{(\tanh x - \epsilon)^2} + \frac{(B + \epsilon 2C)\tanh x - (H + 2C)}{(\tanh x - \epsilon)^2}\,, \qquad x \in \mathbb{R}\,, \tag{4.12}$$

whose immediate factorization leads to the second pair (4.7)-(4.8).

The calculations of the contribution and signature of each function are now straightforward. In the Appendix we give an analytic argument for the values of the signature. The results are given in the following set of relations:

$$\begin{aligned} \Lambda(f_\epsilon) &= i(2\phi_\epsilon(H) + \frac{B}{\phi_\epsilon(H)}\tanh x), &\qquad \Gamma(f_\epsilon) &= -\epsilon, \\ \Lambda(g_\epsilon) &= -2i\sqrt{-H + \epsilon B}\,(\tanh x + \epsilon), &\qquad \Gamma(g_\epsilon) &= -1\,. \end{aligned} \tag{4.13}$$

Our strategy is as follows: we form different products of two of these factor functions such as $f_{\epsilon_1}^{\gamma(H)} g_{\epsilon_2}^{\sigma(H)}$, $f_1^{\gamma(H)} f_{-1}^{\sigma(H)}$, $g_1^{\gamma(H)} g_{-1}^{\sigma(H)}$ and choose the exponents to make the $\tanh x$ dependence in (4.13) cancel against each other and, at the same time, to satisfy the antiperiodic condition. For all these products we have to carefully treat the exponents which could take values in the complex plane in general. There are however products that we can deal with without worrying too much about this issue. Indeed, as we have shown in the derivation of the signature of $f_{\pm 1}$ (see Appendix), we have the useful result $a_{-1}(x, H) > 0$. This means that the image of $f_{-1} = a_{-1} + ibp$ is confined to the half-plane $\mathrm{Re}(z) > 0$ of the complex plane. In this region, we can simply use the principal determination of the logarithm (which is continuous everywhere except on the half-line $\{z : \mathrm{Re}(z) \leq 0, \mathrm{Im}(z) = 0\}$) to define the exponent $f_{-1}^{\gamma(H)}$ as a continuous function of (x, p) (in the region where $H < -B$). Also this means that we can use (3.14) to calculate its signature: $\Gamma(f_{-1}^{\gamma(H)}) = 1$. In order to get a ladder function, let us thus define the following products:

$$A_\epsilon = f_{-1}^{\gamma_\epsilon(H)} g_\epsilon \tag{4.14}$$

with

$$\gamma_\epsilon(H) = \frac{2\phi_{-1}(H)\sqrt{-H + \epsilon B}}{B}\,. \tag{4.15}$$

We use (3.5) and (3.9) to deduce the formula

$$\Lambda(A_\epsilon) = \gamma_\epsilon(H)\Lambda(f_{-1}) + \Lambda(g_\epsilon) = -i\epsilon\,\frac{4\sqrt{H^2 - B^2}}{\sqrt{-H + B} + \sqrt{-H - B}} \equiv -i\epsilon\alpha(H)\,. \tag{4.16}$$

Now, the properties of the signature (3.12) and (3.14) together with (4.13), applied for the chosen factor functions in (4.14), allows us to check that $\Gamma(A_\epsilon) = -1$. Thus, we

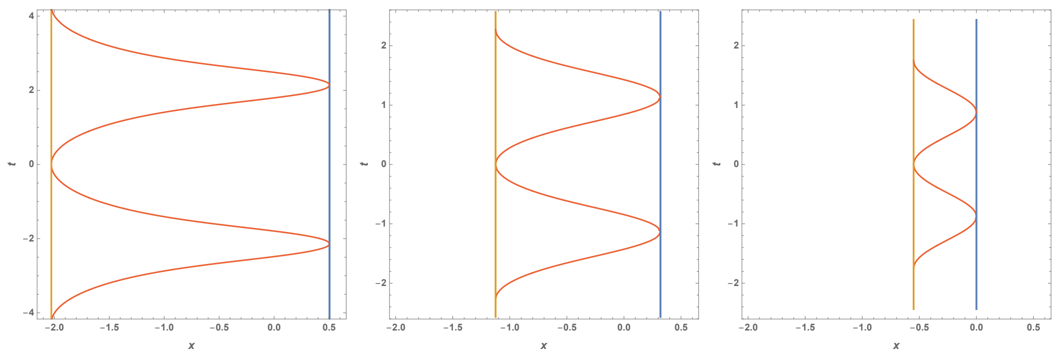

Figure 2: Plot of the motion $x(t)$ in the Rosen-Morse II potential with $B = 2$, $C = 4$, and energy values: $E = -2.2$ (left), $E = -3$ (center) $E = -4$ (right). The left and right vertical lines are for the values x_- and x_+ bounding the motion. The motion in the graphic is restricted to two oscillations.

conclude that we have an algebraically valid ladder function for the region $H < -|B|$ with the equality $\alpha(H) = n\omega(H)$, where n must be odd. In the next subsection we show that $n = 1$ by considering the case $B = 0$. This means that we have found a set of fundamental ladder functions given by $A^\pm = A_{\pm 1}$. In fact, we have independently checked that the value of the natural frequency is indeed given in (4.16) by a direct evaluation of the integral (2.5) for the RMII potential:

$$\omega(H) = \frac{2\pi}{\int_{x_-}^{x_+} \frac{dx'}{\sqrt{H - V(x')}}} = \frac{4\sqrt{H^2 - B^2}}{\sqrt{-H + B} + \sqrt{-H - B}} \,. \tag{4.17}$$

As a consequence, according to the expression of the constant of motion (2.7) and (4.14), the motion of a particle in the MRII potential (for $B > 0$ and $\epsilon = -1$) is given by

$$\begin{aligned} \gamma_{-1}(E) \arctan\left[\frac{p(x,E)}{\phi_{-1}(E)\tanh x + b/(2\phi_{-1}(E))}\right] + \\ \arctan\left[\frac{-2p(x,E)\sqrt{-E-B}}{(B-2C)\tanh x - B - 2(E+C)}\right] - \omega(E)t = \theta_0 \,, \end{aligned} \tag{4.18}$$

where θ_0 is a constant fixing the initial time. A plot of some examples of this motion for some particular values of the energy E is shown in Fig. 2.

4.2 The Pöschl-Teller potential

The (hyperbolic) Pöschl-Teller (PT) potential is a particular case of the Rosen-Morse potential when $B = 0$:

$$H = p^2 - \frac{C}{\cosh^2 x} \,, \qquad x \in \mathbb{R}. \tag{4.19}$$

In this section, we want to check that the ladder functions of the RMII potentials in the previous subsection, when taking the limit $B \to 0$, agree with the known ladder functions of PT [10, 19]. In this way, we will also prove the claim made at the end of the subsection that we have fundamental ladder functions for our potentials ($n = 1$ in (4.16)). The realization of the GHA of this system is considered at the quantum and classical levels in [19, 20]. We quote the classical ladder functions obtained there:

$$A^{\pm}_{PT}(x,p) = \mp ip\cosh x + \sqrt{-H}\sinh x\,. \tag{4.20}$$

The value of the function $\alpha(H)$ obtained in that reference is

$$\alpha_{PT}(H) = \omega_{PT}(H) = 2\sqrt{-H}. \tag{4.21}$$

In this case, the functions (4.20) constitute a system of fundamental ladder functions.

We now show that the expression given in the previous section for the ladder functions

$$A^{\pm} \equiv A_{\pm 1} = f_{-1}^{\gamma_{\pm 1}} g_{\pm 1}$$

agrees with (4.20) in the limit $B \to 0$.

Firstly, after some computations, we can write the expression for $f_{\pm 1}$ as follows

$$f_{\pm 1} = \phi_{\mp 1}\cosh x + \phi_{\pm 1}\sinh x + ip\cosh x\,. \tag{4.22}$$

We deduce from this and from (4.5) that for $B = 0$ ($\phi_{-1} = 0$, $\phi_{+1} = \sqrt{-H}$),

$$f_{-1}(x, B=0) = (\sqrt{-H} + ip)\cosh x\,. \tag{4.23}$$

We then write the expression for g_ϵ for $B = 0$ from (4.7) as

$$g_\epsilon(x, B=0) = \frac{\epsilon\, C}{\sqrt{-H}} + \frac{\sqrt{-H} - ip}{\tanh x - \epsilon}\,. \tag{4.24}$$

The value of the exponent (4.15) is then seen to be

$$\gamma_\epsilon(B=0) = \frac{\sqrt{-H}}{\phi_{+1}(B=0)} = 1\,. \tag{4.25}$$

So, finally the calculation of $A^{\pm}$ reduces to:

$$A_\epsilon = f_{-1}(B=0) g_\epsilon(B=0) = -\frac{C}{\sqrt{-H}}(\sqrt{-H}\sinh x - i\epsilon p\cosh x)\,. \tag{4.26}$$

We conclude from this and (4.20) that indeed our expressions in the limit $B = 0$ corresponds to the known results (4.20) up to a function of H:

$$A^{\pm}(B=0) = -\frac{C}{\sqrt{-H}} A^{\pm}_{PT}. \tag{4.27}$$

Thus, we have

$$\alpha_{RM}(B=0) = n\omega(B=0) = \omega_{PT} = 2\sqrt{-H}. \tag{4.28}$$

From this, we deduce that $n = 1$ for RMII. Remark that, in the simpler case of the PT system, the ladder functions (4.20) can be obtained simply in the form $f_{+1}(x,p) = A^{-}(x,p)$, making use of one factor function (4.22).

5 The curved Kepler-Coulomb system

The Hamiltonian associated to the curved Kepler-Coulomb problem in the radial coordinate r can be written as [6, 7, 9, 21, 22, 23] (sometimes this is also referred to as the Eckart [24] or Hulthén [25, 26] potential):

$$H_\kappa = p^2 + V_\kappa(r) = p^2 - \frac{B\sqrt{\kappa}}{\tanh\sqrt{\kappa}\,r} + \frac{\ell^2\kappa}{\sinh^2\sqrt{\kappa}\,r}, \tag{5.1}$$

where ℓ is the angular momentum and κ is a curvature parameter. Depending on the sign of κ this expression will include the KC problem on the sphere ($\kappa < 0$), the plane ($\kappa = 0$), or hyperboloid ($\kappa > 0$). The corresponding formulas for the values $\kappa = -1$, $\kappa = 0$, $\kappa = 1$, are respectively

$$\begin{aligned} H_1 &= p^2 - \frac{B}{\tanh r} + \frac{\ell^2}{\sinh^2 r}, && 0 < r < +\infty\,, \\ H_0 &= p^2 - \frac{B}{r} + \frac{\ell^2}{r^2}, && 0 < r < +\infty\,, \\ H_{-1} &= p^2 - \frac{B}{\tan r} + \frac{\ell^2}{\sin^2 r}, && 0 < r < \pi\,. \end{aligned} \tag{5.2}$$

In Fig. 2 it is shown the form of the potentials for different values of κ. In order to have the shape of a well for $\kappa \geq 0$, we will assume that the potential is attractive ($B > 0$) and

$$2\ell^2\sqrt{\kappa} < B, \qquad 0 \leq \kappa\,.$$

Unless otherwise stated, we will restrict in this section to the case $\kappa > 0$. We also remark that from the curved KC Hamiltonian H_1 in (5.2) we can formally get the RMII Hamiltonian $H_{\rm RM}$ by means of a complex displacement:

$$H_{\rm RM}(p,x) = H_1(p, x + i\frac{\pi}{2})\,. \tag{5.3}$$

We computed directly the physical frequency $\omega(H)$ for arbitrary κ, and the explicit formula is given by

$$\omega(H) = \frac{4\sqrt{\kappa}\sqrt{H^2 - B^2\kappa}}{\sqrt{-H + B\sqrt{\kappa}} - \sqrt{-H - B\sqrt{\kappa}}}\,. \tag{5.4}$$

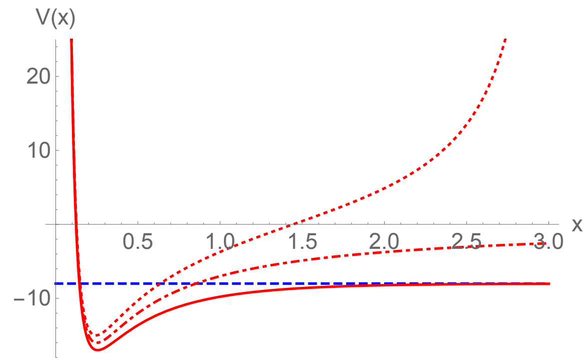

Figure 3: Plot of the curved Kepler-Coulomb potential with the parameters $B = 8$, $\ell^2 = 1$ for $\kappa = 1$ (continuous), $\kappa = 0$ (dashed-dotted), $\kappa = -1$ (dotted). The dashed blue horizontal line corresponds to $E = -B$.

Therefore, we would like to find ladder functions with associated bracket function $\alpha(H) = \omega(H)$ as given by (5.4). As before, following our procedure, we introduce the four factor functions f_ϵ and g_ϵ with $\epsilon = \pm 1$. The first factor function f_ϵ is defined as in (4.2), but now

$$b(r) = \sinh\sqrt{\kappa}\,r, \quad a_\epsilon(r,H) = b(r)\left(\frac{\tilde{\phi}_\epsilon(H)}{\tanh\sqrt{\kappa}\,r} - \frac{B\sqrt{\kappa}}{2\tilde{\phi}_\epsilon(H)}\right), \tag{5.5}$$

with

$$\tilde{\phi}_\epsilon^2 = \frac{-H + \epsilon\sqrt{H^2 - B^2\kappa}}{2}. \tag{5.6}$$

The factor function g_ϵ is defined by the formula (4.7), where

$$d_\epsilon(r) = \frac{-1}{\sqrt{\kappa}(\coth\sqrt{\kappa}\;r + \epsilon)}, \tag{5.7}$$

$$c_\epsilon(r,H) = d_\epsilon(r)\left(\frac{-(B\sqrt{\kappa} + \epsilon 2\ell^2\kappa)\coth\sqrt{\kappa}\;r + \epsilon B\sqrt{\kappa} - 2(H + \ell^2\kappa)}{2\sqrt{-H + \epsilon B\sqrt{\kappa}}}\right). \tag{5.8}$$

In this case, the factorization condition (3.1) is satisfied, respectively, with

$$\delta_{f_\epsilon}(H) = -\ell^2\kappa + \tilde{\phi}_\epsilon(H)^2, \tag{5.9}$$

and

$$\delta_{g_\epsilon}(H) = \frac{B^2\kappa + 4l^2(l^2\kappa + H)}{4(-H + \epsilon B\sqrt{\kappa})}. \tag{5.10}$$

At this point, we remark the similarity of the previous formulas of the factor functions for KC for $\kappa = 1$, with those for the Rosen-Morse II systems, in agreement with the above comment (5.3) on the relation between these Hamiltonians.

The calculation of the contribution and signature proceed in exactly the same way as in the previous section and we obtain:

$$\begin{aligned}
&\Lambda(f_\epsilon) = i\sqrt{\kappa}(2\tilde{\phi}_\epsilon(H) - \frac{B\sqrt{\kappa}}{\tilde{\phi}_\epsilon(H)}\coth\sqrt{\kappa}\, r), \qquad && \Gamma(f_\epsilon) = \epsilon, \\
&\Lambda(g_\epsilon) = 2i\sqrt{\kappa}\sqrt{-H+\epsilon B\sqrt{\kappa}}\,(\epsilon - \coth\sqrt{\kappa}\, r), && \Gamma(g_\epsilon) = -1\,.
\end{aligned} \tag{5.11}$$

Notice the different sign in the signature of f_ϵ. This reflects the fact that all the analytic properties of f_ϵ are reversed from the previous case, so that here we have $a_1(x,H) > 0$. We will then form the products:

$$A_\epsilon = f_1^{\tilde{\gamma}_\epsilon(H)} g_\epsilon, \tag{5.12}$$

where

$$\tilde{\gamma}_\epsilon(H) = -\frac{2\tilde{\phi}_1(H)\sqrt{-H+\epsilon B\sqrt{\kappa}}}{B\sqrt{\kappa}}\,. \tag{5.13}$$

From this and the same computations as in the last section, we deduce that

$$\Lambda(A_\epsilon) = -i\epsilon\frac{4\sqrt{\kappa}\sqrt{H^2-B^2\kappa}}{\sqrt{-H+B\sqrt{\kappa}} - \sqrt{-H-B\sqrt{\kappa}}} \equiv -i\epsilon\alpha(H), \tag{5.14}$$

together with the signature

$$\Gamma(A_\epsilon) = -1\,. \tag{5.15}$$

Thus, here we have also obtained fundamental ladder functions (5.12) with $\omega(H) = \alpha(H)$. The motion (for $B > 0$ and $\epsilon = -1$) is given by

$$\begin{aligned}
&\tilde{\gamma}_{-1}(E)\arctan\left[\frac{p(x,E)}{\frac{\tilde{\phi}_1(E)}{\tanh x} - \frac{B}{2\tilde{\phi}_1(E)}}\right] + \\
&\qquad \arctan\left[\frac{2p(x,E)\sqrt{-E-B}}{(-B+2\ell^2)\coth x - B - 2(E+\ell^2)}\right] - \omega(E)t = \theta_0\,.
\end{aligned} \tag{5.16}$$

Some examples of motion in the curved KC potential for different values of the energy are represented in Fig. 4.

6 The cases $\ell = 0$ and $\kappa \to 0$

6.1 The limit $\ell \to 0$

We will start with the expression (5.12) and choose the parameter $\epsilon = -1$ which corresponds to the ladder function A^-, according to the sign in (5.14). We will compare the limit $\ell \to 0$ with that found by the action-angle method [27], thus we start with the expression

$$A^- \equiv f_1^{\tilde{\gamma}_{-1}(H)} g_{-1}, \tag{6.1}$$

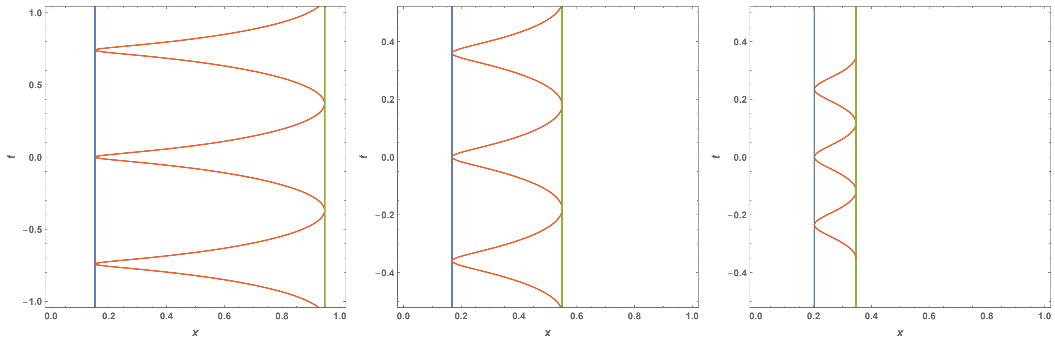

Figure 4: Plot of the motion $x(t)$ in the curved CK potential with $B = 8$, $\ell^2 = 1$, and energy values: $E = -10$ (left), $E = -13$ (center) $E = -16$ (right). The left and right vertical lines are for the values x_- and x_+ bounding the motion. The motion in the graphic is limited to three oscillations.

where

$$g_{-1} = \frac{1}{\sqrt{\kappa}(1 - \coth\sqrt{\kappa}\, r)} \left(\frac{(-B\sqrt{\kappa} + 2\ell^2\kappa)\coth\sqrt{\kappa}\, r - B\sqrt{\kappa} - 2(H + \ell^2\kappa)}{2\sqrt{-H - B\sqrt{\kappa}}} + ip \right) \tag{6.2}$$

and

$$f_1 = \sinh\sqrt{\kappa}\, r \left(\frac{\tilde{\phi}_1(H)}{\tanh\sqrt{\kappa}\, r} - \frac{B\sqrt{\kappa}}{2\tilde{\phi}_1(H)} + ip \right), \tag{6.3}$$

being the exponent

$$\tilde{\gamma}_{-1}(H) = -\frac{2\tilde{\phi}_1(H)\sqrt{-H - B\sqrt{\kappa}}}{B\sqrt{\kappa}}. \tag{6.4}$$

After some computations and taking $\ell \to 0$, we find

$$g_{-1} = -\frac{B}{2\sqrt{-H - B\sqrt{\kappa}}} \left(\frac{1 + \coth\sqrt{\kappa}\, r}{1 - \coth\sqrt{\kappa}\, r} + \frac{2H}{B\sqrt{\kappa}\,(1 - \coth\sqrt{\kappa}\, r)} + 2\, i\, p \frac{\sqrt{-H - B\sqrt{\kappa}}}{B\sqrt{\kappa}\,(1 - \coth\sqrt{\kappa}\, r)} \right). \tag{6.5}$$

If we write the other factor in expression (6.1) in the exponential form

$$f_1^{\tilde{\gamma}_{-1}(H)} = |f_1| e^{i\phi(r,p)}, \tag{6.6}$$

we obtain the same expression of the ladder function A^+ as in [27].

6.2 The limit $\kappa \to 0$

If we take the limit $\kappa \to 0$ of the ladder function A^- defined in (6.1), we should obtain the corresponding ladder function of the Kepler-Coulomb system in flat space:

$$H_0 = p^2 + V_0(r) = p^2 - \frac{B}{r} + \frac{\ell^2}{r^2}, \qquad 0 < r < +\infty. \tag{6.7}$$

We can write eq. (6.1) in the following form

$$A^+ \equiv \left(|f_1|\, e^{i\,\arctan \frac{b_1 p}{a_1}} \right)^{\tilde\gamma_{-1}(H)} \left(c_{-1}(r,H) + i d_{-1}(r)\, p\right). \tag{6.8}$$

After taking the limit $\kappa \to 0$ of this expression, we arrive to the final result

$$A^\pm = F(H_0) \left(-\frac{B}{2\sqrt{-H_0}} + \sqrt{-H_0}\, r \mp i\, p\, r \right) e^{\pm\, i\, \chi(r,p,H_0)}\,, \tag{6.9}$$

where $F(H_0)$ is a function of H_0 and

$$\chi(r,p,H_0) = -\frac{2\sqrt{-H_0}}{B}\, p\, r\,. \tag{6.10}$$

The limit $\kappa \to 0$ the frequency (5.4) becomes

$$\omega(H_0) = \frac{4}{B}\,(-H_0)^{3/2}\,. \tag{6.11}$$

The factorization $\kappa \to 0$ in terms of these ladder functions becomes

$$-\,\ell^2 = A^+ A^- + \lambda = r^2 p^2 - B\, r - r^2 H_0\,, \qquad \lambda = \frac{B^2}{4\, H_0}\,, \tag{6.12}$$

where $H_0 = -e$ takes negative values. They satisfy the Poisson brackets

$$\{A^+, A^-\} = \frac{i\, B}{\sqrt{-H_0}}\,, \qquad \{H, A^\pm\} = \mp\, i\, \omega(H_0)\, A^\pm\,. \tag{6.13}$$

All these expressions are in full agreement with those published in [11].

7 Concluding remarks

In this paper, we have characterized the ladder functions corresponding to the one dimensional systems known as Rosen-Morse II and curved Kepler-Coulomb. These two families of one dimensional systems are the classical version of some of the factorizable solvable systems in quantum mechanics [12]. In fact, they were the only ones for which ladder functions were not yet computed (except the trigonometric counterparts which can be dealt in the same way, their explicit solutions will be published elsewhere); therefore this work constitutes de completion of a program on the motion and algebraic properties of classical one dimensional systems. One important remark that should be mentioned is that there is a close connection of ladder functions and action–angle variables [13].

The ladder functions here found are much more complicated than those of the systems already known. This is reasonable since, for instance, the motion of the KC system in curved space is much more complicated to describe than that in flat space (which may be found in standard textbooks [?]). Therefore, we had to introduce new methods in order to obtain these new ladder functions. The main results of this paper are summarised in (i) the general formulas (4.14) and (5.12) for the ladder functions and (ii) the implicit equations for the motion (4.18) and (5.16) of these systems. We have checked that the formulas here obtained are consistent with the previous ones known for the Pöschl-Teller and flat Kepler-Coulomb potentials by means of appropriate limits.

There is one further comment concerning the freedom of the two possible choices $\epsilon = \pm 1$ for the final ladder functions A_ϵ, due to the factor functions g_ϵ. One way to make the right choice (at least for RMII with $B > 0$ and KC with $\kappa \geq 0$), is to pay attention to the complex character of $g_\epsilon(x, H)$: this function should be complex for $H < -B$, such that if $g_\epsilon = c_\epsilon - i d_\epsilon p$ as given in (4.7), then its complex conjugate should be $g_\epsilon^* = c_\epsilon + i d_\epsilon p$ in order to describe bounded motions. However, the unbounded motions for $H > -B$, are described by factor functions which satisfy $g_\epsilon(x, H) = g_\epsilon(x, H)^*$ (up to a global sign) so that they will essentially be real. This is what happens if $\epsilon = -1$ for both cases RMII and KC above mentioned. This change of character depending on the value of H for the ladder functions was satisfied for all the other simpler cases discussed in [10]. In this respect, it is also known that in quantum mechanics the character of the symmetry algebra depends on the value of the Hamiltonian operator, hence this algebra may change from compact to non compact when the energy values belong to the discrete or to the continuous spectrum.

The one dimensional systems, although not realistic in most physical problems, have considerable interest. For instance, integrable systems can be separated into a set of one dimensional problems. In particular, many superintegrable systems in higher dimensions can be separated in some classes of one dimensional systems which have ladder functions. In this case, the ladder functions together with the so called 'shift functions' allow to get the symmetries and then, prove the superintegrability in a straightforward way [16, 17].

The ladder functions are the classical analog of the ladder operators in quantum mechanics. Such ladder operators are used in the construction of quantum coherent states, while in the classical context the ladder functions allow to characterize the classical motion. Therefore, the knowledge of such functions and operators constitute a natural approach to connect classical and quantum systems [13, 19, 28]. The construction of ladder operators for the quantum RMII and KC systems as well as their coherent states will be investigated in the near future.

Acknowledgments

This work was partially supported by the Spanish MINECO (MTM2014-57129-C2-1-P) and Junta de Castilla y León and FEDER project (BU229P18, VA057U16, VA137G18). Ş. Kuru acknowledges Ankara University and the warm hospitality at Department of Theoretical Physics, University of Valladolid, where part of this work has been done. V. H.

acknowledges the support of research grant from NSERC of Canada.

References

[1] A.O. Barut and A. Böhm, Dynamical groups and mass formula, Phys. Rev. B, **139**, 1107-1112 (1965).

[2] Y. Dothan, M. Gell-Mann and Y. Ne'eman, Series of Hadron energy levels as representations of non-compact groups, Phys. Lett., **17**, 148-151 (1965).

[3] Y. Dothan, Finite-dimensional spectrum-generating algebras, Phys. Rev. D, **2**, 2944-2954 (1970).

[4] F. Cooper, A. Khare and U. Sukhatme, Supersymmetry in Quantum Mechanics, World Scientific, Singapore (2000).

[5] N. Rosen and P. Morse, On the vibrations of polyatomic molecules, Phys. Rev **42**, 210-217 (1932).

[6] A.F.C. Stevenson, Note on the "Kepler Problem" in a spherical space, and the factorization method of solving eigenvalue problems, Phys. Rev., **59**, 842-843 (1941).

[7] L. Infeld and A. Schild, A Note on the Kepler problem in a space of constant negative curvature, Phys. Rev., **67**, 121-122 (1945).

[8] A. Ballesteros and F.J. Herranz, Maximal superintegrability of the generalized Kepler-Coulomb system on N-dimensional curved spaces, J. Phys. A: Math. Theor., **57**, 245203 (12pp) (2009).

[9] M.F. Rañada and M. Santander, Superintegrable systems on the 2D sphere and the hyperbolic plane, J. Math. Phys., **40**, 5026-5057 (1999).

[10] Ş. Kuru and J. Negro, Factorizations of one-dimensional classical systems, Ann. Phys., **323**, 413-431 (2008).

[11] Ş. Kuru and J. Negro, Classical spectrum generating algebra of the Kepler-Coulomb system and action-angle variables, Phys. Lett. A, **376**, 260-264 (2012).

[12] L. Infeld and T.E. Hull, The factorization method, Rev. Mod. Fis., **23**, 21-68 (1951).

[13] R. Campoamor-Stursberg, M. Gadella, Ş. Kuru and J. Negro, Action-angle variables, ladder operators and coherent states, Phys. Lett. A, **376**, 2515-2521 (2012).

[14] I. Marquette and V. Hussin, Generalized Heisenberg algebras, SUSYQM and degeneracies: infinite well and Morse potential, SIGMA, **7**, 024 (16pp) (2011).

[15] F. Bagarello, E.M.F. Curado and J.P. Gazeau, Generalized Heisenberg algebra and (non linear) pseudo-bosons, J. Phys. A: Math. Theor., **51**, 155201 (15pp) (2018).

[16] J.A. Calzada, Ş. Kuru and J. Negro, Superintegrable Lissajous systems on the sphere, Eur. Phys. J. Plus, **129**, 164 (15pp) (2014).

[17] A. Ballesteros, F.J. Herranz, Ş. Kuru and J. Negro, The anisotropic oscillator on curved spaces: A new exactly solvable model, Ann. Phys., **373**, 399-423 (2016).

[18] M. Gadella, Ş. Kuru and J. Negro, The hyperbolic step potential: Anti-bound states, SUSY partners and Wigner time delays, Ann. Phys., **379**, 86-101 (2017).

[19] S. Cruz y Cruz, Ş. Kuru and J. Negro, Classical motion and coherent states for Pöschl Teller potentials, Phys. Lett. A, **372**, 1391-1405 (2008).

[20] Ş. Kuru and J. Negro, Dynamical algebras for Pöschl-Teller Hamiltonian hierarchies, Ann. Phys., **324**, 2548-2560 (2009).

[21] E. Schrödinger, A method of determining quantum-mechanical eigenvalues and eigenfunctions, Proc. R. Ir. Acad., **46**, 9-16 (1940).

[22] A. Ballesteros and F.J. Herranz, Superintegrability on three-dimensional Riemannian and relativistic spaces of constant curvature, SIGMA, **2**, 010 (22pp) (2006).

[23] C. Quesne, Quantum oscillator and Kepler-Coulomb problems in curved spaces: Deformed shape invariance, point canonical transformations, and rational extensions, J. Math. Phys., **57**, 102101 (20pp) (2016).

[24] C. Eckart, The penetration of a potential barrier by electrons, Phys. Rev., **35**, 1303 (1930).

[25] L. Hulthén, Uber die eigenlosunger der Schrdinger-gleichung des deuterons, Ark. Mat. Astron. Fys., **28 A**, 5 (1942).

[26] S. Flügge, Practical Quantum Mechanics, Springer-Verlag, Berlin (1971).

[27] Ş. Kuru and J. Negro, Ladder functions for curved KC systems by the action-angle method, preprint Universidad de Valladolid (2018).

[28] M. Angelova, A. Hertz and V. Hussin, Trajectories of generalized quantum states for systems with finite discrete spectrum and classical analogs, AIP Conference Proceedings, **1488**, 122-129 (2012).

Appendix: Signature of f_ϵ and g_ϵ for RMII)

We will determine the signature of the functions f_ϵ and g_ϵ defined by (4.2) and (4.7) with a simple argument by taking into account the fact that $\Gamma(f_\epsilon) = \frac{a_\epsilon(x_+,H)}{a_\epsilon(x_-,H)} = \pm 1$ and a similar expression for $\Gamma(g_\epsilon)$ because of the factorization condition.

The point is to show that for $\epsilon = 1$, $a_1'(x,H)$ is never equal to zero while for $\epsilon = -1$ $a_{-1}(x,H)$ is never equal to zero, where $'$ denotes the differentiation with respect to x. Thus, in the case $\epsilon = 1$, the function is strictly increasing or decreasing, which implies it is one to one so that $\frac{a_\epsilon(x_+,H)}{a_\epsilon(x_-,H)} \neq 1$. In the case $\epsilon = -1$, the function is always positive (or negative), so that $\frac{a_\epsilon(x_+,H)}{a_\epsilon(x_-,H)} \neq -1$.

For $\epsilon = 1$, we see from the definition (4.3) of a_ϵ that $a_1'(x,H) = 0$ if and only if $\phi_1(H)\cosh x + \frac{B}{2\phi_1(H)}\sinh x = 0$. This would imply that for a value of x we have $\tanh x = -\frac{2\phi_1^2}{B}$. But since $-1 < \tanh(x) < 1$ this is only satisfied if:

$$\frac{2\phi_1(H)^2}{B} < 1. \tag{7.1}$$

If we develop this equation using eq. (4.6) for ϕ_1 we see that it cannot be satisfied and thus $a_1'(x,H)$ does not cancel and $\Gamma(f_1) = -1$.

For $\epsilon = -1$, the condition for the cancelation of $a_{-1}(x,H)$ is the same: $\frac{2\phi_1(H)^2}{B} < 1$. This can be seen from the fact that $a_{-1}(x,H) = a_1'(x,H)$, which may be shown by using the relation $\phi_1\phi_{-1} = 2B$. We thus have $\Gamma(f_{-1}) = 1$. Finaly, we obtain $a_{-1}(x,H) > 0$ by evaluating $a_{-1}(x,H)$ at $x = 0$.

For the function g_ϵ, the argument is simpler since $c_\epsilon'(x,H) = \frac{\sqrt{-H+\epsilon B}(\tanh^2 x - 1)}{(\tanh x - \epsilon)^2} = 0$ implies $\tanh^2 x = 1$ or $\epsilon B - H = 0$ which is excluded. Thus because $c_\epsilon(x,H)$ is one to one, $\Gamma(g_\epsilon) = -1$.