UNIVERSIDADE DE LISBOA

FACULDADE DE CIÊNCIAS

Ciências ULisboa

Systems of Iterative Functional Equations: Theory and Applications

Doutoramento em Matemática

Especialidade: Análise Matemática

Maria Cristina Gonçalves Silveira de Serpa

Tese orientada por:

Professor Doutor Jorge Sebastião de Lemos Carvalhão Buescu

Documento especialmente elaborado para a obtenção do grau de doutor

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Abstract

We formulate a general theoretical framework for systems of iterative functional equations between general spaces. We find general necessary conditions for the existence of solutions such as compatibility conditions (essential hypotheses to ensure problems are well-defined). For topological spaces we characterize continuity of solutions; for metric spaces we find sufficient conditions for existence and uniqueness. For a number of systems we construct explicit formulae for the solution, including affine and other general non-linear cases. We provide an extended list of examples. We construct, as a particular case, an explicit formula for the fractal interpolation functions with variable parameters.

Conjugacy equations arise from the problem of identifying dynamical systems from the topological point of view. When conjugacies exist they cannot, in general, be expected to be smooth. We show that even in the simplest cases, e.g. piecewise affine maps, solutions of functional equations arising from conjugacy problems may have exotic properties. We provide a general construction for finding solutions, including an explicit formula showing how, in certain cases, a solution can be constructively determined.

We establish combinatorial properties of the dynamics of piecewise increasing, continuous, expanding maps of the interval such as description/enumeration of periodic and pre-periodic points and length of pre-periodic itineraries. We include a relation between the dynamics of a family of circle maps and the properties of combinatorial objects such as necklaces and words. We provide some examples. We show the relevance of this for the representation of rational numbers.

There are many possible proofs of Fermat's little theorem. We exemplify those using necklaces and dynamical systems. Both methods lead to generalizations. A natural result from these proofs is a bijection between aperiodic necklaces and circle maps.

The representation of numbers plays an important role in much of this work. Starting from the classical base p representation we present other type of representation of numbers: signed base p representation, Q-representation and finite base p representation of rationals. There is an extended p representation that generalizes some of the listed representations.

We consider the concept of bold play in gambling, where the game has a unique win pay-off. The probability that a gambler reaches his goal using the bold play strategy is the solution of a functional equation. We compare with the timid play strategy and extend to the game with multiple pay-offs.

Keywords: functional equation, conjugacy equation, fractal interpolation, representation of numbers, bold play

Resumo

Na primeira parte deste trabalho é formulado um enquadramento teórico para determinados sistemas de equações funcionais iterativas entre espaços gerais. Formulam-se condições necessárias para a existência de solução, tais como as condições de compatibilidade (hipóteses essenciais para assegurar que os problemas deste tipo estão bem definidos). Estas condições são definidas a partir da informação do sistema nos pontos de contacto (obtidos por condições iniciais ou por resolução parcial do sistema). Para espaços topológicos caracteriza-se a continuidade de soluções (aqui a informação relevante do sistema está nos pontos em contacto e nos pontos limite em contacto); para espaços métricos estabelecem-se condições suficientes de existência e unicidade de solução (as hipóteses necessárias e suficientes têm em conta que a informação contida nas várias equações cobre integralmente e de forma coerente o domínio do sistema e ainda que existe contractividade das equações). Para vários tipos de sistemas constroem-se fórmulas explícitas, incluindo os casos afins, bem como outros casos não lineares gerais. Exibe-se uma extensa lista de exemplos: sistemas contractivos, em especial afins, complexo-conjugados afins, não contínuos, de dimensão superior a 2, com funções argumento não padrão e os gerais não lineares. Os exemplos clássicos incluem, por exemplo, os de de Rham, de Girgensohn e de Zdun. Os mais trabalhados são os do tipo afim. Apresenta-se em primeiro lugar o caso dos sistemas de equações de conjugação (não dependem da variável independente x). Mais geralmente, estudam-se também os sistemas que dependem explicitamente da variável independente x. Constrói-se, como um caso particular, uma fórmula explícita para as funções de interpolação fractal com parâmetros variáveis, que é uma generalização do caso original estabelecido por Barnsley (caso afim). Aqui inclui-se tanto o caso em que os dados a interpolar estão uniformemente distribuídos como o caso contrário. No que se refere aos sistemas de equações de conjugação, estebelece-se a relação existente entre determinadas equações de conjugação e os vulgarmente denominados sistemas de funções iteradas (IFS), que permitem estabelecer características fractais das soluções destas equações. Introduz-se o conceito de sistemas de equações incompletos. Para estes sistemas a informação sobre as soluções não permite determinar uma única solução, mas por vezes ainda assim é possível obter explicitamente a imagem da(s) solução(ões) para um subconjunto do contradomínio.

As equações de conjugação resultam do problema de identificação de sistemas dinâmicos do ponto de vista topológico. Quando existem conjugações, não é de esperar que estas sejam, em geral, suaves. Em primeiro lugar faz-se uma síntese dos métodos existentes para resolução deste tipo de equações. Um dos métodos é um ponto de partida para o desenvolvimento do que se segue. Mostra-se que mesmo em casos muito simples, por exemplo em aplicações afins por troços, existem soluções de equações funcionais que têm propriedades exóticas: podem ser singulares, ou fractais. Apresenta-se uma construção geral que permite obter soluções deste tipo de equações. Esta construção é resultante de sistemas cujos domínios estão definidos por partições. É através destas partições e dos resultados para sistemas de equações funcionais que se obtêm soluções para as funções

de conjugação. Incluem-se fórmulas explícitas, indicando que a solução pode ser determinada construtivamente. Através destes métodos também se indica o caminho a seguir, no caso de se pretender encontrar soluções homeomórficas de equações de conjugação topológica. Isto porque, para a teoria dos sistemas dinâmicos, os sistemas que têm soluções homeomórficas têm relevância para analisar aplicações topologicamente conjugadas. Em geral existem múltiplas soluções; no caso de homeomorfismos é possível existirem duas soluções: uma crescente e uma decrescente. Em determinadas situações é possível obter um conjunto, designado de conjunto aglutinador de uma equação de conjugação, que engloba todas as possíveis soluções destas equações. O clássico *Cantor dust* é um exemplo deste tipo de conjunto.

Depois de uma exposição teórica fundamentada e generalizada sobre os sistemas de equações funcionais iterativos em estudo são dadas algumas aplicações de natureza dinâmica, combinatorial, de teoria dos números e matemática recreativa.

As aplicações das equações funcionais às funções fractais são um tópico com bastantes desenvolvimentos científicos em diversas áreas como sejam a Medicina, a Física, a Engenharia, a Hidrologia, a Sismologia, a Biologia e a Economia. A disponibilização de uma fórmula explícita para este tipo de funções permite o cálculo pontual preciso, que se apresenta como uma vantagem em comparação com os métodos numéricos usualmente utilizados para estimar a solução. Ilustra-se a utilidade de funções fractais com parâmetros variáveis através de gráficos exemplificativos.

Estabelecem-se propriedades combinatoriais da dinâmica de determinadas aplicações contínuas e expansivas por troços, como a descrição e enumeração dos pontos e órbitas periódicas e pré-periódicas e o comprimento dos respectivos itinerários. Inclui-se a relação entre a dinâmica das aplicações do círculo e algumas propriedades de objectos combinatóricos, como sejam os colares e palavras. Estes conceitos são ilustrados com exemplos, em especial de forma recreativa. Mostra-se a relevância deste assunto com a representação dos números racionais.

Existem muitas formas de demonstrar o pequeno teorema de Fermat, de entre as quais estão as que utilizam colares e sistemas dinâmicos. Exemplificam-se algumas destas demonstrações. Estes métodos levam a generalizações. Apresenta-se uma perspectiva histórica da forma como foram surgindo as diversas formulações e demonstrações. Um resultado natural destas demonstrações é a existência de uma bijecção entre o conjunto dos colares aperiódicos e o conjunto das órbitas periódicas de aplicações do círculo.

A representação de números tem um papel importante na maior parte do trabalho aqui desenvolvido. Partindo da representação clássica numa base p, apresentam-se outros tipos de representação de números: representação em base p com sinal (relacionada com a generalização da aplicação tenda), Q-representação (basicamente pode dizer-se que se baseia numa base não uniforme em cada dígito) e representação finita em base p de racionais (resulta da identificação dos pontos e órbitas pré-periódicos de aplicações expansivas por troços). Existe ainda uma extensão deste tipo de representação base p que generaliza algumas destas representações listadas, que advém da generalização da solução dos sistemas de equações não-lineares estudados.

São dadas duas aplicações ao nível da matemática recreativa: a combinatória de palavras em relação com a dinâmica das aplicações do círculo e a estratégia de jogo ousado em jogos com um quadro de prémios fixo definido à partida (como por exemplo o jogo de casino de apostar uma cor na roleta). Na combinatória de palavras incluem-se exemplos de naipes de cartas, de alfabetos de línguas faladas e ilustram-se colares em paralelo com órbitas periódicas.

No que se refere ao jogo do casino, considera-se o conceito de jogo ousado (apostar tudo ou nada) como estratégia de jogo, em que o jogo tem um quadro de prémios definido. Por hipótese, o jogador tem um montante inicial disponível para jogar, vai jogar várias rodadas até ou perder tudo, ou chegar ao seu objectivo. A probabilidade de o jogador atingir o seu objectivo usando a estratégia ousada é a solução de uma equação funcional do tipo estudado numa forma mais geral neste trabalho. Compara-se com a estratégia tímida (tanto em termos analíticos, como em termos gráficos) e generaliza-se formalmente para um jogo que tenha um quadro de prémios múltiplo. A estratégia de jogo tímido é passível de ser utilizada em jogos do tipo raspadinha (cada jogada tem uma aposta fixa de uma unidade que corresponde ao valor de compra de um cartão). Para o jogo com prémios múltiplos modela-se a equação funcional que dá a probabilidade de o jogador alcançar o objectivo. No entanto a resolução deste problema com prémios múltiplos está ainda em aberto, por ser um problema mais geral do que as equações funcionais estudadas neste trabalho.

Palavras chave: equação funcional, equação de conjugação, interpolação fractal, representação de números, jogo ousado

Acknowledgements

The list of acknowledgements is extensive and express my gratitude for those who contributed decisively to this work and somehow made possible and gave me motivation and courage to take this challenge through. The order presented does not reflect the order of importance, but only a funny mathematical scheme.

In the first place I acknowledge my advisor for his full availability, patience and encouragement. He was the ***one*** who always accompanied me on this journey of discovery, with always fruitful discussions and thorough constructive criticism of the work that was being prepared by me. His commitment is extraordinary, when he speaks of Mathematics he always expressed great enthusiasm, specially when we are faced with new results, sometimes surprising ones. He helped me to diversify my approach and disseminate my work around the world and inside the scientific community, encouraging me to participate in various international conferences, of which I had the opportunity to participate when preparing their proceedings. I thank him also to all attentive revisions that he held for the preparatory work that I have elaborated (proceedings and papers) as well as for the text of this thesis. Finally I must express my gratitude for his unconditional support for my scientific development.

I thank my parents, they are the ***two*** people responsible for giving me life. They are my birth and my roots. All I know started in their midst. My identity started with them. My beloved parents are deep in my heart and I will do everything so that they always have a very happy life.

I thank the ***three*** Marys that are always with me and give me support when life is not easy and rejoice with me when there is joy and peace.

I thank the ***four*** institutions which gave me financial, material, infrastructural, logistic, scientific and human resources to support my PhD path: FCT - Fundação para a Ciência e Tecnologia (grant SFRH/BD/77623/2011), Faculdade de Ciências da Universidade de Lisboa (which is my host institution and the one that provides the doctoral program of the Department of Mathematics that I participated in), the CMAF-CIO - Centro de Matemática Aplicações Fundamentais e Investigação Operacional, former CMAF - Centro de Matemática Aplicações Fundamentais (who welcomed me as a scientific collaborator) and the FFCUL - Fundação da Faculdade de Ciências da Universidade de Lisboa (who gave me support, in particular, for my participation in scientific international events).

I thank the ***five*** Professors of the University of Lisbon that discussed with me specific topics of my work: Pedro Duarte, Teresa Faria, Fernando Ferreira, Nuno da Costa Pereira and Henrique Oliveira. I thank also many other Professors of the Department of Mathematics that expressed in various ways their interest in my academic development and success. This university involvement is very important to me.

I thank the organizers of the ***six*** national and international conferences which welcomed me to participate: Recreational Mathematics Colloquium IV, ECIT 2014 - European Conference on Iteration Theory, NOMA'13- International Workshop on Nonlinear Maps and their Applications, ECIT 2012 - European

Conference on Iteration Theory, NOMA'11 - International Workshop on Nonlinear Maps and their Applications and Recreational Mathematics Colloquium II. I thank also all the people I met there and had fruitful discussions with me about Mathematics.

I thank the ***seven*** institutions who welcomed me to present lectures about my work, under national or international conferences, or included in regular seminars of their research centers: Ludus, Uniwersytet Zielonogórski (University of Zielona Góra - Poland), Universidad de Zaragoza - Spain, CMUP - Centro de Matemática da Universidade do Porto, Universidade dos Açores, Universidade de Lisboa and Universidade de Évora.

I thank all faculty colleagues, in particular the occupants of the ***eight*** desks existing in my working room at the Mathematical Department in the Faculdade de Ciências da Universidade de Lisboa. This thanks extends to all colleagues from the beginning of my studies, which shared with me smiles, thoughts, words, emotions, meals, events, homeworks, assignments, seminars, entertainments, etc.. Nice to meet you and thank you for your companionship.

I thank about ***nine*** work colleagues of my previous job for encouraging me to change my life and start from scratch. Thank you for the help you gave me in my former job. This was the step that allowed me to embark on Mathematics and ultimately make this PhD.

I thank all my teachers that in the ***ten*** years of studies in Mathematics in the Faculdade de Ciências da Universidade de Lisboa gave me classes in the graduation, master and PhD degrees. All things that I have learned with them is the fundamental Mathematical knowledge that I have today and are very important to the development of this work.

I thank the ***eleven*** relatives who form my core family. They support what I am, they support what I believe, they care, they give me strength, they hold me. I thank also the other family members that somehow show their interest and encouragement for my personal, emotional and intellectual development. I also thank the family who have passed away and is still in my mind and in my heart.

I thank a ***dozen*** of people who did administrative, technical and secretarial work that support all bureaucratic, logistics and information technology processes that I needed along the realization of this PhD.

Last but not least, I thank the ***thirteen*** very special friends for their companionship, friendship, inspiration, motivation which I receive from them. They will be always in my heart and in my spirit. You left nice deep marks on me. I think you will not forget me as I will never forget you.

For all ***one*** to ***thirteen*** many thanks. Without you none of this work could be done. Thank you for contributing for my personal, emotional, scientific and professional development, in special for the possibility to conclude this PhD.

Contents

Chapter 1

Introduction

This work was done in the sequence of the Master Thesis [80], where the subject in study had a discovery path with led to results from diverse topics of Mathematics. Some of these subjects are connected with the theme of this work and are included in a more developed form in Part II (applications). The Master Thesis considered the dynamics of piecewise expanding maps, with relations to combinatorics, theory of numbers and formal languages. This work now further explores the properties of these maps and develops a main special related topic, the systems of functional equations and, in particular the systems of conjugation equations. In fact, the problem of conjugation in dynamical systems was particularly interesting in terms of finding explicit/constructive solutions. This fact led to widen the subject to more general equations: ones with explicit dependence on the independent variable. The search for a theoretical and general setting made clear the need for consideration of conditions on existence, uniqueness and continuity of solutions in metric spaces. A necessary condition for systems was formally established, the so-called *compatibility conditions*. These conditions, in practice, were already considered, in a case-by-case bases, in the literature of the area, without any previous general theoretical formalism.

The area of systems of functional equations is a vast domain of knowledge in mathematics that has many possible problems and may be studied by different approaches. It is far from being completely theoretically developed. One of the type of systems that can model the behaviour of dynamical systems is the subject of study, e.g. the so-called conjugacy equations. The focus of this work is that of systems of contractive functional equations; for conjugacy equations the focus is that of conjugated maps that are either piecewise contractive or piecewise expansive. In general, the solutions obtained have exotic properties such as singularity and fractal self-similarities.

With a background theory established, there are a wealth of possible applications: fractal interpolation, dynamical systems, representation of numbers and recreational mathematics.

Fractal interpolation functions are a subject of recent development in terms of mathematical applications to other sciences. We mention for instance areas as

Medicine, Physics, Engineering, Hydrology, Seismology, Biology and Economics. This work contributes with explicitly defined fractal interpolation functions with variable parameters, providing an analytic approach and point evaluation. The usual approach was up to now only possible recurring to numerical methods.

In practice, the conjugacy equations are motivated by dynamical problems and are related to combinatorics and representation of numbers. These topics are natural applications of the theoretical work performed.

For a substantial part of this work the representation of real numbers in a base plays a fundamental role, and is one way to give an explicit definition of solutions of systems of functional equations or conjugacy equations. In addition to the classical base p representation, there exist alternative representations of numbers that are, in a modest way, a contribution to Number Theory.

One last application is related with recreational mathematics: the bold play strategy for the roulette game at casinos. The probability of win for an initial available amount of money is the solution of a special case of a system of functional equations studied in Part I. The timid play strategy (for the gambler of roulette) is a much simpler mathematical problem and provides a nice comparison with bold play.

Part I

Iterative functional equations

General problem

The objects of study in this text, and theoretically in this first part, are called systems of iterative functional equations as formulated in by the book *Iterative Functional Equations* by Kuczma, Choczewski and Ger [48]. The reason of the term iterative is derived from the fact that most of the solutions encountered gives rise to an iterative procedure. In fact, the term functional equation is also generally used for delay differential equations (see, e.g. [90]). So with the term iterative there is no confusion. Iterative equations are also referred to as equations of rank 1 (see, e.g. [48]). In [48] the focus is in single equations; here we study sets of equations in a system that must have a coherent solution (one equation cannot be in contradiction with any other equation).

The study of systems of iterative functional equations began in earnest with de Rham's example [74]: two equations with no explicit dependence on the independent variable x; the solution is constructible and may be expressed in terms of the binary representation of numbers. Zdun [103] generalized this setting for systems with more than two equations and Girgensohn [36] generalized for affine systems with explicit dependence on x. The affine case has been subject of recent developments, namely in the context of fractal functions [6] by Wang-Yu [97] and by Serpa and Buescu [84]. In these few particular cases one may find in the literature explicit formulae for the construction of the solution of a specific system of iterative functional equations. A substantial part of the literature generally focuses on what we may term "exotic" properties of the solutions, like lack of regularity, singularity and fractal properties.

Let X and Y be non-empty sets and $p \geq 2$ an integer. The problem considered in this work is a system of functional equations

$$\varphi\left(f_j\left(x\right)\right) = F_j\left(x, \varphi\left(x\right)\right), \; x \in X_j, \; j = 0, 1, \dots p-1, \tag{1.1}$$

where $X_j \subset X$, $f_j : X_j \to X$, $F_j : X_j \times Y_j \to Y$ are given functions, and $\varphi : \cup_{j=0}^{p-1} X_j = X \to Y$ is the unknown function.

If each F_j does not depend explicitly on x, i.e., the system (1.1) is of the form

$$\varphi\left(f_j\left(x\right)\right) = F_j\left(\varphi\left(x\right)\right), \; x \in X_j, \; j = 0, 1, \dots p-1, \tag{1.2}$$

then the equations of system (1.1) are conjugacies. Before studying the general case, we start, in Chapter 2, with this type of systems.

Chapter 2

Systems of conjugacy equations

2.1 Compatibility conditions

Independently of specifics of the sets X and Y, we emphasize the existence of compatibility conditions for the system (1.2). These are necessary conditions for the existence of solutions, were introduced in the general setting in [84,85] and are stated in the following Proposition.

Proposition 1. *[84] If $\varphi : X \to Y$ is a solution of (1.2), then it must satisfy*

$$\forall x_1 \in X_i, \forall x_2 \in X_j, f_i(x_1) = f_j(x_2) \Rightarrow F_i(\varphi(x_1)) = F_j(\varphi(x_2)), \qquad (2.1)$$

for $i, j = 0, 1, \ldots, p-1$.

Let

$$A := \{x_1 \in X : \exists\, x_2 \in X, \exists\, i, j = 0, 1, \ldots, p-1, (i, x_1) \neq (j, x_2), f_i(x_1) = f_j(x_2)\}.$$

The elements of A are called the ***contact points*** of system (1.2). Note that when all f_i are injective, this set A reduces to

$$\{x_1 \in X : \exists\, x_2 \in X, \exists\, i, j = 0, 1, \ldots, p-1, i \neq j, f_i(x_1) = f_j(x_2)\},$$

as defined in [84], since in that context all f_i are injective.

Definition 2. [85] Consider a system of equations (1.2) where for all $x \in A$, $\varphi_x \equiv \varphi(x)$ has been previously determined (by partially solving the system or by initial conditions). We say that

$$\forall x_1, x_2 \in A, f_i(x_1) = f_j(x_2) \Rightarrow F_i(\varphi_{x_1}) = F_j(\varphi_{x_2}), \qquad (2.2)$$

are the ***compatibility conditions*** of system (1.2).

Note that, considering in (2.1) $\varphi_{x_1} \equiv \varphi(x_1)$ and $\varphi_{x_2} \equiv \varphi(x_2)$ as parameters, the compatibility conditions (2.2) are effectively independent of φ. In fact, they are necessary conditions on the f_i and F_j for the existence of solutions.

Remark 3. In cases where system (1.2) is such that, for all $i \neq j$, $f_i(X) \cap f_j(X) = \emptyset$, the compatibility conditions are vacuously satisfied.

Compatibility conditions are necessary but not sufficient for the existence of solutions. In each specific context more conditions must be added to ensure existence and uniqueness of solutions.

We next concentrate in the context of metric spaces.

2.2 Existence and uniqueness

The minimal structure we require to derive meaningful results for functional equations is that of a metric space.

Definition 4. Let (M, d_1), (N, d_2) be metric spaces and $f : M \to N$.

a) We say f is a ***contraction***, or a ***contracting map***, if there is some non-negative real number $0 < \lambda < 1$ such that for all x and y in M, $d_2(f(x), f(y)) \leq \lambda d_1(x, y)$. In this case λ is called the ***contraction factor*** and f is called a λ-***contraction***.

b) We say f is an ***expansion***, or an ***expanding map***, if there is some non-negative real number $\lambda > 1$ such that for all x and y in M, $d_2(f(x), f(y)) \geq \lambda d_1(x, y)$. In this case λ is called the ***expansion factor*** and f is called a λ-***expansion***.

The following Proposition is immediate and its proof is omitted.

Proposition 5. *[86] Let (M, d_1), (N, d_2) be metric spaces and $f : M \to N$.*

i) If f is a contraction map then f is continuous;

ii) If f is an injective λ-contraction map with contraction factor λ, then $\exists f^{-1} : f(M) \to M$ such that

$$\forall x, y \in f(M) : d_1(f^{-1}(x), f^{-1}(y)) \geq \frac{1}{\lambda} d_2(x, y);$$

i.e., f^{-1} is an injective $1/\lambda$-expansion;

iii) If f is an injective λ-expansion map with expansion factor λ, then $\exists f^{-1} : f(M) \to M$ such that

$$\forall x, y \in f(M) : d_1(f^{-1}(x), f^{-1}(y)) \leq \frac{1}{\lambda} d_2(x, y);$$

i.e., f^{-1} is an injective $1/\lambda$-contraction.

Our results below are expressed in terms of contraction maps. A slightly more general class of maps (α-contractions, as defined in [65]) could be used instead.

Definition 6. We say $f : M \to N$ is a φ-contraction if for all x and y in M, $d_2(f(x), f(y)) \leq \alpha(d_1(x, y))$, where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a comparison function, i.e., $(\alpha^n(t))$ converges to zero for all $t \in \mathbb{R}_+$ and α is increasing.

Let (X, d_1) be a bounded metric space and (Y, d_2) be a complete metric space. We now give an existence and uniqueness result for solutions $\varphi : X \to Y$ of the system of functional equations

$$\varphi(f_i(x)) = F_i(\varphi(x)), \; i = 0, 1, \ldots, p-1, \; x \in X_i, \tag{2.3}$$

where $p \geq 2$ is an integer. The next lemma is a typical exercise in Functional Analysis.

Lemma 7. *[86] Let $B = \{\varphi : X \to Y, \varphi \text{ is bounded}\}$ equipped with the metric $d(\varphi, \psi) = \sup_{x \in X} d_2(\varphi(x), \psi(x))$. The space B is a complete metric space.*

Proof. Let $(\varphi_n)_{n \in \mathbb{N}}$ be a Cauchy sequence of elements of B. Since, for fixed $x \in X$, $\{\varphi_n(x)\}$ is a Cauchy sequence in Y, it is convergent. So the pointwise limit $\varphi(x) = \lim_{n \to \infty} \varphi_n(x)$ exists and defines a function $\varphi : X \to Y$.

We show that the Cauchy sequence $(\varphi_n)_{n \in \mathbb{N}}$ is convergent in B.

For any $\varepsilon > 0$ there exists N_ε such that

$$\sup_{x \in X} d_2(\varphi_n(x), \varphi_m(x)) \leq \frac{\varepsilon}{2}, \; \forall n, m \geq N_\varepsilon.$$

Since $\varphi_n(x)$ is Cauchy in Y and Y is by hypothesis complete, for each $x \in X$ there exists $\varphi(x) \in Y$ such that $\varphi_n(x) \to \varphi(x)$ for fixed $x \in X$, that is, φ_n is pointwise convergent. Then for any $\varepsilon > 0$ there exists an integer $M_{\varepsilon,x}$ such that

$$d_2(\varphi_m(x), \varphi(x)) < \frac{\varepsilon}{2}, \; \forall n, m \geq M_{\varepsilon,x}.$$

By the triangle inequality, for any $x \in X$ and any $n, m \geq 1$,

$$d_2(\varphi_n(x), \varphi(x)) \leq d_2(\varphi_n(x), \varphi_m(x)) + d_2(\varphi_m(x), \varphi(x)).$$

If $n, m > N_\varepsilon$ and $m > M_{\varepsilon,x}$, then the right hand side is bounded above by ε. Therefore the left hand side is bounded by ε for all $x \in X$. Indeed, given $x \in X$, we can always choose m in the right hand side to be greater than both N_ε and $M_{\varepsilon,x}$. This implies that

$$d(\varphi_n, \varphi) = \sup_{x \in X} d_2(\varphi_n(x), \varphi(x)) < \varepsilon, \; \forall n \geq N_\varepsilon,$$

which means that $\varphi_n \to \varphi$ uniformly.

We recall the characterization of bounded sets in a metric space as those which are contained in a ball of finite radius. More precisely, given a metric space (M, ρ), a set S is bounded if and only if $\exists m \in M, r \in \mathbb{R} : \forall s \in S : \rho(m, s) \leq r$.

Let $n \in \mathbb{N}$. Since φ_n is bounded

$$\exists y_n \in Y, r_n \in \mathbb{R}, \forall y \in \varphi_n(X) : d_2(y, \bar{y}) \leq r_n.$$

The pointwise limit function φ is bounded if $\varphi(X)$ is bounded, that is

$$\exists y_* \in X, r_* \in \mathbb{R} : \forall y \in \varphi(X) : d_2(y, y_*) \leq r_*. \tag{2.4}$$

Let now $\varepsilon > 0$ and N be such that $d(\varphi_n, \varphi) < \varepsilon$ for $n \geq N$. Select an $n \geq N$ and take, in equation (2.4), $y_* = y_n$. Since $\forall y \in \varphi(X)\, \exists x \in X, y = \varphi(x)$, we have

$$\begin{aligned} d_2(y, y_n) &= d_2(\varphi(x), y_n) \\ &\leq d_2(\varphi(x), \varphi_n(x)) + d_2(\varphi_n(x), y_n) \\ &< \varepsilon + r_n = R. \end{aligned}$$

Thus the function φ is bounded.

This shows that the Cauchy sequence $(\varphi_n)_{n\in\mathbb{N}}$ is convergent in B and that B is complete, as stated. □

Theorem 8. *[86] Let $f_i : X_i \to X$ be a family of injective functions such that*
(i) $\forall i \neq j,\ f_i(X_i) \cap f_j(X_j) = \emptyset$,
(ii) $\cup_{i=0}^{p-1} f_i(X_i) = X$
and
(iii) $F_i : Y_i \to Y$ is a λ_i-contraction and $\overline{Y_i} = Y$, for all $i = 0, 1, \ldots, p-1$.
Then there exists a unique bounded solution $\varphi : X \to Y$ of system (2.3).

Proof. Let B be the complete metric space defined in Lemma 7. Observe that, since F_i is a λ_i-contraction, and thus Lipschitz, F_i has an extension by continuity $\tilde{F}_i$ to $Y = \overline{Y_i}$. Since each f_i is invertible onto its image, system (2.3) is equivalent to

$$\varphi(x) = \tilde{F}_i\left(\varphi\left(f_i^{-1}(x)\right)\right),\ i = 0, 1, \ldots, p-1,\ x \in f_i(X_i).$$

Let $T : B \to B$ be the operator defined by $T\{\varphi\}(x) = \tilde{F}_i\left(\varphi\left(f_i^{-1}(x)\right)\right)$, $x \in f_i(X_i)$, $i \in \{0, 1, \ldots, p-1\}$.

Since f_i, $\tilde{F}_i$ are bounded, $T(\varphi) = \tilde{F}_i \circ \varphi \circ f_i^{-1}$ is also bounded. Let $\lambda := \max\limits_{i\in\{0,1,\ldots,p-1\}} \lambda_i$. Then

$$\begin{aligned} d(T\varphi, T\psi) &= \max_{i\in\{0,1,\ldots,p-1\}} \sup_{x\in f_i(X_i)} \left(\tilde{F}_i\left(\varphi\left(f_i^{-1}(x)\right)\right), \tilde{F}_i\left(\psi\left(f_i^{-1}(x)\right)\right)\right) \\ &= \max_{i\in\{0,1,\ldots,p-1\}} \sup_{t\in X_i} \left(\tilde{F}_i(\varphi(t)), \tilde{F}_i(\psi(t))\right) \\ &\leq \max_{i\in\{0,1,\ldots,p-1\}} \sup_{t\in X_i} \lambda_i\, d_2(\varphi(t), \psi(t)) \\ &\leq \lambda\, d(\varphi, \psi). \end{aligned}$$

By the Banach Fixed Point Theorem T has a unique fixed point $\varphi_0 \in B$. □

Remark 9. A version of this result may be stated for the case where (i) does not hold. In that case, (i) must be replaced by the corresponding compatibility conditions at contact points. Condition (i) in Theorem 8 implies that contact points do not exist, so that these compatibility conditions are in this case vacuously satisfied.

We remark that, when f and F are expansions in each subset, the corresponding systems are formed by conjugacy equations, f_i and F_j being expansion maps. If, moreover, f_i and F_j are invertible (or at least injective) it is possible to convert the systems to other equivalent systems where now the corresponding f_i and F_j are the inverses of the original functions, and therefore, according to Proposition 5, are contraction maps. This latter form is more suitable for standard applications because of fixed point theorems with respect to this type of maps, as will be developed in chapter 4. In the particular case where

$$f(x) = px \,(\mathrm{mod}\, 1)\,, \tag{2.5}$$

solutions of conjugacy equations of type (1.2) may be obtained by an iterative method and, in specific cases, explicit formulas for solutions may be obtained. In this case the

$$f_i \equiv f_{|\left[\frac{i}{p},\frac{i+1}{p}\right)}$$

are expansion maps. The corresponding contraction maps $f_i^{-1} : [0,1) \to [0,1)$ are

$$f_i^{-1}(x) = \frac{x+i}{p},\ i = 0,1,\ldots,p-1,\ x \in [0,1)\,. \tag{2.6}$$

The importance of solving systems of functional equations in order to obtain solutions of conjugacy equations (whose definition depend on partitions) will become clear in Chapter 4.

There also exist conditions that ensure solutions of (1.2) are continuous. These will be given in the more general context of systems with explicit dependence on the independent variable (Chapter 3).

2.3 Explicit solutions

Contractive systems of conjugacy equations are studied and explicit and constructive solutions are obtained. In the following we present some examples of these constructions. In some of the examples below it will be convenient to use the partial sums of a sequence.

Definition 10. For each sequence $\{\xi_j\}_{j\in\mathbb{N}}$, we define the sequence $\{s_n\}_{n\in\mathbb{N}_0}$ by $s_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $\forall n \in \mathbb{N}$ and $s_0 = 0$.

2.3.1 Contractive systems of two equations

De Rham [74] (1956): $Y = \mathbb{C}$, f_k are of the form

$$f_k(x) = \frac{x+k}{2}$$

and

$$\begin{cases} F_0(y), & \text{contractive} \\ F_1(y), & \text{contractive.} \end{cases}$$

The solution of this system is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{2^n}\right) = \lim_{\nu\to\infty} F_{\xi_1}\circ F_{\xi_2}\circ\cdots\circ F_{\xi_\nu}(\xi),\ \xi\in Y.$$

2.3.2 Contractive systems of p equations

Zdun [103] (2001): Y is a complete metric space, f_k are of the form

$$f_k(x) = \frac{x+k}{p} \tag{2.7}$$

and $F_k(y)$ are α-contractive, $0 \le k \le p-1$, with $\alpha : [0,\infty) \to [0,\infty)$ an increasing function such that its sequence of iterates tends pointwise to 0 on $[0,\infty)$. The solution is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{p^n}\right) = \lim_{\nu\to\infty} F_{\xi_1}\circ F_{\xi_2}\circ\cdots\circ F_{\xi_\nu}(\xi),\ \xi\in Y. \tag{2.8}$$

2.3.3 Affine systems

Affine systems are special cases of 2.3.1 and 2.3.2.

(i) [85] Suppose for all $k \in \{0,1,\ldots,p-1\}$, f_k are of the form (2.7) and $F_k(y)$ are contractive functions of the form $F_k(y) = \alpha\beta^k y + \gamma k$. This is a particular case of 2.3.2. The solution is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{p^n}\right) = \sum_{n=0}^{\infty}\alpha^n\beta^{s_n}\gamma\xi_{n+1}. \tag{2.9}$$

Proof. For $x=0$, the equation labelled by zero gives $\varphi(0)=0$, which agrees with (2.9). For $x=\xi_1/p$,

$$\varphi\left(\frac{\xi_1}{p}\right) = F_{\xi_1}(\varphi(0)) = \alpha\beta^{\xi_1}\varphi(0)+\gamma\xi_1 = \gamma\xi_1.$$

By (2.9), $\varphi(\xi_1/p) = \alpha^0\beta^{s_0}\gamma\xi_1 = \gamma\xi_1$.

Let $m\in\mathbb{N}$. Suppose formula (2.9) is valid for

$$x = \sum_{n=1}^{m}\frac{\xi_n}{p^n},$$

i.e., since $\xi_{n+1} = 0$ for $n \geq m$, we have

$$\varphi\left(\sum_{n=1}^{m}\frac{\xi_n}{p^n}\right) = \sum_{n=0}^{m}\alpha^n\beta^{s_n}\gamma\xi_{n+1}.$$

Then

$$\begin{aligned}
\varphi\left(\sum_{n=1}^{m+1}\frac{\xi_n}{p^n}\right) &= \varphi\left(\frac{\xi_1+\sum_{n=2}^{m+1}\frac{\xi_n}{p^{n-1}}}{p}\right) = \varphi\left(\frac{\xi_1+\sum_{n=1}^{m}\frac{\xi_{n+1}}{p^n}}{p}\right)\\
&= F_{\xi_1}\left(\varphi\left(\sum_{n=1}^{m}\frac{\xi_{n+1}}{p^n}\right)\right) = F_{\xi_1}\left(\sum_{n=0}^{m}\alpha^n\beta^{s_{n+1}-\xi_1}\gamma\xi_{n+1}\right)\\
&= \alpha\beta^{\xi_1}\sum_{n=0}^{m}\alpha^n\beta^{s_{n+1}-\xi_1}\gamma\xi_{n+1}+\gamma\xi_1 = \sum_{n=0}^{m}\alpha^{n+1}\beta^{s_{n+1}}\gamma\xi_{n+1}+\gamma\xi_1\\
&= \sum_{n=0}^{m+1}\alpha^n\beta^{s_n}\gamma\xi_{n+1}.
\end{aligned}$$

By induction and with a limiting procedure the proof is concluded. □

(ii) De Rham [74] (1956): f_k are of the form (2.7), $Y = \mathbb{C}$ and

$$\begin{cases} F_0(y) = ay \\ F_1(y) = (1-a)\,y + a, \end{cases}$$

$0 < a < 1$. This is a special case of (i) with $p = 2$, $\alpha = \gamma = a$ and $\beta = (1-a)/a$. The solution is obtained as a special case of (2.9) and is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{2^n}\right) = \sum_{n=0}^{\infty}a^{n+1}\left(\frac{1-a}{a}\right)^{s_n}\xi_{n+1}. \tag{2.10}$$

2.3.4 Complex conjugate affine systems

For the complex conjugate affine case it will be convenient to introduce some notation.

Definition 11. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{C}$. We define the ***alternating complex sequence*** of $(u_n)_{n\in\mathbb{N}}$ as the sequence given by

$$u_n^\sharp = \begin{cases} \bar{u}_n, & \text{if } n \text{ is odd} \\ u_n, & \text{if } n \text{ is even,} \end{cases}$$

where for each $y \in \mathbb{C}$, $\bar{y}$ denotes the complex conjugate of y.

Remark 12. The alternating complex sequence $u_n^\sharp$ is also applicable to a constant sequence $u_n \equiv u$ in the obvious way.

(i) Suppose $Y = \mathbb{C}$, for all $k \in \{0, 1, \dots, p-1\}$, f_k are of the form (2.7) and $F_k(y)$ are contractive functions of the form $F_k(y) = \overline{\alpha\beta^k y + \gamma k}$ (which is the complex conjugate of case (i) in section 2.3.3). Then the solution is given by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \left(\alpha\beta^{\xi_k}\right)_k^\sharp \gamma_n^\sharp \xi_n. \tag{2.11}$$

Note that in the alternating complex notation $F_k(y) = \left(\alpha\beta^k y + \gamma k\right)_1^\sharp$.

Proof. For $x = 0$, the equation labelled by zero gives $\varphi(0) = 0$, which agrees with formula (2.11). For $x = \xi_1/p$,

$$\varphi\left(\frac{\xi_1}{p}\right) = F_{\xi_1}(\varphi(0)) = \left(\alpha\beta^{\xi_1}\varphi(0) + \gamma\xi_1\right)_1^\sharp = (\gamma\xi_1)_1^\sharp = \gamma_1^\sharp \xi_1.$$

By (2.11)

$$\varphi\left(\frac{\xi_1}{p}\right) = \sum_{n=1}^{\infty} \prod_{k=1}^{n-1} \left(\alpha\beta^{\xi_k}\right)_k^\sharp \gamma_n^\sharp \xi_n = \gamma_1^\sharp \xi_1.$$

Let $m \in \mathbb{N}$. Suppose formula (2.11) is valid for

$$x = \sum_{n=1}^{m} \frac{\xi_n}{p^n},$$

i.e., since $\xi_{n+1} = 0$ for $n \geq m$,

$$\varphi\left(\sum_{n=1}^{m} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{m} \prod_{k=1}^{n-1} \left(\alpha\beta^{\xi_k}\right)_k^\sharp \gamma_n^\sharp \xi_n.$$

Then

$$\varphi\left(\sum_{n=1}^{m+1}\frac{\xi_n}{p^n}\right)=\varphi\left(\frac{\xi_1+\sum_{n=2}^{m+1}\frac{\xi_n}{p^{n-1}}}{p}\right)=\varphi\left(\frac{\xi_1+\sum_{n=1}^{m}\frac{\xi_{n+1}}{p^n}}{p}\right)$$

$$=F_{\xi_1}\left(\varphi\left(\sum_{n=1}^{m}\frac{\xi_{n+1}}{p^n}\right)\right)=F_{\xi_1}\left(\sum_{n=1}^{m}\prod_{k=1}^{n-1}\left(\alpha\beta^{\xi_{k+1}}\right)_k^\sharp\gamma_n^\sharp\xi_{n+1}\right)$$

$$=\left(\alpha\beta^{\xi_1}\sum_{n=1}^{m}\prod_{k=1}^{n-1}\left(\alpha\beta^{\xi_{k+1}}\right)_k^\sharp\gamma_n^\sharp\xi_{n+1}+\gamma\xi_1\right)_1^\sharp$$

$$=\sum_{n=1}^{m}\prod_{k=1}^{n-1}\left(\alpha\beta^{\xi_1}\right)_1^\sharp\left(\alpha\beta^{\xi_{k+1}}\right)_{k+1}^\sharp\gamma_{n+1}^\sharp\xi_{n+1}+\gamma_1^\sharp\xi_1$$

$$=\sum_{n=1}^{m+1}\prod_{k=1}^{n-1}\left(\alpha\beta^{\xi_k}\right)_k^\sharp\gamma_n^\sharp\xi_n.$$

By induction and with a limiting procedure the proof is concluded. □

(ii) De Rham [74] (1956): f_k are of the form (2.7), $Y\subset\mathbb{C}$ and

$$\begin{cases}F_0(y)=a\bar{y}\\ F_1(y)=(1-a)\,\bar{y}+a.\end{cases}$$

This is a particular case of (i) with $p=2$, $\alpha=\gamma=\bar{a}$ and $\beta=(1-\bar{a})/\bar{a}$. The solution is obtained as a special case of (2.11) and is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{p^n}\right)=\sum_{n=1}^{\infty}\prod_{k=1}^{n-1}\left(\left(\frac{1-\bar{a}}{\bar{a}}\right)^{\xi_k}\right)_k^\sharp\bar{a}_{n+k}^\sharp\xi_n. \tag{2.12}$$

(iii) von Koch [46] curve (1904): particular case of (ii) with $a=1/2+i\sqrt{3}/6$, see de Rham in [74].

Since

$$1-\bar{a}=\frac{1}{2}-i\frac{\sqrt{3}}{6}=\bar{a},$$

we have for all $k\in\mathbb{N}$,

$$\left(\left(\frac{1-\bar{a}}{\bar{a}}\right)^{\xi_k}\right)_k^\sharp=1.$$

Then the solution is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{p^n}\right)=\sum_{n=1}^{\infty}\prod_{k=1}^{n-1}\left(\frac{1}{2}-i\frac{\sqrt{3}}{6}\right)_{n+k}^\sharp\xi_n=\sum_{n=1}^{\infty}\prod_{k=1}^{n-1}\left(\frac{1}{2}+i\,(-1)^{n+k+1}\,\frac{\sqrt{3}}{6}\right)\xi_n.$$

The von Koch curve may also be obtained as the image of the solution of the system where $Y \subset \mathbb{R}^2$, f_k are of the form (2.7) for $0 \leq k \leq 3$ and

$$\begin{cases} F_0\left(y_1, y_2\right) = \left(\frac{y_1}{3}, \frac{y_2}{3}\right) \\ F_1\left(y_1, y_2\right) = \left(\frac{y_1}{6} - \frac{\sqrt{3}}{6} y_2 + \frac{1}{3}, \frac{\sqrt{3}}{6} y_1 + \frac{y_2}{6}\right) \\ F_2\left(y_1, y_2\right) = \left(\frac{y_1}{6} + \frac{\sqrt{3}}{6} y_2 + \frac{1}{2}, -\frac{\sqrt{3}}{6} y_1 + \frac{y_2}{6} + \frac{\sqrt{3}}{6}\right) \\ F_3\left(y_1, y_2\right) = \left(\frac{y_1}{3} + \frac{2}{3}, \frac{y_2}{3}\right), \end{cases}$$

by transforming the IFS explicit formulae for the similarity transformations given in [42] into this suitable system of functional equations.

(iv) Cesàro [20] (1896) and Pólya [70] (1913) - Peano curves [67] (1890): particular case of (ii) with

$$\left| a - \frac{1}{2} \right| = \frac{1}{2},$$

as claimed by de Rham in [74], where (2.12) applies.

2.3.5 Non-continuous systems

Usually, classical problems have continuous solutions. However non-continuity cases may also be studied. We next give a result for a special case. The proof is similar to a theorem of Zdun [22], although our degree of generality is much greater. Our constructions use less hypotheses and the conclusions are therefore weaker; in particular, continuity of the solution is not ensured. The result gives an explicit formula for solutions in terms of base p representation of numbers

$$x = \sum_{i=1}^{\infty} \frac{\xi_i}{p^i}. \tag{2.13}$$

Note that since φ is not necessarily continuous, this representation for the arguments of the function may bring some constraints. The reason is that some real numbers have two different base p representations. The non-uniqueness of the base p representation occurs when $\xi_m \neq 0$ and $\xi_i = 0$ for all $i > m$. In this case, given a (finite) representation

$$x = \sum_{i=1}^{m} \frac{\xi_i}{p^i},$$

the second (infinite) representation is given by

$$x = \sum_{i=1}^{m-1} \frac{\xi_i}{p^i} + \frac{\xi_m - 1}{p^m} + \sum_{i=m+1}^{\infty} \frac{p-1}{p^i}.$$

Remark 13. This last expression represents x as the limit of the increasing sequence of numbers

$$x_n = \sum_{i=1}^{m-1} \frac{\xi_i}{p^i} + \frac{\xi_m - 1}{p^m} + \sum_{i=m+1}^{n} \frac{p-1}{p^i},$$

with $x = \lim_{n\to\infty} x_n$. However, if φ is not a continuous function, $\varphi(x) = \lim_{n\to\infty} \varphi(x_n)$ may not hold. That is why in what follows, in further results that depend on a base p representation of numbers, and in the case of a double representation, the explicit constructive formulae apply only for the finite representation, as will become clear from the proofs.

Theorem 14. *[86] Consider the hypotheses of Theorem 8 and f_i given by (2.6). Then system (2.3) has exactly one bounded solution. This solution is given by*

$$\varphi\left(\sum_{i=1}^{\infty} \frac{k_i}{p^i}\right) = \lim_{\nu\to\infty} F_{k_1} \circ \cdots \circ F_{k_\nu}(\xi), \tag{2.14}$$

where $\xi \in \cap_i X_i$.

Proof. By Theorem 8, there exists a unique bounded solution of (2.3). By definition of f_i, for $x \in [0,1)$,

$$f_{k_1} \circ \cdots \circ f_{k_\nu}(x) = \sum_{i=1}^{\nu} \frac{k_i}{p^i} + \frac{x}{p^\nu},$$

$$\lim_{\nu\to\infty} f_{k_1} \circ \cdots \circ f_{k_\nu}(x) = \sum_{i=1}^{\infty} \frac{k_i}{p^i}.$$

By recurrence, we obtain for $x \in [0,1)$

$$\varphi\left(f_{k_1} \circ \cdots \circ f_{k_\nu}(x)\right) = F_{k_1} \circ \cdots \circ F_{k_\nu}(\varphi(x)).$$

Letting $\nu \to +\infty$, it follows that

$$\varphi\left(\sum_{i=1}^{\infty} \frac{k_i}{p^i}\right) = \lim_{\nu\to\infty} F_{k_1} \circ \cdots \circ F_{k_\nu}(\xi), \tag{2.15}$$

where $\xi = \varphi(x) \in X$, $x \in [0,1)$. Note that, since all F_k are contractions, the limit in (2.15) exists and is independent of ξ. We may choose $\xi = \varphi(x)$ the (unique) fixed point of F_0 since from (2.3) and (2.6) it follows immediately that $\varphi(f_0(0)) = \varphi(0) = F_0(\varphi(0))$ is in $\varphi(X)$. $\square$

2.4 IFS - Iterated Function Systems

Problems of iterative functional equations are related to iterated function systems (IFS) as defined by Falconer [31] in the following way.

Definition 15. A finite family of contractions $\{S_1, S_2, ..., S_m\}$, on a set $Q \subset \mathbb{R}^n$ with $m \geq 2$ is called an ***iterated function system*** or ***IFS***. We denote such an IFS by $\{Q : S_1, S_2, ..., S_m\}$.

The theory of IFS relies on the following fundamental result.

Theorem 16. *([31], Thm. 9.1) Let $\mathcal{S}_Q$ be an IFS on a closed set $D \subset \mathbb{R}^n$. Then $\mathcal{S}_Q$ has a unique attractor A, that is, a unique non-empty compact set A such that*

$$A = \overset{m}{\underset{i=1}{\cup}} S_i(A).$$

Moreover

$$A = \overset{m}{\underset{k=0}{\cap}} S^k(E)$$

for every non-empty compact set $E \in D$ such that $S_i(E) \subset E$ for all i.

We will use also a related result.

Proposition 17. *([31], Prop. 9.7) Consider the IFS consisting of contractions $\{S_1, S_2, ..., S_m\}$ on a closed subset D of $\mathbb{R}^n$ such that*

$$b_i |x - y| \leq |S_i(x) - S_i(y)|, \; x, y \in D,$$

with $0 < b_i < 1$ for each i. Assume that the (non-empty compact) attractor A satisfies

$$A = \overset{m}{\underset{i=1}{\cup}} S_i(A),$$

with this union disjoint. Then A is totally disconnected and $\dim_H A \geq s$ where

$$\sum_{i=1}^{m} b_i^s = 1.$$

Here $\dim_H$ represents the Hausdorff dimension which depends on covering a set by small sets and on the Hausdorff measure (the full definition is in [31]). In this work we will not further develop this notion.

Assume a system of functional equations of the form (2.3) is given, defined by invertible expansion maps $\{f_i\}$, $\{F_j\}$. In order to convert from the functional equation setting to the IFS setting, observe that by Prop. 5 $\{f_i^{-1}\}$, $\{F_j^{-1}\}$ are contractions. Then, given $E \subset \mathbb{R}^n$ and families f_i and F_j of invertible expansion maps, we define $W_F(E) = \cup_{i=0}^{p-1} F_i^{-1}(E)$ and $W_f(E) = \cup_{i=0}^{p-1} f_i^{-1}(E)$. We may therefore state the following theorem:

Theorem 18. *[86] Consider a system of functional equations (2.3), where f_i, F_i are bijective expansion maps.*

(i) If X, Y are closed subsets of $\mathbb{R}^n$ and $\cup_{i=0}^{p-1} f_i^{-1}(X_i) = X$ then the image D of the unique solution of (2.3) is the attractor of the IFS $\{Y : F_i^{-1}, i = 0, 1, \ldots, p-1\}$.

(ii) If, in addition, there exist $b_i \in (0,1)$, $\forall i \in \{0, 1, \ldots, p-1\}$ such that

$$b_i |x - y| \leq \left| F_i^{-1}(x) - F_i^{-1}(y) \right|, \; x \in Y, \tag{2.16}$$

and D satisfies $D = \cup_{i=0}^{p-1} F_i^{-1}(D)$, with this union disjoint, then D is totally disconnected.

Proof. The system is equivalent to a contraction system of functional equations and has a unique solution by Theorem 8. In order to construct the image of the solution, we observe that for $i \in \{0, 1, \ldots, p-1\}$, $\varphi(f_i(X_i)) = F_i(\varphi(X_i))$. By an iterative process, we see that

$$\varphi(f_{k_1} \circ \cdots \circ f_{k_\nu}(X_{k_\nu})) = F_{k_1} \circ \cdots \circ F_{k_\nu}(\varphi(X_{k_\nu})), \forall k_1, \ldots, k_\nu \in \{0, 1, \ldots, p-1\}.$$

Since each f_{k_ν} is a bijection, this last expression is equivalent to

$$\varphi(X) = F_{k_1} \circ \cdots \circ F_{k_\nu}\left(\varphi\left(f_{k_\nu}^{-1} \circ \cdots \circ f_{k_1}^{-1}(X)\right)\right), \forall k_1, \ldots, k_\nu \in \{0, 1, \ldots, p-1\}.$$

We want to find sets Γ and Λ such that $\Gamma = \varphi(\Lambda)$ and $W_F(\Gamma) = \varphi(W_f(\Lambda))$. By Theorem 16 there exists a unique Λ such that $\Lambda = W_f(\Lambda)$. Now $W_F(\varphi(\Lambda)) = \varphi(\Lambda)$. By the same result there is a unique Γ such that $W_F(\Gamma) = \Gamma$. Moreover, Λ is the fixed point (attractor) of the IFS $\left\{X : f_i^{-1}, i = 0, 1, \ldots, p-1\right\}$ and Γ is the fixed point (attractor) of the IFS $\left\{Y : F_i^{-1}, i = 0, 1, \ldots, p-1\right\}$.

The statement in (ii) is an immediate consequence of Proposition 17. □

Definition 19. A topological space is a ***Cantor space*** if and only if it is non-empty, perfect, compact, totally disconnected, and metrizable.

Remark 20. A Cantor space is unique up to homeomorphism, allowing us to refer to "the" topological Cantor set independently of particular geometrical constructions (see Hocking-Young [39,Thm 2.97]).

Remark 21. Eventually, the sets D in (ii) of Theorem 18 may be Cantor spaces. However this is not assured and in general is not true: Theorem 18 does not guarantee that the set D in (ii) is perfect. It is possible that D is a finite set of isolated points. A trivial example is the IFS $\left\{Y : F_i^{-1}, i = 0, 1, \ldots, p-1\right\}$ being a single point, case where the solution of (2.3) is constant.

Note that Equation (2.16) together with contractivity of F_i^{-1} implies that the F_i^{-1} are maps of bounded distortion.

2.5 Incomplete systems

Theorem 18 is particularly interesting in the case where $W_F(Y) = \cup_{i=0}^{p-1} F_i^{-1}(Y) \neq Y$. We are also interested in the case where $W_f(X) \neq X$. We define this as an incomplete system, and for convenience formulate its definition in both the expanding and contracting versions.

Definition 22. [86] We say a system (2.3) is an **incomplete system** of equations if either

(i) all f_k are contraction maps and $\emptyset \neq \cup_{i=0}^{p-1} f_i(X_i) \subsetneq X$;

or

(ii) all f_k are expansion maps and $\emptyset \neq \cup_{i=0}^{p-1} f_i^{-1}(X_i) \subsetneq X$.

Example 23. Consider a system (2.3) where f_i are defined as in (2.6). The corresponding system where some (but not all) of the p equations are missing is an incomplete system.

We now introduce a result on incomplete systems of equations which includes Example 23.

Theorem 24. *[86] Consider the incomplete system of functional equations*

$$\varphi\left(f_{j_m}(x)\right) = F_{j_m}\left(\varphi(x)\right), \; j_m \in \{0, 1, \ldots, p-1\}, \; m = 1, \ldots, s, \; x \in [0,1), \tag{2.17}$$

where $j_m \neq j_n \,(m \neq n)$*, with* $0 < s < p - 1$*, where the* f_{j_m} *given by (2.6) and the* $F_{j_m} : Y \to Y$ *are* λ_{j_m}*-contractions.*

Then

(i) There exists a totally disconnected sub-domain C *of any solution of the system where the images are uniquely determined.*

(ii) If, in addition, the equation with index 0 *is part of the system (2.17), this solution is given by*

$$\varphi\left(\sum_{i=1}^{\infty} \frac{k_i}{p^i}\right) = \lim_{\nu \to \infty} F_{k_1} \circ \cdots \circ F_{k_\nu}(\xi),$$

where $\xi \in \cap_i X_i$*.*

Proof. Note that the equation with index 0 gives the fixed point ξ of F_0 needed for formula (2.14). Consider the base p representation (2.13) of $x \in [0,1)$. Then $C_{j_1, j_2, \ldots, j_s} = \{x \in [0,1) : k_i \in \{j_1, j_2, \ldots, j_s\}, i = 1, \ldots, \infty\}$ is the set whose elements are those numbers in $[0,1)$ for which the missing indices are absent from the base p expansion. This is easily seen to be a subset of a Cantor set, and thus is totally disconnected. □

Chapter 3

Systems with explicit dependence on x

3.1 Compatibility conditions

We restate the existence of compatibility conditions now for systems of the form (1.1), i.e. for systems with explicit dependence on the independent variable x. These are necessary conditions for the existence of solutions and were introduced in section 2.1 for systems of conjugacy equations, independently of specifics of the sets X and Y. The generalization is stated in the following.

Proposition 25. *[84] If $\varphi : X \to Y$ is a solution of (1.1), then it must satisfy*

$$\forall x_1 \in X_i, \forall x_2 \in X_j, f_i(x_1) = f_j(x_2) \Rightarrow F_i(x_1, \varphi(x_1)) = F_j(x_2, \varphi(x_2)), \quad (3.1)$$

for $i, j = 0, 1, \ldots, p-1$.

Let

$$A := \{x_1 \in X : \exists\, x_2 \in X, \exists\, i, j = 0, 1, \ldots, p-1, (i, x_1) \neq (j, x_2), f_i(x_1) = f_j(x_2)\}.$$

The elements of A are called the ***contact points*** of system (1.1). Note that when all f_i are injective, this set A reduces to

$$\{x_1 \in X : \exists\, x_2 \in X, \exists\, i, j = 0, 1, \ldots, p-1, i \neq j, f_i(x_1) = f_j(x_2)\},$$

as defined in [84], since in that context all f_i are injective.

Definition 26. [85] Consider a system of equations (1.1) where for all $x \in A$, $\varphi_x \equiv \varphi(x)$ has been previously determined (by partially solving the system or by initial conditions). We say that

$$\forall x_1, x_2 \in A, f_i(x_1) = f_j(x_2) \Rightarrow F_i(x_1, \varphi_{x_1}) = F_j(x_2, \varphi_{x_2}), \quad (3.2)$$

are the ***compatibility conditions*** of system (1.1).

Note that, considering in (3.1) $\varphi_{x_1} \equiv \varphi(x_1)$ and $\varphi_{x_2} \equiv \varphi(x_2)$ as parameters, the compatibility conditions (3.2) are effectively independent of φ. In fact, they are necessary conditions on the f_i and F_j for the existence of solutions.

Remark 27. In cases where system (1.1) is such that, for all $i \neq j$, $f_i(X) \cap f_j(X) = \emptyset$, the compatibility conditions are vacuously satisfied.

Compatibility conditions are necessary but not sufficient for the existence of solutions. In each specific context more conditions must be added to ensure existence and uniqueness of solutions. For the next example, compatibility conditions are explicitly determined.

Example 28. [84] Let $X = [0,1]$ and consider the system of functional equations

$$\begin{cases} \varphi(f_0(x)) = F_0(x, \varphi(x)), & x \in [0,1] \\ \varphi(f_1(x)) = F_1(x, \varphi(x)), & x \in [0,1]. \end{cases} \tag{3.3}$$

Suppose

$$f_0(x) = \frac{x}{2},\ f_1(x) = \frac{x+1}{2}.$$

Here $X_0 = X_1 = [0,1]$. Since

$$f_0(X_0) \cap f_1(X_1) = \left\{\frac{1}{2}\right\}$$

and

$$f_0(1) = \frac{1}{2} = f_1(0),$$

the compatibility condition is

$$F_0(1, \varphi(1)) = F_1(0, \varphi(0)). \tag{3.4}$$

If we suppose, additionally, that $F_i(x,y) = \alpha_i(x)\, y + q_i(x)$, then the compatibility condition is

$$\alpha_0(1)\,\varphi(1) + q_0(1) = \alpha_1(0)\,\varphi(0) + q_1(0).$$

Solving the first equation of (3.3) for $x = 0$ and the second for $x = 1$, we obtain for the images of the contact points

$$\varphi(0) = \frac{q_0(0)}{1 - \alpha_0(0)},\ \varphi(1) = \frac{q_1(1)}{1 - \alpha_1(1)}.$$

The compatibility condition on F_0, F_1 is

$$\frac{\alpha_0(1)\, q_1(1)}{1 - \alpha_1(1)} + q_0(1) = \frac{\alpha_1(0)\, q_0(0)}{1 - \alpha_0(0)} + q_1(0).$$

The autonomous version of this problem (that is, where F_0, F_1 do not depend explicitly on x) was studied by de Rham [74], who showed existence and uniqueness of solution for the corresponding system of functional equations under the assumption $F_0(\varphi(1)) = F_1(\varphi(0))$ which corresponds to (3.4) in this special case. More general cases were studied, for instance, by Girgensohn [36] and by Zdun and Ciepliński [22,103].

3.2 Existence, uniqueness and continuity

We now concentrate in the context of metric spaces. Suppose now that (X, d_1) is a bounded metric space and (Y, d_2) is a complete metric space.

Theorem 29. *[85] Let $f_i : X_i \to X$ be a family of injective functions such that*
(i) $\forall i \neq j,\ f_i(X_i) \cap f_j(X_j) = \emptyset$,
(ii) $\cup_{i=0}^{p-1} f_i(X_i) = X$
and $F_i : X \times Y_i \to Y$ is, with respect to the second coordinate, a λ_i-contraction and $\overline{Y_i} = Y$, for all $i = 0, 1, \ldots, p-1$.
Then there exists a unique bounded solution $\varphi : X \to Y$ of system (1.1).

Proof. Let B be the complete metric space defined in Lemma 7. Observe that, since F_i is a λ_i-contraction and thus Lipschitz in the second coordinate, for each x, $F_i(\cdot, y)$ has an extension by continuity $\tilde{F}_i$ to $X \times Y = X \times \overline{Y_i}$. Since each f_i is invertible over its image, system (1.1) is equivalent to

$$\varphi(x) = \tilde{F}_i\left(f_i^{-1}(x), \varphi\left(f_i^{-1}(x)\right)\right),\ i = 0, 1, \ldots, p-1,\ x \in f_i(X_i).$$

Let $T : B \to B$ be the operator defined by $T\{\varphi\}(x) = F_i\left(f_i^{-1}(x), \varphi\left(f_i^{-1}(x)\right)\right)$, $x \in f_i(X_i)$, $i \in \{0, 1, \ldots, p-1\}$. Since f_i, $\tilde{F}_i$ are bounded, T is also bounded. Let $\lambda := \max\limits_{i \in \{0,1,\ldots,p-1\}} \lambda_i$. Then

$$\begin{aligned}
&d(T\varphi, T\psi) \\
&= \max_{i \in \{0,1,\ldots,p-1\}} \sup_{x \in f_i(X_i)} \left(\tilde{F}_i\left(f_i^{-1}(x), \varphi\left(f_i^{-1}(x)\right)\right), \tilde{F}_i\left(f_i^{-1}(x), \psi\left(f_i^{-1}(x)\right)\right)\right) \\
&= \max_{i \in \{0,1,\ldots,p-1\}} \sup_{t \in X_i} \left(\tilde{F}_i(t, \varphi(t)), \tilde{F}_i(t, \psi(t))\right) \\
&\leq \max_{i \in \{0,1,\ldots,p-1\}} \sup_{t \in X_i} \lambda_i d_2(\varphi(t), \psi(t)) \\
&\leq \lambda d(\varphi, \psi).
\end{aligned}$$

By the Banach Fixed Point Theorem T has a unique fixed point $\varphi_0 \in B$. $\square$

When the system (1.1) is such that $\exists i \neq j$: $f_i(X) \cap f_j(X) \neq \emptyset$, the compatibility conditions are necessary for the existence of solution. In this case, if the compatibility conditions are satisfied, the images of elements of $f_i(X) \cap f_j(X)$ by a solution φ of system (1.1) may be obtained either from the equation labelled by i or from the equation labelled by j.

Remark 30. A version of this result may be stated for the case where (i) does not hold. In that case, (i) must be replaced by the corresponding compatibility conditions at contact points. Condition (i) in Theorem 29 implies that contact points do not exist, so that the compatibility conditions are in this case vacuously satisfied.

We now provide sufficient conditions for solutions of systems (1.1) to be continuous. We deal first with the setting of metric spaces. Let (X, d_1) and (Y, d_2) be metric spaces.

Proposition 31. *[85] Let $i \in \{0,1,\ldots,p-1\}$ and $x_2 \in X_i$. If $X = \cup_{i=0}^{p-1} f_i(X_i)$ and $\varphi : X \to Y$ is a solution of (1.1) such that*

$$\begin{gathered} \forall j \in \{0,1,\ldots,p-1\}, \forall x_1 \in X_j, \forall \epsilon > 0, \exists \delta > 0 \\ d_1(f_j(x_1), f_i(x_2)) < \delta \Rightarrow d_2(F_j(x_1, \varphi(x_1)), F_i(x_2, \varphi(x_2))) < \epsilon, \end{gathered} \tag{3.5}$$

then φ is continuous at $f_i(x_2)$.

Proof. Recall that a function f is continuous at $x_2 \in X$ if for every $x_1 \in X$ and $\epsilon > 0$ there exists a $\delta > 0$ such that $d_1(x_1, x_2) < \delta \Rightarrow d_2(f(x_1), f(x_2)) < \epsilon$. Let $\varphi : X \to Y$ be a solution of (1.1). The goal is to prove the continuity condition for each $x_2 \in X$:

$$\forall x_1 \in X, \forall \epsilon > 0, \exists \delta > 0, d_1(x_1, x_2) < \delta \Rightarrow d_2(\varphi(x_1), \varphi(x_2)) < \epsilon.$$

Since in system (1.1) the arguments of φ are of the form $f_i(x)$, replacing (x_1, x_2) by $(f_i(x_1), f_j(x_2))$, we obtain condition (3.5). □

Equivalently we state the result in terms of limits.

Proposition 32. *[85] Let $i \in \{0,1,\ldots,p-1\}$ and $x_2 \in X_i$. If $X = \cup_{i=0}^{p-1} f_i(X_i)$ and $\varphi : X \to Y$ is a solution of (1.1) such that*
$\forall j \in \{0,1,\ldots,p-1\}, \forall x_1 \in X,$

$$\lim_{\substack{x \to x_1 \\ x \in X_j}} f_j(x) = f_i(x_2) \Rightarrow \lim_{\substack{x \to x_1 \\ x \in X_j}} F_j(x, \varphi(x)) = F_i(x_2, \varphi(x_2)), \tag{3.6}$$

then φ is continuous at $f_i(x_2)$.

Proof. Let $\tilde{x}_1, \tilde{x}_2 \in X$. Since $X = \cup_{i=0}^{p-1} f_i(X_i)$, there exists $i, j \in \{0,1,\ldots,p-1\}$, $x_1, x_2 \in X$ such that $\tilde{x}_1 = f_i(x_1)$, $\tilde{x}_2 = f_j(x_2)$. By (1.1)

$$d_2(\varphi(\tilde{x}_1), \varphi(\tilde{x}_2)) = d_2(F_i(x_1, \varphi(x_1)), F_j(x_2, \varphi(x_2))). \tag{3.7}$$

In this setting, if (3.6) holds, equality (3.7) implies the continuity of φ. □

More practical conditions of continuity may be stated in terms of f_i, F_j. Note that continuity of all f_i, F_j, although necessary, is not sufficient to ensure continuity of the solution of system (1.1). The following Proposition provides sufficient conditions for continuity.

Proposition 33. *[85] Suppose all f_i, F_j are continuous and $X = \cup_{i=0}^{p-1} f_i(X_i)$. Let $i \in \{0,1,\ldots,p-1\}$ and $x_2 \in X_i$. If $\varphi : X \to Y$ is a solution of (1.1) such that*
$\forall j \in \{0,1,\ldots,p-1\}, i \neq j, \forall x_1 \in X,$

$$\lim_{\substack{x \to x_1 \\ x \in X_j}} f_j(x) = f_i(x_2) \Rightarrow \lim_{\substack{x \to x_1 \\ x \in X_j}} F_j(x, \varphi(x)) = F_i(x_2, \varphi(x_2)), \tag{3.8}$$

then φ is continuous at $f_i(x_2)$.

Proof. Continuity of f_i, F_j implies condition (3.6) whenever $i = j$; condition (3.8) is condition (3.6) for $i \neq j$. □

Similarly to the definition of contact points, and regarding Proposition 33, we define the **limit-contacting points** which are those in the set

$$A_0 = \left\{ x_1 \in X : \exists\, i, j = 0, 1, \ldots p-1, i \neq j, \exists\, x_2 \in X_i, \lim_{\substack{x \to x_1 \\ x \in X_j}} f_j(x) = f_i(x_2) \right\}$$

and the **contacting points** which are those in the set

$$\tilde{A}_0 = \left\{ x_2 \in X : \exists\, i, j = 0, 1, \ldots p-1, i \neq j, \exists\, x_1 \in X_j, \lim_{\substack{x \to x_1 \\ x \in X_j}} f_j(x) = f_i(x_2) \right\}.$$

Given a system of equations (1.1), if f_i and F_j are continuous and condition (3.8) is verified for both the limit-contacting and the contacting points then any solution φ is necessarily continuous.

More generally, for topological spaces the condition which ensures the continuity of solutions is given in the following result.

Proposition 34. *[85] Let X and Y topological spaces. If $X = \cup_{i=0}^{p-1} f_i(X_i)$ and $\varphi : X \to Y$ is a solution of (1.1) such that $\forall S \subset \{0, 1, \ldots, p-1\} \times X$,*

$$\{F_i(x, \varphi(x)) : (i, x) \in S\} \text{ is open } \Rightarrow \{f_i(x) : (i, x) \in S\} \text{ is open,} \tag{3.9}$$

then φ is continuous.

Proof. By definition, a function φ is a continuous solution of (1.1) if and only if for every open set U in Y the inverse image $\varphi^{-1}(U)$ is open in X.

Let U be an open set in Y. Define

$$S_U = \{(i, x) \in \{0, 1, \ldots, p-1\} \times X : \exists y \in U, y = F_i(x, \varphi(x))\}.$$

By definition of inverse image $\varphi^{-1}(U) = \{\tilde{x} \in X : \varphi(\tilde{x}) \in U\}$. Since by hypothesis $X = \cup_{i=0}^{p-1} f_i(X_i)$, for each $\tilde{x} \in X$ there exist $i \in \{0, 1, \ldots, p-1\}$ and $x \in X_i$ such that $\tilde{x} = f_i(x)$. Since φ is by hypothesis a solution of (1.1), $\varphi(\tilde{x}) = F_i(x, \varphi(x))$ and

$$\begin{aligned}
\varphi^{-1}(U) &= \{f_i(x) : i \in \{0, 1, \ldots, p-1\}, x \in X_i, F_i(x, \varphi(x)) \in U\} \\
&= \{f_i(x) : (i, x) \in S_U\}.
\end{aligned}$$

Note that when $\varphi^{-1}(U) = \{f_i(x) : (i, x) \in S_U\}$, the set U is equal to $\{F_i(x, \varphi(x)) : (i, x) \in S_U\}$. Replacing S_U by S the continuity condition is given by (3.9). □

As for metric spaces, more practical conditions of continuity may be stated in terms of the continuity of f_i, F_j.

Proposition 35. *[85] Suppose all f_i, F_j are continuous and $X = \cup_{i=0}^{p-1} f_i(X_i)$. Let $i \in \{0, 1, \ldots, p-1\}$ and $x_2 \in X_i$. If $\varphi : X \to Y$ is a solution of (1.1) such that $\forall j \in \{0, 1, \ldots, p-1\}, i \neq j, \forall x_1 \in X, \forall W$ neighbourhood of $F_j(x_1, \varphi(x_1))$, $\exists V$ neighbourhood of $f_j(x_1)$ such that*

$$f_i(x_2) \in V \Rightarrow F_i(x_2, \varphi(x_2)) \in W, \tag{3.10}$$

then φ is continuous at $f_i(x_2)$.

Proof. Continuity in a metric space may be formulated using only open sets, without explicit use of the metric [73]. Condition (3.10) corresponds to (3.5) in this context, and therefore is its analogue in the category of topological spaces. □

The definition of contacting and limit contacting points via A_0 and $\tilde{A}_0$ translate immediately to their analogues in topological spaces. Thus given a system of equations (1.1) in topological spaces, if f_i and F_j are continuous and condition (3.10) is verified for the limit-contacting points and for the contacting points then, if φ is a solution, φ is continuous.

Example 36. [85] Let $X = [0, 1]$ and consider the system of functional equations

$$\begin{cases} \varphi(f_0(x)) = F_0(x, \varphi(x)), & x \in [0, 1] \\ \varphi(f_1(x)) = F_1(x, \varphi(x)), & x \in (0, 1] \\ \varphi(f_2(x)) = F_2(x, \varphi(x)), & x \in [0, 1]. \end{cases} \tag{3.11}$$

Let $\alpha = \sin(1)/2$ and suppose

$$f_0(x) = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right), \ f_1(x) = 1 - \frac{\sin(x)}{2x}, \ f_2(x) = \alpha x + 1 - \alpha.$$

Here $X_0 = X_2 = [0, 1]$, $X_1 = (0, 1]$. Since

$$f_1(X_1) \cap f_2(X_2) = \left(\frac{1}{2}, 1 - \alpha\right] \cap [1 - \alpha, 1] = \{1 - \alpha\}$$

and

$$f_1(1) = 1 - \alpha = f_2(0),$$

the compatibility condition will result from imposing

$$F_1(1, \varphi(1)) = F_2(0, \varphi(0)). \tag{3.12}$$

Theorem 29 requires that the system satisfies the disjointness condition: $f_i(X_i) \cap f_j(X_j) = \emptyset$ for $i \neq j$. If system (3.11) verifies the compatibility conditions derived from (3.12), then it is equivalent to the system

$$\begin{cases} \varphi(f_0(x)) = F_0(x, \varphi(x)), & x \in (0, 1] \\ \varphi(f_1(x)) = F_1(x, \varphi(x)), & x \in (0, 1] \\ \varphi(f_2(x)) = F_2(x, \varphi(x)), & x \in [0, 1] \end{cases}$$

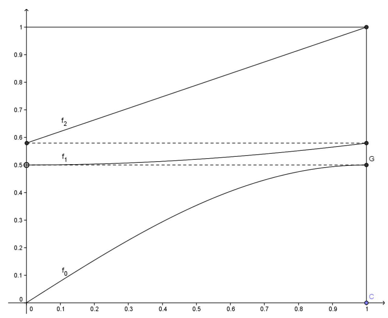

Figure 3.1: Functions f_0, f_1 and f_2

which satisfies the disjointness conditions: $f_i\left(X_i\right) \cap f_j\left(X_j\right) = \emptyset$ if $i \neq j$. Both systems satisfy $\cup_{i=0}^{p-1} f_i\left(X_i\right) = X = [0,1]$.

We observe that

$$\lim_{\substack{x \to 0 \\ x \in X_1}} f_1\left(x\right) = \frac{1}{2} = f_0\left(1\right),$$

and all f_i, $i = 0,1,2$ are continuous in their domains. By condition (3.8), to obtain a continuous solution, F_0 and F_1 must satisfy

$$\lim_{\substack{x \to 0 \\ x \in X_1}} F_1\left(x, \varphi\left(x\right)\right) = F_0\left(1, \varphi\left(1\right)\right) \tag{3.13}$$

In this case, the continuity conditions in Proposition 33 are the continuity of F_0, F_1 and (3.13). Note that the compatibility condition (3.12), together with the continuity of f_1, f_2, F_1, F_2, ensures the continuity condition (3.8) between the two equations with indices 1 and 2.

Consider the special case where the F_i have the form $F_i\left(x,y\right) = \alpha_i\left(x\right) y + q_i\left(x\right)$, which is the relevant one for fractal interpolation; see [5,84,97], where solutions are called fractal functions (see Barnsley [4,6]). Solving the first equation of (3.11) for $x = 0$ and the last for $x = 1$, we obtain for images of the contact points

$$\varphi\left(0\right) = \frac{q_0\left(0\right)}{1 - \alpha_0\left(0\right)}, \; \varphi\left(1\right) = \frac{q_1\left(1\right)}{1 - \alpha_1\left(1\right)}.$$

The compatibility condition results from

$$\alpha_1\left(1\right)\varphi\left(1\right) + q_1\left(1\right) = \alpha_2\left(0\right)\varphi\left(0\right) + q_2\left(0\right),$$

which upon substituting $\varphi\left(0\right)$ and $\varphi\left(1\right)$ yields

$$\alpha_1\left(1\right)\frac{q_1\left(1\right)}{1 - \alpha_1\left(1\right)} + q_1\left(1\right) = \alpha_2\left(0\right)\frac{q_0\left(0\right)}{1 - \alpha_0\left(0\right)} + q_2\left(0\right). \tag{3.14}$$

On the other hand, condition (3.13) becomes

$$\lim_{\substack{x \to 0 \\ x \in X_1}} \left(\alpha_1(x)\varphi(x) + q_1(x)\right) = \alpha_0(1)\varphi(1) + q_0(1),$$

or, since we want φ to be continuous,

$$\varphi(0) \lim_{\substack{x \to 0 \\ x \in X_1}} \alpha_1(x) + \lim_{\substack{x \to 0 \\ x \in X_1}} q_1(x) = \alpha_0(1)\varphi(1) + q_0(1).$$

Substituting $\varphi(0)$ and $\varphi(1)$,

$$\frac{q_0(0)}{1-\alpha_0(0)} \lim_{\substack{x \to 0 \\ x \in X_1}} \alpha_1(x) + \lim_{\substack{x \to 0 \\ x \in X_1}} q_1(x) = \alpha_0(1)\frac{q_1(1)}{1-\alpha_1(1)} + q_0(1). \quad (3.15)$$

We thus see that condition (3.14) is a necessary condition for the existence of solution of system (3.11). If additionally $\{F_i\}$ is a family of contraction functions with respect to the second coordinate, then by Theorem 29 there exists a unique bounded solution. On the other hand, if the F_j are only supposed to be continuous, φ is a solution of (3.11) and (3.15) is satisfied, then, since the f_i are continuous, φ is continuous by Proposition 33.

3.3 Constructive solutions

We treat systems (1.1) separately according to the type of functions F_i. They may be affine (section 2.3.3) or more generally non-linear (section 3.3.2).

Definition 37. A system (1.1) for which each F_j is affine relative to the second variable we call it an **affine system**.

In general, as we will see below, it is possible to construct explicit solutions for C-contractive systems.

Definition 38. Given $F : X \times Y \to Y$, we will say that F is a **C-contraction** if it is continuous in the first argument and contractive in the second.

After a presentation of constructive general formulae for different general cases we provide a list of possible examples/particular cases (section 3.3.3).

3.3.1 Affine systems

We first state general existence and uniqueness results for systems where the functions F_j are affine in the second variable with variable parameters (first variable) and $f_j(x) = (x + j)/p$. In this cases the solution admits an explicit constructive formula in terms of a base p representation of numbers.

The following theorem is authored by Girgensohn (see [36]). He gives the solution for affine systems with constant parameters (the coefficient of variable φ for each F_j is constant).

Theorem 39. *(Girgensohn) Fix $p \in \{2, 3, 4, \ldots\}$, let $r_k : [0, 1] \to \mathbb{R}$ be continuous, $|s_k| < 1$ for $0 \le k \le p - 1$ and assume*

$$\frac{s_{k-1}}{1 - s_{p-1}} r_{p-1}(1) + r_{k-1}(1) = \frac{s_k}{1 - s_0} r_0(0) + r_k(0). \tag{3.16}$$

Then there exists exactly one bounded $\varphi : [0, 1] \to \mathbb{R}$ which satisfies the system

$$\varphi\left(\frac{x + k}{p}\right) = s_k \varphi(x) + r_k(x),\ x \in [0, 1],\ 0 \le k \le p - 1.$$

The function φ is continuous and given in terms of the base p expansion

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n}$$

by

$$\varphi(x) = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} s_{\xi_k}\right) r_{\xi_n}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{p^k}\right).$$

This result provides the solution to problems of fractal interpolation functions, namely those studied by Barnsley and Harrington [5] with constant parameter functions.

Remark 40. Note that in the theorem above the usual combinatorial convention for the empty product ($\prod_{k=1}^{0} x_k = 1$) is used; see e.g. [56, p. 12]. In what follows we use both this convention and the empty sum convention ($\sum_{k=1}^{0} x_k = 0$).

Theorem 39 does not provide the solution to the more general problem defined by the affine system of functional equations (5.4) with variable parameters (see in chapter 5 equation (5.5)). We now state the general existence and uniqueness result for this type of systems, including the explicit formula for the solution.

Theorem 41. *[84] Fix $p \in \{2, 3, 4, \ldots\}$, let $r_k : [0, 1] \to \mathbb{R}$, and $s_k : [0, 1] \to [0, 1]$ be continuous functions such that $|s_k(x)| < 1$, $\forall x \in [0, 1]$, and assume that for any $k \in \{0, 1, \ldots, p - 2\}$, condition*

$$s_k(1) \frac{r_{p-1}(1)}{1 - s_{p-1}(1)} + r_k(1) = s_{k+1}(0) \frac{r_0(0)}{1 - s_0(0)} + r_{k+1}(0) \tag{3.17}$$

is satisfied. Then there exists exactly one bounded $\varphi : [0, 1] \to \mathbb{R}$ which satisfies the system

$$\varphi\left(\frac{x + k}{p}\right) = s_k(x) \varphi(x) + r_k(x),\ x \in [0, 1],\ 0 \le k \le p - 1. \tag{3.18}$$

The function φ is continuous and is given in terms of the base p expansion

$$x=\sum_{n=1}^{\infty}\frac{\xi_n}{p^n},$$

by

$$\varphi\left(x\right)=\sum_{n=1}^{\infty}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+m}}{p^k}\right)\right)r_{\xi_n}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{p^k}\right). \tag{3.19}$$

Proof. Assume there is a bounded solution φ of (3.18). From the first and the last equations of system (3.18) we obtain, respectively, the values of $\varphi\left(0\right)$ and $\varphi\left(1\right)$:

$$\varphi\left(0\right)=\frac{r_0\left(0\right)}{1-s_0\left(0\right)},\ \varphi\left(1\right)=\frac{r_{p-1}\left(1\right)}{1-s_{p-1}\left(1\right)}.$$

With these values, the compatibility conditions are written as conditions (3.17), and are therefore necessary conditions for the existence of solution. Let now $x\in[0,1]$ with base p expansion

$$x=\sum_{n=1}^{\infty}\xi_n p^{-n}.$$

For each $m\in\mathbb{N}$, define γ_m by

$$\gamma_m=\sum_{k=1}^{\infty}\frac{\xi_{k+m}}{p^k}.$$

We obtain

$$\begin{aligned}\gamma_{m+1}&=\sum_{k=1}^{\infty}\frac{\xi_{k+m+1}}{p^k}=\sum_{k=2}^{\infty}\frac{\xi_{k+m}}{p^{k-1}}=\sum_{k=1}^{\infty}\frac{\xi_{k+m}}{p^{k-1}}-\frac{\xi_{m+1}}{p^0}\\&=p\sum_{k=1}^{\infty}\frac{\xi_{k+m}}{p^k}-\xi_{m+1}=p\gamma_m-\xi_{m+1}\end{aligned}$$

and therefore $\gamma_m=p^{-1}\left(\gamma_{m+1}+\xi_{m+1}\right)$.

We want to show the explicit solution is given by

$$\varphi\left(\sum_{n=1}^{\infty}\frac{\xi_n}{p^n}\right)=\sum_{n=1}^{\infty}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\gamma_m\right)\right)r_{\xi_n}\left(\gamma_n\right). \tag{3.20}$$

To that effect, we shall prove by induction that, for $M\in\mathbb{N}_0$,

$$\varphi\left(x\right)=\left(\prod_{m=1}^{M}s_{\xi_m}\left(\gamma_m\right)\right)\varphi\left(\gamma_M\right)+\sum_{n=1}^{M}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\gamma_m\right)\right)r_{\xi_n}\left(\sum_{k=1}^{M-n}\frac{\xi_{k+n}}{p^k}+\frac{\gamma_M}{p^{M-n}}\right), \tag{3.21}$$

from which (3.20) follows by a limiting procedure, provided the corresponding series is convergent. For $M=0$ and $\gamma_0 = x$ formula (3.21) gives $\varphi(x) = \varphi(\gamma_0)$.

For the induction step, suppose (3.21) is satisfied for a positive integer M. Then

$$\begin{aligned}
\varphi(x) &= \left(\prod_{m=1}^{M} s_{\xi_m}(\gamma_m)\right)\varphi(\gamma_M) + \sum_{n=1}^{M}\left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_n}\left(\sum_{k=1}^{M-n}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_M}{p^{M-n}}\right)\\
&= \left(\prod_{m=1}^{M} s_{\xi_m}(\gamma_m)\right)\varphi\left(\frac{\gamma_{M+1}+\xi_{M+1}}{p}\right)\\
&\quad + \sum_{n=1}^{M}\left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_n}\left(\sum_{k=1}^{M-n}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_{M+1}+\xi_{M+1}}{p^{M+1-n}}\right)\\
&= \left(\prod_{m=1}^{M} s_{\xi_m}(\gamma_m)\right)\left(s_{\xi_{M+1}}(\gamma_{M+1})\,\varphi(\gamma_{M+1}) + r_{\xi_{M+1}}(\gamma_{M+1})\right)\\
&\quad + \sum_{n=1}^{M}\left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_n}\left(\sum_{k=1}^{M+1-n}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_{M+1}}{p^{M+1-n}}\right)\\
&= \left(\prod_{m=1}^{M} s_{\xi_m}(\gamma_m)\right) s_{\xi_{M+1}}(\gamma_{M+1})\,\varphi(x_{M+1}) + \left(\prod_{m=1}^{M} s_{\xi_m}(\gamma_m)\right) r_{\xi_{M+1}}(\gamma_{M+1})\\
&\quad + \sum_{n=1}^{M}\left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_n}\left(\sum_{k=1}^{M+1-n}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_{M+1}}{p^{M+1-n}}\right)\\
&= \left(\prod_{m=1}^{M} s_{\xi_m}(\gamma_m)\right) s_{\xi_{M+1}}(\gamma_{M+1})\,\varphi(\gamma_{M+1})\\
&\quad + \left(\prod_{m=1}^{M+1-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_{M+1}}\left(\sum_{k=1}^{M+1-M-1}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_{M+1}}{p^{M+1-M-1}}\right)\\
&\quad + \sum_{n=1}^{M}\left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_n}\left(\sum_{k=1}^{M+1-n}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_{M+1}}{p^{M+1-n}}\right)\\
&= \left(\prod_{m=1}^{M+1} s_{\xi_m}(\gamma_m)\right)\varphi(\gamma_{M+1})\\
&\quad + \sum_{n=1}^{M+1}\left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m)\right) r_{\xi_n}\left(\sum_{k=1}^{M+1-n}\frac{\xi_{k+n}}{p^k} + \frac{\gamma_{M+1}}{p^{M+1-n}}\right),
\end{aligned}$$

finishing the induction step. Thus we conclude that (3.21) is satisfied for all $M \in \mathbb{N}$.

We now show that the series obtained by letting $n \to \infty$ is convergent. Set

$$R = \sup_{\substack{x\in[0,1]\\ k=0,1,\dots p-1}} r_k(x)$$

and

$$S = \sup_{\substack{x \in [0,1] \\ k=0,1,\dots p-1}} s_k(x).$$

Since s_k and r_k are continuous in $[0,1]$, both R and S are attained as maxima and are therefore finite. Our hypothesis on the s_k further ensures that $S < 1$. Therefore

$$\left| \sum_{n=1}^{\infty} \left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m) \right) r_{\xi_n}(\gamma_n) \right| \leq \sum_{n=1}^{\infty} S^{n-1} R = \frac{R}{1-S} < +\infty$$

and the series on the right hand-side of (3.20) is absolutely convergent.

We therefore have

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n} \right) = \sum_{n=1}^{\infty} \left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m) \right) r_{\xi_n}(\gamma_n),$$

which is exactly representation (3.20). We conclude that the solution of system (3.18) is unique.

Existence of a solution is shown by verifying that $\varphi(x)$ is well-defined (that is, its definition is not ambiguous). If x is irrational, it is clear that the function is well-defined. If x is rational it has two base p representations. The case of non-uniqueness of base p representation occurs when $\xi_k = 0$ for all $k > m$, and $\xi_m \neq 0$. In this case, given a representation

$$x = \sum_{n=1}^{m} \frac{\xi_n}{p^n},$$

the second representation is given by

$$x = \sum_{n=1}^{m-1} \frac{\xi_n}{p^n} + \frac{\xi_m - 1}{p^m} + \sum_{n=m+1}^{\infty} \frac{p-1}{p^n}.$$

In fact,

$$\frac{\xi_m}{p^m} = \frac{\xi_m - 1}{p^m} + \sum_{n=m+1}^{\infty} \frac{p-1}{p^n}.$$

The function φ is well-defined if both representations lead to the same result in the right hand-side of (3.19). By definition $\xi_n = 0$, for $n > m$. Computing the function in both cases we obtain

$$\varphi\left(\sum_{n=1}^{M} \frac{\xi_n}{p^n} \right) = \sum_{n=1}^{\infty} \left(\prod_{m=1}^{n-1} s_{\xi_m}(\gamma_m) \right) r_{\xi_n}\left(\sum_{k=1}^{M-n} \frac{\xi_{k+n}}{p^k} \right),$$

$$\begin{aligned}
&\varphi\left(\sum_{n=1}^{M-1}\frac{\xi_n}{p^n}+\frac{\xi_M-1}{p^M}+\sum_{n=M+1}^{\infty}\frac{p-1}{p^n}\right)\\
&=\sum_{n=1}^{\infty}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\gamma_m\right)\right)r_{\xi_n}\left(\sum_{k=1}^{M-n-1}\frac{\xi_{k+n}}{p^k}+\frac{\xi_M-1}{p^{M-n}}+\sum_{k=M-n+1}^{\infty}\frac{p-1}{p^k}\right)\\
&=\sum_{n=1}^{\infty}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\gamma_m\right)\right)r_{\xi_n}\left(\sum_{k=1}^{M-n-1}\frac{\xi_{k+n}}{p^k}+\frac{\xi_M-1}{p^{M-n}}+\frac{1}{p^{M-n}}\right)\\
&=\sum_{n=1}^{\infty}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\gamma_m\right)\right)r_{\xi_n}\left(\sum_{k=1}^{M-n}\frac{\xi_{k+n}}{p^k}\right),
\end{aligned}$$

showing that both representations coincide.

Finally, we conclude that φ is continuous because the right hand-side of (3.19) is a uniformly convergent series of continuous functions. □

Remark 42. We observe that the explicit formula (3.19) in Theorem 41 may also be derived from Lemma 44 by setting $a_{ij} = 1/p$ and letting $n \to \infty$ in (3.25)-(3.27). This Lemma is stated later in the text, after the introduction of the FIF - fractal interpolation functions context.

Condition (3.17) corresponds to the compatibility conditions of the system (3.18). This indicates that we may reduce system (3.18) to another where $f_i(X) \cap f_j(X) = \emptyset$. A similar result may then be formulated without compatibility conditions in the list of hypothesis. The interval $[0,1]$ is replaced by $[0,1)$, the compatibility conditions become vacuously satisfied and continuity is not ensured. Similarly to section 2.3.5, in the case of a double representation, the explicit constructive formula applies only for the finite representation.

Theorem 43. *[85] Fix $p \in \{2,3,4,\ldots\}$, let $r_k : [0,1) \to \mathbb{R}$, and $s_k : [0,1) \to [0,1)$ be continuous functions such that $|s_k(x)| < 1$, $\forall x \in [0,1)$. Then there exists exactly one bounded $\varphi : [0,1) \to \mathbb{R}$ which satisfies the system*

$$\varphi\left(\frac{x+k}{p}\right) = s_k(x)\,\varphi(x) + r_k(x)\,,\ x \in [0,1)\,,\ 0 \le k \le p-1.$$

For each $x \in [0,1)$ the function φ is given in terms of the base p expansion

$$x = \sum_{n=1}^{\infty}\frac{\xi_n}{p^n}$$

by

$$\varphi(x) = \sum_{n=1}^{\infty}\left(\prod_{m=1}^{n-1}s_{\xi_m}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+m}}{p^k}\right)\right)r_{\xi_n}\left(\sum_{k=1}^{\infty}\frac{\xi_{k+n}}{p^k}\right).$$

Note that this result does not suppose compatibility conditions as hypotheses, since they are vacuously satisfied. On the other hand, continuity of the solution is not ensured. In order to ensure continuity of the solution φ in Theorem 43 we must add the continuity condition mentioned in Proposition 33:

$$\lim_{x\to1^-} s_k(x) \frac{\lim\limits_{x\to1^-} r_{p-1}(x)}{1-\lim\limits_{x\to1^-} s_{p-1}(x)} + \lim_{x\to1^-} r_k(x) = s_{k+1}(0)\frac{r_0(0)}{1-s_0(0)} + r_{k+1}(0).$$

Theorem 41 was stated as a way to define constructively fractal interpolation functions as solutions of a model formulated by Barnsley [3] and generalized by Wang and Yu [97]. This model was constructed in such a way that the interpolation functions are required to be continuous (for more details see chapter 5). One way to generalize this procedure is by allowing the interpolation functions to be discontinuous. The solutions could be, for instance, "dust clouds"; in chapter 4 examples of this kind are constructed where the F_j do not depend explicitly on x.

The definition of fractal interpolation functions (FIF) given in chapter 5 is performed for a set of data

$$D = \left\{(x_i, y_i) \in I \times J \subset \mathbb{R}^2 : i = 0, 1, 2, \ldots, N\right\}, \tag{3.22}$$

where

$$x_0 = 0 < x_1 = \frac{1}{p} < x_2 = \frac{2}{p} < \cdots < x_j = \frac{j}{p} < \cdots < x_p = 1$$

is a uniform partition of the interval $I = [x_0, x_N]$. However, in applications requiring experimental data acquisition, it may not be possible to ensure uniformity of the sampling interval. It therefore becomes of practical importance to construct analogous results for the non-uniform case $x_j \neq j/p$. It is still possible to obtain an explicit solution, but the construction is more elaborate. In fact, this new explicit formula is based on a generalization of base p expansions. We define the p-symbol Q-representation (Prats'ovytyi and Kalashnikov [71]):

$$\triangle^Q_\nu = \triangle^Q_{\nu_1\nu_2\cdots\nu_n\cdots} := \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n-1} a_{\nu_k}\right) r_{\nu_n},$$

where $a_0, a_1, \ldots, a_{p-1} \in (0,1)$ are such that $\sum_{j=0}^{p-1} a_j = 1$, $r_0 = 0$, $r_j = \sum_{k=0}^{j} a_{k-1}$, for $j \in \{1, \ldots, p\}$, $\nu = (\nu_n)_{n\in\mathbb{N}}$ is a sequence in the set $\{0, 1, \ldots, p-1\}$, and $Q = \{a_0, a_1, \ldots, a_{p-1}\}$ (observe that this notation for Q-representations differs slightly from the one in [71], with adaptations for coherence with the rest of this section). In section 7.2.3 the role of this representation in the context of conjugacy equations and its connection with the classical base p representation are stressed.

The construction of the FIF with D defined in (3.22) is performed next. Let L_j be the affine map satisfying

$$L_j(x_0) = x_{j-1},\ L_j(x_N) = x_j,\ j = 1, 2, \ldots, N.$$

Let $-1 < \alpha_j < 1$ and $F_j : I \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that, for $j = 1, 2, \ldots, N$,

$$|F_j(x, \xi_1) - F_j(x, \xi_2)| \leq |\alpha_j| \, |\xi_1 - \xi_2| \, , \; x \in I, \; \xi_1, \xi_2 \in \mathbb{R},$$

and

$$F_j(x_0, y_0) = y_{j-1}, \; F_j(x_N, y_N) = y_j.$$

As shown in [5], the FIF associated with $\{(L_j(x), F_j(x, y))\}_{j=1}^N$ is the unique function $f : I \to \mathbb{R}$ satisfying

$$\varphi(L_j(x)) = F_j(x, \varphi(x)), \; j = 1, 2, \ldots N. \tag{3.23}$$

The Barnsley classical setting [3] is the special case where $F_j(x, y) = \alpha_j y + q_j(x)$. The Wang-Yu [97] model with variable parameters corresponds to

$$F_j(x, y) = \alpha_j(x) y + q_j(x), \; i \in \{1, 2, \ldots, N\}. \tag{3.24}$$

Theorem 45 provides the explicit fractal interpolation function in terms of the $N-1$-symbol Q-representation of numbers for the Wang-Yu model. This result generalizes Theorem 41, dealing with the comparatively simpler case of all the L_j having slope $1/p$, corresponding to the classical base p expansion of the argument in (3.23).

Wang-Yu [97] provided notations for compositions of the L_j: for any $x \in I = [0, 1]$, let $L_{i_1 i_2 \ldots i_n}(x) = L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_n}(x)$ and $L_{i_1 i_2 \ldots i_n}(I) = L_{i_1} \circ L_{i_2} \circ \cdots \circ L_{i_n}(I)$, where $i_j \in \{1, 2, \ldots, N\}$, $j = 1, 2, \ldots n$. Define a shift operator σ by $\sigma(i_1 i_2 \ldots i_n) = (i_2 i_3 \ldots i_n)$. Let σ^k denote the k-fold composition of σ with itself such that $L_{\sigma^k(i_1 i_2 \ldots i_n)}(x) = L_{i_{k+1} \ldots i_n}(x)$ for $1 \leq k \leq n-1$, while $L_{\sigma^n(i_1 i_2 \ldots i_n)}(x) = x$. Using successive iteration and induction, they were able to prove the following Lemma.

Lemma 44. *[97, Lemma 3.1] Let $\varphi(x)$ the FIF defined by (3.23). For $x \in I$, $\forall i_j \in \{1, 2, \ldots, n\}$, $j = 1, 2, \ldots n$,*

$$L_{i_1 i_2 \ldots i_n}(x) = \left(\prod_{j=1}^{n} a_{i_j} \right) x + \sum_{k=1}^{n} \left(\prod_{j=1}^{k-1} a_{i_j} \right) x_{i_k - 1}, \tag{3.25}$$

$$\begin{aligned} \varphi(L_{i_1 i_2 \ldots i_n}(x)) = & \left[\prod_{k=1}^{n} \alpha_{i_k} \left(L_{\sigma^k(i_1 i_2 \ldots i_n)}(x) \right) \right] \varphi(x) \\ & + \sum_{r=1}^{n} \left[\prod_{k=1}^{r-1} \alpha_{i_k} \left(L_{\sigma^k(i_1 i_2 \ldots i_n)}(x) \right) \right] q_{i_r} \left(L_{\sigma^r(i_1 i_2 \ldots i_n)}(x) \right), \end{aligned} \tag{3.26}$$

where

$$L_{\sigma^k(i_1 i_2 \ldots i_n)}(x) = \left(\prod_{j=1}^{n-k} a_{i_{k+j}} \right) x + \sum_{l=1}^{n-k} \left(\prod_{j=1}^{l-1} a_{i_{k+j}} \right) x_{i_{k+l} - 1}. \tag{3.27}$$

Theorem 45. *[85] Let φ be the FIF solution of (3.23), with $L_i(x) = a_i x + x_{i-1}$, $1 \leq i \leq N$ and the F_j given by (3.24). Then φ is given in terms of the Q-expansion*

$$x = \sum_{k=1}^{\infty} \left(\prod_{j=1}^{k-1} a_{i_j} \right) x_{i_k - 1}$$

by

$$\varphi(x) = \sum_{r=1}^{\infty} \left[\prod_{k=1}^{r-1} \alpha_{i_k} \left(\sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} a_{i_{k+j}} \right) x_{i_{k+l}-1} \right) \right] q_{i_r} \left(\sum_{l=1}^{\infty} \left(\prod_{j=1}^{l-1} a_{i_{r+j}} \right) x_{i_{r+l}-1} \right).$$

Proof. This follows immediately, in the present context, from an application of Lemma 44 and letting $n \to \infty$ in (3.25)-(3.27). □

A similar result may be obtained for systems as in Theorem 41, adapting the model.

3.3.2 General non-linear systems

Fractal Interpolation Functions are one possible application of the theory of systems of iterative functional equations of the form (3.18). Constructive solutions for systems where each F_k is affine are known from the literature. The problem of variable parameters in this affine case was treated in section 3.3.1. The next step is that of finding explicit solutions for more general problems, where each F_k is non-linear. A suitable contractiveness condition is required in order to ensure the existence of solution, as stated in Theorem 29.

In order to obtain a constructive solution for general non-linear contractive systems we need to define a family of functions $\{F_{k,x}\}$. Since in (1.1) each F_k is a function of two variables x and $\varphi(x)$, we define, for each k and each x, the function $F_{k,x} : Y \to Y$ by $F_{k,x}(y) := F_k(x, y)$.

For each sequence $\{\xi_j\}_{j \in \mathbb{N}}$, define

$$\varrho_j = \sum_{n=1}^{\infty} \frac{\xi_{n+j}}{p^n}. \tag{3.28}$$

Theorem 46. *[85] Let Y be a complete metric space. Fix $p \in \{2, 3, 4, \ldots\}$ and let $F_k : [0,1] \times Y \to Y$ be a family of C-contractions. Assume that the compatibility conditions for the system*

$$\varphi\left(\frac{x+k}{p}\right) = F_k(x, \varphi(x)), \; x \in [0,1], \; 0 \leq k \leq p-1 \tag{3.29}$$

are satisfied. Then there exists exactly one bounded $\varphi : [0,1] \to Y$ which satisfies system (3.29). The function φ is given in terms of the base p expansion

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n}$$

by

$$\varphi(x) = \lim_{\nu\to\infty} F_{\xi_1,\varrho_1} \circ F_{\xi_2,\varrho_2} \circ \cdots \circ F_{\xi_\nu,\varrho_\nu}(\eta),\ \eta \in Y, \tag{3.30}$$

where the ϱ_j are given by (3.28).

Proof. Existence and uniqueness follow from Theorem 29.

The case $x = 0$ is singular, since $\varphi(0)$ is a fixed point of F_k in (3.29) with $k = 0$, so it is dealt with separately. For $x = 0$,

$$\varphi(0) = F_0(0, \varphi(0)) \equiv F_{0,0}(\varphi(0)),$$

i.e., $\varphi(0)$ is the unique fixed point of $F_{0,0}$ as stated. On the other hand

$$0 = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n},$$

i.e., the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ which represents zero in base p is such that $\xi_n = 0$, $\forall n \in \mathbb{N}$. Thus, for this sequence,

$$\varrho_j = \sum_{k=1}^{\infty} \frac{\xi_{k+j}}{p^k} = 0,\ \forall j \in \mathbb{N}.$$

Since $\varphi(0)$ is the unique fixed point of $F_{0,0}$, $\forall \xi \in Y$,

$$\begin{aligned} &\lim_{\nu\to\infty} F_{\xi_1,\varrho_1} \circ F_{\xi_2,\varrho_2} \circ F_{\xi_3,\varrho_3} \circ \cdots \circ F_{\xi_\nu,\varrho_\nu}(\eta) \\ &= \lim_{\nu\to\infty} F_{0,0} \circ F_{0,0} \circ F_{0,0} \circ \cdots \circ F_{0,0}(\xi) = \varphi(0), \end{aligned}$$

and formula (3.30) holds for $x = 0$.

The proof for the general case proceeds by induction. For

$$x = \frac{\xi_1}{p},$$

the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ is such that $\xi_k = 0$, $\forall k > 1$ and $\varrho_j = 0$, $\forall j \in \mathbb{N}$. Then

$$\begin{aligned} \varphi\left(\frac{\xi_1}{p}\right) &= F_{\xi_1}(0, \varphi(0)) \equiv F_{\xi_1,\varrho_1}(\varphi(0)) \\ &= F_{\xi_1,\varrho_1}\left(\lim_{\nu\to\infty} F_{\xi_2,\varrho_2} \circ F_{\xi_3,\varrho_3} \circ \cdots \circ F_{\xi_\nu,\varrho_\nu}(\eta)\right) \\ &= \lim_{\nu\to\infty} F_{\xi_1,\varrho_1} \circ F_{\xi_2,\varrho_2} \circ F_{\xi_3,\varrho_3} \circ \cdots \circ F_{\xi_\nu,\varrho_\nu}(\eta), \end{aligned}$$

where $\eta \in Y$, and formula (3.30) holds for $x = \xi_1/p$.

For the induction step, let $m \in \mathbb{N}$. Suppose that (3.30) is valid for $x_m = \sum_{n=1}^{m} \xi_n/p^n$. Consider

$$x_{m+1} = \sum_{n=1}^{m+1} \frac{\xi_n}{p^n}.$$

The sequence $\{\xi_k\}_{k\in\mathbb{N}}$ corresponding to x_{m+1} is such that $\xi_k = 0$, $\forall k > m+1$ and

$$\varrho_1 = \sum_{n=1}^{\infty} \frac{\xi_{n+1}}{p^n} = \sum_{n=2}^{\infty} \frac{\xi_n}{p^{n-1}},$$

$\varrho_j = 0$, $\forall j > m+1$. Then

$$\begin{aligned}
\varphi\left(\sum_{n=1}^{m+1} \frac{\xi_n}{p^n}\right) &= \varphi\left(\frac{\xi_1 + \sum\limits_{n=2}^{m+1} \frac{\xi_n}{p^{n-1}}}{p}\right) = \varphi\left(\frac{\xi_1 + \varrho_1}{p}\right) \\
&= F_{\xi_1}\left(\varrho_1, \varphi\left(\varrho_1\right)\right) \equiv F_{\xi_1,\varrho_1}\left(\varphi\left(\varrho_1\right)\right) \\
&= F_{\xi_1,\varrho_1}\left(\lim_{\nu\to\infty} F_{\xi_2,\varrho_2} \circ F_{\xi_3,\varrho_3} \circ \cdots \circ F_{\xi_\nu,\varrho_\nu}\left(\eta\right)\right) \\
&= \lim_{\nu\to\infty} F_{\xi_1,\varrho_1} \circ F_{\xi_2,\varrho_2} \circ F_{\xi_3,\varrho_3} \circ \cdots \circ F_{\xi_\nu,\varrho_\nu}\left(\eta\right),
\end{aligned}$$

where $\eta \in Y$. Therefore formula (3.30) holds for $x_{m+1} = \sum_{n=1}^{m+1} \xi_n / p^n$, concluding the proof of the induction step. By induction and through a limiting procedure we obtain (3.30) for all $x \in [0,1]$. □

Remark 47. As is clear from the induction step, in the case where x has a double Q-representation (necessarily one of them finite and the other infinite), this construction yields the value of φ corresponding to the finite representation, as stated in Remark 13.

Theorem 46 provides a constructive solution in terms of a base p representation. This is possible because the f_k are of the form

$$f_k\left(x\right) = \frac{x+k}{p}.$$

A more general formula for a constructive solution is obtained by generalizing the argument functions f_k. A special case of this has already occurred, in the generalization of the affine FIF in Theorem 45. In the general case, the solution is given in terms of a representation of objects by an infinite composition of the f_k functions.

Definition 48. [85] For each family of functions $f_k : X_k \to X$ we define K_f as the set of $x \in X$ such that $x = \lim\limits_{\nu\to\infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(\xi\right)$, for some sequence $\{\xi_j\}_{j\in\mathbb{N}}$ in $\{0, 1, \ldots, p-1\}$ and $\xi \in \cap_{i=0}^{p-1} X_i$.

In the proof of Theorem 51 we will use, for convenience, $x = \lim\limits_{\nu\to\infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(x_k\right)$, for some sequence $\{\xi_j\}_{j\in\mathbb{N}}$ in $\{0, 1, \ldots, p-1\}$ and, for each k, x_k is the fixed point of f_k, such that $x_k \in \cap_{i=0}^{p-1} X_i$.

Definition 49. For each sequence $\{\xi_j\}_{j\in\mathbb{N}}$, $k \in \{0, 1, \ldots, p-1\}$ and $\zeta \in X$ we define

$$\varrho_{j,k}\left(\zeta\right) = \lim_{\nu\to\infty} f_{\xi_{1+j}} \circ f_{\xi_{2+j}} \circ \cdots \circ f_{\xi_{\nu+j}}\left(\zeta\right).$$

Theorem 50. *[85] Let X be a complete bounded metric space, Y be a complete metric space, p be an integer ≥ 2, $f_k : X_k \to X$ be a family of injective and contractive functions such that $\cup_{k=0}^{p-1} f_k(X_k) = X$ and $F_k : X \times Y \to Y$ be a family of C-contractions. Assume that the compatibility conditions for the system*

$$\varphi(f_k(x)) = F_k(x, \varphi(x)), \; x \in X, \; 0 \leq k \leq p-1 \tag{3.31}$$

are satisfied. Then there exists exactly one bounded $\varphi : X \to Y$ which satisfies the system (3.31). For the sub-domain K_f, the function φ is given by

$$\varphi(x) = \lim_{\nu\to\infty} F_{\xi_1,\varrho_{1,k}(\zeta)} \circ F_{\xi_2,\varrho_{2,k}(\zeta)} \circ \cdots \circ F_{\xi_\nu,\varrho_{\nu,k}(\zeta)}(\eta), \tag{3.32}$$

where $x = \lim_{\nu\to\infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}(\xi)$, $\zeta \in X$, $\eta \in Y$ and $\xi \in \cap_{i=0}^{p-1} X_i$.

Proof. Existence and uniqueness follow from Theorem 29.

Let $x \in K_f$. By definition 48 there exists a representative sequence $\{\xi_j\}_{j\in\mathbb{N}}$ in $\{0, 1, \ldots, p-1\}$ and a $x_k \in \cap_{j=0}^{p-1} X_j$ such that $x = \lim_{\nu\to\infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}(x_k)$. Associated to this sequence also exist the respective $\varrho_{j,k}(\zeta)$ sequences of functions.

The case where x is a fixed point of some f_k is singular for the same reasons as in the proof of Theorem 46, so must be dealt with separately. For each $k \in \{0, 1, \ldots, p-1\}$, let x_k be the fixed point of f_k. Then

$$\varphi(x_k) = \varphi(f_k(x_k)) = F_k(x_k, \varphi(x_k)) \equiv F_{k,x_k}(\varphi(x_k)),$$

i.e., $\varphi(x_k)$ is the unique fixed point of F_{k,x_k}. On the other hand

$$x_k = \lim_{\nu\to\infty} \underbrace{f_k \circ \cdots \circ f_k}_{\nu}(\zeta), \forall \zeta \in X,$$

i.e., the sequence $\{\xi_j\}_{j\in\mathbb{N}}$ which represents x_k is such that $\xi_n \equiv k$, $\forall n \in \mathbb{N}$. Then for that sequence

$$\varrho_{j,k}(\zeta) = \lim_{\nu\to\infty} \underbrace{f_k \circ \cdots \circ f_k}_{\nu}(\zeta) = x_k, \; \forall j \in \mathbb{N}, \; \forall \zeta \in X.$$

Since $\varphi(x_k)$ is the unique fixed point of F_{k,x_k}, $\forall \zeta \in X$, $\forall \eta \in Y$

$$\begin{aligned}&\lim_{\nu\to\infty} F_{\xi_1,\varrho_{1,k}(\zeta)} \circ F_{\xi_2,\varrho_{2,k}(\zeta)} \circ F_{\xi_3,\varrho_{3,k}(\zeta)} \circ \cdots \circ F_{\xi_\nu,\varrho_{\nu,k}(\zeta)}(\eta) \\ &\quad = \lim_{\nu\to\infty} \underbrace{F_{k,x_k} \circ \cdots \circ F_{k,x_k}}_{\nu}(\xi) = \varphi(x_k),\end{aligned}$$

and formula (3.32) is valid for x_k.

The proof for the general case proceeds by induction. For $x = f_{\xi_1}(x_k)$, the sequence $\{\xi_j\}_{j\in\mathbb{N}}$ is such that $\xi_j = k$, $\forall j > 1$ and

$$\varrho_{j,k}(\zeta) = \lim_{\nu\to\infty} \underbrace{f_k \circ \cdots \circ f_k}_{\nu}(\zeta) = x_k, \; \forall j \in \mathbb{N}, \; \forall \zeta \in X, \; \forall j > 1.$$

Then

$$\begin{aligned}\varphi\left(f_{\xi_1}\left(x_k\right)\right) &= F_{\xi_1}\left(x_k, \varphi\left(x_k\right)\right) \equiv F_{\xi_1, x_k}\left(\varphi\left(x_k\right)\right) \\ &= F_{\xi_1, x_k}\left(\lim_{\nu\to\infty} F_{\xi_2,\varrho_{2,k}(\zeta)} \circ F_{\xi_3,\varrho_{3,k}(\zeta)} \circ F_{\xi_4,\varrho_{4,k}(\zeta)} \circ \cdots \circ F_{\xi_\nu,\varrho_{\nu,k}(\zeta)}\left(\eta\right)\right) \\ &= \lim_{\nu\to\infty} F_{\xi_1,\varrho_{1,k}(\zeta)} \circ F_{\xi_2,\varrho_{2,k}(\zeta)} \circ F_{\xi_3,\varrho_{3,k}(\zeta)} \circ \cdots \circ F_{\xi_\nu,\varrho_{\nu,k}(\zeta)}\left(\eta\right),\end{aligned}$$

where $\zeta \in X$, $\eta \in Y$, and formula (3.32) is valid for $x = f_{\xi_1}\left(x_k\right)$.

For the induction step, let $\nu \in \mathbb{N}$. Suppose (3.32) is valid for $x = f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_{\nu-1}}\left(x_k\right)$.

For $x = f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(x_k\right)$ the sequence $\{\xi_k\}_{k\in\mathbb{N}}$ is such that $\xi_j = x_k$, $\forall j > \nu$, $\varrho_{1,k}\left(\zeta\right) = f_{k_2} \circ \cdots \circ f_{k_\nu}\left(\zeta\right)$. Then

$$\begin{aligned}&\varphi\left(f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(x_k\right)\right) = \varphi\left(f_{\xi_1}\left(f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(\zeta\right)\right)\right) = \varphi\left(f_{\xi_1}\left(\varrho_{1,k}\left(\zeta\right)\right)\right) \\ &\quad= F_{\xi_1}\left(\varrho_{1,k}\left(\zeta\right), \varphi\left(\varrho_{1,k}\left(\zeta\right)\right)\right) \equiv F_{\xi_1,\varrho_{1,k}(\zeta)}\left(\varphi\left(f_{k_2} \circ \cdots \circ f_{k_\nu}\left(\zeta\right)\right)\right) \\ &\quad= F_{\xi_1,\varrho_{1,k}(\zeta)}\left(\lim_{\nu\to\infty} F_{\xi_2,\varrho_{2,k}(\zeta)} \circ F_{\xi_3,\varrho_{3,k}(\zeta)} \circ F_{\xi_4,\varrho_{4,k}(\zeta)} \circ \cdots \circ F_{\xi_\nu,\varrho_{\nu,k}(\zeta)}\left(\eta\right)\right) \\ &\quad= \lim_{\nu\to\infty} F_{\xi_1,\varrho_{1,k}(\zeta)} \circ F_{\xi_2,\varrho_{2,k}(\zeta)} \circ F_{\xi_3,\varrho_{3,k}(\zeta)} \circ \cdots \circ F_{\xi_\nu,\varrho_{\nu,k}(\zeta)}\left(\eta\right),\end{aligned}$$

where $\zeta \in X$, $\eta \in Y$, and formula (3.32) is valid for $x = f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(x_k\right)$.

By induction and through a limiting procedure we obtain (3.32).

Finally, formula (3.32) is well defined: since existence and uniqueness are ensured, the right hand-side can neither depend on the representative sequence $\{\xi_j\}_{j\in\mathbb{N}}$, nor on the fixed point x_k. □

Remark 51. In the case where there is an f_k whose fixed point is zero, it may be useful to choose the fixed point $x_k = 0$ in (3.32). This is the case of Theorem 46 where, in particular, $x_k = 0$ for $k = 0$.

Theorem 51 refers to the general non-linear case when the system depends explicitly on the independent variable x. The version of this result for systems of conjugacy equations is much simpler, as Corollary 53 shows.

Corollary 52. *[85] Let X be a complete bounded metric space, Y be a complete space, p be an integer ≥ 2, $f_k : X_k \to X$ be a family of injective and contractive functions such that $\cup_{k=0}^{p-1} f_k\left(X_k\right) = X$ and $F_k : Y \to Y$ be a family of C-contractions.*

Assume that the compatibility conditions for the system

$$\varphi\left(f_k\left(x\right)\right) = F_k\left(\varphi\left(x\right)\right),\ x \in X,\ 0 \leq k \leq p-1 \tag{3.33}$$

are satisfied. Then there exists exactly one bounded $\varphi : X \to Y$ which satisfies the system (3.33). For the sub-domain K_f, the function φ is given by

$$\varphi\left(x\right) = \lim_{\nu\to\infty} F_{\xi_1} \circ F_{\xi_2} \circ \cdots \circ F_{\xi_\nu}\left(\eta\right),$$

where $x = \lim_{\nu\to\infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}\left(\xi\right)$, $\eta \in Y$ and $\xi \in \cap_{i=0}^{p-1} X_i$.

3.3.3 Particular cases

The results in section 3.3.2 cover particular cases of systems (1.1) whose solutions are functions already known from the literature as well as new non-linear cases, as described below. The conjugacy system examples are in chapter 2.

In some of the examples below it will be convenient to use the partial sums as defined in Definition 10.

3.3.3.1 Affine systems of p equations with one variable parameter

(i) Girgensohn [36] (1993): f_k are of the form (2.7), $r_k : [0,1] \to \mathbb{R}$ are continuous, $|s_k| < 1$ and

$$F_k(x,y) = s_k \varphi(x) + r_k(x),$$

whose solution is given by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{m=1}^{n-1} s_{\xi_m}\right) r_{\xi_n}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{p^k}\right).$$

(ii) inspired in van der Waerden [96] functions (1930):

$$V_{a,p}(x) = \sum_{n=0}^{\infty} a^n \left(\operatorname{dist}\left(p^n \cdot \frac{p}{2} x, \mathbb{Z}\right) - \frac{1}{4}\right).$$

A special case of (i), where

$$F_k(x,y) = (-1)^{pk} ay + (-1)^k \frac{2x-1}{4},$$

$|a| < 1$, whose solution is given by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} a^n (-1)^{ps_{n-1}+\xi_n} \left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{2p^k} - \frac{1}{4}\right).$$

(iii) inspired in Weierstrass [15] functions (1875):

$$W_{a,p,\theta}(x) = \sum_{n=0}^{\infty} a^n \sin(p^n \cdot p\pi x + \theta).$$

A special case of (i), where

$$F_k(x,y) = (-1)^{pk} ay + (-1)^k \sin(\pi x + \theta),\ 0 \le k \le p-1,$$

$a, \theta \in \mathbb{R}$, $|a| < 1$, whose solution is given by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} a^n (-1)^{ps_{n-1}+\xi_n} \sin\left(\pi \sum_{k=1}^{\infty} \frac{\xi_{k+n}}{p^k} + \theta\right).$$

More generally,
(iv) inspired in Knopp [45] (1918) and Behrend [8] functions (1949):

$$\varphi(x) = \sum_{n=1}^{\infty} a^n g(p^n x),$$

where g is a continuous 1-periodic function. A particular case of (i), where

$$F_k(x, y) = ay + g\left(\frac{x+k}{p}\right),\ 0 \le k \le p-1,$$

whose solution is given by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} a^n g\left(p^{n-1} \sum_{k=0}^{\infty} \frac{\xi_{k+n}}{p^{k+n}}\right).$$

3.3.3.2 Affine systems of p equations with both variable parameters

Serpa and Buescu [84] (2015): f_k are of the form (2.7), $r_k : [0,1] \to \mathbb{R}$, and $s_k : [0,1] \to [0,1]$ are continuous functions such that $|s_k(x)| < 1$, $\forall x \in [0,1]$,

$$F_k(x, y) = s_k(x)\,\varphi(x) + r_k(x),$$

whose solution is given by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{m=1}^{n-1} s_{\xi_m}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+m}}{p^k}\right)\right) r_{\xi_n}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{p^k}\right).$$

3.3.3.3 Higher-dimensional systems

Higher-dimensional fractal interpolation functions [85]: The theory of fractal interpolation functions in $\mathbb{R}$ [4,6,97] may be extended in the obvious way to the interpolation functions $\varphi : X \subset \mathbb{R}^n \to \mathbb{R}^m$, with families of m-dimensional affine contractive maps $\{F_k\}$ and n-dimensional affine contractive maps $\{f_k\}$. Theorem 51 applies immediately with $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$, stating existence and uniqueness and providing a constructive formula for the solution.

3.3.3.4 Systems with non-standard argument functions

Definition 53. We call f_k the ***argument functions*** of the system (1.1).

(i) fractal functions by Barnsley [3] (1986) and Wang-Yu [97] (2013): case (i) of section 3.3.3.1 and the case of section 3.3.3.2 generalized with all f_k affine, but not necessarily of the standard form (2.7) ($f_k \equiv L_k$, see Theorem 45).

(ii) “Inverted” generalized tent map [85]: the argument functions are of the form

$$f_k(x) = \begin{cases} \frac{x+k}{p}, & \text{if } k \text{ is even} \\ \frac{-x+k+1}{p}, & \text{if } k \text{ is odd.} \end{cases} \tag{3.34}$$

In systems with these argument functions, formula (3.32) applies with

$$x = \lim_{\nu \to \infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}(x_k) = \sum_{n=1}^{\infty} \frac{(-1)^{s_{n-1}}(\xi_n + \delta_{\xi_n})}{p^n}, \tag{3.35}$$

where $\delta_n = 0$ if n is even, $\delta_n = 1$ if n is odd.

Proof. First we prove by induction in ν that for each $x \in X$,

$$f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}(x) = \frac{(-1)^{s_\nu}}{p^\nu} x + \sum_{n=1}^{\nu} \frac{(-1)^{s_{n-1}}(\xi_n + \delta_{\xi_n})}{p^n}. \tag{3.36}$$

By definition $f_{\xi_1}(x) = \left((-1)^{\xi_1} x + \xi_1 + \delta_{\xi_1}\right)/p$. Suppose (3.36) is valid for $\nu \in \mathbb{N}$. Then

$$\begin{aligned}
&f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_{\nu+1}}(x) \\
&\quad = f_{\xi_1}\left(\frac{(-1)^{s_{\nu+1}-\xi_1}}{p^{\nu+1}} x + \sum_{n=1}^{\nu} \frac{(-1)^{s_n-\xi_1}\left(\xi_{n+1} + \delta_{\xi_{n+1}}\right)}{p^n}\right) \\
&\quad = \frac{(-1)^{\xi_1}\left(\frac{(-1)^{s_{\nu+1}-\xi_1}}{p^{\nu+1}} x + \sum_{n=1}^{\nu} \frac{(-1)^{s_n-\xi_1}\left(\xi_{n+1}+\delta_{\xi_{n+1}}\right)}{p^n}\right) + \xi_1 + \delta_{\xi_1}}{p} \\
&\quad = \frac{(-1)^{s_{\nu+1}}}{p^{\nu+1}} x + \sum_{n=1}^{\nu} \frac{(-1)^{s_n}\left(\xi_{n+1} + \delta_{\xi_{n+1}}\right)}{p^{n+1}} + \frac{\xi_1 + \delta_{\xi_1}}{p} \\
&\quad = \frac{(-1)^{s_{\nu+1}}}{p^{\nu+1}} x + \sum_{n=1}^{\nu+1} \frac{(-1)^{s_{n-1}}(\xi_n + \delta_{\xi_n})}{p^n}.
\end{aligned}$$

Then (3.36) is satisfied for all $\nu \in \mathbb{N}$. By a limiting procedure we obtain (3.35). $\square$

(iii) Minkowski Question Mark function [61] (1911): $Y = [0, 1]$. The function $? \equiv \varphi$ is explicitly defined by Salem [76] by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{s_n-1}}.$$

This function conjugates the Farey map

$$F(x) = \begin{cases} \frac{x}{1-x}, & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1-x}{x}, & \text{if } \frac{1}{2} \le x \le 1, \end{cases}$$

with the tent map

$$f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le \frac{1}{2} \\ 2 - 2x, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

By the results further presented in chapter 4 on conjugation equations, since φ is an increasing homeomorphism, it is the solution of the system of equations

$$\begin{cases} \varphi\left(2x\right) = \frac{\varphi(x)}{1-\varphi(x)}, & \text{if } 0 \leq x \leq \frac{1}{2} \\ \varphi\left(2-2x\right) = \frac{1-\varphi(x)}{\varphi(x)}, & \text{if } \frac{1}{2} \leq x \leq 1, \end{cases}$$

whose contractive form is

$$\begin{cases} \varphi\left(\frac{x}{2}\right) = \frac{\varphi(x)}{1+\varphi(x)}, & \text{if } 0 \leq x \leq 1 \\ \varphi\left(\frac{2-x}{2}\right) = \frac{1}{1+\varphi(x)}, & \text{if } 0 \leq x \leq 1. \end{cases}$$

In this system, for each $k = 0, 1$, $F_k\left(x, y\right) \equiv F_k\left(y\right) = y^{-k+1}\left(1+y\right)^{-1}$ is a non-linear function with no explicit dependence on x and f_k is not on the standard form (2.7), but (3.34). Now Corollary 53 may be applied to obtain the solution.

Final Remarks. We observe that most of these results are valid for contractive operators on (suitable subsets of) metric spaces, and the corresponding results may be applied in other contexts where the conditions are satisfied. The examples supplied are not supposed to constitute a comprehensive list but only a starting point.

We also note that some of these results generalize to the case of systems with an infinite number of equations. More precisely, results on existence and uniqueness, continuity, compatibility and the constructions of Theorem 51 extend to this more general setting. This is not true, however, for the explicit constructive formulae for the solutions, e.g. (3.30) in Theorem 46 and its analogues in other results.

Chapter 4

Conjugacy equations

In this chapter we are in the context of single equations, a special type of equations as the reader will see. In spite of this fact, the work done in the previous chapter about systems will be needed for what is going to be done.

Let X and Y be non-empty sets. In this chapter we consider a functional equation in the form of a conjugacy

$$\varphi \circ f = F \circ \varphi, \tag{4.1}$$

where $\varphi : X \to Y$ is the unknown function, and $f : X \to X$, $F : Y \to Y$.

Note that here the focus is in a single equation, in contrast with the previous chapter where the study was about systems. Here there is no explicit dependence of x (neither in f nor in F), which is the reason why equation (4.1) is called a conjugacy.

The interest in equations such as (4.1) is as conjugacy equations, not necessarily topological. If X, Y are topological spaces and φ is a homeomorphism, then φ is a topological conjugacy, which is used to classify dynamics from the topological point of view.

Definition 54. Let X and Y be non-empty defined in topological spaces. The maps $f : X \to X$, $F : Y \to Y$ are said to be ***topological conjugate*** if there exists a homeomorphism $\varphi : X \to Y$ satisfying the functional equation (4.1). The homeomorphism φ is called a ***topological conjugacy***.

In the sequel, for each function f we denote by $\mathcal{M}_f$ the family of all functions F such that are topologically conjugate with f.

4.1 Methods to obtain solutions

The conjugacy equation (4.1) (where φ is the unknown function) may be approached from several view points and methods and is the subject of much research in the field of functional equations. The main results already obtained for this kind of equation are those for invertible functions f, F (see [47]), where

f is a scalar or a linear operator on the range of φ (see the Schröder equation in [47,48]). Other cases of interest arise when f, F are continuous functions with real domain and real range, strictly increasing and fixed-point free (see [49]), or f strictly decreasing continuous and F continuous (maybe non-monotonic) (see [89]). In a book about iterative functional equations, Kuczma et al. [48, Chapter 8] mention several methods of finding solutions as the linearization, the change of variables and, in particular, methods for specific type of equations:

(i) The Schröder equation,
(ii) The Böttcher equation,
(iii) The Abel equation,
(iv) The equation of permutable functions,
(v) Conjugate/commuting formal series and analytic functions.

In particular, they give examples of piecewise defined functions which are conjugated with each other by continuous functions, such as the trapezoid functions, $t_a : [0,1] \to [0,1]$ defined by

$$t_a(x) = \begin{cases} \frac{x}{a}, & x \in [0,a] \\ 1, & x \in [a, 1-a] \\ \frac{1-x}{a}, & x \in [1-a, 1], \end{cases}$$

where $a \in (0, 1/2)$; the hat functions $h_{u,v} : [0,1] \to [0,1]$ defined by

$$h_{u,v}(x) = \begin{cases} \frac{v}{u} x, & x \in [0,u] \\ \frac{v}{1-u}(1-x), & x \in [u,1], \end{cases}$$

where $u, v \in (0,1]$, $u \neq 1$.

A special conjugacy equation studied by Poincaré in the complex plane is the equation which conjugates $\phi(z) = \sum_{n \geq 0} \phi_n z^{n+1}$, $\phi_0 = q$ with the function $z \mapsto qz$. We may cite also Bajraktarević [2], who studied conditions for the existence of solutions of certain types of conjugacy equations. Uniqueness in the case where f and F are continuous functions in intervals of $\mathbb{R}$ is studied by Ciepliński and Zdun in [22]. Pelyukh and Sharkovskii devoted a book [68] to the method of invariants to study functional equations.

Usual references of one dimensional dynamics [14,57,58] treat the case where f is continuous. In the context of dynamical systems, Katok and Hasselblatt [44] described the following methods for finding conjugacies:

(i) The fundamental domain method, for proving structural stability of an interval map with attracting and repelling points at the ends, as well as describing the moduli for smooth conjugacy. This method works for some systems with highly dissipative behaviour, but cannot be used for systems with nontrivial recurrence behaviour.

(ii) The majorization method, for the local analytic linearization method, which determines a power formal series conjugating two maps. This method depends on the local character of the problem.

(iii) The coding method which is used to construct topological conjugacies between arbitrary expanding circle maps of the same degree, or between the full

2-shift and the invariant sets of the quadratic and the "horseshoe" map. It is very powerful in global and semi-local hyperbolic systems, and it is particularly effective in low-dimensional situations, where it often works without hyperbolic assumptions.

(iv) The contraction mapping method, which rewrites the equation as a problem of a fixed point of a certain contractive operator on an appropriate space of functions. This method requires hyperbolic behaviour and is restricted to finding topological conjugacies.

(v) Construction of an iteration process commonly referred to as "Newton method" and often called KAM method after Kolmogorov, Arnold and Moser. This method reduces the problem to an implicit function method, rather than the fixed point problem in the contraction mapping method.

Consider, in particular and in more detail, the contraction mapping method (iv), as described by Katok and Hasselblatt [44]: let $g_p : x \to px \pmod 1$. The goal is to solve the conjugacy equation $g_p \circ \varphi = \varphi \circ f$, reformulating the problem of finding such a map φ as a fixed-point problem for a contracting operator in an appropriate function space. It is assumed that 0 is a fixed point of f. The difficulty lies in the noninvertibility of the map g_p. Let $\mathcal{C}$ be the space of all continuous maps φ from the interval $[0, 1]$ such that $\varphi(0) = 0$, $\varphi(1) = 1$, endowed with the uniform metric. The conjugacy equation is restated as $\varphi = \mathcal{F}(\varphi)$, where is the operator $\mathcal{F}$ on $\mathcal{C}$ defined by

$$(\mathcal{F}\varphi)(x) = \begin{cases} \frac{1}{p}\varphi\left(\{f(x)\}\right) + \frac{m}{p} \pmod 1, & \text{for } a_1^m \leq x < a_1^{m+1}, 0 \leq m < p, \\ 1, & \text{for } x = 1, \end{cases}$$

where $\{x\}$ means the fractional part of x and $\{a_n^m\}$ are obtained via method (iii) and will be all rational numbers on the interval $[0, 1]$ whose denominators are powers of p, such that $\varphi(a_n^m) = m/p^n$. In other words, they apply the m^{th} branch $(x + m)/p$ of the inverse to g_p on the interval Γ_1^m (an element of a nested sequence of partitions obtained by method (iii)). Finally, if f is a piecewise expanding map, using the branches (laps) of f, it is possible to prove that the unique solution φ is an homeomorphism.

Given this brief overview of the literature for finding solutions for conjugacy equations, the goal for this chapter is to study equations (4.1) when both X, Y have finite partitions where some specific properties hold. We establish the adequate setting to obtain results on the existence and on the image of possible solutions, in the case where X, Y are metric spaces. In fact, we extend the ideas of method (iv) from Katok and Hasselblatt [44] to systems (4.1) where both F and f are piecewise defined. The scope of the study goes far beyond obtaining homeomorphic solutions. For the case of piecewise continuous functions there may exist both continuous and non-continuous solutions. Nevertheless, we are looking for solutions of the problem (4.1) irrespective of regularity. That is why we do not need to have a topological conjugacy: a conjugacy is enough. In some conditions it is possible to find explicit formulae for solutions.

To do so we need some preparatory results on systems of functional equations. The geometry of the graph of solutions is closely related to a pair of

IFS (iterated function systems), which are usually associated with the construction of fractal sets in the plane (see chapter 3). We add some examples that corroborate the fractal nature of some solutions as could have been expected.

We first concentrate, in the following section, in the simple piecewise affine case. Inspired by this case, we extend the study given a generic construction leading to results on existence, non-uniqueness, explicitly defined solutions and we establish a link to the IFS context. This construction benefits from the study performed in Chapter 2 regarding systems of functions. We conclude the chapter with some specific examples illustrating the fractal nature of some solutions implied by our results.

4.2 The piecewise affine example

One possible interest of conjugacy equations is to simplify the study of a family of maps by considering the simplest possible cases while preserving topological dynamical properties. In a first and simple case piecewise affine and expansion interval maps will be studied.

We focus our attention on particular cases of equation (4.1), which correspond to a conjugacy equation involving the piecewise affine case. From the functional point of view the results of de Rham [74] and their generalization by Girgensohn [36] are needed. Since this last generalization provides an explicit solution in terms of the base p expansion of numbers, it is possible to construct explicitly a solution of our equation.

Definition 55. (see [14]) A map $f : I \to I$ is a ***horseshoe map*** if it has more than one lap and each lap is mapped onto the whole of I.

The next Lemma is a generalization of a result of Ciepliński and Zdun [22] to non-continuous functions (only piecewise continuity is required).

Lemma 56. *[81] Let $p \geq 2$, be an integer, $f : [\alpha_1, \beta_1] \to [\alpha_1, \beta_1]$, $F : [\alpha_2, \beta_2] \to [\alpha_2, \beta_2]$ be piecewise continuous functions, respectively with laps $I_i = [x_i, x_{i+1}]$, $J_i = [y_i, y_{i+1}]$, $i \in \{0, 1, \dots, p-1\}$. Consider the conjugacy equation*

$$h\left(f\left(x\right)\right) = F\left(h\left(x\right)\right),\ x \in [\alpha_1, \beta_1], \tag{4.2}$$

where $h : [\alpha_1, \beta_1] \to [\alpha_2, \beta_2]$ is the unknown function.

Suppose f, F are horseshoe maps, and φ is a monotone and surjective solution of (4.2). If φ is increasing, then $\varphi\left(x_i\right) = y_i$, and $\varphi\left[I_i\right] = J_i$ for $i \in \{0, 1, \dots, p-1\}$. If φ is decreasing, then $\varphi\left(x_i\right) = y_{p-i}$, and $\varphi\left[I_i\right] = J_{p-i-1}$ for $i \in \{0, 1, \dots, p-1\}$.

Proof. Since f is a horseshoe map and φ is a monotone and surjective solution of equation (4.2), we have for each $i \in \{0, 1, \dots, p-1\}$,

$$\begin{aligned} \varphi\left(f\left(x_i\right)\right) = F\left(\varphi\left(x_i\right)\right) &\Rightarrow F\left(\varphi\left(x_i\right)\right) \in \left\{\varphi\left(\alpha_1\right), \varphi\left(\beta_1\right)\right\} \\ &\Rightarrow F\left(\varphi\left(x_i\right)\right) \in \left\{\alpha_2, \beta_2\right\} \\ &\Rightarrow \varphi\left(x_i\right) \in \left\{y_0, y_1, \dots, y_p\right\}. \end{aligned}$$

Suppose for each $i \in \{0,1,\ldots,p-1\}$, $\varphi(x_i) = \varphi(x_{i+1})$. Since φ is monotone, $\varphi[I_i]$ is a single point, as is $F[\varphi[I_i]]$. Again by equation (4.2) and surjectivity of φ we obtain $F[\varphi[I_i]] = \varphi[f[I_i]] = \varphi[\alpha_1,\beta_1] = [\alpha_2,\beta_2]$. Then $\varphi(x_i) \neq \varphi(x_{i+1})$, $i \in \{0,1,\ldots,p-1\}$.

If φ is increasing, then $\varphi(x_0) < \varphi(x_1) < \cdots < \varphi(x_a)$, implying $\varphi(x_i) = y_i$, because $\varphi(x_i) \in \{y_0,y_1,\ldots,y_p\}$. Then $\varphi[I_i] = J_i$, $i \in \{0,1,\ldots,p-1\}$. If φ is decreasing, then $\varphi(x_0) > \varphi(x_1) > \cdots > \varphi(x_p)$, implying $\varphi(x_i) = y_{p-i}$, because $\varphi(x_i) \in \{y_0,y_1,\ldots,y_p\}$. Then $\varphi[I_i] = J_{p-i-1}$, $i \in \{0,1,\ldots,p-1\}$. $\square$

We will restrict attention to the family $\mathcal{M}$ of piecewise monotone and expanding interval maps $f:[0,1]\to[0,1]$ where there exists a partition $0 = a_0 < a_1 < \cdots < a_r = 1$, with $r \geq 2$, of $[0,1]$ such that $f_{|[a_{i-1},a_i]}$, for $i = 1,2,\ldots,r$, is a monotone piecewise C^1 function for which there exists $\lambda > 1$ such that $|f'(x)| \geq \lambda$, for every $x \in (a_{i-1},a_i)$.

Note that the expansivity condition does not necessarily require that the function be differentiable. The definition of $\mathcal{M}$ may be weakened to the expansivity condition given in Definition 4.

Let $f \in \mathcal{M}$ with partition $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ of $[0,1]$ such that $f_{|[\alpha_{i-1},\alpha_i]}$, for $i = 1,2,\ldots,p$, is an increasing, continuous and expanding function satisfying $f(\alpha_{i-1},\alpha_i) = (0,1)$, for every $i = 1,2,\ldots,p$. We will see, in the piecewise affine case, that f is topologically conjugate to the map

$$g_p(x) = \begin{cases} px \,(\mathrm{mod}\, 1), & \text{if } x \in [0,1) \\ 1, & \text{if } x = 1, \end{cases}$$

i.e., there exists a homeomorphism φ such that

$$\varphi \circ g_p = f \circ \varphi. \tag{4.3}$$

Note that in [44], $E_k : x \to kx$ (mod 1).

Let $p \geq 2$. Define $\mu_0,\mu_1,\ldots,\mu_{a-1} \in (0,1)$ by $\mu_i = \alpha_{i+1} - \alpha_i$ for $i \in \{0,1,\ldots,p-1\}$. Clearly $\sum\limits_{j=0}^{p-1} \mu_j = 1$.

In the affine case, f is given by

$$f(x) = \begin{cases} \frac{1}{\mu_i}x - \frac{\alpha_i}{\mu_i}, & \text{if } x \in [\alpha_{i-1},\alpha_i)\,,\ i \in \{0,1,\ldots,p-1\} \\ 1, & \text{if } x = 1, \end{cases}$$

and it is possible to construct an explicit solution of (4.3) using the following results.

Theorem 57. *[81] Any monotone increasing and surjective solution of the conjugation equation $h \circ g_p = f \circ h$ satisfies the functional equation*

$$\varphi(x) = \mu_i \varphi(ax - i) + \alpha_i, \ \textit{for each } i \in \{0,1,\ldots,p-1\}\,,\ x \in \left[\frac{i}{p}, \frac{i+1}{p}\right]. \tag{4.4}$$

Proof. Let M be an increasing and surjective solution of equation (4.4). Then

$$M\left(px-i\right)=\frac{1}{\mu_i}M\left(x\right)-\frac{\alpha_i}{\mu_i},\ i\in\left\{0,1,\ldots,p-1\right\},\ x\in\left[\frac{i}{p},\frac{i+1}{p}\right].$$

By direct computation

$$\begin{aligned} M\circ g_p\left(x\right) &= \begin{cases} M\left(ax-i\right), & \text{if } x\in\left[\frac{i}{p},\frac{i+1}{p}\right),\ i\in\left\{0,1,\ldots,p-1\right\} \\ 1, & \text{if } x=1. \end{cases} \\ &= \begin{cases} \frac{1}{\mu_i}M\left(x\right)-\frac{\alpha_i}{\mu_i}, & \text{if } x\in\left[\frac{i}{p},\frac{i+1}{p}\right),\ i\in\left\{0,1,\ldots,p-1\right\} \\ 1, & \text{if } x=1, \end{cases} \end{aligned}$$

and

$$f\circ M\left(x\right)=\begin{cases} \frac{1}{\mu_i}M\left(x\right)-\frac{\alpha_i}{\mu_i}, & \text{if } M\left(x\right)\in\left[\alpha_{i-1},\alpha_i\right),\ i\in\left\{0,1,\ldots,p-1\right\} \\ 1, & \text{if } M\left(x\right)=1. \end{cases}$$

By Lemma 57, if M is monotone increasing and surjective, then for each $i\in\{0,1,\ldots,p-1\}$ $M\left(x\right)\in[\alpha_{i-1},\alpha_i)$ is equivalent to $x\in[i/p,\left(i+1\right)/p)$. □

Remark 58. An analogous result holds for decreasing and surjective solutions as a consequence of Lemma 57.

Example 59. [81] For $p=3$, equation (4.4) is

$$\varphi\left(x\right)=\begin{cases} \mu_0\varphi\left(3x\right), & \text{if } x\in\left[0,\frac{1}{3}\right] \\ \mu_1\varphi\left(3x-1\right)+\alpha_1, & \text{if } x\in\left[\frac{1}{3},\frac{2}{3}\right] \\ \mu_2\varphi\left(3x-2\right)+\alpha_2, & \text{if } x\in\left[\frac{2}{3},1\right]. \end{cases}$$

Lemma 60. *[81] Let $p\geq 2$ be an integer. Consider the system of functional equations*

$$\varphi\left(\frac{k+x}{p}\right)=F_k\varphi\left(x\right), \tag{4.5}$$

where $k\in\{0,1,2,\ldots,p-1\}$, F_k are contractions mappings and $\varphi:[0,1]\to[0,1]$ is the unknown function.

Then the system of functional equations (4.5) is equivalent to the functional equation

$$\varphi\left(x\right)=F_k\varphi\left(px-k\right),x\in\left[\frac{k}{p},\frac{k+1}{p}\right],\ k\in\left\{0,1,2,\ldots,p-1\right\}.$$

Proof. This result is proved by a simple change of variable: $y=\left(k+x\right)/p$. □

We now return to the study of the general equation (4.3).

Lemma 61. *[81] Any monotone increasing and surjective solution of the conjugation equation $\varphi \circ g_p = f \circ \varphi$ satisfies the functional equation*

$$\varphi(x) = \mu_i \varphi(px - i) + \alpha_i,\ i \in \{0, 1, \ldots, p-1\},\ x \in \left[\frac{i}{p}, \frac{i+1}{p}\right],$$

i.e.,

$$\varphi\left(\frac{x+i}{p}\right) = \mu_i \varphi(x) + \alpha_i,\ i \in \{0, 1, \ldots, p-1\},\ x \in \left[\frac{i}{p}, \frac{i+1}{p}\right].$$

The condition

$$\frac{r_{k-1}}{1 - r_{p-1}} s_{p-1}(1) + s_{k-1}(1) = \frac{r_k}{1 - r_0} s_0(0) + s_k(0),\ 1 \le k \le p-1$$

in this case assumes the form

$$\frac{\mu_{i-1}}{1 - \mu_{p-1}} \alpha_{p-1} + \alpha_{i-1} = \frac{\mu_i}{1 - \mu_0} 0 + \alpha_i,\ 1 \le i \le p-1,$$

which is equivalent to

$$\mu_{i-1} = \alpha_i - \alpha_{i-1},\ 1 \le i \le p-1,$$

coinciding with the original hypothesis.

The next result is proved by Zdun [103].

Lemma 62. *Let $F_0, \ldots, F_{p-1} : J \to J$ be strictly increasing contraction maps satisfying $F_0(a) = a,\ F_{p-1}(b) = b,\ F_{k+1}(a) = F_k(b)$ for $k = 0, \ldots, p-2$ and some $a, b \in J,\ a \neq b$. If $f_0, \ldots, f_{p-1} : [0,1] \to [0,1]$ are continuous strictly increasing maps satisfying $f_0(0) = 0,\ f_{p-1}(1) = 1,\ f_{k+1}(0) = f_k(1)$, for $k = 0, \ldots, p-2$ and*

$$|f_k(x) - f_k(y)| < |x - y|,\ x \neq y,\ x, y \in [0,1],\ k = 0, \ldots, p-2,$$

then the system

$$\varphi(f_k(x)) = F_k(\varphi(x)),\ x \in [0,1],\ k = 0, 1, \ldots p-1$$

has a unique homeomorphic solution.

Applying Theorem 39, we obtain the following explicit solution in terms of the base p expansion of numbers.

Theorem 63. *[81] Let $p \ge 2$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$, and $\mu_0, \mu_1, \ldots, \mu_{p-1} \in (0,1)$, such that $\mu_i = \alpha_{i+1} - \alpha_i$, for $i \in \{0, 1, \ldots, p-1\}$. Given f and g_p defined above, there exists exactly one increasing homeomorphism $\varphi : [0,1] \to [0,1]$ such that $\varphi \circ g_p = f \circ \varphi$, defined by*

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n-1} \mu_{\xi_i}\right) \alpha_{\xi_n}. \tag{4.6}$$

Proof. By Lemma 62 and Theorem 39, there exists exactly one bounded monotone increasing and surjective $\varphi : [0,1] \to [0,1]$ such that $\varphi \circ g_p = f \circ \varphi$ which is defined by formula (4.6). Lemma 63 shows that function φ is a homeomorphism. □

Remark 64. An analogous result may be proved for the case of a decreasing homeomorphism.

4.3 A general construction

After a study of a special simple case of conjugacy equation (4.1), it is now appropriate to give a general approach to this type of problem. We will be particularly interested in studying (4.1) in the case where both X, Y have finite partitions

$$\begin{cases} X = X_0 \cup X_1 \cup \ldots \cup X_{p-1}, & X_i \cap X_j = \emptyset \ (i \neq j) \\ Y = Y_0 \cup Y_1 \cup \ldots \cup Y_{q-1}, & Y_i \cap Y_j = \emptyset \ (i \neq j), \end{cases} \tag{4.7}$$

whose properties are not specified at this point.

In this setting the conjugacy equation (4.1) may be represented by

$$\begin{cases} \varphi\left(f_0\left(x\right)\right), & x \in X_0 \\ \varphi\left(f_1\left(x\right)\right), & x \in X_1 \\ \vdots & \\ \varphi\left(f_i\left(x\right)\right), & x \in X_i \\ \vdots & \\ \varphi\left(f_{p-1}\left(x\right)\right), & x \in X_{p-1} \end{cases} = \begin{cases} F_0\left(\varphi\left(x\right)\right), & \varphi\left(x\right) \in Y_0 \\ F_1\left(\varphi\left(x\right)\right), & \varphi\left(x\right) \in Y_1 \\ \vdots & \\ F_j\left(\varphi\left(x\right)\right), & \varphi\left(x\right) \in Y_j \\ \vdots & \\ F_{q-1}\left(\varphi\left(x\right)\right), & \varphi\left(x\right) \in Y_{q-1}, \end{cases} \tag{4.8}$$

where $f_i := f_{|X_i}$, $i = 0, 1, \ldots, p-1$ and $F_j := F_{|Y_j}$, $j = 0, 1, \ldots, q-1$. In conjunctive normal form (see e.g. [100]) equation (4.8) is the condition

$$\bigwedge_{i=0}^{p-1} \bigvee_{j=0}^{q-1} \varphi\left(f_i\left(x\right)\right) = F_j\left(\varphi\left(x\right)\right), \ x \in X_i, \ \varphi\left(x\right) \in Y_j. \tag{4.9}$$

Thus, solving the original functional equation (4.1) is equivalent, in terms of the given partitions, to solving $p \cdot q$ functional equations, each corresponding to each of the combinations of indices (i, j), and ensuring that the images under φ are in the appropriate subsets (Y_j). We summarize the discussion in the following Lemma.

Lemma 65. *[86] The conjugacy (4.1) is equivalent, with respect to the partitions (4.7), to the condition* $\forall i \in \{0, 1, \ldots, p-1\}, \forall x \in X_i, \exists j \in \{0, 1, \ldots, q-1\}$*:*

$$\varphi\left(f_i\left(x\right)\right) = F_j\left(\varphi\left(x\right)\right), \ \varphi\left(x\right) \in Y_j. \tag{4.10}$$

One way to find a solution of equation (4.1) is to solve one system of functional equations, where for each i we fix $j \equiv j(i)$.

Example 66. [86] If $q \geq p$ and $j(i)$ is the identity, then the system of equations (4.10) is

$$\varphi(f_i(x)) = F_i(\varphi(x)), \ x \in X_i, \ \varphi(x) \in Y_i, \ i = 0, 1, \ldots, p-1.$$

If $j(i) \equiv i_0$, then the system of equations (4.10) is

$$\varphi(f_i(x)) = F_{i_0}(\varphi(x)), \ x \in X_i, \ \varphi(x) \in Y_{i_0}, \ i = 0, 1, \ldots, p-1.$$

We state an immediate result which will be useful for future reference.

Proposition 67. *[86] Assume $\varphi : X \to Y$ is a solution of (4.1) such that $\exists\, \sigma : \{0, 1, \ldots, p-1\} \to \{0, 1, \ldots, q-1\}$, $\forall i \in \{0, 1, \ldots, p-1\}, \varphi(X_i) \subset Y_{\sigma(i)}$. Then φ is a solution of the system of functional equations*

$$\varphi(f_i(x)) = F_{\sigma(i)}(\varphi(x)), \ x \in X_i, \ i = 0, 1, \ldots, p-1. \tag{4.11}$$

Proof. Since by hypothesis the family $\{Y_j\}$ is pairwise disjoint, when solving (4.9) all but

$$\varphi(f_i(x)) = F_{\sigma(i)}(\varphi(x)), \ x \in X_i, \ \varphi(x) \in Y_{\sigma(i)},$$

with $i = 0, 1, \ldots, p-1$ are impossible equations. Then φ is a solution of (4.11). $\square$

The next results provide concrete applications of the above general construction. Proposition 69 extends a result of Ciepliński and Zdun [22] for compact intervals, dropping the continuity requirement and is equivalent to Lemma 57 without explicit mention of the concept of horseshoe map. Here $p = q$.

Proposition 68. *[86] Let $X = [a, b]$, $Y = [c, d]$, and $a = x_0 < x_1 < \cdots < x_p = b$, $c = y_0 < y_1 < \cdots < y_p = d$. Let $X_i = [x_i, x_{i+1}]$, $Y_i = [y_i, y_{i+1}]$, and suppose all $f_{|X_i}$ are homeomorphisms onto X. Suppose further that $\varphi : X \to Y$ is a monotone and surjective solution of (4.1).*

(a) If φ is increasing, then $\varphi(x_i) = y_i$ and

$$\varphi[X_i] = Y_i, \ i \in \{0, 1, \ldots, p-1\}.$$

(b) If φ is decreasing, then $\varphi(x_i) = y_{p-i}$ and

$$\varphi[X_i] = Y_{p-i-1}, \ i \in \{0, 1, \ldots, p-1\}.$$

This result allows the construction of monotone and surjective solutions of the conjugacy equation (4.1).

Proposition 69. *[86] Let $X = [a, b]$, $Y = [c, d]$, and $a = x_0 < x_1 < \cdots < x_p = b$, $c = y_0 < y_1 < \cdots < y_p = d$. Let $X_i = [x_i, x_{i+1}]$, $Y_i = [y_i, y_{i+1}]$, and suppose all $f_{|X_i}$ are homeomorphisms onto X. Suppose further that $\varphi : X \to Y$ is a monotone and surjective solution of (4.1).*

If φ is increasing, then φ is a solution of the system of functional equations

$$\varphi(f_i(x)) = F_i(\varphi(x)),\ x \in X_i,\ i = 0, 1, \ldots, p-1. \tag{4.12}$$

If φ is decreasing, then φ is a solution of the system of functional equations

$$\varphi(f_i(x)) = F_{p-i-1}(\varphi(x)),\ x \in X_i,\ i = 0, 1, \ldots, p-1. \tag{4.13}$$

Proof. If φ is a solution of a system of equations from (4.10), then it is a solution of equation (4.1). Suppose $\varphi : X \to Y$ is a monotone increasing and surjective solution of (4.1). By Proposition 69, equation (4.1) is equivalent to the system (4.12). For the decreasing case, the proof is similar and the corresponding system reverses the order of indices. □

As a consequence of this result we derive a description of the homeomorphic solutions of topological conjugacies between piecewise defined functions with real interval domains.

Corollary 70. *[86] Consider a topological conjugacy equation (4.1), where both X, Y are real intervals and both equipped with partition of order p. If all $f_{|X_i}$ are homeomorphisms onto X, then a homeomorphic solution φ of (4.1) is either a solution of the system of functional equations (4.12), or a solution of the system of functional equations (4.13). In the first case φ is increasing, and in the second case φ is decreasing.*

In the sequence of Theorem 14 and considering condition (4.10), we give a general result about the solutions of conjugacy equations where f is defined by (2.5).

Theorem 71. *[86] Consider the conjugacy equation (4.1), with $f(x) = px \,(mod\, 1)$ and F piecewise invertible and expanding, with p laps. Then the solutions of (4.1) are of the form*

$$\varphi\left(\sum_{i=1}^{\infty} \frac{k_i}{p^i}\right) = \lim_{\nu\to\infty} F^{-1}_{r_{\nu,k_1}} \circ \cdots \circ F^{-1}_{r_{1,k_\nu}}(\xi), \tag{4.14}$$

where for each $\tau \in \{0, 1, 2, \ldots, p-1\}$, $\{r_{n,\tau}\}_{n\in\mathbb{N}}$ is a sequence of integers belonging to $\{0, 1, 2, \ldots, q-1\}$ and ξ is the fixed point of $F^{-1}_{r_{1,0}}$.

Remark 72. The converse of Theorem 72 is not true. Not all functions of the form (4.14) are solutions to (4.1), since each equation in (4.10) has domains which have to be respected.

Proof. By Lemma 66 solving conjugacy equation (4.1) is equivalent to solving (4.10). Now (4.10), in the corresponding contracting form, is written as a set of individual conjugacy equations

$$\varphi\left(\frac{x+i}{p}\right) = F_j^{-1}\left(\varphi\left(x\right)\right),\ \varphi\left(x\right) \in Y_j,$$

where $i = 0, 1, 2, \ldots, p-1$ and $j = 0, 1, 2, \ldots, q-1$.

The proof is constructive. We define p sequences of numbers $\{r_{n,\tau}\}_{n\in\mathbb{N}}$ belonging to the set $\{0, 1, 2, \ldots, q-1\}$ by the following procedure.

$$\begin{array}{cc|cccccc} & & & & \tau & & \\ & & 0 & 1 & 2 & \cdots & p-1 \\ & 1 & r_{1,0} & r_{1,1} & r_{1,2} & \cdots & r_{1,p-1} \\ \text{iteration} & 2 & r_{2,0} & r_{2,1} & r_{2,2} & \cdots & r_{2,p-1} \\ & 3 & r_{3,0} & r_{3,1} & r_{3,2} & \cdots & r_{3,p-1} \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

In the initial step we consider the image of $x = 0$. For $x = 0$ there are at most q possibilities, which correspond to the solutions of the equations

$$\varphi\left(0\right) = F_{j_0}^{-1}\left(\varphi\left(0\right)\right), \varphi\left(0\right) \in Y_{j_0},\ j_0 = 0, 1, 2, \ldots, q-1, \tag{4.15}$$

i.e., $\varphi\left(0\right)$ is the unique fixed point of one of the maps $F_{j_0}^{-1}$, provided $\varphi\left(0\right) \in Y_{j_0}$. Define $r_{1,0} \equiv j_0$ as the selection performed in (4.15).

In the next step we consider the images of the points of the form $x_{k_1} = k_1/p$, with $k_1 = 1, 2, \ldots, p-1$. Once the image of zero is defined, for each $k_1 \in \{1, 2, \ldots, p-1\}$, there are at most q possibilities ($j_1 = 0, 1, 2, \ldots, q-1$):

$$\varphi\left(\frac{k_1}{p}\right) = F_{j_1}^{-1}\left(\varphi\left(0\right)\right),\ \varphi\left(0\right) \in Y_{j_1},\ j_1 = 0, 1, 2, \ldots, q-1. \tag{4.16}$$

Define $r_{1,k_1} \equiv j_1$ as the selection defined in (4.16). Next we obtain images of numbers of the form

$$\frac{k_1}{p} + \frac{k_2}{p^2} = \frac{\frac{k_2}{p} + k_1}{p}, \tag{4.17}$$

where there are at most q possibilities, corresponding to the solutions of the equations

$$\varphi\left(\frac{\frac{k_2}{p} + k_1}{p}\right) = F_{j_2}^{-1}\left(\varphi\left(\frac{k_2}{p}\right)\right),\ \varphi\left(\frac{k_2}{p}\right) \in Y_{j_2},\ j_2 = 0, 1, 2, \ldots, q-1. \tag{4.18}$$

Note that the input needed in equations (4.18) is $\varphi\left(k_2/p\right)$. This has already been determined by (4.16) and (4.15) in terms of k_1; the procedure for k_2 is formally the same, replacing k_1 by k_2. We obtain

$$\varphi\left(\frac{k_1}{p} + \frac{k_2}{p^2}\right) = F_{j_2}^{-1}\left(F_{r_{1,k_2}}^{-1}\left(\varphi\left(0\right)\right)\right),\ F_{r_{1,k_2}}^{-1}\left(\varphi\left(0\right)\right) \in Y_{j_2},\ j_2 = 0, 1, 2, \ldots, q-1. \tag{4.19}$$

This means that, when obtaining the image of the numbers given by (4.17), we first had to obtain $\varphi(k_2/p)$ by r_{1,k_2} (k_2 now plays the same role as k_1 in the previous step). In this step we similarly fix, for each $k_1 \in \{1, 2, \ldots, p-1\}$, $r_{2,k_1} \equiv j_2$ as the selection defined in (4.19). Continuing this process we obtain

$$\varphi\left(\frac{\frac{k_2}{p} + \frac{k_3}{p^2} + k_1}{p}\right) = F_{j_3}^{-1}\left(\varphi\left(\frac{k_2}{p} + \frac{k_3}{p^2}\right)\right),\ \varphi\left(\frac{k_2}{p} + \frac{k_3}{p^2}\right) \in Y_{j_3}$$

$$\varphi\left(\frac{k_1}{p} + \frac{k_2}{p^2} + \frac{k_3}{p^3}\right) = F_{j_3}^{-1}\left(F_{r_{2,k_2}}^{-1}\left(F_{r_{1,k_3}}^{-1}\left(\varphi(0)\right)\right)\right),\ \varphi\left(\frac{k_2}{p} + \frac{k_3}{p^2}\right) \in Y_{j_3},$$

with $r_{3,k_1} \equiv j_3$. Note that to obtain the image of

$$\frac{k_1}{p} + \frac{k_2}{p^2} + \frac{k_3}{p^3},$$

we first look at k_3, then at k_2 and finally at k_1.

Proceeding inductively, we define the sequences $\{r_{n,\tau}\}_{n\in\mathbb{N}}$, where for $\nu \in \mathbb{N}$ and $\tau \in \{0, 1, \ldots, p-1\}$

$$\varphi\left(\sum_{i=1}^{\nu} \frac{k_i}{p^i}\right) = F_{r_{\nu,k_1}}^{-1} \circ \cdots \circ F_{r_{1,k_\nu}}^{-1}(\xi),\ F_{r_{\nu-1,k_2}}^{-1} \circ \cdots \circ F_{r_{1,k_\nu}}^{-1}(\xi) \in Y_{r_{\nu,k_1}},$$

where ξ is the fixed point of $F_{r_{1,0}}^{-1}$. By a limiting procedure we obtain the desired result.

In case $\{r_{n,\tau}\}_{n\in\mathbb{N}}$ are constant sequences, then φ is also a solution of a system of type (2.3), where possibly the indices of functions F_k^{-1} are not the original, or not in the original order, but still $k \in \{0, 1, 2, \ldots, q-1\}$. The indices are the constants appearing in the sequences $\{r_{n,\tau}\}_{n\in\mathbb{N}}$. In this case the conclusion follows from Theorem 14. □

Remark 73. If $\{r_{n,\tau}\}_{n\in\mathbb{N}}$ are constant sequences then the solution is self-similar in the sense of [31].

In this context we may find solutions with fractal-shaped graphs.

Theorem 74. *[86] Consider the conjugacy equation (4.1), where f and F are, in each sub-domain, bijective and expansion maps respectively in X, Y, closed subsets of $\mathbb{R}^n$. Then in the sub-domain given by the attractor of the IFS $\Lambda = \{X : f_k^{-1}, k = 0, 1, \ldots, p-1\}$ the solutions of (4.1) have images in the attractor of the IFS $\Gamma = \{Y : F_k^{-1}, k = 0, 1, \ldots, p-1\}$. Moreover, if $\Lambda = X$, then the image of each solution is contained in the attractor of Γ.*

The proof follows from the proof of Theorem 18.

Definition 75. [86] In the conditions of Theorem 75, we call

$$\Lambda \times \Gamma = \left\{X : f_i^{-1}, i = 0, 1, \ldots, p-1\right\} \times \left\{Y : F_i^{-1}, i = 0, 1, \ldots, p-1\right\}$$

the **agglutinator set** of the corresponding conjugacy equation.

Example 76. [86] The well-known Cantor dust (see Figure 4.1) is the agglutinator set of the conjugacy equation $\varphi \circ f = F \circ \varphi$, where

$$f(x) = F(x) = \begin{cases} 3x, & 0 \le x \le \frac{1}{3} \\ 3x - 2, & \frac{2}{3} \le x \le 1. \end{cases}$$

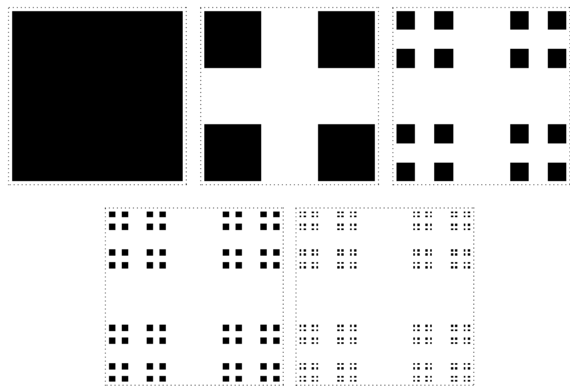

Figure 4.1: The construction of the Cantor dust.

In this context a conjugacy equation between maps that are, in each subdomain, invertible and expanding, is a generator of 2-iterated function system (2-IFS). The solutions resulting from solving these equations are determined inside the product of two IFS ($\Lambda \times \Gamma \subset X \times Y$), the corresponding agglutinator set.

4.4 Examples

To illustrate the contents of chapter 4 we refer to Dubuc [29, 30], who collected several examples of solutions of equations of type (4.1), where, more generally, F also depends explicitly on x:

$$\varphi(f(x)) = F(x, \varphi(x)). \tag{4.20}$$

Dubuc shows that solutions of (4.20) are typically fractal functions, such as the Weierstrass, Knopp, van der Waerden, Hildebrandt and von Koch-Mandelbrot functions, the dragon of Harter-Heightway, the pyramid of von Koch and the flaming crown. For a special case of equations (4.20), Girgensohn [36] (Theorem 39) gives an explicit continuous solution result.

This result was extended in [84] (Theorem 41) to variable coefficients $s_i(x)$. We next generalize Theorem 41 to the case where continuity is dropped; in this case there are no conditions analogous to (3.17). Note that, as detailed in section 2.3.5, in the case of a double representation of the argument of φ, the explicit constructive formulae apply only for the finite representation.

Theorem 77. *[86] Fix $p \in \{2,3,4,\ldots\}$, let $r_i : [0,1) \to \mathbb{R}$ be bounded, $|s_i| < 1$ for $0 \le i \le p-1$. Then there exists exactly one bounded $\varphi : [0,1) \to \mathbb{R}$ which satisfies the system*

$$\varphi\left(\frac{x+i}{p}\right) = s_i \varphi(x) + r_i(x),\ x \in [0,1),\ 0 \le i \le p-1.$$

The function φ is given in terms of the base p expansion

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n},$$

by

$$\varphi(x) = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} s_{\xi_k}\right) r_{\xi_n}\left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{p^k}\right).$$

Although this is a weaker result in terms of regularity, it extends to a wider class of functions, in general non-continuous. In the case of conjugacy equations (4.1) with real interval domain, solutions may be obtained by systems of type (1.2) and a particular result is sufficient.

Theorem 78. *[86] Fix $p \in \{2,3,4,\ldots\}$, let $r_i, s_i \in \mathbb{R}$, $|s_i| < 1$ for $0 \le i \le p-1$. Then there exists exactly one bounded $\varphi : [0,1) \to \mathbb{R}$ which satisfies the system*

$$\varphi\left(\frac{x+i}{p}\right) = s_i \varphi(x) + r_i,\ x \in [0,1),\ 0 \le i \le p-1.$$

The function φ is given in terms of the base p expansion

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n},$$

by

$$\varphi(x) = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} s_{\xi_k}\right) r_{\xi_n}.$$

Examples of such solutions are fractal functions exhibiting a self-similarity but with a non-continuous structure, for instance Sierpiński-type triangles.

These are examples of dynamical systems where the functions f and F are piecewise affine maps, considering $p \ge 2$, $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_p = 1$ and $\mu_0, \mu_1, \ldots, \mu_{p-1} \in (0,1)$, such that $\mu_i = \alpha_{i+1} - \alpha_i$, for $i \in \{0,1,\ldots,p-1\}$, f defined by (2.5) and

$$F(x) = \frac{1}{\mu_i} x - \frac{\alpha_i}{\mu_i},\ x \in [\alpha_i, \alpha_{i+1}),\ i \in \{0,1,\ldots,p-1\}.$$

Note that

$$\sum_{i=0}^{p-1} \mu_i = 1.$$

Other exotic shapes found in graphs of solutions of this conjugation equation between piecewise affine maps are shown in Figures 4.2-4.4.

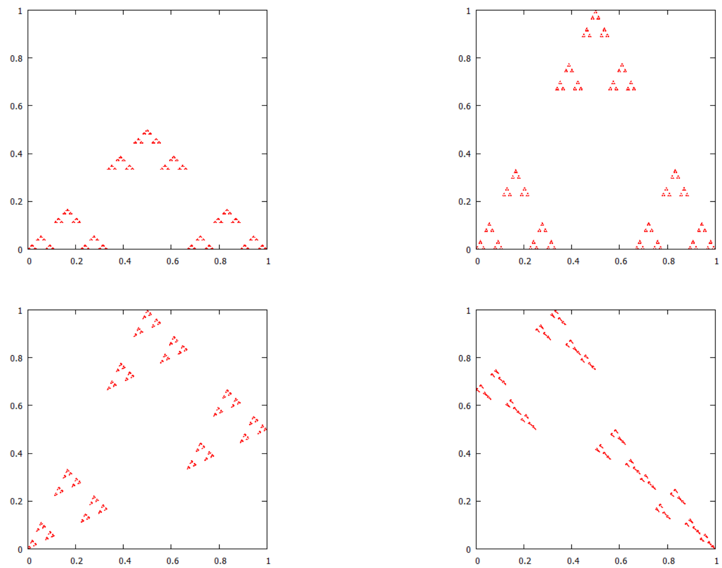

Figure 4.2: Solutions of affine systems: Sierpiński-type triangle-shaped.

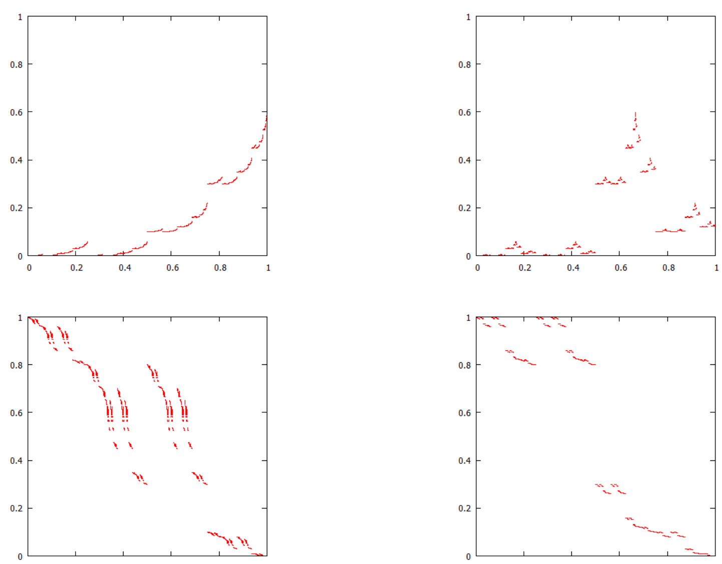

Figure 4.3: Solutions of affine systems: broken line-shaped.

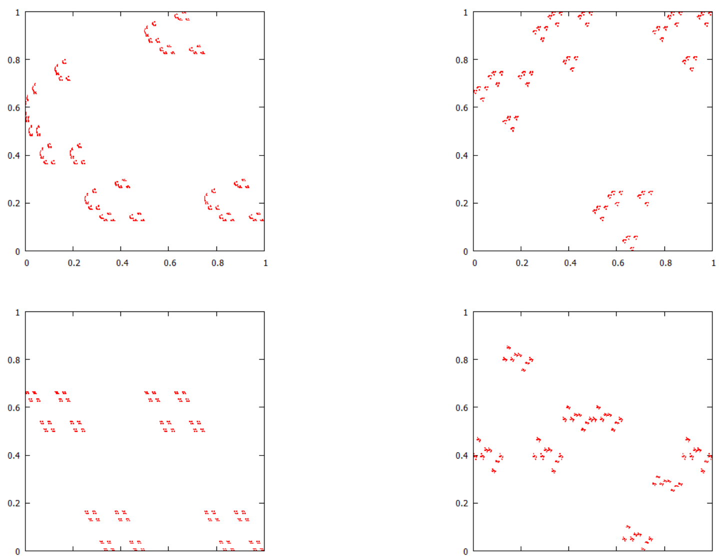

Figure 4.4: Solutions of affine systems: dust cloud-shaped.

Part II

Applications

Chapter 5

Fractal Interpolation

5.1 FIF - Fractal Interpolation Functions

Given a set of data

$$\triangle = \left\{ (x_i, y_i) \in I \times J \subset \mathbb{R}^2 : i = 0, 1, 2, \ldots, N \right\}, \tag{5.1}$$

where $x_0 < x_1 < \cdots < x_N$ is a partition of the interval $I = [x_0, x_N]$, Barnsley [4] constructed fractal interpolation functions (FIF) considering the associated iterated function system $\left\{\mathbb{R}^2; w_n, n = 1, 2, \ldots, N\right\}$, where the maps w_n are affine transformations with the special form

$$w_n \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_n & 0 \\ c_n & d_n \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_n \\ f_n \end{pmatrix},$$

constrained by the data according to

$$w_n \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} x_{n-1} \\ y_{n-1} \end{pmatrix}, \text{ and } w_n \begin{pmatrix} x_N \\ y_N \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \; n = 1, 2, \ldots, N.$$

This system of affine transformations has one free parameter, which is usually chosen to be d_n. All other parameters a_n, c_n, e_n, f_n are determined by d_n.

The problem of finding a fractal interpolation function can be formulated in terms of functional relations, as first stated by Barnsley [3]. His setting for computing FIF is the following.

Let $\triangle$ be as defined in (5.1). Let L_j be the affine map satisfying

$$L_j\left(x_0\right) = x_{j-1}, \; L_j\left(x_N\right) = x_j, \; j = 1, 2, \ldots, N. \tag{5.2}$$

Let $-1 < \alpha_j < 1$ and $F_j : I \times \mathbb{R} \to \mathbb{R}$ be continuous functions such that, for $j = 1, 2, \ldots, N$,

$$\left|F_j\left(x, \xi_1\right) - F_j\left(x, \xi_2\right)\right| \leq \left|\alpha_j\right| \left|\xi_1 - \xi_2\right|, \; x \in I, \; \xi_1, \xi_2 \in \mathbb{R},$$

and

$$F_j\left(x_0, y_0\right) = y_{j-1},\ F_j\left(x_N, y_N\right) = y_j. \tag{5.3}$$

As shown in [5], the FIF associated with $\left\{\left(L_j\left(x\right), F_j\left(x, y\right)\right)\right\}_{j=1}^{N}$ is the unique function $f : I \to \mathbb{R}$ satisfying

$$f\left(L_j\left(x\right)\right) = F_j\left(x, f\left(x\right)\right),\ j = 1, 2, \ldots N. \tag{5.4}$$

The classical setting studied in [5] is the special case where $F_j\left(x, y\right) = \alpha_j y + q_j\left(x\right)$. Wang and Yu [97] recently introduced a class of systems with variable parameters which generates fractal interpolation functions (FIF). Using a normalized interval $I = [0, 1]$, the problem is generalized by allowing the parameters α_j to depend on x:

$$F_j\left(x, y\right) = \alpha_j\left(x\right) y + q_j\left(x\right). \tag{5.5}$$

Fractal interpolation functions are a very useful and efficient way to construct models that approximate data exhibiting irregular structure. Many areas of the applied sciences explore this interpolation method numerically, modelling in general with affine FIF. We mention for instance areas as Medicine, with studies about cardiac arrhythmia [21], tumor perfusion [24], electroencephalograms [77] and neural networks [88]; Physics or Engineering, with works on turbulence [7], wind speed prediction [99], Hydrology [72], Seismology [51], speech signals [95] and PN waveform generators [35]; Biology or Economics, with studies on DNA or stock prices [41,98]. FIF can be related with other topics within Mathematics such as estimation of fractal dimension [69] or reproducing kernel Hilbert spaces [16].

5.2 Explicit solutions

Much of the research performed on FIF is devoted to the construction of algorithms, numerical simulations and evaluations of algebraic and analytic properties (e.g. study of fractal dimension, stability or smoothness). We present here an explicit functional form of the FIF with variable parameters in terms of base p representation of numbers. In order to perform this we construct solutions of systems of functional equations of type (5.4). The study of these systems is not complete and is the subject of ongoing research (see, e.g. [103]).

As pointed out by Zhai, Kuzma and Rector in 2010 [102], “the ordinary fractal interpolation cannot get the value of any arbitrary point directly, which has not been found in the existing literature”. Because of this lack in the literature they created a new fractal-interpolation method called PPA (Pointed Point Algorithm) based on Iterated Function Systems (IFS) to interpolate the signals with the expected interpolation error, obtaining a method to overtake the problem of direct point estimation in fractal interpolation. The goal is to get accuracy, in particular, in the reconstruction of the seismic profile, leading to a significant improvement over other trace interpolation methods.

To overcome this problem of accuracy, we introduce an analytical and explicit solution for FIF which gives the exact value at every point in terms of a series.

Consider the system of functional equations defined by (5.4) and (5.5). Generalizing Girgensohn's result, we can provide an explicit solution to this problem.

By (5.2) the functions L_j may be explicitly defined by

$$L_j\left(x\right) = \frac{x_j - x_{j-1}}{x_N - x_0}x + x_{j-1} - \frac{x_j - x_{j-1}}{x_N - x_0}x_0.$$

In the construction that follows the base p representation of numbers will play an essential role. In this setting, the usual notation in systems similar to these is to use p as the number of equations in the system, and to label the equations with the integers between 0 and $p-1$. For this reason we denote N by p, and relabel the equations accordingly.

To normalize the problem set $I = [0,1]$. Fix $p \in \{2,3,4,\ldots\}$, and suppose the sub-intervals all have the same length:

$$x_0 = 0 < x_1 = \frac{1}{p} < x_2 = \frac{2}{p} < \cdots < x_j = \frac{j}{p} < \cdots < x_p = 1. \tag{5.6}$$

For the normalized problem we denote the functions L_j by

$$f_j\left(x\right) = \frac{x+j}{p},\ j \in \{0,1,\ldots,p-1\},\ x \in [0,1],$$

and the functions α_j, q_j, respectively by r_k, s_k.

In order to obtain a fractal interpolation function, we consider the problem

$$\varphi\left(L_j\left(x\right)\right) = \alpha_j\left(x\right)\varphi\left(x\right) + q_j\left(x\right),\ j = 0,1,2,\ldots p-1,$$

where the partition of $[0,1]$ is given by (5.6) and

$$L_j\left(x\right) = \frac{x+j}{p},\ j \in \{0,1,\ldots,p-1\},\ x \in [0,1].$$

Since by hypothesis $L_j\left(1\right) = L_{j+1}\left(0\right)$, the compatibility conditions (3.2) imply $F_j\left(1,\varphi\left(1\right)\right) = F_{j+1}\left(0,\varphi\left(0\right)\right)$ for $j = 0,1,\ldots,p-2$. Then

$$F_j\left(1,\varphi\left(1\right)\right) = F_{j+1}\left(0,\varphi\left(0\right)\right)$$

or equivalently

$$\alpha_j\left(1\right)\varphi\left(1\right) + q_j\left(1\right) = \alpha_{j+1}\left(0\right)\varphi\left(0\right) + q_{j+1}\left(0\right).$$

Since $\varphi\left(0\right) = \alpha_0\left(0\right)\varphi\left(0\right) + q_0\left(0\right)$ and $\varphi\left(1\right) = \alpha_{p-1}\left(1\right)\varphi\left(1\right) + q_{p-1}\left(1\right)$,

$$\varphi\left(0\right) = \frac{q_0\left(0\right)}{1-\alpha_0\left(0\right)},\ \varphi\left(1\right) = \frac{q_{p-1}\left(1\right)}{1-\alpha_{p-1}\left(1\right)},$$

which implies

$$F_j\left(1, \frac{q_{p-1}\left(1\right)}{1-\alpha_{p-1}\left(1\right)}\right) = F_{j+1}\left(0, \frac{q_0\left(0\right)}{1-\alpha_0\left(0\right)}\right).$$

In fact hypothesis (5.3) that ensures the compatibility conditions are satisfied and provides the additional information

$$\frac{q_0(0)}{1-\alpha_0(0)} = y_0,\ \frac{q_p(1)}{1-\alpha_p(1)} = y_p. \tag{5.7}$$

We thus conclude that, for the normalized problem, Theorem 41 provides the explicit solution for the problem of finding the fractal interpolation function given by Wang and Yu [97].

The fundamental aim of this section is to provide an explicit solution for a type of fractal interpolation function formulated by Wang and Yu [97]. This work may be summarized in the following result.

Theorem 79. *[84] Let $\triangle$ be as defined in (5.1), verifying (5.6). Let L_j be the affine map satisfying $L_j(0) = x_j$, $L_j(1) = x_{j+1}$, $j = 0, 1, 2, \ldots, N-1$. Let $|\alpha_j(x)| < 1$, $\forall x \in [0,1]$, $q_j : [0,1] \to \mathbb{R}$ be continuous functions and $F_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ be defined by (5.5), such that*

$$F_{j+1}(0, y_0) = y_{j+1} = F_j(1, y_N), \tag{5.8}$$

for $j = 0, 1, 2, \ldots, N-1$.

The FIF associated with $\{(L_j(x), F_j(x,y))\}_{j=0}^{N-1}$ is the unique function $f : I \to \mathbb{R}$ satisfying system (5.4), defined for each x in terms of its base N expansion

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{N^n}$$

by

$$f(x) = \sum_{n=1}^{\infty} \left(\prod_{m=1}^{n-1} \alpha_{\xi_m} \left(\sum_{k=1}^{\infty} \frac{\xi_{k+m}}{N^k} \right) \right) q_{\xi_n} \left(\sum_{k=1}^{\infty} \frac{\xi_{k+n}}{N^k} \right). \tag{5.9}$$

Remark 80. Note that the relabelling of indices to $j = 0, 1, 2, \ldots, N-1$ implies that conditions (5.3) are now rewritten as (5.8).

This result provides an analytic approach to the construction of FIF, in contrast with the purely numerical approach used in most applications. When modelling a set of data with FIF, the first step is to identify a suitable system of equations. The problem of optimal parameter identification within such model is still an ongoing research topic, see e.g. [55]. Only after this optimization procedure is performed may the formula of Theorem 80 be applied.

We remark that there are alternative FIF for specific behaviour of data, such as for graphs with polar shapes [25].

When applying Theorem 80, condition (5.6) may be a practical restriction on the data to be interpolated, because it requires a uniform distribution of the data points. The alternative is to use Theorem 45 but with the disadvantage of being a more laborious computation. In this case, it is helpful to consider the relation between base p expansion and Q-representation as detailed further in section 7.2.3.

5.3 Examples

We now illustrate the previous results by constructing examples of FIF, as given in [84], which interpolate the same set of data points but use different types of basis functions. We choose a system of 10 functional equations with 11 (fixed) interpolation points and the following function sets: for $j = 0, 1, 2, \ldots, 9$, $q_j(x) = c_j x + d_j$, and the following cases:

A. Constant functions: $\alpha_j(x) = \alpha_j = a_j + b_j$, $|a_j + b_j| < 1$;

B. Affine functions: $\alpha_j(x) = a_j x + b_j$, $|a_j + b_j| < 1$;

C. Polynomial functions: $\alpha_j(x) = a_j x^{\theta_j} + b_j$, $|a_j + b_j| < 1$, $\theta_j \in \mathbb{N}$;

D. Sinusoidal functions: $\alpha_j(x) = a_j \sin(l_j x) + b_j$, $|a_j + b_j| < 1$, $l_j \in \mathbb{R}$;

E. Damped exponential functions: $\alpha_j(x) = a_j e^{-\theta_j x} + b_j$, $|a_j + b_j| < 1$, $\theta_j \in \mathbb{R}^+$;

F. Exponentially damped sinusoidal functions: $\alpha_j(x) = a_j e^{-\theta_j x} \sin(l_j x) + b_j$, $|a_j + b_j| < 1$, $\theta_j \in \mathbb{R}^+$, $l_j \in \mathbb{R}$.

Thus case A is the standard Barnsley FIF setting, which we want to compare to the variable-coefficient cases. The chosen coefficients a_j, b_j, l_j and θ_j are such that $|\alpha_j(x)| < 1, \forall x \in [0, 1]$ as required, and the remaining coefficients c_j, d_j are determined by the compatibility conditions. Conditions (5.7) are also verified.

The set of parameters is chosen according to Table 5.1.

j	0	1	2	3	4	5	6	7	8	9	10
x_j	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y_j	0	0	0.7	0.8	0.4	0.5	0.8	0.7	0.2	0.2	0
a_j	0.5	0.4	0.1	0.2	0.0	0.6	0.2	0.4	0.1	0.2	
b_j	0.1	0.5	0.1	0.1	0.8	0.1	0.7	0.1	0.6	0.3	
θ_j	0	1	6	5	7	6	1	5	2	4	
l_j	$\frac{9}{2}\pi$	$\frac{13}{2}\pi$	$\frac{21}{2}\pi$	$\frac{29}{2}\pi$	$\frac{25}{2}\pi$	$\frac{25}{2}\pi$	$\frac{9}{2}\pi$	$\frac{29}{2}\pi$	$\frac{9}{2}\pi$	$\frac{37}{2}\pi$.	

Table 5.1: Set of parameters for constructing of the FIF.

For each case Theorem 41 allows us to compute, for each x, the value of $\varphi(x)$ by formula (3.19) or (5.9). We thus obtain the graphs in Figures 5.1-5.6.

Comparing these graphs, some qualitative conclusions may be drawn. In every case, there is some degree of self-similarity, although the extreme spikiness of the standard FIF is generally smoothed out. This is already clear in the affine case (Fig. 5.2) and even more so, naturally, in the polynomial case (Fig. 5.3). In the case of sinusoidal functions (Fig. 5.4), this smoothing is masked by the introduction of high-frequency components. In the damped exponential case this does not happen, and in the damped sinusoidal case the high-frequency components are exponentially killed off.

This overview suggests that FIF with variable parameters may be a useful tool for modelling phenomena which exhibit irregular structure. They have significantly more flexibility than standard FIF, and therefore may have a much broader range of applicability. Moreover, the existence of the explicit formula (3.19) or (5.9) makes them very easy to compute and explore numerically.

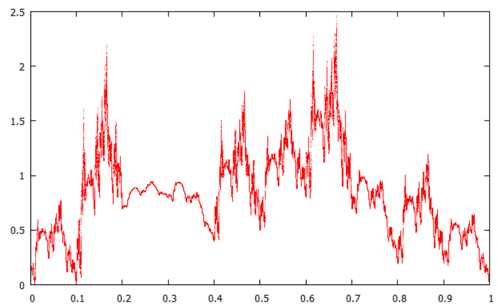

Figure 5.1: Graph of the solution for the standard Barnsley FIF (case A).

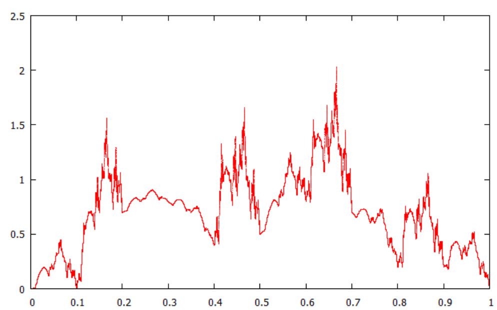

Figure 5.2: Graph of the solution for affine variable parameters (case B).

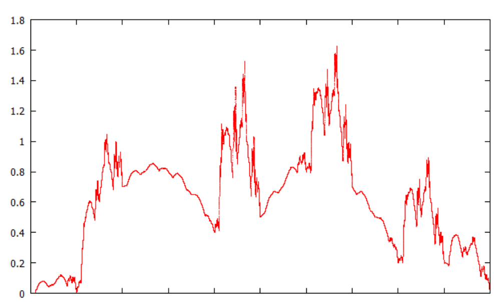

Figure 5.3: Graph of the solution for polynomial variable parameters (case C).

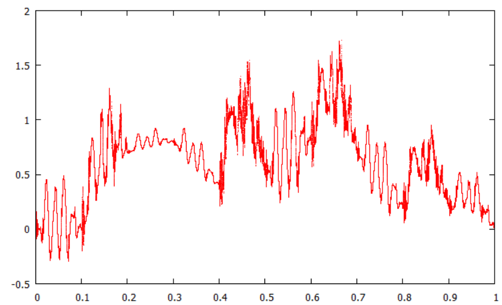

Figure 5.4: Graph of the solution for sinusoidal variable parameters (case D).

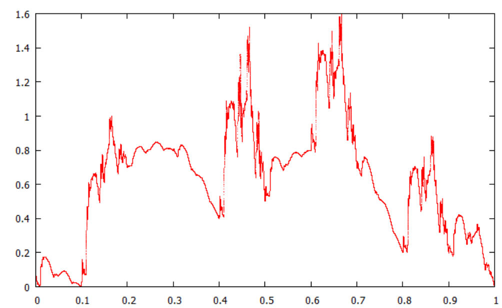

Figure 5.5: Graph of the solution for exponentially damped parameters (case E).

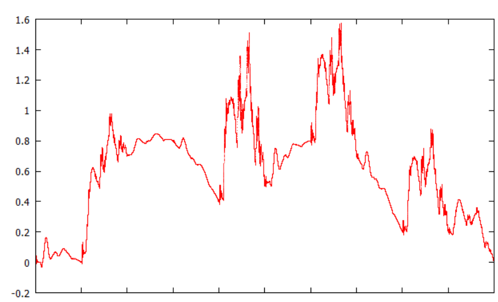

Figure 5.6: Graph of the solution for exponentially damped sinusoidal parameters (case F).

Of course, in a modelling context only the nature of real data can dictate the suitable functional form for the parameters.

Chapter 6

Dynamical systems

We establish combinatorial properties of the dynamics of piecewise increasing, continuous, expanding maps of the interval such as description of periodic and pre-periodic points, primitiveness of truncated itineraries and length of pre-periodic itineraries. We include a relation between the dynamics of a family of circle maps and the properties of combinatorial objects as necklaces and words. We identify in a natural way each periodic orbit with an aperiodic necklace. We show the relevance of this combinatorial approach for the representation of rational numbers and for the orbit structure of pre-periodic points.

The study of piecewise expanding interval maps in terms of the enumeration of periodic orbits and points may be extremely complex. The chaotic behaviour of maps is a complicating factor. One way to simplify the work is to study these topics in the simplest case, i.e., the piecewise affine case. For this, it is suitable to consider a topological conjugacy function since it preserves many dynamical properties. For this purpose we introduce some definitions and notation.

Chaotic maps have three main properties: unpredictability, indecomposability and an element of regularity (see e.g. [26]). The first comes from sensitive dependence on initial conditions, the second is due to the impossibility of decomposing into two or more subsystems and the last one arises from the density of periodic points. However, unpredictability is compatible with the deterministic nature of chaotic systems. In fact, it is possible to study the long time behaviour of this kind of systems using tools from ergodic theory (see e.g. [17]). Using combinatorics and symbolic dynamics, we devote particular attention to a family of chaotic maps for which it is possible to obtain partial predictability for long-term behaviour.

6.1 Definitions

Definition 81. Let X be a topological space and $f : X \to X$ a map. Given a point $x \in X$, the sequence $(x, f(x), f(f(x)), \ldots, f^n(x), \ldots)$ is the ***orbit*** of x under f. A ***fixed point*** of f is a point such that $f(x) = x$. The ***set of fixed***

points of f is denoted by $\mathrm{Fix}(f)$. A ***periodic point*** is a point x such that $f^n(x) = x$ for some $n \in \mathbb{N}$, that is, a point in $\mathrm{Fix}(f^n)$. Such n is a ***period*** of x and its orbit is a n-***periodic orbit***. The smallest such n is called the ***prime period*** of x. The number of periodic points of f of period n is denoted by $\mathrm{Per}_n(f)$, that is, the number of fixed points for f^n. A point $x \in X$ is ***pre-periodic*** if there exists $k \geq 1$ such that $f^k(x)$ is periodic and the corresponding periodic orbit does not contain $x, \ldots, f^{k-1}(x)$.

We will denote, respectively, by $\mathcal{N}_n(f)$ and $\mathcal{O}_n(f)$ the set of periodic points and of periodic orbits (both of prime period n) of f and by $N_n(f)$ and $O_n(f)$ the corresponding cardinalities.

Adapting the symbolic dynamics construction of Milnor and Thurston [60] we have the following formal definitions.

Definition 82. A mapping f from the half-open interval J to itself has a ***piece-wise*** property P if J can be subdivided into finitely many intervals $J_0, \cdots, J_{p-1}$, on each of which f has property P. We denote $f_i := f_{|J_i}$, $i \in \{0, 1, \ldots, p-1\}$. If P refers to monotonicity, each such maximal interval of property P is a ***lap*** of f.

Remark 83. Examples of maps with piecewise properties were important in chapter 4, in terms of contractivity, injectivity, continuity and monotonicity.

Definition 84. The ***address*** $A(x)$ of a point $x \in J$ is the formal symbol j if x belongs to the lap J_j. The ***itinerary*** $I(x)$ is the sequence of addresses $\left(A(x), A(f(x)), A\left(f^2(x)\right), \cdots\right)$ of the successive images of x. $I_m(x)$ denotes the m-length string corresponding to the truncation $I(x)$ at the m^{th} symbol, and we call it m-***truncated itinerary***.

In definitions 86 and 87 we introduce some concepts of formal languages defined in [9,75].

Definition 85. An ***alphabet*** A is a finite non-empty totally ordered set of symbols (letters). A ***word*** or ***string*** over an alphabet A is a finite sequence of symbols taken from A. We denote by $\sum$ the set of words over an alphabet. A necklace is the equivalence class of a word under a circular shift.

Definition 86. A word $u \in \sum$ is ***primitive***, or ***aperiodic***, if

$$u = z^n \Rightarrow n = 1 \text{ (and hence } u = z\text{)},$$

where $z \in \sum$ and z^n means the n-times catenation of the word z.

Example 87. The Morse-Thue sequence is composed by primitive blocks $u = (x_0 x_1 \cdots x_p)$. This is true for $p = 2^k$, as proved in [1].

A ***Lyndon word*** $u \in \sum$ is a word which is primitive and the smallest one in its necklace with respect to the lexicographic ordering $(\prec)$. We say a Lyndon word is the representative word of a necklace. A necklace is called aperiodic if its representative word is primitive and periodic otherwise.

Definition 88. [37] For $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$, where for $i \neq j$, $p_i \neq p_j$ the ***Möbius function*** $\mu(n)$ is defined as follows:

(i) $\mu(1) = 1$;
(ii) $\mu(n) = 0$ if any $\alpha_j \geq 2$;
(iii) $\mu(n) = (-1)^k$ if $\alpha_1 = \alpha_2 = \dots = \alpha_k = 1$.

Remark 89. The original notation of this function was a_n.

Adapting the notation in [59] we will denote, respectively, by $\mathcal{S}(p,n)$, $\mathcal{M}(p,n)$ and $\mathcal{L}(p,n)$ the set of primitive words, necklaces and Lyndon words of length n over an alphabet with p symbols and by $S(p,n)$, $M(p,n)$ and $L(p,n)$ the corresponding cardinalities. A classical result (see e.g. [59]) states that these are given by

$$M(p,n) = L(p,n) = \frac{1}{n}\sum_{d|n} \mu(d)\, p^{n/d},\ S(p,n) = \sum_{d|n} \mu(d)\, p^{n/d}. \tag{6.1}$$

6.2 Piecewise affine interval maps

6.2.1 Periodic and pre-periodic orbits and points

We will study throughout the dynamics of the piecewise affine interval map

$$g_p(x) = px \pmod 1, \tag{6.2}$$

see, e.g. [34].

Remark 90. This map can be seen as a rigid uniformly expanding degree p circle map via the canonical lifting $\pi(x) : x \mapsto e^{2\pi i x}$.

Denoting by $I_j = [j/p, (j+1)/p)$, the interval map g_p may be written as $g_p(x) = px - j$ if $x \in I_j$, where j ranges from 0 to $p-1$. We associate the symbol j to $x \in I_j$, so that Definitions 83 and 85 apply in the obvious way.

The following theorem (see [50]) is a consequence of counting periodic points of (6.2).

Theorem 91. *Let $p, n \in \mathbb{N}$, with $p, n \geq 2$. Then*

$$O_n(g_p) = \frac{1}{n}\sum_{d|n} \mu(d)\, p^{n/d},\ N_n(g_p) = \sum_{d|n} \mu(d)\, p^{n/d}. \tag{6.3}$$

We now describe some properties of the periodic orbits and points of g_p.

For each $0 \leq j \leq p-1$, we denote by $g_{p|j}$ the map that is equal to g_p defined in the lap (interval of monotonicity) denoted by I_j. For each $m \in \mathbb{N}$, $p \geq 2$ and $x \in [0,1)$ there exists a unique sequence $I_m(x) = (j_1, j_2, \cdots, j_m)$ such that $g_p^m(x) = g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x)$. The sequence is called, as stated in definition 85, the m-truncated itinerary of x.

Lemma 92. *Let $x \in [0,1)$ such that $I_m(x) = (j_1, j_2, \cdots, j_m)$. Then*

$$g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) = p^m x - \sum_{k=1}^{m} j_k p^{m-k}.$$

Proof. The proof proceeds by induction on m. For $m = 1$ it is immediate.

For the induction step, suppose $g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) = p^m x - \sum_{k=1}^{m} j_k p^{m-k}$. Then

$$\begin{aligned} g_{p|j_{m+1}} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) &= g_{p|j_{m+1}} \circ g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) \\ &= g_{p|j_{m+1}} \left(p^m x - \sum_{k=1}^{m} j_k p^{m-k} \right) \\ &= p \left(p^m x - \sum_{k=1}^{m} j_k p^{m-k} \right) - j_{m+1} \\ &= p^{m+1} x - \sum_{k=1}^{m+1} j_k p^{m+1-k}, \end{aligned}$$

that is, $g_{p|j_{m+1}} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) = p^{m+1} x - \sum_{k=1}^{m+1} j_k p^{m+1-k}$. □

Proposition 93. *Let x be a m-periodic point of g_p, i.e., $g_p^m(x) = x$ with truncated itinerary $I_m(x) = (j_1, \cdots, j_m)$. Then*

$$x = \frac{\sum_{k=1}^{m} j_k p^{m-k}}{p^m - 1}. \tag{6.4}$$

Proof. If the truncated itinerary is $I_m(x) = (j_1, \cdots, j_m)$, then $g_p^m(x) = g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x)$. Now (6.4) is an immediate consequence of Lemma 93. □

Conversely we have the following result.

Proposition 94. *Let $m \in \mathbb{N}$, $p \geq 2$ and $x \in [0,1)$ such that for $k \in \{1, \ldots, m\}$ and $j_k \in \{0, 1, \ldots, p-1\}$ equation (6.4) holds. Then x is a m-periodic point of g_p with truncated itinerary $I_m(x) = (j_1, \cdots, j_m)$.*

Proof. We first prove $x \in [j_1/p, (j_1+1)/p)$. For the lower bound we have

$$\frac{\sum_{k=1}^{m} j_k p^{m-k}}{p^m - 1} \geq \frac{j_1 p^{m-1}}{p^m} + \frac{\sum_{k=2}^{m} j_k p^{m-k}}{p^m} \geq \frac{j_1}{p}.$$

For the upper bound, since $j_k \leq p-1, \forall k \in \{1,\ldots,m\}$ we have

$$\frac{\sum\limits_{k=1}^{m} j_k p^{m-k}}{p^m-1} - \frac{j_1+1}{p} = \frac{\sum\limits_{k=2}^{m} j_k p^{m-k+1} - p^m + j_1 + 1}{p\left(p^m-1\right)} \leq \frac{\sum\limits_{k=2}^{m} \left(p-1\right) p^{m-k+1} - p^m + p}{p\left(p^m-1\right)} = 0.$$

Thus

$$\frac{\sum\limits_{k=1}^{m} j_k p^{m-k}}{p^m-1} \leq \frac{j_1+1}{p},$$

with equality if and only if $j_k = p-1, \forall k \in \{1,\ldots,m\}$. In this case $x=1$, contradicting $x \in [0,1)$. Thus

$$\frac{\sum\limits_{k=1}^{m} j_k p^{m-k}}{p^m-1} < \frac{j_1+1}{p},$$

proving that $x \in \left[j_1/p, \left(j_1+1\right)/p\right)$.

Suppose $r \in \{0,1,\ldots,m-1\}$ and for any positive integer $s \leq r$, $g_p^s\left(x\right) \in \left[\left(j_{s+1}\right)/p, \left(j_{s+1}+1\right)/p\right)$. We want to show $g_p^{r+1}\left(x\right) \in \left[\left(j_{r+2}\right)/p, \left(j_{r+2}+1\right)/p\right)$. This is equivalent to the fact that x has a truncated itinerary $I_{r+1}\left(x\right) = \left(j_1,\cdots,j_{r+1}\right)$. By Lemma 93 and by definition of g_p

$$\begin{aligned} g_p^{r+1}\left(x\right) &= g_{p|j_{r+1}} \circ g_{p|j_r} \circ \cdots \circ g_{p|j_1}\left(x\right) = g_{p|j_{r+1}}\left(p^r x - \sum_{k=1}^{r} j_k p^{r-k}\right) \\ &= p\left(p^r x - \sum_{k=1}^{r} j_k p^{r-k}\right) - j_r = p^{r+1}x - \sum_{k=1}^{r} j_k p^{r-k+1} - j_r. \end{aligned}$$

Since x satisfies (6.4), we obtain

$$\begin{aligned} g_p^{r+1}\left(x\right) &= p^{r+1}\frac{\sum\limits_{k=1}^{m} j_k p^{m-k}}{p^m-1} - \sum_{k=1}^{r} j_k p^{r-k+1} - j_r \\ &= \frac{\sum\limits_{k=r+2}^{m} j_k p^{m-k+r+1} + \sum\limits_{k=1}^{r} j_k p^{r-k+1} + j_r}{p^m-1}. \end{aligned}$$

For the lower bound we have

$$\begin{aligned} g_p^{r+1}\left(x\right) &\geq \frac{\sum\limits_{k=r+2}^{m} j_k p^{m-k+r+1} + \sum\limits_{k=1}^{r} j_k p^{r-k+1} + j_r}{p^m} \\ &= \frac{j_{r+2}}{p} + \frac{\sum\limits_{k=r+3}^{m} j_k p^{m-k+r+1} + \sum\limits_{k=1}^{r} j_k p^{r-k+1} + j_r}{p^m} \geq \frac{j_{r+2}}{p}. \end{aligned}$$

For the upper bound, since $j_k \leq p-1, \forall k \in \{1, \ldots, m\}$ we have

$$\begin{aligned} g_p^{r+1}(x) - \frac{j_{r+2}+1}{p} &= \frac{\sum\limits_{k=r+2}^{m} j_k p^{m-k+r+1} + \sum\limits_{k=1}^{r} j_k p^{r-k+1} + j_r}{p^m - 1} - \frac{j_{r+2}+1}{p} \\ &\leq \frac{\sum\limits_{k=r+2}^{m} (p-1) p^{m-k+r+2} + \sum\limits_{k=1}^{r} (p-1) p^{r-k+2}}{p^m - 1} \\ &\quad + \frac{(p-1)p - (p-1)p^m - p^m + p}{p^m - 1} = 0. \end{aligned}$$

Similarly equality is true if and only if $j_k = p-1, \forall k \in \{1, \ldots, m\}$, which is not verified because $x \in [0,1)$. Thus $g_p^{r+1}(x) < (j_{r+2}+1)/p$.

To show that x is n-periodic, observe that, since we know the truncated itinerary, by Lemma 93

$$\begin{aligned} g_{p|j_n} \circ \cdots \circ g_{p|j_1}(x) = p^n x - \sum_{k=1}^{n} j_k p^{n-k} &= p^n \frac{\sum\limits_{k=1}^{n} j_k p^{n-k}}{p^n - 1} - \sum_{k=1}^{n} j_k p^{n-k}, \\ &= \frac{\sum\limits_{k=1}^{n} j_k p^{n-k}}{p^n - 1} = x, \end{aligned}$$

which finishes the proof. □

Proposition 95. *The n-periodic points of g_p are $j/(p^n-1)$, for $j = 0, \cdots, p^n - 1$.*

This is a consequence of the last result. It is also proved in [34].

Lemma 96. *Let $p, s, m \in \mathbb{N}$ such that $p \geq 2$ and let $x \in [0,1)$ be such that*

$$I_{sm}(x) = \left(\underbrace{j_1, \ldots, j_m}_{1}, \underbrace{j_1, \ldots, j_m}_{2}, \ldots, \underbrace{j_1, \ldots, j_m}_{s} \right).$$

Then

$$\underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{s} \circ \cdots \circ \underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{1}(x) = p^{sm} x - \sum_{k=1}^{m} j_k \sum_{r=1}^{s} p^{rm-k}. \quad (6.5)$$

Proof. The proof proceeds by induction on s. For $s = 1$ the result is Lemma

93. Suppose (6.5) is verified for a given $s \in \mathbb{N}$. Then

$$\begin{aligned}
&\underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{s+1} \circ \cdots \circ \underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{1}(x) \\
&\qquad = g_{p|j_m} \circ \cdots \circ g_{p|j_1}\left(p^{sm}x - \sum_{k=1}^{m} j_k \sum_{r=1}^{s} p^{rm-k}\right) \\
&\qquad = p^m\left(p^{sm}x - \sum_{k=1}^{m} j_k \sum_{r=1}^{s} p^{rm-k}\right) - \sum_{k=1}^{m} j_k p^{m-k} \\
&\qquad = p^{(s+1)m}x - \sum_{k=1}^{m} j_k \sum_{r=1}^{s+1} p^{rm-k},
\end{aligned}$$

which concludes the proof. $\square$

Suppose that x is a periodic orbit of g_p of period $2m$ and that its $2m$-truncated itinerary is the catenation of two identical m-blocks of symbols. The next result states that x is necessarily m-periodic.

Lemma 97. *Let x be a $2m$-periodic point of g_p such that $g_p^{2m}(x) = x$ with truncated itinerary $I_{2m}(x) = (j_1, \cdots, j_m, j_1, \cdots, j_m)$. Then $g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) = x$, i.e., x is m-periodic.*

Proof. By Lemma 97 we have

$$\begin{aligned}
&g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1} \circ g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) \\
&\qquad = p^{2m}x - \sum_{k=1}^{m} j_k\left(p^{2m-k} + p^{m-k}\right).
\end{aligned}$$

Then

$$x = \frac{\sum_{k=1}^{m} j_k\left(p^{2m-k} + p^{m-k}\right)}{p^{2m}-1} = \frac{\sum_{k=1}^{m} j_k p^{m-k}}{p^m - 1},$$

and thus x is m-periodic. $\square$

Generalizing Lemma 98 we have

Proposition 98. *Let $s \in \mathbb{N}$ and let x be a sm-periodic point of g_p such that $g_p^{sm}(x) = x$ with truncated itinerary*

$$I_{sm}(x) = \left(\underbrace{j_1, \ldots, j_m}_{1}, \underbrace{j_1, \ldots, j_m}_{2}, \ldots, \underbrace{j_1, \ldots, j_m}_{s}\right),$$

that is,

$$g_p^{sm}(x) = \underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{s} \circ \cdots \circ \underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{2} \circ \underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{1}(x).$$

Then $g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) = x$, i.e., x is m-periodic.

Proof. By Lemma 97 we have

$$\underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{s} \circ \cdots \circ \underbrace{g_{p|j_m} \circ \cdots \circ g_{p|j_1}}_{1}(x) = p^{sm}x - \sum_{k=1}^{m} j_k \sum_{r=1}^{s} p^{rm-k}.$$

Then

$$x = \frac{\sum\limits_{k=1}^{m} j_k \sum\limits_{r=1}^{s} p^{rm-k}}{p^{sm}-1}.$$

On the other hand, a direct computation shows that

$$(p^m - 1)\left(\sum_{k=1}^{m} j_k \sum_{r=1}^{s} p^{rm-k}\right) = (p^{sm}-1)\sum_{k=1}^{m} j_k p^{m-k}.$$

It follows that

$$x = \frac{\sum\limits_{k=1}^{m} j_k \sum\limits_{r=1}^{s} p^{rm-k}}{p^{sm}-1} = \frac{\sum\limits_{k=1}^{m} j_k p^{m-k}}{p^m - 1},$$

implying that x is actually m-periodic and $g_{p|j_m} \circ \cdots \circ g_{p|j_2} \circ g_{p|j_1}(x) = x$. □

Remark 99. Proposition 99 implies that if a m-truncated itinerary of a periodic point with prime period m is considered as a word of a formal language, then it is a primitive word in the sense of Definition 87.

We may summarize these results in the following theorem.

Proposition 100. *Consider the map g_p. Its m-periodic points are of the form*

$$x = \frac{\sum\limits_{k=1}^{m} j_k p^{m-k}}{p^m - 1},$$

where $p \geq 2$ is an integer and $0 \leq j_k < p$, and the pre-periodic points are of the form

$$x = \frac{\sum\limits_{k=1}^{m} j_k p^{m-k-n}}{p^m - 1} + \sum_{k=1}^{n} i_k p^{-k},$$

where $0 \leq j_k < p$ is an integer and $0 \leq i_k < p$.

6.2.2 Identification of dynamical and combinatorial objects

In view of equations (6.1) and (6.3) it is possible to establish bijections between the corresponding sets. In fact a natural way to construct these bijections suggests itself, since by remark 100 combinatorial primitiveness in the sense of Definition 87 is a property of the truncated itineraries of periodic points.

Definition 101. Let $p, n \in \mathbb{N}$, such as $p, n \geq 2$. We define the maps

$$\begin{array}{rccc} B_\bullet : & \mathcal{S}(p,n) & \to & \mathcal{N}_n(g_p) \\ & u = u_1 \cdots u_n & \mapsto & \frac{\sum_{k=1}^{n} u_k p^{n-k}}{p^n - 1} \end{array},$$

$$\begin{array}{rccc} B_o : & \mathcal{L}(p,n) & \to & \mathcal{O}_n(g_p) \\ & u & \mapsto & \left[\left(B_\bullet(u), g_p(B_\bullet(u)), g_p^2(B_\bullet(u)), \ldots, g_p^{n-1}(B_\bullet(u))\right)\right] \end{array},$$

where $\left[\left(x, g_p(x), g_p^2(x), \ldots, g_p^{n-1}(x)\right)\right]$ denotes a n-periodic orbit of g_p.

Theorem 102. *The maps $B_\bullet$ and B_o are bijections.*

Proof. The map $B_\bullet$ is bijective by Propositions 94 and 95.

For B_o we first prove injectivity. Let $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ be distinct Lyndon words of $\mathcal{L}(p;n)$. By definition they represent different necklaces, that is, they correspond to truncated itineraries which are not a cyclic permutation of each other. Since $B_\bullet$ is a bijection, $B_\bullet(u) \neq B_\bullet(v)$. Also $B_\bullet(u)$ is different from any of $B_\bullet(u)$, $g_p(B_\bullet(u))$, $g_p^2(B_\bullet(u))$, ..., $g_p^{n-1}(B_\bullet(u))$. Thus B_o is injective.

We now prove surjectivity of B_o. Let $(x_1, x_2, \ldots, x_n) \in \mathcal{O}_n(g_p)$. Then $x_1, x_2, \ldots, x_n$ are periodic points of g_p of prime period n, i.e., $x_1, x_2, \ldots, x_n \in \mathcal{N}_n(g_p)$. Since all points belong to the same periodic orbit, the respective truncated itineraries $I_n(x_i)$ are related by cyclic permutations. Let v be the Lyndon word corresponding to one of these n-truncated itineraries $I_n(x_i)$. Let $y \in \{x_1, x_2, \ldots, x_n\}$ the point which has this truncated itinerary. Then

$$B_o(v) = \left[\left(B_\bullet(y), g_p(B_\bullet(y)), g_p^2(B_\bullet(y)), \ldots, g_p^{n-1}(B_\bullet(y))\right)\right]$$

which coincides with $[(x_1, x_2, \ldots, x_n)]$. Thus B_o is surjective, and therefore a bijection. $\square$

The existence of a bijection between $\mathcal{M}(p,n)$ and $\mathcal{O}_n(g_p)$ is now a consequence of the bijection B_o.

Definition 103. Let $p, n \in \mathbb{N}$, such as $p, n \geq 2$. We define the bijection

$$\begin{array}{rccc} B : & \mathcal{M}(p,n) & \to & \mathcal{O}_n(g_p) \\ & [u] & \mapsto & \left[\left(B_\bullet(u), g_p(B_\bullet(u)), g_p^2(B_\bullet(u)), \ldots, g_p^{n-1}(B_\bullet(u))\right)\right] \end{array}.$$

Theorem 104. *The map B is a bijection.*

Proof. It suffices to observe that the map B is induced by B_o on the corresponding equivalence classes. $\square$

Remark 105. As a consequence of a Lemma 2.4 in [33] these bijections preserve order (the usual order for real numbers and the lexicographic order of itineraries).

Theorem 105 establishes a relation between equivalence classes (aperiodic necklaces and periodic orbits). Theorem 103 refines this, by specifying the meaning of the bijection to the elements of the equivalence classes (primitive words and periodic points). In fact, it is equivalent to define the relation between the set of periodic orbits and the set of aperiodic necklaces or between the set of periodic points and the set of primitive words.

6.3 Piecewise expanding maps: enumeration of periodic and pre-periodic points

In this section we show how the study of periodic and pre-periodic points of maps topologically conjugate to the ones in the previous sections is greatly simplified by our results. In fact, we reduce the study of maps belonging to topologically conjugate families to the study of the piecewise affine case. Topological conjugacy ensures that properties related to periodic and pre-periodic points are preserved.

Definition 106. We denote by $\mathcal{M}_f$ the equivalence class of maps topologically conjugate with f.

Let $F \in \mathcal{M}_{g_p}$ a piecewise expanding map and denote $F_i := F_{|Y_i}$ for the partition $Y = \bigcup_{i=0}^{p-1} Y_i$.

We now recall the work developed in chapter 4 regarding the conjugacy equation (4.1). An application of Corollary 71 allows us to topologically conjugate the piecewise affine map f given by (2.5), (6.2), i.e., $f \equiv g_p$ with a piecewise expanding map F.

It easily shown (see e.g. Block and Coppel [13]) that the periodic points and orbits of topologically conjugate maps are in 1-1 correspondence. We state this result formally for future reference:

Lemma 107. *Suppose $f : X \to X$ and $F : Y \to Y$ are topologically conjugate maps, and let φ be the corresponding conjugacy. Then a point $x \in X$ is a periodic point of f of period n if and only if the point $\varphi(x) \in Y$ is a periodic point of period n for F.*

The following theorem about periodic points is a consequence of Lemma 108, Proposition 101, formula (2.8) and Corollary 71.

Corollary 108. *The m-periodic points of F are of the form*

$$x = \varphi\left(\frac{\sum_{k=1}^{m} j_k p^{m-k}}{p^m - 1}\right),$$

where $m \geq 1$, $0 \leq j_1, j_2, \ldots, j_m < p$ and s are integers and $\varphi : [0,1] \to [0,1]$ is the homeomorphism

$$\varphi\left(\sum_{k=1}^{\infty} \frac{\xi_k}{p^k}\right) = \lim_{\nu\to\infty} F_{\xi_1}^{-1} \circ \cdots \circ F_{\xi_\nu}^{-1}(\xi), \text{ for every } \xi \in X. \tag{6.6}$$

Remark 109. It is interesting to compare this result with Theorem 72, where a general formula for conjugacy equations is given, irrespective of continuity of solutions. In Corollary 109 φ is a homeomorphism, a particular case of Theorem 72. It is Corollary 71 that enables us to specify the map φ. It gives two possibilities for homeomorphic solutions, one increasing and one decreasing. In Corollary 109 we consider the increasing solution, since it is the natural one (it preserves the order of indices).

A similar result is given for pre-periodic points.

Proposition 110. *The pre-periodic points of F are of the form*

$$x = \varphi\left(\frac{\sum_{k=1}^{m} j_k p^{m-k-n}}{p^m - 1} + \sum_{k=1}^{n} i_k p^{-k}\right).$$

where $m, n \geq 1$, $0 \leq j_1, j_2, \ldots, j_m < p$, $-p < i_1, i_2, \ldots, i_n < p$ are integers and $\varphi : [0,1] \to [0,1]$ is the homeomorphism

$$\varphi\left(\sum_{k=1}^{\infty} \frac{\xi_k}{p^k}\right) = \lim_{\nu\to\infty} F_{\xi_1}^{-1} \circ \cdots \circ F_{\xi_\nu}^{-1}(\xi)\,, \text{ for every } \xi \in X.$$

In the sequel of Corollary 109 and Proposition 111 it is possible to enumerate the periodic and pre-periodic points of a piecewise expanding map $F \in \mathcal{M}_{g_p}$ with a given itinerary. For a periodic point the procedure is the following:

(i) From Proposition 94 obtain a periodic point x of g_p of the form (6.4);
(ii) Represent x as a base p expansion;
(iii) Compute formula (6.6).

For pre-periodic points the procedure is the same adapting (i).

However, this procedure is not the most efficient method to get this enumeration. In fact the identification between the two representations of numbers given in Lemma 112 gives a better way to do this.

Lemma 111. *Let $\{\xi_k\}_{k\in\mathbb{N}}$ be a sequence of integers belonging to $\{0, 1, \ldots, p-1\}$. If*

$$(\xi_1, \xi_2, \ldots, \xi_k, \ldots) = (i_1, i_2, \ldots, i_n, j_1, j_2, \ldots, j_m, j_1, j_2, \ldots, j_m, j_1, j_2, \ldots, j_m, \ldots)\,,$$

then

$$\sum_{k=1}^{\infty} \frac{\xi_k}{p^k} = \frac{\sum_{k=1}^{m} j_k p^{m-k-n}}{p^m - 1} + \sum_{k=1}^{n} i_k p^{-k}. \tag{6.7}$$

Proof. The proof proceeds by representing the series of the left hand side of

(6.7) with a finite sum of series:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\xi_k}{p^k} &= \sum_{k=1}^{n}\frac{i_k}{p^k}+\sum_{l=0}^{\infty}\sum_{k=lm+n+1}^{lm+n+m}\frac{j_k}{p^k}=\sum_{k=1}^{n}\frac{i_k}{p^k}+\sum_{l=0}^{\infty}\sum_{k=1}^{m}\frac{j_{k+lm+n}}{p^{k+lm+n}}\\
&= \sum_{k=1}^{n}\frac{i_k}{p^k}+\sum_{l=0}^{\infty}\sum_{k=1}^{m}\frac{j_k}{p^{k+lm+n}}=\sum_{k=1}^{n}\frac{i_k}{p^k}+\sum_{k=1}^{m}\sum_{l=0}^{\infty}\frac{j_k p^{-k-n}}{(p^m)^l}\\
&= \sum_{k=1}^{n}\frac{i_k}{p^k}+\sum_{k=1}^{m}\frac{j_k p^{-k-n}}{1-p^{-m}}=\sum_{k=1}^{n}\frac{i_k}{p^k}+\sum_{k=1}^{m}\frac{j_k p^{m-k-n}}{p^m-1}.
\end{aligned}$$

□

Now the enumeration of the periodic and pre-periodic points is immediate.

Theorem 112. *The m-periodic point of F with truncated itinerary $I_m(x) = (j_1, \cdots, j_m)$ is the fixed point of $F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}$.*

Proof. By Propositions 101, 111 and Lemmas 108, 112 the m-periodic point of F with truncated itinerary $I_m(x) = (j_1, \cdots, j_m)$ is

$$x = \lim_{k\to\infty} \underbrace{F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}}_{1} \circ \cdots \circ \underbrace{F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}}_{k}(\xi), \text{ for every } \xi \in X.$$

Since each F_k^{-1} is a contraction, $F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}$ is also a contraction. Then for any $\xi \in X$, x is the fixed point of $F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}$. □

Theorem 113. *The pre-periodic point of F with itinerary*

$$I(x) = (i_1, \ldots, i_n, j_1, \ldots, j_m, j_1, \ldots, j_m, j_1, \ldots, j_m, \ldots)$$

is $F_{i_1}^{-1} \circ \cdots \circ F_{i_n}^{-1}(x_{j_1,\ldots,j_m})$, where $x_{j_1,j_2,\ldots,j_m}$ is the fixed point of $F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}$.

Proof. By Propositions 101, 111 and Lemmas 108, 112 the pre-periodic point of F with truncated itinerary $I(x) = (i_1, \ldots, i_n, j_1, \ldots, j_m, j_1, \ldots, j_m, j_1, \ldots, j_m, \ldots)$ is

$$x = \lim_{k\to\infty} F_{i_1}^{-1} \circ \cdots \circ F_{i_n}^{-1} \circ \underbrace{F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}}_{1} \circ \cdots \circ \underbrace{F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}}_{k}(\xi), \text{ for every } \xi \in X.$$

Since each F_k^{-1} is a contraction, also $F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}$ is a contraction. Let $x_{j_1,j_2,\ldots,j_m}$ be the fixed point of $F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}$. Then for any $\xi \in X$,

$$\lim_{k\to\infty} \underbrace{F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}}_{1} \circ \cdots \circ \underbrace{F_{j_1}^{-1} \circ \cdots \circ F_{j_m}^{-1}}_{k}(\xi) = x_{j_1,j_2,\ldots,j_m}.$$

Since each F_k^{-1} is continuous, $F_{i_1}^{-1} \circ \cdots \circ F_{i_n}^{-1}$ is also continuous and we conclude $x = F_{i_1}^{-1} \circ \cdots \circ F_{i_n}^{-1}(x_{j_1,\ldots,j_m})$. □

In the following we provide an example where we compute three periodic/pre-periodic points of a piecewise non-affine expanding map, given the itinerary of each point.

Example 114. Let $F : [0,1] \to [0,1]$ be defined by

$$F(x) = \begin{cases} x^2 + 2x, & x \in I_0 = \left[0, \sqrt{2} - 1\right) \\ \frac{1}{2}x^2 + x - \frac{1}{2}, & x \in I_1 = \left[\sqrt{2} - 1, 1\right]. \end{cases}$$

This map is topologically conjugate with g_2.

(i) Let x be the 3-periodic point of F with truncated itinerary $I_3(x) = (0,1,1)$. By Theorem 113 x is the fixed point of $F_0^{-1} \circ F_1^{-1} \circ F_1^{-1}$.

The piecewise definition of F gives

$$F_0(x) = x^2 + 2x, \quad F_1(x) = \frac{1}{2}x^2 + x - \frac{1}{2},$$

$$F_0^{-1}(x) = \sqrt{x+1} - 1, \quad F_1^{-1}(x) = \sqrt{2x+2} - 1.$$

Then

$$\begin{aligned} F_1^{-1}\left(F_1^{-1}(x)\right) &= \sqrt{2\left(\sqrt{2x+2} - 1\right) + 2} - 1 = \sqrt{2\sqrt{2x+2}} - 1 \\ &= \sqrt{2}\sqrt[4]{2x+2} - 1, \end{aligned}$$

$$F_0^{-1}\left(F_1^{-1}\left(F_1^{-1}(x)\right)\right) = \sqrt{\sqrt{2}\sqrt[4]{2x+2}} - 1 = \sqrt[4]{2}\sqrt[8]{2x+2} - 1.$$

Since x is the fixed point of $F_0^{-1} \circ F_1^{-1} \circ F_1^{-1}$, x is the solution of equation $x = F_0^{-1}\left(F_1^{-1}\left(F_1^{-1}(x)\right)\right)$. This equation is

$$x = \sqrt[4]{2}\sqrt[8]{2x+2} - 1, \tag{6.8}$$

whose solution, belonging to the domain of F, is $x = 2^{3/7} - 1$.

Here is a confirmation of this fact:

$$\begin{aligned} x &= 2^{3/7} - 1 \in I_0; \\ F\left(2^{3/7} - 1\right) &= 2^{6/7} - 1 \in I_1; \\ F^2\left(2^{3/7} - 1\right) &= 2^{5/7} - 1 \in I_1; \\ F^3\left(2^{3/7} - 1\right) &= 2^{3/7} - 1 \in I_0. \end{aligned}$$

(ii) Let x be the pre-periodic point of F with itinerary $I(x) = (1,1,0,0,0,0,\ldots)$. By Theorem 114 $x = F_1^{-1}F_1^{-1}(x_0)$. The fixed point of F_0^{-1} is $x_0 = 0$. Then

$$x = F_1^{-1}F_1^{-1}(0) = \sqrt{2}\sqrt[4]{2 \times 0 + 2} - 1 = 2^{3/4} - 1.$$

We confirm this fact:

$$\begin{aligned} x &= 2^{3/4} - 1 \in I_1; \\ F\left(2^{3/4} - 1\right) &= \sqrt{2} - 1 \in I_1; \\ F^2\left(2^{3/4} - 1\right) &= 0 \in I_0; \\ F^3\left(2^{3/4} - 1\right) &= 0 \in I_0. \end{aligned}$$

(iii) Let x be the pre-periodic point x of F with itinerary $I(x) = (1, 1, 1, 0, 1, 0, 1, 0\ldots)$. By Theorem 114 $x = F_1^{-1} \circ F_1^{-1}(x_{1,0})$. Since $x_{1,0}$ is the fixed point of $F_1^{-1} \circ F_0^{-1}$, $x_{1,0}$ it is the solution of equation $x = F_1^{-1}\left(F_0^{-1}(x)\right)$. Since

$$F_1^{-1}\left(F_0^{-1}(x)\right) = \sqrt{2\left(\sqrt{x+1} - 1\right) + 2} - 1,$$

$x_{1,0}$ is the solution of equation

$$x = \sqrt{2\left(\sqrt{x+1} - 1\right) + 2} - 1,$$

belonging to the domain of F, which is $x_{1,0} = 2^{2/3} - 1$.

Then

$$\begin{aligned} x &= F_1^{-1} \circ F_1^{-1}\left(2^{2/3} - 1\right) = \sqrt{2}\sqrt[4]{2\left(2^{2/3} - 1\right) + 2} - 1 \\ &= 1 + 4\sqrt[6]{2} - \sqrt{2} \end{aligned}$$

is the pre-periodic point x of F with itinerary $I(x) = (1, 1, 1, 0, 1, 0, 1, 0\ldots)$.

We confirm this fact:

$$\begin{aligned} x &= 2^{11/12} - 1 \in I_1; \\ F\left(2^{11/12} - 1\right) &= 2^{5/6} - 1 \in I_1; \\ F^2\left(2^{11/12} - 1\right) &= 2^{2/3} - 1 \in I_1; \\ F^3\left(2^{11/12} - 1\right) &= 2^{1/3} - 1 \in I_0; \\ F^4\left(2^{11/12} - 1\right) &= 2^{2/3} - 1 \in I_1; \\ F^5\left(2^{11/12} - 1\right) &= 2^{1/3} - 1 \in I_0. \end{aligned}$$

Chapter 7

Number Theory

7.1 Fermat's Little Theorem

The map (6.2) presented in section 6.2 is rich with respect to possible results, not only for dynamical systems and functional equations, but also for number theory. In fact, Fermat's Little Theorem may be proved through this map and many other proofs are possible. Here we present a historical overview of this theorem and some generalizations. This section includes the paper [79] published in "*Gazeta da Matemática*" of SPM - Sociedade Portuguesa da Matemática.

7.1.1 Proofs

Among the areas of mathematics in which Fermat engaged, number theory have been the one that most involved him. Of special significance was a result that became known as Little Theorem of Fermat. Although today several proofs are known, the construction of a rigorous proof took almost a century (through Euler, in 1736). Nevertheless, it seems to have been Leibniz the first to prove in an unpublished work. This theorem was stated in a letter that Fermat wrote to a correspondant, Bernhard Frenicle of Bessy, in 1640, where he said that he would not send the proof for fear of being too long (see [19,64]).

Theorem 115. ***(Fermat's Little Theorem)*** *If p is a prime number and a is an arbitrary integer not divisible by p, then p divides $a^{p-1} - 1$.*

This theorem can be proved using various mathematical techniques and various areas of mathematics. As a result of number theory, the most natural proof arises from the manipulation of the divisibility of numbers.

The first published proof is due to Euler (see [12]).

Proof. By the binomial theorem

$$(a+1)^p \equiv a^p + 1 \pmod{p}, \tag{7.1}$$

since, in fact,

$$\tbinom{p}{k} \equiv 0 \pmod{p},$$

for $0 < k < p$. Subtracting $a+1$ from both sides of the congruence (7.1),

$$(a+1)^p - (a+1) \equiv a^p - a \pmod{p}. \tag{7.2}$$

By induction, it can be seen firstly that $1^p - 1$ is divisible by p.

Suppose that $a^p - a$ is divisible by p. By (7.2), $(a+1)^p - (a+1)$ is divisible by p. This completes the proof by induction for

$$a^p \equiv a \pmod{p}. \tag{7.3}$$

Thus, multiplying the latter congruence by the multiplicative inverse of $a \pmod{p}$, we obtain the result in the classic form

$$a^{p-1} \equiv 1 \pmod{p}, \tag{7.4}$$

where a and p are relatively prime. □

Remark 116. Note that the formulation (7.3) is a little more general than the original (7.4), because in this case it is not necessary that a and p be relatively prime, that is, one can consider any $a \in \mathbb{N}$. However, they are still two equivalent formulations, so sometimes the Little Fermat's Theorem is stated of the form (7.3).

In 2008 Bishop published a proof (see [12]) using techniques available at the time of Fermat, but that is original in that it had not yet been given this approach.

Proof. For the proof of (7.4) p is supposed to be a prime odd number, because for the only even prime the proof is trivial. Let

$$f(x) = x^{p-1} - 1. \tag{7.5}$$

Performing the Taylor series development of the function around $x = 1$ the following expansion is obtained:

$$f(x) = (p-1)(x-1) + \frac{1}{2!}(p-1)(p-2)(x-1)^2 + \cdots + \frac{1}{(p-1)!}(p-1)!\,(x-1)^{p-1}.$$

Note that this series may also be obtained by applying the binomial theorem to the expression

$$(1 + (x-1))^{p-1} - 1.$$

Consider now the values of x that are divisible by p. For these values of x, $f(x)$ is equal to a multiple of p minus 1, which therefore is not divisible by p.

Consider now $x = kp + c$, with $c \in \mathbb{Z}$ and $0 < c < p$. Then by equation (7.5) $\pmod{p}$,

$$f(kp + c) \equiv f(c) \pmod{p}.$$

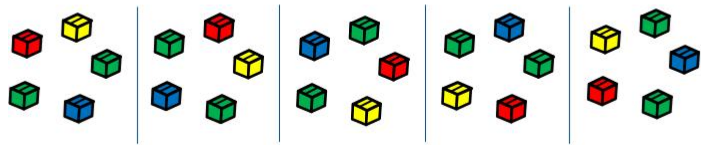

Figure 7.1: Illustration of the box counting proof of Fermat's Little Theorem.

Thus it is only necessary to consider the values of x such that $0 < x < p$.

The proof of the theorem proceeds by induction in $n = kp + c$. The base case is $x = 1$. Note that $f(1) = 0$ and therefore is divisible by p. Suppose that $f(n)$ is divisible by p, with $0 < n < p - 1$, that is, $n^{p-1} - 1$ is divisible by p. Notice that $f(n+1)$ is divisible by p.

Then

$$f(n+1) = (p-1)\,n + \frac{1}{2!}(p-1)(p-2)\,n^2 + \cdots + \frac{1}{(p-1)!}(p-1)!n^{p-1}.$$

Since

$$\binom{p-1}{k} = \frac{(p-1)\cdots(p-k)}{k!} \equiv (-1)^k \pmod{p},$$

then

$$f(n+1) \equiv -n + n^2 - n^3 + \cdots + (-1)^k\, n^{p-1} \pmod{p}.$$

Therefore $f(n+1)$ is congruent with the sum of a geometric progression with ratio $-n$, which, using the formula for the sum thereof, is

$$f(n+1) \equiv \frac{-n + n^p}{1+n} \pmod{p}.$$

Factoring this latter congruence,

$$f(n+1) \equiv \frac{n\left(-1 + n^{p-1}\right)}{1+n} \pmod{p}.$$

Since $0 < n < p - 1$, $1 + n$ is not divisible by p. By hypothesis $n^{p-1} - 1$ is divisible by p, which proves the result by induction. □

Although these proofs do not offer great difficulty, there is an alternative which is much more intuitive and accessible (even for non-mathematicians) that comes from object counting. Indeed, it was in 1872 that Petersen (see [91]) presented the proof that we now describe.

"Let p boxes, arranged in a circle, to be coloured with a colours. There are in total a^p possible colouring ways, a ways of colouring if all boxes stay with the same colour. The remaining colouring possibilities $a^p - a$ can be grouped into sets of p elements, since the p possible rotations of these colours are all different. Whereby, $p|a^p - a$."

This proof uses concepts that today are studied more systematically, with terminology already established. Thus, the concepts involved are necklaces and words formed from letters of an alphabet. Here necklaces represent equivalence classes (rotations), the words represent forms of colouring and letters are the available colours in the palette (alphabet). The precise notion of necklace appeared explicitly in a paper [54] of MacMahon in 1892. These mathematical objects appear in a relatively new area of mathematics, the combinatorics on words. A decisive contribution in this area came from volumes written under the pseudonym Lothaire, the first appeared [53] in 1983 and whose third volume [52] was published in 2005. However, as a precursor of this issue are the research works of patterns repetitions in words made by Axel Thue (papers [93] and [92] respectively from 1906 and 1912). For more details on the history of this branch of mathematics see [9].

		
Pierre de Fermat (1601-1665)	Leonhard Euler (1707-1783)	Julius Petersen (1839-1910)

Recently, an alternative has been presented as a proof using dynamical systems (see, e.g. [34], [40] and [50]). In schematic terms the dynamic proof is as follows.

For coherence with the notation in this section we define the map g_a, with a as the parameter, since here, frequently, p is a prime number (see definition of map g_p in section 6.2).

Proof. Consider the map $g_a : [0,1] \to [0,1]$ defined in (6.2) now defined by

$$g_a(x) = \begin{cases} ax \,(\text{mod } 1)\,, & \text{se } x \neq 1 \\ 1, & \text{se } x = 1. \end{cases} \tag{7.6}$$

It is easy to see that if a a positive integer, g_a has a fixed points, for example, by observing the graph of the map (Figure 7.2).

Also is not difficult to show that $\forall a, b \in \mathbb{N}, g_a(g_b) = g_{ab}$, and in particular that $\forall a, n \in \mathbb{N}, g_a^n = g_{a^n}$.

Moreover, the periodic points with period p are the fixed points of $g_a^p = g_{a^p}$. It follows that there are exactly $a^p - a$ periodic points of g_a whose minimum period is p. And this implies that $p | (a^p - a)$ as required. $\square$

Another possible proof, more algebraic in flavour comes from Group Theory. Basically the idea is analogous to the above proof, in the sense that the order of an element of a finite group divides the order of the group (see, e.g. [63]).

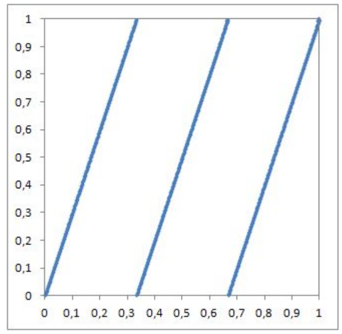

Figure 7.2: Graph of g_3.

In summary, these three latter statements explore a way to count mathematical objects that can be grouped into sets of p elements, where p is a prime number. Of these three, the proof elaborated by Petersen is the most simple and intuitive as possible.

7.1.2 Generalizations

Euler, author of the first known proof of Fermat's Little Theorem, was also the author of the first and best-known generalization which is also commonly referred to as Euler's theorem (see, e.g. [19] and [37]). In this result the Euler function (presented around 1760) is used.

Definition 117. ***Euler's totient function*** is the function $\phi(m)$ which counts the number of positive integers $\leq m$, that are relatively prime with m, that is, m.d.c. $(n, m) = 1$.

Theorem 118. *(**Euler's Theorem**) Let a, n be integers such that m.d.c. $(a, n) = 1$. Then*

$$a^{\phi(n)} \equiv 1 \ (mod\ n). \tag{7.7}$$

Note that, in the particular case when n is prime, (7.7) is Fermat's Little Theorem. However, Euler's theorem is valid for any positive integer n.

Proof. [19] First it is assumed that $n = p^k$, with p prime, $p \nmid a$ and $k > 0$ and is proved by induction for k that

$$a^{\phi\left(p^k\right)} \equiv 1 \ (\mathrm{mod}\ p^k). \tag{7.8}$$

For $k = 1$, (7.8) reduces to $a^{\phi(p)} \equiv 1 \ (\mathrm{mod}\ p)$, which is Fermat's Little Theorem.

Suppose that (7.8) holds for some k. We want to prove that it holds also for $k + 1$.

Since Euler's function is multiplicative, $(\phi\left(p^{k+1}\right) = p\phi\left(p^k\right))$,

$$a^{\phi\left(p^{k+1}\right)} = a^{p\phi\left(p^k\right)} = \left(a^{\phi\left(p^k\right)}\right)^p .$$

By (7.8) it is known that there is an integer q such that $a^{\phi\left(p^k\right)} = 1 + qp^k$. Using the binomial theorem

$$\begin{aligned} a^{\phi\left(p^{k+1}\right)} &= \left(a^{\phi\left(p^k\right)}\right)^p = \left(1+qp^k\right)^p \\ &= 1 + \binom{p}{1} qp^k + \binom{p}{2}\left(qp^k\right)^2 + \cdots + \binom{p}{p-1}\left(qp^k\right)^{p-1} + \left(qp^k\right)^p \\ &\equiv 1 + \binom{p}{1} qp^k \pmod{p^{k+1}}. \end{aligned}$$

But $p|\binom{p}{1}$, whereby $p^{k+1}|\binom{p}{1}qp^k$. Applying this result to the last congruence

$$a^{\phi\left(p^{k+1}\right)} \equiv 1 \pmod{p^{k+1}}$$

which concludes the proof by induction for (7.8).

Consider m.d.c. $(a, n) = 1$ and the factorization of n in prime numbers $n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$. For each $i \in \{1, 2, \ldots, r\}$ apply the congruence (7.8):

$$a^{\phi\left(p_i^{k_i}\right)} \equiv 1 \pmod{p_i^{k_i}}.$$

Since $\phi(n)$ is divisible by $\phi\left(p_i^{k_i}\right)$, rising each member of these congruences to the power $\phi(n)/\phi\left(p_i^{k_i}\right)$ we obtain

$$a^{\phi(n)} \equiv 1 \pmod{p_i^{k_i}}.$$

Since the $p_i^{k_i}$ are relatively prime,

$$a^{\phi(n)} \equiv 1 \pmod{p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}}$$

which is the same as (7.7). □

A big generalization of Fermat's Little Theorem was given by Gauss, who discovered a way to build congruences with any positive integers a and n. Now it is not required that a and n be relatively prime. According to Dickson [27] this result was known from a Gaussian posthumous article published in 1863 which stated that if $N = p_1^{e_1} \cdots p_s^{e_s}$, where $p_1, \cdots, p_s$ are distinct primes, then

$$F(a, N) = a^N - \sum_{i=1}^{s} a^{N/p_i} + \sum_{i<j} a^{N/p_i p_j} - \sum_{i<j<k} a^{N/p_i p_j p_k} + \ldots + (-1)^s a^{N/p_1 \ldots p_s}$$

is divisible by N, for the particular case of a being a prime. In the years 1882 and 1883 four direct proofs of this result were given by Kantor, Weyr, Luke and Pellet. However this form of presentation is neither elegant, nor very practical.

In his magisterial work *Disquisitiones arithmeticae*[1], published in 1801 (see [18] and [64]), Gauss is responsible for the development of language and notations in number theory and in particular the congruences of algebra. This work begins with a definition:

"If a number a divides a difference between two numbers b and c, then we say that b and c are congruent, otherwise are incongruent; and a is itself called a module."

The notation used by Gauss is used today (in this example, $b \equiv c \,(\text{mod } a)$) and enabled the construction of an algebra with respect to the relation $\equiv$. In the first chapters Gauss introduces a new way of calculation, the theory of congruences, which quickly gained general acceptance, and its terminology was important for the current number theory.

One of the most prominent Gauss students was Möbius (see [62]). He was Gauss's theoretical astronomy student at the University of Göttingen. Despite the fact that his major works were in the areas of analytic geometry and topology, Möbius made an important contribution to consolidating the generalization made by Gauss for Fermat's Little Theorem. In fact, as Dickson reports on [27], there are two relevant contributions (1832): the Möbius function and the Möbius inversion formula. The first allows writing the result with a much more compact formula, and the second is important with regard to the construction of proofs which are based on counting objects.

Recalling the Definition 89 of the Möbius function, we may state a related result.

Theorem 119. *(**Möbius inversion formula**) Let F e f two arithmetic functions related by the formula*

$$F\left(n\right) = \sum_{d|n} f\left(d\right). \tag{7.9}$$

Then

$$f\left(n\right) = \sum_{d|n} \mu\left(d\right) F\left(\frac{n}{d}\right) = \sum_{d|n} \mu\left(\frac{n}{d}\right) F\left(d\right). \tag{7.10}$$

Using the Möbius function the result of Gauss can then be expressed as follows (the result is valid even if a is not prime).

Theorem 120. *For any a, n positive integers,*

$$\sum_{d|n} \mu\left(d\right) a^{n/d} \equiv 0 \ (mod\ n). \tag{7.11}$$

Curiously this formulation was not used nor by Gauss, nor by Möbius, but by the Austrian mathematician Gegenbauer (1900).

In the following we present a proof (suggested by Prof. Nuno da Costa Pereira).

[1] Translations in French were made in 1807 (Paris) and in English in 1966 (Yale University Press)

Proof. Suppose, first, that n is divisible only by a prime number p. Being $n = p^\alpha$,

$$\sum_{d|n} \mu(d)\, a^{n/d} = a^{p^\alpha} - a^{p^{\alpha-1}} = a^{p^{\alpha-1}} \left(a^{p^\alpha - p^{\alpha-1}} - 1 \right).$$

Since $\phi\left(p^k\right) = p^k - p^{k-1}$, the number of integers in $[1, p^\alpha]$ relatively prime with p^α is $\phi(p^\alpha) = p^\alpha - p^{\alpha-1}$. It can then be written

$$a^{p^\alpha} - a^{p^{\alpha-1}} = a^{p^{\alpha-1}} \left(a^{\phi(p^\alpha)} - 1 \right).$$

If $p \nmid a$, by Euler's Theorem, $a^{\phi(p^\alpha)} \equiv 1 \pmod{p^\alpha}$. Assuming that p is a divisor of a also $p^{p^{\alpha-1}} | a^{p^{\alpha-1}}$ and since $p^{\alpha-1} \geq 2^{\alpha-1} \geq \alpha$, $p^\alpha | a^{p^{\alpha-1}}$. Thus in both cases we conclude that $p^\alpha | a^{p^{\alpha-1}} \left(a^{\phi(p^\alpha)} - 1 \right)$ and thus

$$a^{p^\alpha} - a^{p^{\alpha-1}} \equiv 0 \pmod{p^\alpha}.$$

We now proceed to the general case where n decomposes into a product of prime factors of the form $n = p_1^{\alpha_1} \cdots p_m^{\alpha_m}$. Fixed $k \in \{1, \ldots, m\}$, let $n_k = n/p_k^{\alpha_k}$. For each divisor d of n_k denote $b_d = a^{\frac{n_k}{d}}$. Then

$$\sum_{d|n} \mu(d)\, a^{n/d} = \sum_{d|n_k} \mu(d)\, a^{n/d} + \sum_{d|n_k} \mu(dp_k)\, a^{n/dp_k} = \sum_{d|n_k} \mu(d) \left(b_d^{p_k^{\alpha_k}} - b_d^{p_k^{\alpha_k - 1}} \right)$$

and by the part of the statement already established

$$b_d^{p_k^{\alpha_k}} - b_d^{p_k^{\alpha_k - 1}} \equiv 0 \pmod{p_k^{\alpha_k}}$$

if $d | n_k$.

It follows that

$$\sum_{d|n} \mu(d)\, a^{n/d} \equiv 0 \pmod{p_k^{\alpha_k}}$$

and therefore also

$$\sum_{d|n} \mu(d)\, a^{n/d} \equiv 0 \pmod{n},$$

since $p_k^{\alpha_k}$ are relatively prime. □

		
Carl Friedrich Gauss (1777-1855)	August Möbius (1790-1868)	Leopold Gegenbauer (1849-1903)

Other proofs of this known generalization of Fermat's Little Theorem due to Gauss-Gegenbauer involve the Möbius inversion formula. The idea is to identify relationships between objects of counting functions with the property (7.9). Hence, by (7.10), we obtain the explicit determination of the cardinality count formula of a set of objects that, at the outset, it is known that it is possible to aggregate in groups (or equivalence classes) of n elements. So results the congruence module n. Examples of proofs of this type are those that resort to necklaces and those resulting from dynamical system (7.6).

This result may also be presented in a form of combinatorics. We now introduce the appropriate terminology.

Adopting the notation followed by J. Matoušek e J. Nešetřil [56] and in a original formulation [79] we state the following.

Definition 121. [79] Denote by $C_j^{\{1,2,\cdots,k\}}$ the set of all subsets of $\{1,2,\cdots,k\}$ with j elements.

Thus, $\delta \in C_j^{\{1,2,\cdots,k\}}$ is a set of j distinct elements of $\{1,2,\cdots,k\}$. Without loss of generality, we write $\delta \in C_j^{\{1,2,\cdots,k\}}$ in the form $\delta = \{\delta_1, \delta_2, \ldots, \delta_j\}$, where the δ_i, with $i = 1, \ldots, j$, are the elements of δ.

Theorem 122. *[79] Let a and n be positive integers, and let the factorization of n in prime numbers be $n = p_1^{i_1} p_2^{i_2} \cdots p_k^{i_k}$ and let $P = p_1 p_2 \cdots p_k$. Then*

$$\sum_{j=0}^{k} \sum_{\delta \in C_j^{\{p_1,p_2,\cdots,p_k\}}} (-1)^{k+j}\, a^{\frac{n}{P}\delta_1\delta_2\cdots\delta_j} \equiv 0 \ (mod\ n).$$

Proof. The double summation is simply the sum of all divisors of P. Further noting that $(-1)^k = \mu(P)$ and $(-1)^j = \mu(\delta_1\delta_2\cdots\delta_j)$, this takes the form

$$\sum_{d|P} \mu(dP)\, a^{\frac{n}{P}d}.$$

Replacing d by P/d,

$$\sum_{d|P} \mu(d)\, a^{\frac{n}{d}}.$$

However, since $\mu(d) = 0$, if $d|n$ and $d \nmid P$ this sum is equal to

$$\sum_{d|n} \mu(d)\, a^{\frac{n}{d}}.$$

By (7.11) we obtain the desired result. □

Example 123. [82] For $n = 105$ one has the following congruence valid for all $a \in \mathbb{N}$:

$$a^{105} - a^{35} - a^{21} - a^{15} + a^7 + a^5 + a^3 - a \equiv 0 \pmod{105}.$$

The advantage of this latter formulation is to provide an algorithm for construction of congruences. This comprises the steps [79]:

(i) All summands have the factor $a^{n/p}$. This corresponds to decreasing by one the exponent of each prime factor;

(ii) the exponent of each summand is multiplied by a divisor of P. There are so many summands as there are divisors of P;

(iii) the sign of the summand depends on the number of prime factors that were multiplied in step (ii). The summand whose divisor of P is itself P gets a positive sign. Each elimination of a prime factor introduces a sign change.

As a final remark is interesting to note the diversity of possible generalizations of Fermat's Little Theorem. The book of Dickson [27] on the history of number theory is very illustrative. Thus, while the result of Gauss-Gegenbauer seemed to close the question of generalising the Fermat Theorem it is itself amenable to generalization.

As an example of generalization of the Gauss-Gegenbauer theorem there is a result of Axer, published in 1911, which includes polynomials of sums rather than simple powers of sums (see [27]).

7.2 Representation of real numbers

7.2.1 Base p representation and dyadic rationals

The classical representation of real numbers used through this thesis is defined as follows.

Definition 124. The ***base p representation*** of real numbers is given by

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n}, \tag{7.12}$$

where $\xi_n = 0, \ldots, p-1$.

This representation plays an essential role in the subject of this work, as viewed in Part I, namely to obtain explicit and constructive solutions of systems of functional equations. However, as noted in section 2.3.5, some real numbers have two different base p representations. The non-uniqueness of the base p representation occurs when $\xi_m \neq 0$ and $\xi_i = 0$ for all $i > m$. In this case, given a (finite) representation

$$x = \sum_{i=1}^{m} \frac{\xi_i}{p^i},$$

the second (infinite) representation is given by

$$x = \sum_{i=1}^{m-1} \frac{\xi_i}{p^i} + \frac{\xi_m - 1}{p^m} + \sum_{i=m+1}^{\infty} \frac{p-1}{p^i}.$$

According to Remark 13, this last expression represents x as the limit of the increasing sequence of numbers

$$x_n = \sum_{i=1}^{m-1} \frac{\xi_i}{p^i} + \frac{\xi_m - 1}{p^m} + \sum_{i=m+1}^{n} \frac{p-1}{p^i}.$$

The use of this representation has to be handled carefully when it is applied to non-continuous solutions of systems of iterative equations (see section 2.3.5).

Definition 125. A ***dyadic fraction*** or ***dyadic rational*** is a rational number whose denominator is a power of two, i.e., a number of the form $a/2^b$ where a is an integer and b is a natural number. These are precisely the numbers whose binary numeral system expansion is finite.

For instance, there is a bijective relation between the Farey sets and the dyadic rationals (see e.g. [66]).

Definition 126. The n^{th} order ***Farey set*** $\mathcal{F}_n$ is defined by recursion: $\mathcal{F}_0 = \{0/1, 1/1\}$. The $\mathcal{F}_n$ is obtained by adding to the $\mathcal{F}_{n-1}$ all the ***Farey sums*** $\nu_1 \oplus \nu_2 = (a_1 + a_2) / (b_1 + b_2)$ of two consecutive elements $\nu_i = a_i/b_i$ of $\mathcal{F}_{n-1}$.

The union of all the $\mathcal{F}_n$'s is the set of all rational numbers in $[0,1]$. Analogously, consider a sequence of sets $\mathcal{B}_n$ defined again recursively. Setting $\mathcal{B}_0 = \mathcal{F}_0$. The construction of the sequence if the same as for the $\mathcal{F}_n$'s, replacing the Farey sums by the ***barycentric sums*** $\nu_1 \boxplus \nu_2 = (a_1 + a_2)/2$. We obtain an increasing sequence $\mathcal{B}_0 \subset \mathcal{B}_1 \subset \mathcal{B}_2 \subset \cdots$, whose union is the set of all dyadic rationals in $[0,1]$. For every $n \geq 0$, there exists a unique order-preserving bijection from $\mathcal{F}_n$ to $\mathcal{B}_n$. The union of these bijections is again a bijection which extends uniquely by continuity to an order-preserving bijection $\Phi : [0,1] \to [0,1]$, which, according to Panti [66] is the Minkowski question mark function defined in (iii) of section 3.3.3.4.

In systems of functional equations (1.1) with two equations, the explicit solutions may also be given in terms of dyadic rationals. For example, the explicit formula (2.10) for the solution of the de Rham example (case (ii) of section 2.3.3) in [43] is given by

$$\varphi\left(\sum_{n=0}^{\infty} 2^{-\gamma_n}\right) = \sum_{n=0}^{\infty} \left(\frac{1-a}{a}\right)^n a^{\gamma_n},$$

with $\gamma_0 < \gamma_1 < \gamma_2 < \cdots < \gamma_n \in \mathbb{N}$.

7.2.2 Signed base p representation

In the example (ii) in section 3.3.3.4 it is useful to consider a ***signed base*** p ***representation*** given by

$$x = \sum_{n=1}^{\infty} \frac{(-1)^{s_{n-1}} \left(\xi_n + \delta_{\xi_n}\right)}{p^n},$$

where $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of integers (digits) in $\{0, 1, 2, \ldots, p-1\}$, $\{s_n\}_{n\in\mathbb{N}_0}$ is the sequence defined by $s_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $\forall n \in \mathbb{N}$, $s_0 = 0$ and $\delta_n = 0$ if n is even, $\delta_n = 1$ if n is odd. This representation results from "inverting" a generalized tent map. Note that the classical base p representation may be viewed as resulting from "inverting" the piecewise affine expansion map g_p. We do not prove neither that all numbers $x \in [0, 1]$ are representable by this form, neither the uniqueness of a number $x \in [0, 1]$ in this representation.

7.2.3 Q-representation

Turbin and Prats'ovytyi [94] introduced a more general representation with respect to classical representation of real numbers, where the length of the intervals defining the expansion is not uniform.

Let $p \geq 2$ be a fixed positive integer, and $q_0, q_1, \ldots, q_{p-1} \in (0, 1)$, such that $\sum_{j=0}^{p-1} q_j = 1$. Let $r_0 = 0$, $r_j = \sum_{k=0}^{j} q_{k-1}$, for $j \in \{1, 2, \ldots, a\}$, $A = \{0, 1, 2, \ldots, p-1\}$, and $Q = \{q_0, q_1, \ldots, q_{p-1}\}$.

Theorem 127. *(Turbin and Prats'ovytyi) For any number $x \in [0, 1]$, there exists a sequence of numbers $\nu = (\nu_n) \in A$ such that*

$$x = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\nu_k} \right) r_{\nu_n}. \tag{7.13}$$

Remark 128. Obviously, for any real number x there exists an expansion

$$x = [x] + \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\nu_k(x)} \right) r_{\nu_n(x)}. \tag{7.14}$$

Definition 129. Given $x \in \mathbb{R}$, the representation by series (7.13) or (7.14) is called ***p-symbol Q-representation*** or ***p-symbol Q-expansion*** of x. For $x \in [0, 1]$ we use the notation given by

$$\triangle_{\nu}^{Q} = \triangle_{\nu_1\nu_2\cdots\nu_n\cdots}^{Q} := \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\nu_k} \right) r_{\nu_n}. \tag{7.15}$$

Remark 130. [81] An algorithm to find a p-symbol Q-representation of a number $x \in [0, 1]$ is:

0. Let $n = 1$ and $x_1 = x$.

1. Find $\nu_n \in \{0, 1, \ldots, p-1\}$, such that $r_{\nu_n} \leq x_n < r_{\nu_n+1}$. If $x = r_{\nu_n}$, set $\nu_N = 0$, for $N > n$, and the process is finished. Else go to step 2.

2. Find the difference $x_n - r_{\nu_n}$ and divide by q_{ν_n}:

$$x_{n+1} = \frac{x_n - r_{\nu_n}}{q_{\nu_n}}.$$

3. Perform step 1 for $n + 1$.

The sequence $\nu = (\nu_n) \in A$ determines the p-symbol Q-representation of x.

We now show that representations (7.12) and (7.15) are related by a homeomorphism which is a solution of an equation of the form (4.3).

Theorem 131. *[81] There is a homeomorphism φ such that the image of each $x \in [0,1]$ by φ is the p-symbol Q-expansion of x, where the sequence of numbers $(\nu_n) \in A$ of the image obtained is the same as the base p representation of x, i.e., $\nu_n = \xi_n$, $\forall n \in \mathbb{N}$. Moreover, φ is the unique bounded solution of the system of equations*

$$\varphi\left(\frac{x+k}{p}\right) = q_k \varphi(x) + r_k,\ x \in [0,1],\ 0 \leq k \leq p-1. \tag{7.16}$$

Proof. By definition of q_k and r_k, condition (3.16) is satisfied. Applying Theorem 39 there exists a unique bounded solution of (7.16) given, in terms of the base p representation, by

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \sum_{n=1}^{\infty} \left(\prod_{k=1}^{n-1} q_{\xi_k}\right) r_{\xi_n}. \tag{7.17}$$

□

This result may be viewed as a special case of Lemma 63 which shows that the function φ is a homeomorphism. Girgensohn's result (Theorem 39) is more general, since in its statement the r_k are allowed to be continuous functions of x, $r_k : [0,1] \to \mathbb{R}$, instead of constants.

Remark 132. [81] The function φ in (7.17) may be given in the equivalent forms

$$\varphi\left(\sum_{n=1}^{\infty} \frac{\xi_n}{p^n}\right) = \triangle_{\xi}^{Q},$$

$$\varphi^{-1}\left(\triangle_{\xi}^{Q}\right) = \sum_{n=1}^{\infty} \frac{\xi_n}{p^n}.$$

In a similar fashion, the function that transforms p-symbol Q-expansions in other p-symbol Q-expansions was given by Prats'ovytyi and Kalashnikov [71]. If we denote by $\triangle_{\nu}^{Q[q,r]} = \triangle_{\nu_1\nu_2\cdots\nu_n\cdots}^{Q[q,r]}$ the p-symbol Q-expansion in terms of q_j, r_j, and ν, defined above, the system of functional equations

$$\psi\left(\triangle_{k\nu_1\nu_2\cdots\nu_n\cdots}^{Q[q,r]}\right) = t_k \psi\left(\triangle_{\nu_1\nu_2\cdots\nu_n\cdots}^{Q[q,r]}\right) + s_k,\ k \in \{0,1,\ldots,p-1\}$$

has a unique bounded solution ψ defined by

$$\psi\left(\triangle_{\nu}^{Q[q,r]}\right) = \triangle_{\nu}^{Q[t,s]}. \tag{7.18}$$

We note that our results imply immediately construction (7.18). In fact, let $p \geq 2$ be an integer, and let $(p, Q[q,r])$ and $(p, Q[t,s])$ be bases of p-symbol

Q-representations. Remark 133 implies that the diagram

$$\begin{array}{ccc} \sum_{n=1}^{\infty}\frac{\xi_n}{p^n} & \xrightarrow[\mathrm{id}_{[0,1]}]{} & \sum_{n=1}^{\infty}\frac{\xi_n}{p^n} \\ \varphi_{q,r}\downarrow & & \downarrow\varphi_{t,s} \\ \triangle_{\xi}^{Q[q,r]} & \xrightarrow[\psi]{} & \triangle_{\xi}^{Q[t,s]} \end{array}$$

is commutative, where the indices of φ correspond to the parameters of the system (7.16) for which φ is the solution.

Thus the required homeomorphism ψ in (7.18) is given by

$$\psi = \varphi_{t,s} \circ \varphi_{q,r}^{-1}.$$

In this way, Theorem 39 allows us to construct an alternative, equivalent representation of the homeomorphism ψ.

Remark 133. Similarly to the previous listed representations, this representation results from "inverting" a piecewise affine non-uniform interval expansion map.

As presented in sections 3.3.1 (Theorem 45) and 5.2, this representation is important for defining explicitly FIF when the data is not uniformly distributed.

7.2.4 General p representations

Observe that all representations of numbers given so far result from "inverting" piecewise affine expansion maps. In fact, all these representations are particular cases of a more general method to obtain p representations of numbers, the general construction of Theorem 51, such that

$$x = \lim_{\nu\to\infty} f_{\xi_1} \circ f_{\xi_2} \circ \cdots \circ f_{\xi_\nu}(x_k),$$

where $\{f_k\}$ is a family of p injective contraction maps defined in a complete bounded metric space X, each x_k is the fixed point of f_k and $\{\xi_n\}_{n\in\mathbb{N}}$ is a sequence of integers (digits) in $\{0, 1, 2, \ldots, p-1\}$. This construction generalizes not only to real numbers, but also to higher-dimensional objects belonging to other complete bounded metric spaces.

7.2.5 Finite base p representation of rationals

The ***finite base p representation of rationals*** is based on the periodic and pre-periodic points of the piecewise affine map (6.2).

The usual representation of real numbers in base p is of the form

$$\sum_{k=s}^{+\infty} \alpha_k p^{-k}, \tag{7.19}$$

where $s \in \mathbb{Z}$.

Through the pre-periodic points of g_p it is possible to find a finite representation of rational numbers in terms of a base p.

Theorem 134. *[83] Let $p \geq 2$ an integer and $x \in \mathbb{Q}$. Then there exist $m \geq 1$, $0 \leq j_1, j_2, \ldots, j_m < p$, $n \geq 1$, $-p < i_1, i_2, \ldots, i_n < p$ and s integers such that*

$$x = \frac{\sum_{k=1}^{m} j_k p^{m-k-n}}{p^m - 1} + \sum_{k=s}^{n} i_k p^{-k}. \tag{7.20}$$

Proof. First consider $x \in \mathbb{Q} \cap [0,1)$. By the properties of g_p, the set of rational numbers in $[0,1)$ is the set of pre-periodic points of the this map. If x is periodic then the second summand is zero. If it is not periodic then it is a n^{th} pre-image of a periodic point. Computing the pre-images we get

$$x = \frac{\sum_{k=1}^{m} j_k p^{m-k-n}}{p^m - 1} + \sum_{k=1}^{n} i_k p^{-k}.$$

For a number $x \in \mathbb{Q}$ we split x in the integer and decimal parts. The integer part has its usual representation

$$\sum_{k=s}^{0} \alpha_k p^{-k},$$

where s is the non-positive integer in (7.19). The sum of the two parts gives the result. $\square$

This finite representation of rational numbers in base p is similar to a formulation done by Hehner and Horspool (see [38]). The fundamental difference between these two representations is in the form of distributing positive and negative summands. In our case the first summand is always positive while in their case it is always negative. For them the second summand is positive and for us it can be positive or negative. One of the advantages of this kind of representation referred by Hehner and Horspool is, not only being a finite representation, but the existence of easy algorithms for addition, subtraction and multiplication.

Remark 135. Formula (7.20) implies that each rational number may be represented as a fraction of the form

$$x = \frac{b}{p^n \left(p^m - 1\right)},$$

where $p \geq 2$ is a positive integer, $n, m \in \mathbb{N}$ and $b \in \mathbb{Z}$. For each fixed p, the integers b, n, m depend of x.

Corollary 136. *[83] Let $p \geq 2$ be a positive integer. Then every positive integer has multiples of the form $p^n \left(p^m - 1\right)$, where $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$. We say $p^n \left(p^m - 1\right)$ is the p-***base least multiple** *(p-blm) of a number if it is the least possible multiple of this form.*

From the dynamical point of view the main advantage of having a number written in our finite representation is that it provides complete information about its orbit. Therefore it is useful to represent rational numbers in this form. Lemma 112 gives directly a formula to transform the classical base p representation in the finite representation. We now give an alternative algorithm to convert the usual representation q_1/q_2 of rational numbers into this new formulation. Without loss of generality we consider a number in the interval $[0, 1)$.

Let $q_1/q_2 \in \mathbb{Q} \cap [0, 1)$ an irreducible fraction.

Step 1: find the p-base least multiple of q_2.

Step 2: multiply the numerator and the denominator by the adequate number such that

$$\frac{q_1}{q_2} = \frac{P}{p^n\,(p^m - 1)}.$$

Step 3: perform successive divisions by $(p^m - 1)$, p^n, p^{n-1}, etc.

The advantage of this finite representation of rational numbers is that it exhibits the complete itinerary of rational numbers through the map g_p. Step 1 implies that no more than n iterates will be needed for the itinerary reach its periodic part (a m-periodic orbit).

Example 137. Let $p = 6 = 2 \cdot 3$ and $x = 771/860$ a point in $[0, 1]$ in the irreducible form. The algorithm gives

$$x = \frac{771}{860} = \frac{3 \cdot 257}{2^2 \cdot 5 \cdot 43} = \frac{3^3 \cdot 257}{2^2 \cdot 3^2 \cdot 5 \cdot 43} = \frac{3^3 \cdot 257}{6^2 \cdot 215}.$$

Note that $215 = 6^3 - 1$. Then

$$x = \frac{6939}{6^2 \cdot (6^3 - 1)}.$$

Performing successive divisions,

$$\begin{aligned}
\frac{6939}{6^2 \cdot (6^3 - 1)} &= \frac{59}{6^2\,(6^3 - 1)} + \frac{32\left(6^3 - 1\right)}{6^2\,(6^3 - 1)} = \frac{59}{6^2\,(6^3 - 1)} + \frac{32}{6^2} \\
&= \frac{\frac{59}{6^2}}{6^3 - 1} + \frac{32}{6^2} = \frac{\frac{36+23}{6^2}}{6^3 - 1} + \frac{30 + 2}{6^2} \\
&= \frac{\frac{36}{6^2} + \frac{23}{6^2}}{6^3 - 1} + \frac{30}{6^2} + \frac{2}{6^2} = \frac{1 + \frac{18+5}{6^2}}{6^3 - 1} + \frac{5}{6} + \frac{2}{6^2} \\
&= \frac{1 \cdot 6^0 + \frac{3}{6} + \frac{5}{6^2}}{6^3 - 1} + \frac{5}{6^1} + \frac{2}{6^2}.
\end{aligned}$$

The dynamics of this point by the map g_6 is then completely determined by this expansion. The itinerary of x is $(2, 5, 5, 3, 1, 5, 3, 1, 5, 3, 1, 5, 3, 1, \ldots, 5, 3, 1, \ldots)$.

Chapter 8

Recreational Mathematics

In this chapter we present two possible applications of the subjects presented in previous chapters to the field of Recreational Mathematics. The playful aspects of mathematics stand when there is imagination to work on the fun things and games that are hiding behind theoretical results. Here we present in section 8.1 a recreational approach of section 6.2 (presented in the Recreational Mathematics Colloquium II [78]) and in section 8.2 we study a casino gamble which has a strategy, called bold play, that may be mathematically modelled by the systems of functional equations of Part I (presented in the Recreational Mathematics Colloquium IV [87]). For this game we compare the timid play strategy and generalize the two pay-off game to multiple pay-offs.

8.1 Words and necklaces

8.1.1 Formal languages

We recall concepts of formal languages, such as necklaces and Lyndon words. We provide examples, some classical and others visually more attractive and playful. In parallel, in terms of dynamical systems, we consider a particular circle map defined in 6.2. Associating all the listed concepts we construct simultaneously aperiodic necklaces and periodic orbits of the defined circle map, that is, we show, by examples, that given a finite alphabet and a the previously defined circle map, there is a natural bijection between the set of all the aperiodic necklaces of length n over this alphabet and the set of all the n-periodic orbits of this circle map.

We are going to present a relation between concepts of two different areas of mathematics: combinatorics on words and dynamical systems. Our aim is to give a light approach emphasizing the playful side of the subject. The relation between concepts is based on results from dynamical systems presented in section 6.2 and number theory summarized in section 7.2.

For completeness we recall the definition of Möbius function (see Definition 89).

We restate some concepts of formal languages and combinatorics on words defined in [9] and [75] and give some illustrative examples.

Definition 138. An **alphabet** is a finite non-empty set of symbols.

We mention some common examples.

Example 139. The binary alphabet $\{0, 1\}$.

Example 140. The 10-digit alphabet $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

Example 141. The English language alphabet $\{a, b, c, \dots, x, y, z\}$ as well as other language alphabets.

Example 142. The alpha-numeric alphabet $\{0, 1, \dots, 8, 9, a, b, c, \dots, x, y, z\}$.

Example 143. The hexadecimal alphabet $\{0, 1, \dots, 8, 9, a, b, c, d, e, f\}$.

We now show other alternative alphabets.

Definition 144. The suit of cards alphabet $\{\spadesuit, \clubsuit, \heartsuit, \diamondsuit\}$.

Definition 145. The 3-geometric forms alphabet $\{\square, \triangle, \bigcirc\}$.

Example 146. The 7-colours alphabet $\{\blacksquare, \blacksquare, \blacksquare, \blacksquare, \blacksquare, \blacksquare, \blacksquare\}$.

Example 147. The zodiac alphabet which include the symbols of Aries, Taurus, Gemini, Cancer, Leo, Virgo, Libra, Scorpio, Sagittarius, Capricorn, Aquarius and Pisces.

Definition 148. A **word** or **string** over an alphabet $\sum$ is a finite sequence of symbols taken from $\sum$. The **catenation** of two words is the word formed by juxtaposing the two words together, i.e., writing the first word immediately followed by the second word, with no space in between. A **factorization** of a word u is any sequence $u_1, \dots, u_t$ such that $u = u_1 \dots u_t$.

For a pair (u, v) of words we define four relations:

1. u is a **prefix** of v if there exists a word z such that $v = uz$, and $\text{pref}_k(v)$ is the prefix of v of length k;

2. u is a **suffix** of v if there exists a word z such that $v = zu$, and $\text{suf}_k(v)$ is the suffix of v of length k;

3. u is a **factor** of v if there exists words z and z' such that $v = zuz'$;

4. If $v = uz$ we write $u = vz^{-1}$ or $z = u^{-1}v$, and say that u is the **right quotient** of v by z, and that z is the **left quotient** of v by u.

Definition 149. Consider the cyclic permutation $c : \sum \to \sum$ defined by

$$c(u) = \mathrm{pref}_1(u)^{-1}\, u\, \mathrm{pref}_1(u)$$

for $u \in \sum$. We say that two words u and v are **conjugates** if, and only if, there exists a k such that $v = c^k(u)$.

Remark 150. It is easily shown that conjugacy is an equivalence relation.

As in [9] we name each equivalence class under conjugation by **necklace**.

Definition 151. Let $u = a_1 \dots a_n$, with $a_i \in \sum$. A period of u is an integer p such that

$$a_{p+i} = a_i \text{ for } i = 1, \dots, n - p. \tag{8.1}$$

The smallest p satisfying (8.1) is called the **period** of u, and it is denoted by $p(u)$. A word with a given period is called periodic word, otherwise is aperiodic. The words in the necklace $\left[\mathrm{pref}_{p(u)}(u)\right]$ are called **cyclic roots** of u. We say that a word $u \in \sum$ is **primitive** if it is not a proper integer power of any of its cyclic roots; its necklace is an **aperiodic necklace**. A **Lyndon word** $u \in \sum$ is a word which is primitive and the smallest one in its necklace with respect to the lexicographic ordering.

Remark 152. We note that primitive words are aperiodic words and an equivalent condition of primitiveness is

$$\forall z \in \sum : u = z^n \Rightarrow n = 1 \text{ (i.e., } u = z).$$

As noted in [9], considering an alphabet with a symbols, the number of aperiodic necklaces of length n is given by $\mathcal{N}_n(a)$, where

$$\mathcal{N}_n(a) = \frac{1}{n}\sum_{d|n} a^d \mu\left(\frac{n}{d}\right).$$

Example 153. In Figure 8.1 we exhibit all possible necklaces of length 4 from an alphabet of two symbols and corresponding words.

Example 154. Consider the suit of cards alphabet with the order $\spadesuit \prec \clubsuit \prec \heartsuit \prec \diamondsuit$. The necklaces from the Figure 8.2 correspond, respectively, to the Lyndon words: $\spadesuit\spadesuit\clubsuit\spadesuit\diamondsuit$, $\spadesuit\clubsuit\clubsuit\heartsuit\diamondsuit$, $\clubsuit\diamondsuit\heartsuit\diamondsuit\heartsuit$ and $\clubsuit\heartsuit\diamondsuit\heartsuit\diamondsuit$.

8.1.2 Necklaces in a circle map

In this section the necessary definitions from dynamical systems are those given in Definition 82.

We turn again our attention to the circle map g_p as defined in (6.2).

Example 155. Consider the graph of g_4 shown in Figure 8.3.

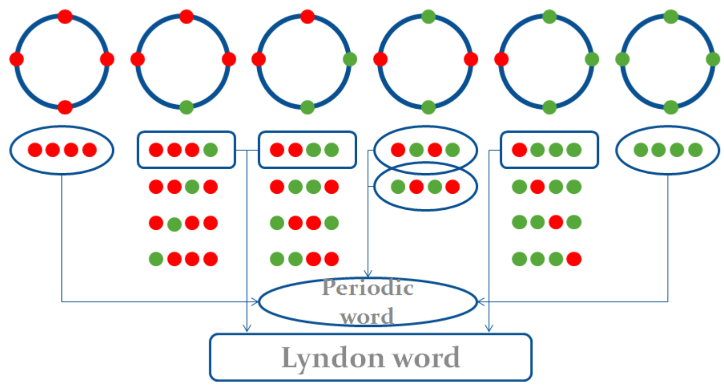

Figure 8.1: Necklaces and words of length 4 from a two symbol alphabet.

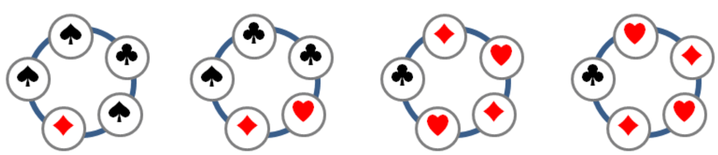

Figure 8.2: Necklaces corresponding to Lyndon words.

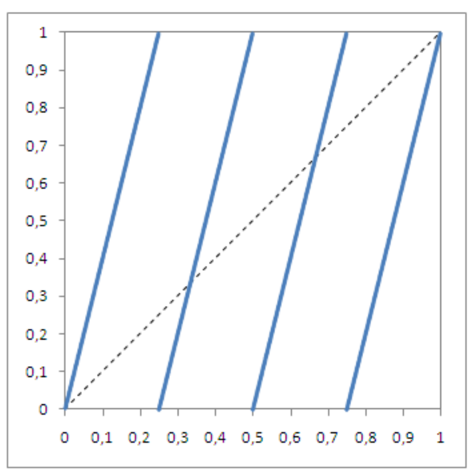

Figure 8.3: Graph of g_4.

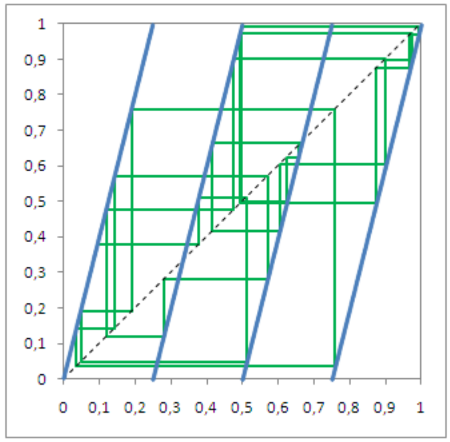

Figure 8.4: Graph of g_4 with the first 25 iterates of a randomly chosen initial condition.

Given any initial condition in $[0,1]$, the corresponding orbit may be constructed by graphical analysis.

Example 156. We exhibit in Figure 8.4 the first 25 iterates of a randomly chosen initial condition. This orbit seems not to be periodic.

In fact, since g_p preserves Lebesgue measure, the probability that a randomly chosen point will have a dense (in fact, uniformly distributed) orbit on $[0,1]$ is 1; see, e.g., [23].

We are interested in the 0-measure set of periodic orbits, since we are going to associate aperiodic necklaces to periodic orbits of a map g_p. This is possible in view of the next result.

Theorem 157. *The number of periodic points of prime period n of g_p is*

$$N_n(p) = \sum_{d|n} p^d \mu\left(\frac{n}{d}\right).$$

Proof. This results directly from Theorem 92 of chapter 6.2. □

Knowing that each periodic orbit of prime period n contains exactly n periodic points of prime period n, we easily conclude that the number of orbits of prime period n of g_p is given by $\mathcal{N}_n(p)/n$. We thus have:

Corollary 158. *The number of periodic orbits of prime period n of g_p is*

$$\mathcal{O}_n(p) = \frac{1}{n} \sum_{d|n} p^d \mu\left(\frac{n}{d}\right).$$

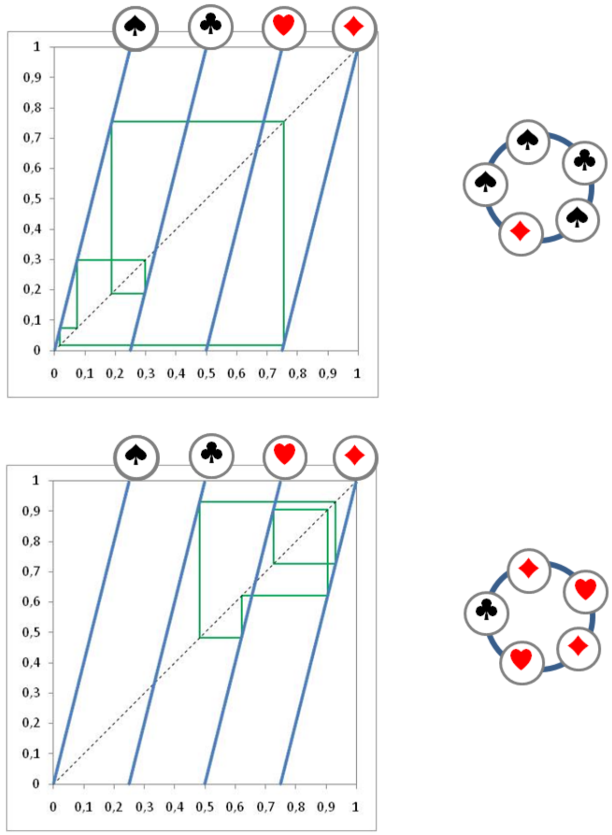

Figure 8.5: Necklaces associated with orbits of g_4.

Thus the cardinalities $\mathcal{N}_n(p)$, of the set of all aperiodic necklaces of length n over an alphabet with p symbols, and $\mathcal{O}_n(p)$ of the set of all periodic orbits of prime period n of g_p, are the same. We next show how a natural bijection between these sets may be constructed.

Example 159. From previous examples of necklaces we associate them to periodic orbits of g_4 as in Figure 8.5.

Visually this method can be identified as a loom that produces aperiodic necklaces. First each line segment should have a different colour. Then just make a string go through an orbit and put a coloured bead every time this touches the line segments of the circle map.

Conversely, as shown in section 6.2, we may derive a formula for obtaining the points of a periodic orbit of g_p from a correspondent necklace or primitive word. We can calculate the points of a periodic orbit of g_p, using the sequence

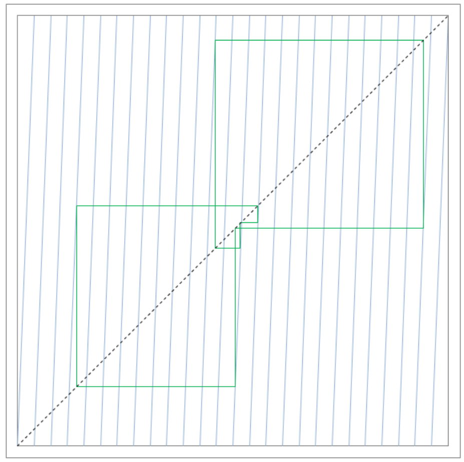

Figure 8.6: The orbit of g_{26} associated to the word "Lyndon".

of letters of a given word and its necklace:

$$x = \frac{\sum_{k=1}^{m} j_k p^{m-k}}{p^m - 1}.$$

Considering the English alphabet, the corresponding circle map is $26x$ (mod 1) and the set of all orbits resulting from the English words would compile an orbit English dictionary. Note that almost all English words are primitive words. It is rare to find exceptions, such as *cuscus*, *gaga*, *mama* and *yoyo*. This ones would be excluded from that dictionary. Let us see an example.

Example 160. The orbit associated to the word *Lyndon* is depicted in Figure 8.6.

Here we presented a light approach to the issue, stressing the visual aspects and omitting the proofs and developments (see section 6.2). However, we think there are possible consequences arising for both areas (combinatorics on words and dynamical systems) yet unexplored.

8.2 The gamble of bold play

In this section we study a casino gamble. We model a gambler making bets at a certain game (such as betting on red on roulette). The goal is to analyse possible strategies (the bold play and comparatively the timid play) and its probabilities of winning.

We extend the concept of bold play in gambling, where the game has a unique win pay-off (it returns twice the original wager). We model a game where the player can bet all his money in each stake. The probability that a gambler reaches his goal using the bold play strategy is the solution of a functional equation. We also consider the timid play strategy in which the bet in each stake is always the same regardless the amount of money the gambler has. We refer to the game of scratch cards where is impossible for a gambler to play a bold play strategy (for simple or multiple pay-offs).

In this chapter we first introduce the classical game strategy called bold play and then extend this concept introducing a multiple set of pay-offs for the game. We also introduce the classical timid play strategy and extend it similarly to a multiple set of pay-offs.

8.2.1 Classical bold play

The classical scenario of bold play is that of a gambler with an initial amount c available to gamble and a goal a for his final fortune. The game is binary, that is, each stake of the game has only two possible outcomes: win or loss. For example, in a typical game at a casino, it depends on which colour, red or black, comes out in the roulette. In case of win the game returns twice the bet. Otherwise the gambler ends with nothing and is forced to retire.

Bold play is defined by the strategy in which, on each stake, the gambler either bets his entire fortune or, if he has more than half of his goal, the amount strictly necessary to reach his target fortune. In fact, only the ratio of the initial money to the target fortune matters, so the study of this strategy is done by first normalizing the problem with $a = 1$ and $x = c/a$. Now the initial amount is x and the goal is 1.

We study this game theoretically, which means that any amount of money is allowed in a single stake, such as fractional or even irrational amounts.

The set of all possible fortunes at a given moment is the interval $[0, 1]$. If $x = 0$ the game ends. This happens when the player has lost all his available money. If $x = 1$ the game also ends, because the target fortune is reached. Denote by p the probability of win in a single stake. In a fair game $p = 1/2$.

Consider $S(x)$, the amount of money spent in a single bet. If the player has a bold play strategy then the function S is defined by

$$S(x) = \min(x, 1-x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} < x \le 1. \end{cases}$$

Let $F(x)$ be the probability of reaching the target fortune with an initial available amount of x. It is immediate to conclude that $F(0) = 0$ and $F(1) = 1$. In the following we provide the setting in order to obtain the functional relation whose solution is the function F.

Proposition 161. *Let $p+q = 1$, $p > 0$, $q > 0$. The probability of success under bold play $F(x)$ for an initial fortune x is the unique solution of the system of equations*

$$\begin{cases} F(x) = pF(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ F(x) = p + qF(2x-1), & \text{if } \frac{1}{2} \le x \le 1, \end{cases} \tag{8.2}$$

Available in stake n	Event and probability	Available in stake $n+1$
$0 \leq x \leq \frac{1}{2}$	wins with probability p	$2x$
$0 \leq x \leq \frac{1}{2}$	loses with probability $q = 1 - p$	0
$\frac{1}{2} < x \leq 1$	wins with probability p	1
$\frac{1}{2} < x \leq 1$	loses with probability $q = 1 - p$	$2x - 1$

Table 8.1: Classical Bold Play: possible stakes - bet and pay-off

constrained to $F(0) = 0$, $F(1) = 1$.

Remark 162. If $p = q = 1/2$, it is immediate that the identity is a solution of (8.2). Note that the solution is unique.

Proof. (see [11]) For the case $p = q = 1/2$, a classical result in probability applies (see e.g. [32]). For $x = 0$, $F(x) = 0$, and equation $F(x) = pF(2x)$ is satisfied. For $x = 1$, $F(x) = 1$, and equation $F(x) = p + qF(2x - 1)$ is verified. For $0 < x \leq 1/2$, under bold play the gambler stakes the amount x. In case of success in the first turn, with probability p, his new fortune is $2x$. Since each turn has independent outcomes, the probability of success in the second turn $F(2x)$ multiplied by p equals the initial probability of success $F(x)$, proving the first equation of (8.2).

For $1/2 \leq x < 1$, the first stake is $1 - x$. In case of success in the first turn, with probability p, his new fortune is 1. In case of loss, with probability q, his new fortune is $2x - 1$. The probability of success after a loss is $F(2x - 1)$. Since each turn has independent outcomes, the probability of success given by $p + qF(2x - 1)$ equals the initial probability of success $F(x)$, proving the second equation of (8.2). Uniqueness of the solution now follows from Theorem 64 given initial conditions. ☐

An informal proof of this result is as follows.

Proof. Suppose $0 \leq x \leq \frac{1}{2}$. In case of a win, then $F(x) = pF(2x)$. Otherwise the player loses and will not reach the goal. A similar reasoning applies to the case $\frac{1}{2} < x \leq 1$. The win may result from getting a gain in a previous turn, or a loss with recovery. The function F is thus the solution of the functional equation (8.2). ☐

A theorem of Dubins-Savage [28] states that in the sub-fair case ($p < 1/2$), bold play strategy maximizes the probability of successfully reaching the goal, i.e., bold play is an optimal strategy. This means that bold play can be substantially better than betting at constant stakes. Nevertheless bold play is not the only optimal strategy.

The solution of equation (8.2) is either a singular function (continuous, increasing and with zero derivative almost everywhere) or the identity. This last possibility corresponds to the fair case ($p = 1/2$) (see [11]).

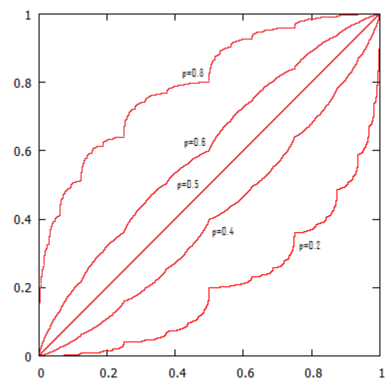

Figure 8.7: Probability of reaching the target fortune playing bold play strategy

The general solution F of equation (8.2) was given explicitly in section 2.3.3 and geometrically by Billingsley (see [11]). The probability of success F under bold play for an initial fortune

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{2^n}$$

is

$$\varphi(x) = \sum_{n=0}^{\infty} p^{n+1} \left(\frac{1-p}{p} \right)^{s_n} \xi_{n+1}, \tag{8.3}$$

where $s_n = \xi_1 + \xi_2 + \cdots + \xi_n$, $\forall n \in \mathbb{N}$ and $s_0 = 0$. We graph the solution for several values of p in Fig. 8.7.

The probability of success of a strategy of bold play gamble satisfies a system of equations of type (4.4). Applying Theorem 39 we have the following alternative explicit expression for the probability of success, in terms of binary representation of real numbers.

Proposition 163. *The probability of success under bold play for an initial fortune*

$$x = \sum_{n=1}^{\infty} \frac{\xi_n}{2^n}$$

is

$$f(x) = \sum_{n=1}^{\infty} \left(\prod_{i=1}^{n-1} \mu_{\xi_i} \right) \alpha_{\xi_n},$$

where $\alpha_0 = 0$*,* $\alpha_1 = p$*,* $\mu_0 = p$*, and* $\mu_1 = q$*.*

This formula is equivalent to (8.3).

8.2.2 Bold play with multiple pay-offs

The new concept for a bold play strategy with n-multiple pay-offs is applicable to games where in each stake there is a probability table for pay-offs:

i	p_i	m_i
0	p_0	m_0
1	p_1	m_1
2	p_2	m_2
$\vdots$	$\vdots$	$\vdots$
$n-1$	p_{n-1}	m_{n-1}

Here p_i is the probability of getting a pay-off m_i and $p = \sum_{i=0}^{n-1} p_i \leq 1$. In the case of classical bold play $p_0 = p$, $m_0 = 2$. The probability of losing the bet is $q = 1 - p$.
The bold play strategy with n-multiple pay-offs may be defined with respect to a reference parameter $m_* \in [m, M]$, where $m = \min_{i\in\{0,1,\dots,n-1\}} \{m_i\}$ and $M = \max_{i\in\{0,1,\dots,n-1\}} \{m_i\}$. In the classical bold play $m_* = m_0 = 2$.

For a n-multiple pay-off bold play we have a similar strategy: on each stake the gambler bets either his entire fortune or, if he has more than a certain amount, the amount of money necessary to reach his target fortune in one bet with a reference gain of m_*. We assume that $m_i > 1$, $\forall i \in \{0, 1, \dots, n-1\}$. What is the amount of money from which the player does not play all his money in a stake?

We focus on the case $m_* = m$. Note that the player's objective is to reach the fortune 1. In case of win, if his bet is x, his pay-off is at least mx. Then he bets all his money if he has less than $1/m$. If he bets S, in case of win he will have available for the next round the amount $x - S + Sm$. To reach his goal he should think of the identity $x - S + Sm = 1$. Then the function $S(x)$ of the amount of money spent in a bet is now given by

$$S(x) = \begin{cases} x, & 0 \leq x \leq \frac{1}{m} \\ \frac{1-x}{m-1}, & \frac{1}{m} < x < 1 \\ 0, & x \geq 1. \end{cases}$$

If the player loses after betting $(1-x)/(m-1)$, he has $(mx-1)/(m-1)$ available for the next stake.

Thus the generalization of the functional equation (8.2) for n-multiple pay-offs bold play is

$$F(x) = \begin{cases} \sum\limits_{i=0}^{n-1} p_i F(m_i x), & 0 \leq x \leq \frac{1}{m} \\ p + qF\left(\frac{mx-1}{m-1}\right), & \frac{1}{m} < x < 1 \\ 1, & x \geq 1. \end{cases}$$

8.2.3 Classical timid play

The scenario of classical timid play is the same as the one described for bold play, where there is a gambler with an available initial amount of money x to play and a goal for his final fortune a. The strategy for each stake is, however, different. Here the gambler always bets the same amount of money in each round. In this case we normalize the game with respect to the amount of money in each bet, i.e., we consider unit bets. Here

$$S(x) = \begin{cases} 0, & x \in \{0, a\} \\ 1, & 0 < x < a, \end{cases} \tag{8.4}$$

and the function F is the solution of the functional equation

$$F(x) = pF(x+1) + qF(x-1), \tag{8.5}$$

with boundary conditions $F(0) = 0$, $F(a) = 1$ (see Billingsley [10]). We now provide the setting to obtain the functional relation (8.5) whose solution is the function F.

Available in stake n	Event and probability	Available in stake $n+1$
$0 < x < a$	wins with probability p	$x+1$
$0 < x < a$	loses with probability $q = 1-p$	$x-1$

Table 8.2: Classical Timid Play: possible stakes - bet and pay-off

The solution of (8.5) (see e.g. Billingsley [10]) is

$$F(x) = \begin{cases} \frac{\left(\frac{q}{p}\right)^x - 1}{\left(\frac{q}{p}\right)^a - 1}, & 0 \le x \le a, \text{ if } p \ne \frac{1}{2} \\ \frac{x}{a}, & 0 \le x \le a, \text{ if } p = \frac{1}{2}. \end{cases} \tag{8.6}$$

Note that, in contrast with the bold play case, where F is singular for $p \ne 1/2$, at first glance the timid play function F is smooth. However, it should be duly taken into account the domain in which the function is defined. In fact, equation (8.6) is only valid considering $F : \mathbb{N} \to \mathbb{N}$. What if $x \notin \mathbb{N}$? F is a step function; for a large enough it approximates a smooth function which graph is as Fig. 8.8 as the following for some values of p.

Actually, extending function F to $\mathbb{R}^+$, we obtain a step function, which is still a solution of the problem. See in Fig. 8.9 the graph of F for $a = 10$ and some values of p.

We remark that in the fair case ($p = 1/2$), $F(x)$ converges uniformly to the identity as $a \to \infty$. This means that, when $a \gg 1$, $F(x)$ is approximately the same for both bold and timid play strategies.

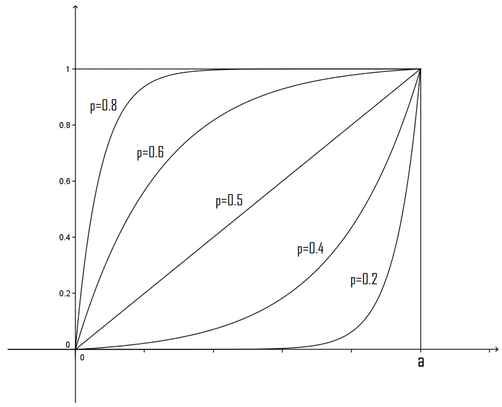

Figure 8.8: Probability (approximately) of reaching the target fortune playing timid play strategy

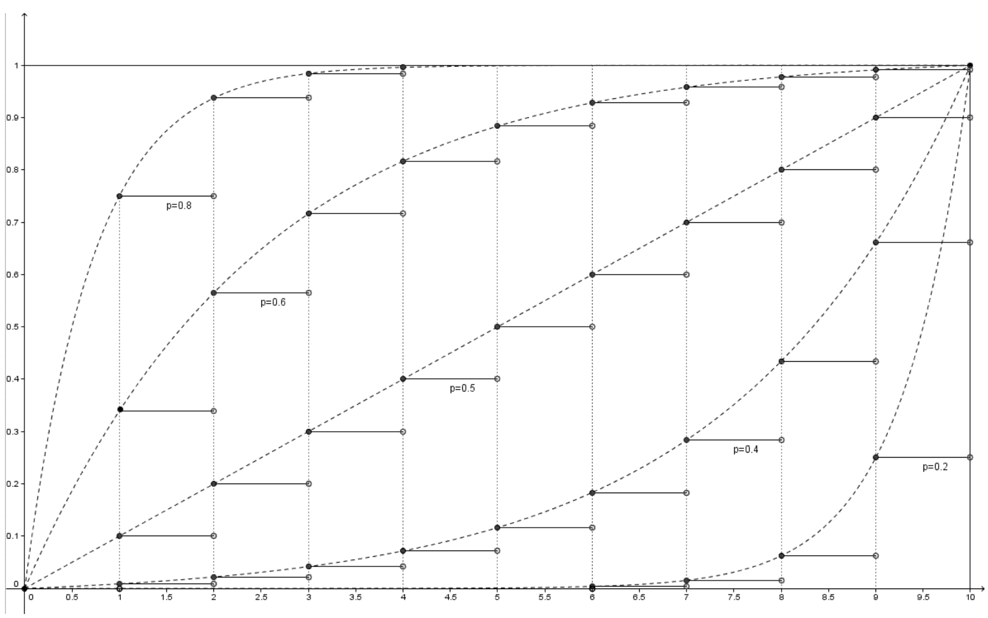

Figure 8.9: Probability of reaching the target fortune playing timid play strategy

8.2.4 Timid play with multiple pay-offs

We now consider the strategy of timid play where the unit bet game has multiple pay-offs (a probability table like the one for bold play). In this case S is still given by (8.4). Extending equation (8.5) we have that F is now the solution of

$$F(x) = \sum_{i=0}^{n-1} p_i F(x + m_i - 1) + qF(x - 1), \tag{8.7}$$

with boundary conditions $F(0) = 0$, $F(a) = 1$. We now give the setting to obtain the functional relation (8.7) whose solution is function $F(x)$.

Available in stake n	Event and probability	Available in stake $n+1$
$0 < x < a$	wins with probability p	$x + m_i + 1$
$0 < x < a$	loses with probability $q = 1 - p$	$x - 1$

Table 8.3: Timid Play with multiple pay-offs: possible stakes - bet and pay-off

The scratch cards game has a probability table for pay-offs similar to bold play with multiple pay-offs. The rules, however are different. The amount of bet is always the same for each scratch card. It is not allowed to spend all the money in a single stake (a card). The spirit of the bold play strategy is invalidated, because when spending a certain amount in this game (more than one card at the same time) there are more than one simultaneous pay-offs which clouds the outcome of the game.

The game of scratch cards is suitable for a strategy of timid play, where a fixed amount is played in each round (the price of one scratch card). The reason is that one can buy one card at a time, and this is pursuing timid play strategy.

For this game rules see, respectively in the United Kingdom and Portugal:

https://www.national-lottery.co.uk/games/scratchcards/prizes

https://www.jogossantacasa.pt/web/JogarLotInst/

An example of table of pay-offs is Table 8.4 corresponding to a scratch card of United Kingdom.

Usually scratch card games have the minimum prize equal to the amount of bet, i.e., $m = 1$. In case of timid play, the problem formulation is still valid for this case. In the bold play case we considered $m > 1$.

8.2.5 Conclusion

In this chapter we review the classical strategies of bold play and timid play in a simple win/lose game. The distribution functions for the player reaching a goal satisfy certain functional equations. The solutions are explicitly constructed and graphically illustrated. The solutions have peculiar characteristics, namely they are singular functions.

Prize amount	Number of prizes at start of game	Approximate odds at start of game: (1 in:)
£3	2,991,733	10
£5	1,833,950	16
£6	579,022	51
£6 (Card 1 £3, Card 2 £3)	579,109	50
£10	579,109	50
£15	289,466	101
£15 (Card 2 £10, Card 3 £5)	193,024	151
£20	193,075	150
£20 (Card 1 £6, Card 2 £3, Card 3 £5, Card 4 £6)	193,057	150
£30	107,455	270
£31 (Card 1 £6, Card 2 £10, Card 3 £15)	72,321	401
£50	24,099	1,202
£50 (Card 1 £20, Card 2 £30)	24,065	1,204
£100	9,695	2,987
£100 (Card 1 £20, Card 2 £30, Card 3 £50)	7,205	4,019
£200	242	119,648
£200 (Card 1 £100, Card 2 £30, Card 3 £50, Card 4 £20)	233	124,269
£1,000	236	122,690
£1,350 (Card 1 £100, Card 2 £1,000, Card 3 £50, Card 4 £200)	75	386,062
£10,000	12	2,412,885
£300,000	8	3,619,328

Table 8.4: Bingo Red Game scratch card of United Kingdom

As mentioned, a theorem of Dubins-Savage [28] states that in the sub-fair case ($p < 1/2$), bold play strategy maximizes the probability that the player successfully reaches the goal. In return, for the gambling house, in this case timid play strategy is more favourable. Observe that, while the gambling house chooses the rules of the game (in general the sub-fair case), the player is the one who chooses the strategy. We observe that when the game has sub-fair rules, there is no winning strategy (i.e. the probability of win is greater than lose). An example of a winning strategy, for another game at casinos, is a counting cards strategy, for which some people, in special mathematicians, were able to make lots of money (see e.g. the book by Thorp [101] "Beat the dealer - a winning strategy for the game of twenty-one"). Nowadays it is no longer possible to use this strategy because casinos, for their own protection, use mixing card machines, to prevent the cards counting strategy to be effective in the game.

The multiple pay-off scenario is modelled in terms of functional equations for both strategies. The subject of explicit and graphic solutions is not developed here. It is conjectured that these solutions are particularly exotic, bearing in mind what are the solutions to the classic case, in special for the bold play strategy. In fact, none of the functional equations for the multiple pay-off scenario may be solved directly by the theory presented in this work and it is a possible future development.

* Original terms (not found in the literature).

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