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Dice Questions Answered

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THE GATTON ACADEMY of Mathematics and Science

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Abstract:

Superstitious discussion of fair and unfair dice has pervaded the tabletop gaming industry since its inception. Many of these are not based on any quantitative data or studies. Consequently, misconceptions have been spread widely. One dice float test video on Youtube currently has 925,000 views (Fisher, 2015a). To combat the flood of misconceptions we investigated the following questions: 1) Are dice cursed? 2) Are D20s (20-sided dice) less fair than D6s (6-sided dice)? 3) Do float tests tell anything about the fairness of dice? 4) Are some dice systems inherently fairer than others? 5) Are density differences or dimensions more critical to dice fairness? 6) What is the best way to test your dice for fairness? 7) How many rolls are needed to detect unfair dice? 8) Are metal dice fairer than plastic dice? Based on tens of thousands of physical dice rolls, billions of simulated dice rolls, and analysis our answer to these questions are as follows. 1) Probably plastic dice are cursed. 2) Yes, D6s are fairer than D20s. 3) Float tests tell you nothing about which side of a die will come up more often. 4) Yes, some dice systems are fairer. 5) Usually dimensions are more important except for large, off-center bubbles. 6) The running chi square goodness of fit test is the best way to test dice that we found. 7) 100 rolls are not enough except possible for loaded dice. 8) Our preliminary conclusion based on limited tests is that metal dice are not fairer than plastic dice.

Background:

Tabletop game industry sales are expected to reach $12B by 2023 (O'Connell, 2019). Many or even most of these games use dice. Hasbro says it sells 50M copies of Yahtzee each year and each of these games have at least 5 dice (Wikipedia). No die is fair and this leads to the suspicion that dice are cursed (Appendix C). Many of these citations are not founded on solid data or analysis and misconceptions abound. Campbell and Dolan (2019) addressed some of these misconceptions. We extend this work with more test data and more analysis. To make this accessible to the average gamer while still being complete, the analysis and some of the data are provided in the appendices. The questions addressed are the following.

1. Are dice cursed?
2. Are D20 dice less fair than D6 dice?
3. Do float tests tell you anything about the fairness of dice?
4. Are some dice systems inherently fairer than others?
5. Are density differences or dimension inaccuracies more important in causing dice unfairness?
6. What is the best way to test your dice for unfairness?
7. How many rolls are needed to detect unfair dice?
8. Are metal dice fairer than plastic dice?

The remainder of this report addresses these questions and provides the basis for answers to the questions.

1. Are dice cursed?

For our 2019 paper and now, we have physically rolled 32 dice 3000 times each. These dice were made of plastic, various metals, and even one made of a mineral. We have rolled D6s including Fate dice, a D7, a D12, and many D20s. Manufacturers included Chessex, Crystal Caste, Eldritch, Game Science, Koplow Games, Level Up dice, Metal Dice Games, Norse Foundry, and Q Workshop.

The most commonly used dice are made of plastic. From our tests of plastic dice an interesting pattern emerged. Of the plastic D20s from various manufacturers that we tested, 12 of 16 had more rolls of the numbers 1 through 10 than of rolls of 11 through 20. The probability of having 12 or more of 16 dice with more low rolls than high rolls of a fair die is 0.0384. That is, it would only occur on average once in 26 times that the die was rolled 3000 times.

Interestingly, of the 6 metal dice tested, 3 rolled low and 3 high. We have no explanation of why plastic dice would roll low while metal dice apparently do not. For a fair D20, the expected or average value of a roll is 10.5. Of the 23 D20s tested 18 had an average roll less than 10.5. The probability of 18 or more out of 23 would only happen once in 189 times.

Finally, 17 of 23 D20s tested unfair in 3000 rolls.

Our answer to the question "Are dice cursed?" is probably at least for plastic dice. The answer depends on your definition of cursed. We did not see average rolls less than 10. The lowest was 10.199.

2. <u>Are D20 dice less fair than D6 dice?</u>

We rolled six D6 dice 3000 times. Of those, 5 tested fair. Of twenty-three D20s tested only 6 tested fair in 3000 rolls. Cubes are easier to manufacture than the D20 icosahedron. Also, the faces of D20s are smaller than those of D6s with the same diameters. However, the manufacturing tolerances for D20s are not smaller than that of D6s. Based on our roll results and these considerations we conclude that on average D6 dice are fairer than D20s.

3. <u>Do float tests tell you anything about the fairness of dice?</u>

Campbell and Dolan (2019) answered this question by float testing dice that had been rolled 3000 times. They found no correlation between the face that turned up in a float test and the highest probability face of rolled dice. The answer is no.

4. <u>Are some dice systems inherently fairer than others?</u>

Figure 1 is the running chi square statistic for a metal Fate die. The 95 percent confidence value for a Fate die is about 6 and is indicated by the red line in the figure. That is, the probability that a fair die will obtain a value of 6 or above is 5 percent. The probability of a fair die producing a value of 15.666 is 0.0004. If you rolled a fair die 3000 times and then repeated that 10,000 times (total of 30,000,000 rolls) only 4 times would it produce a χ^2 value of 15.666 or larger.

Suppose we take that die and roll it 4 times and sum the results. Note that a Fate die has 2 sides with a plus, 2 blank sides, and 2 sides with a minus. A minus is counted as minus 1, and blank face as zero, and a plus side as plus 1. Summing 4 dice would give a total between -4 and +4. In the Fate Core system every dice roll involves rolling 4 Fate dice.

The die in Figure 1 was rolled 3000 times, -1 was obtained 936 times, zero 1101 times, and +1 963 times. Then the best guesses for the side probabilities are 936/3000 = 0.312 for -1, 0.367 for 0, and 0.321 for +1. Based on these assumed probabilities, the probabilities for rolling this die four times and summing the results are given by Table 1. Assume the die is rolled 4 times and summed 3000 times. Then the expected number of times you get each possible result is given in the 3rd column. Assuming the die is fair then the 4-dice χ^2 statistic is about 6.32 while the 95% confidence value is about 15.51 (8 degrees of freedom). Rolling 4 fair dice would have a value greater than 6.32 61% of the time. The die goes from extremely unfair to very fair when rolled in the Fate dice system.

What if this is a result peculiar to this die? As we roll and sum we approach the normal distribution (bell curve). What if the probability distribution was U-shaped distribution instead of the peak at 0 like this die. We repeated the process for a hypothetical die where 3000 rolls yields 1100 minus one results, 800 zero results, and 1100 plus one results. The χ^2 value for this die is 60. This would be an extremely unfair die. A fair die would have a value of 60 or greater only once in 2 billion times according to Excel. This assumes Excel calculates χ^2 accurately. The χ^2 value for this die is 23.9 versus the critical value of 15.5. It is not quite fair but it goes from occurring once in 2 billion times for a fair die to once in 426 times. It appears that the Fate dice system is much fairer than any system that uses a single die.

Our answer to question 4 is yes.

Table 1. 4 dice probabilities for the die of Figure 1.

Roll	p	N
-4	0.009475854	28.42756301
-3	0.04458511	133.7553285
-2	0.117663781	352.9913437
-1	0.199303372	597.910117
0	0.240196138	720.5884142
1	0.205052508	615.1575242
2	0.124549984	373.6499533
3	0.048555804	145.667413
4	0.010617448	31.85234304

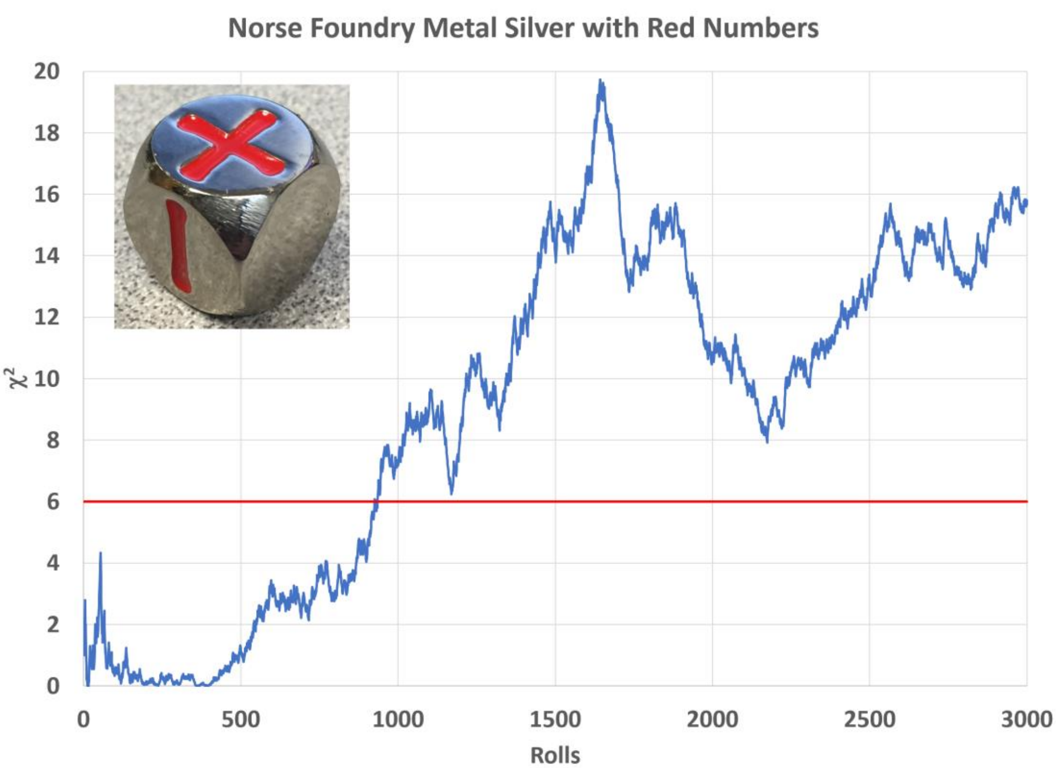

Figure 1. Running chi square statistic for an unfair Fate die.

5. Are density differences or dimension inaccuracies more important in causing dice unfairness?

Dice can be unfair for at least three reasons.

1. Density differences through the body of the die (bubbles),
2. Dimensional errors in manufacture of the die which causes diameter differences or differences in face areas,
3. A combination of the two.

A density difference such as a bubble under one face would make that side of the die lighter and cause the face near the bubble to come up more often and the opposite face less often. On the other hand, dimensional errors have the opposite effect. Think of rolling a brick. The roll that has the lowest center of gravity will occur most often., that is, landing flat. That means that these two opposite faces would occur most often when the brick lands flat. Landing on one of the sides would occur second most often and landing on end rarely. If dimensions are most important you would expect that the sum of observations of opposite faces would have a negative correlation with the corresponding diameter. That is, the short diameter sides would be observed more often than long diameter sides.

For a D20, opposite sides sum to 21. Side 1 is opposite side 20, side 2 is opposite 19, and so on. If density is more important we would expect a negative correlation between plots of high side and low side observations because the side closest to the bubble would tend to have a higher number of observations and the opposite side a lower number. If there were a negative correlation, the slope of the trend line would be negative, that is, decreasing from left to right. Figure 2 shows that is not occurring. This is a very unfair die. Figure 3 repeats the chart for a less unfair die. In neither case is the trend line's slope negative as would be expected if density differences were important.

If dimensions were important the diameters would be negatively correlated with the number of corresponding side observations. We carefully measured the 1-20 diameters, the 2-19 diameters, and so on and plotted them versus the sum of 1-20 observations, the 2-19 observations, and so on. The plot for the Koplow die is shown in Figure 4. Figures 2 and 4 were done for the same die. Figure 4 shows the expected negative correlation (negative slope of the trend line). For dice that tested fair in 3000 rolls, the trend line is very nearly flat as would also be expected.

Based on these results, we conclude that dimensional inaccuracies are more important to dice unfairness. This suggests that dice manufacturers should focus on dimensional inaccuracies to improve the fairness of their dice.

Campbell and Dolan (2019) presumed that a bubble would have a strong effect on dice unfairness. This was based on some analysis and on rolls of a D12 die 3000 times. However, that die was sold at a discount by Game Science. The conclusion that dimensional inaccuracies are more important is based on physical rolls of several dice. Bubbles occurring near a face would doubtless increase dice unfairness. However, preliminary tests of production dice lead to the preliminary conclusion that dimensions are more important.

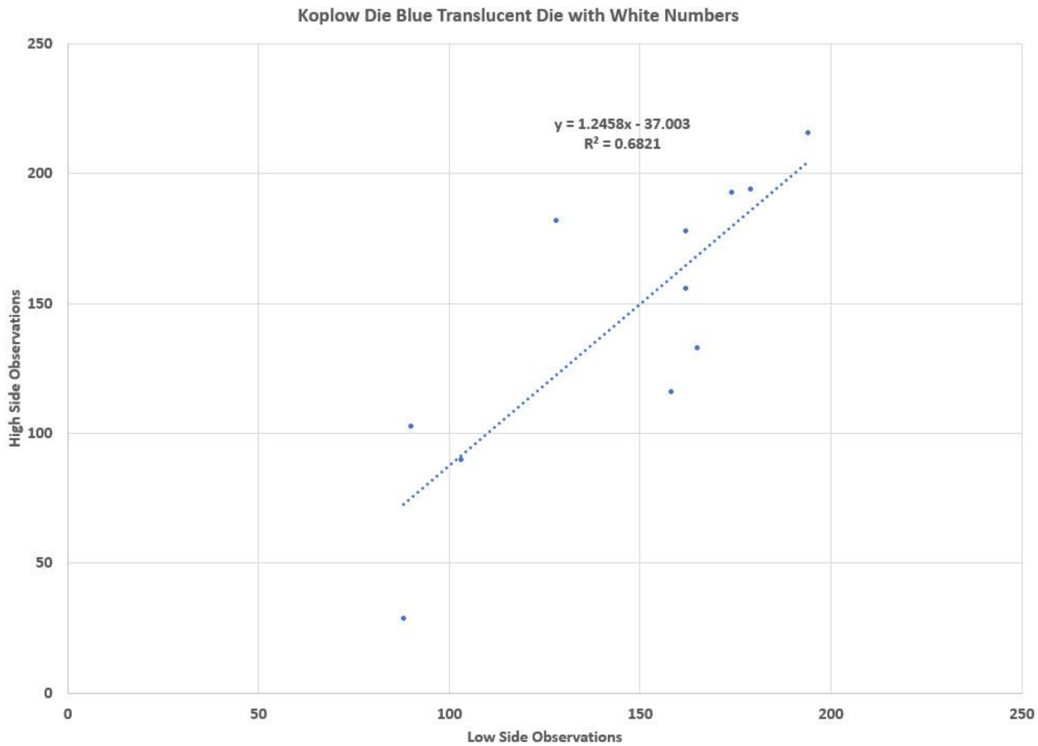

Figure 2. High side vs. low side observations for an unfair die (χ^2 = 268)

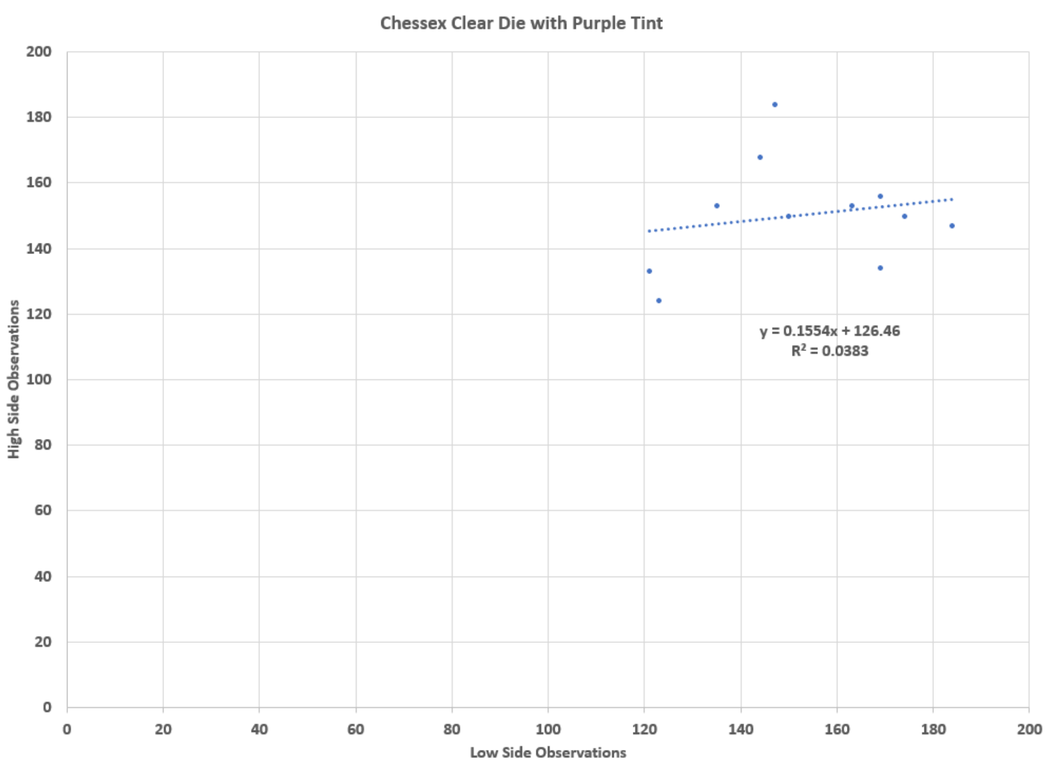

Figure 3. High side vs. low side observations (χ^2 = 40).

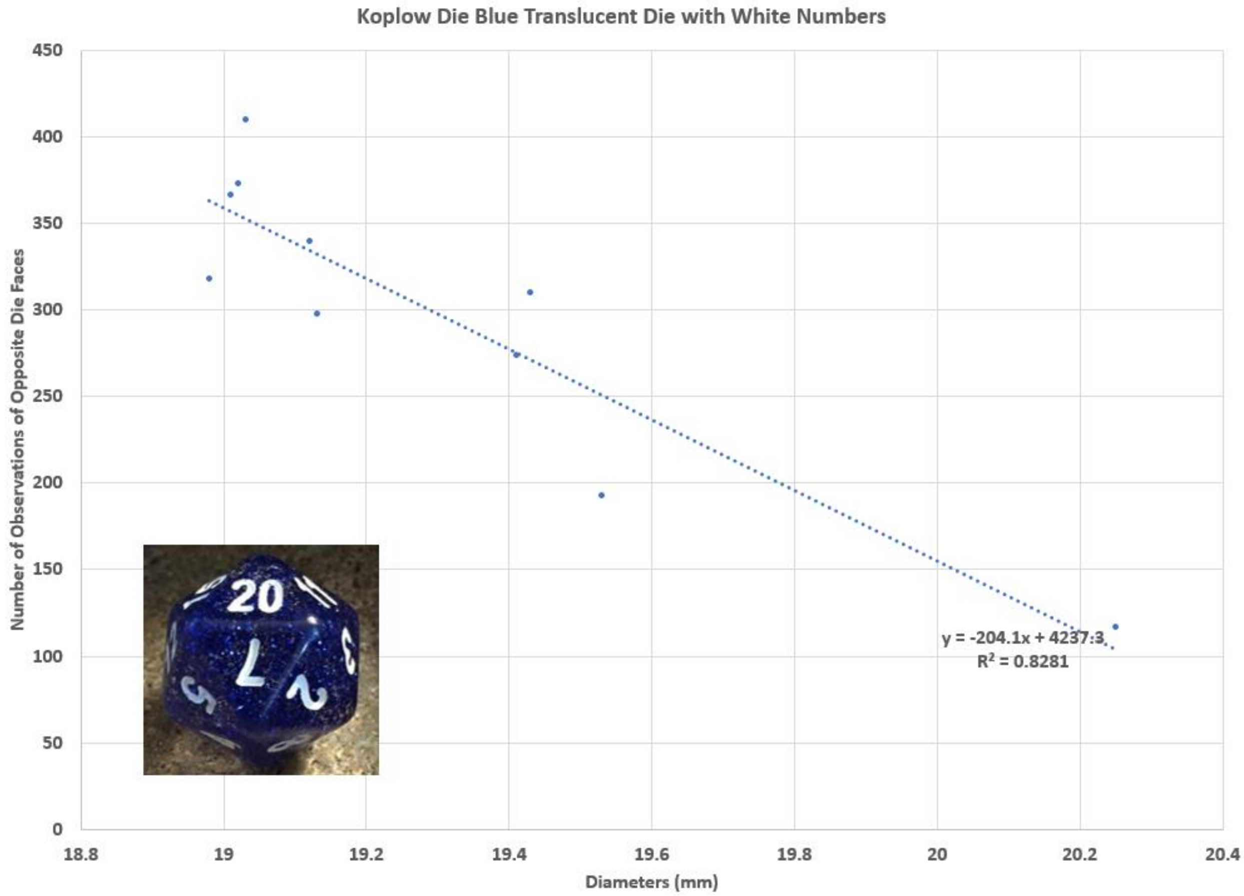

Figure 4. Low side-high side observations vs. corresponding diameters

6. <u>What is the best way to test your dice for unfairness?</u>

If you do not have the time or patience to roll dice a few thousand times, sometimes you can measure the variation in diameters. Dice that test fair in 3000 rolls generally have lower differences between the longest and shorter diameters. Figure 5 clearly demonstrates that. An investment in a good caliper can help you choose the fairest dice. x-axis values were determined by subtracting the minimum diameter from the maximum diameter. The y-axis values are the χ^2values after rolling the dice 3000 times each.

Statistically, we have found the running χ^2 to be our best guide to the fairness of dice. We experimented with a modification of the Kolmogorov-Smirnov test but it did not do as well as the running χ^2 test. By running χ^2 we mean that we calculate the χ^2 statistic for 1 roll, for 2 rolls, and so on up to 3000 rolls. It is easy to show (Appendix B) that for unfair dice the χ^2 statistic has a linear trend. The slope of the trend is given by Equation 1.

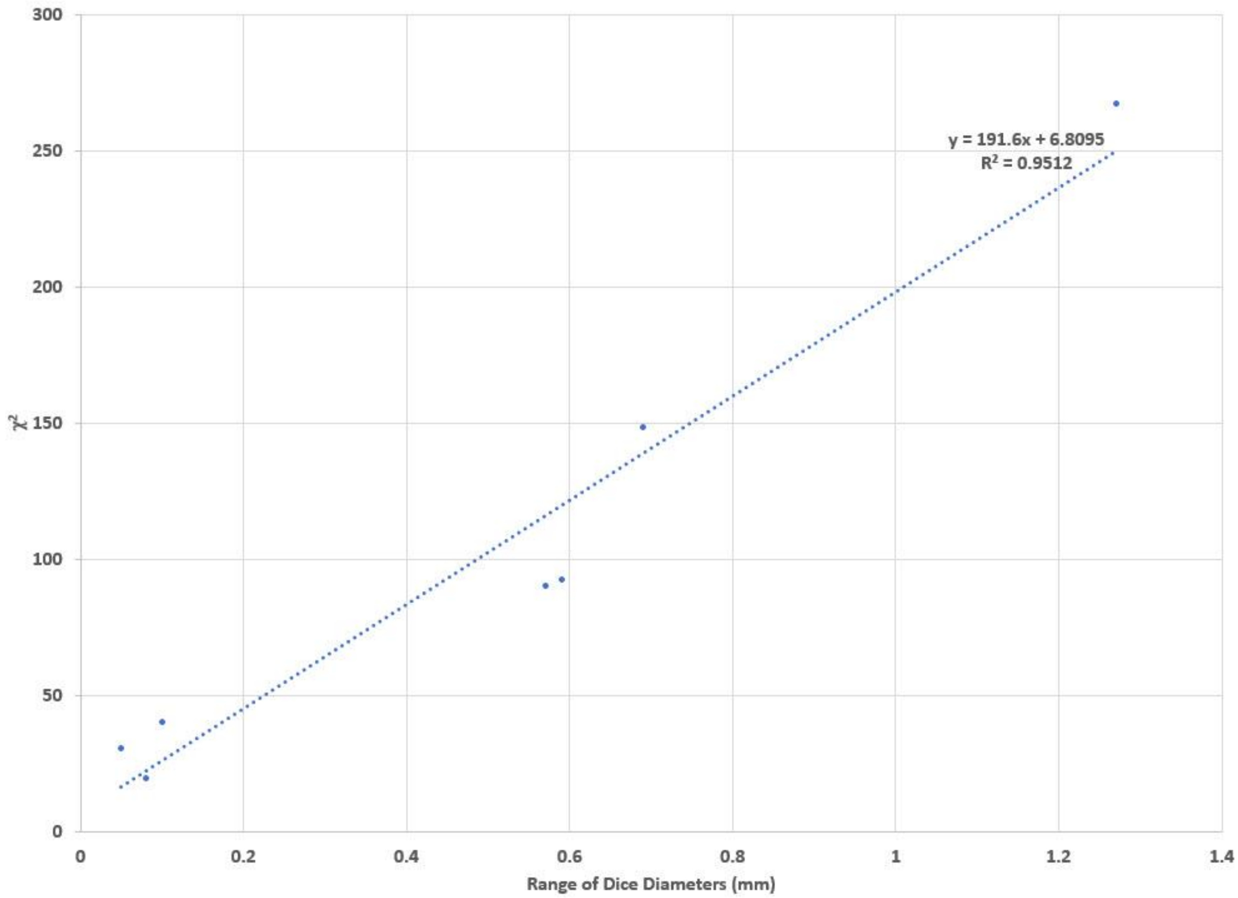

Figure 5. χ^2 vs. dice diameter ranges for D20 dice

$$Slope = s \cdot \sum_{i=1}^{s} \delta_i^2 \qquad (1)$$

$s =$ the number of sides of the die

$\delta_i =$ the difference between a face probability and the fair die face probability

For a D20 die, if the probability of rolling a one for an unfair die is 0.06, then $\delta_1 = 0.06 - 0.05 = 0.01$. If the probability of rolling a two is 0.04 then $\delta_2 = 0.04 - 0.05 = -0.01$ and so on. The $\sum$ symbol just means the sum over the number of sides of the die.

The running χ^2 statistic is calculated using Equation 2.

$$\chi^2 = \sum_{i=1}^{s} \frac{(O_i - E_i)^2}{E_i} \qquad (2)$$

$O_i =$ observations of the ith side of the dice

$E_i =$ expected number of observations of the ith face of the die

This statistic is calculated for one roll up through 3000 rolls. After 1 roll, the statistic is always $s-1$ so that for a D20, after one roll $\chi^2 = 19$. Unfair dice will have a linear trend that can be clearly seen in Figure 6. The more unfair the die the clearer the trend. The running χ^2 statistic for several D20 dice are given in appendix A. Others can be found in Campbell and Dolan (2019).

The best answer we have found to question 6 is the <u>running χ^2 test.</u> It is good because the running χ^2 test can show unfair dice at a glance. Lacking time or patience, <u>measuring diameters</u> of the dice can give an indication of the fairness or unfairness of a die.

While the best way we found to test dice unfairness is the running χ^2 test, our simulations seem to suggest that a distribution -based system has the potential to be more effective though the Kolmogorov-Smirnov method did not seem to offer any advantage over χ^2. This will be illustrated more clearly in the Summary.

7. <u>How many rolls are required to identify unfair dice?</u>

Our standard is 3000 rolls. We believe this is roughly the number of rolls constituting the lifetime of a die. Gamers buy other dice and begin using them. Rolling 3000 times can cause some wear to dice, but for the dice we rolled the most common effect we saw was the fading of dice numbers as the paint wore off. Using Equation 1 and the fact that the running χ^2 begins at 19 for D20s allows us to estimate how many rolls are required to determine a die of a given unfairness.

One Youtube video tested five D20s by rolling them 100 times each on a tabletop and then again in a dice tower (Fisher, 2015b) and concluded that the tower made all the dice fair. Every die we rolled was rolled in a dice tower and only 6 of 23 rolled fair. For a die to roll unfair in 100 rolls, it must be very unfair. As Figure 6 shows, there is a linear trend in the χ^2 statistic and the trend must take the die over the critical value to roll unfair.

The die shown rolls fair until about 250 rolls. Others in Appendix A stay fair for hundreds of rolls. Figure 7 was developed by solving for the number of rolls when the trend crosses the critical χ^2 value. For 100 rolls, the die would have to have an average error for each face of 33 percent. This is a very unfair die which is difficult to achieve unless the die is loaded. The error is proportional to $\frac{1}{\sqrt{N}}$. 3000 rolls can be used to detect an average error of only 6 percent. Many D20s will have this much error.

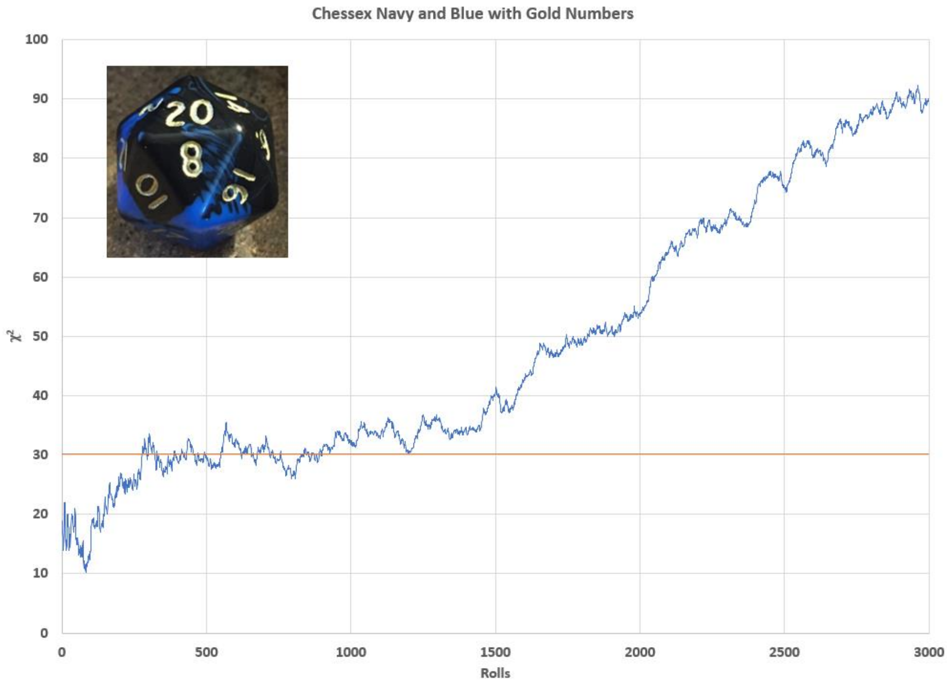

Figure 6. Running χ^2 showing linear trend

Another approach (see Delta's D&D Hotspot, 2011) makes the case for the number of dice rolls in a different, but very convincing way. He explains why using 100 rolls of a D20 (Fisher, 2015b) tells you nothing about the fairness of a die.

A third way to illustrate this is the power curve for dice. This is the probability that you would identify a dice of a given unfairness in a given number of rolls. Figure 8 shows this. The curve was developed by performing 10,000 simulations with different numbers of rolls. We drew 20-dimensional probabilities from the Dirichlet distribution, and multiplied those probabilities by different numbers of rolls and then calculated the χ^2 statistic for each. Appendix D shows how the random variates were generated. We will show in the Summary that this statistic is not χ^2 distributed when the die is unfair.

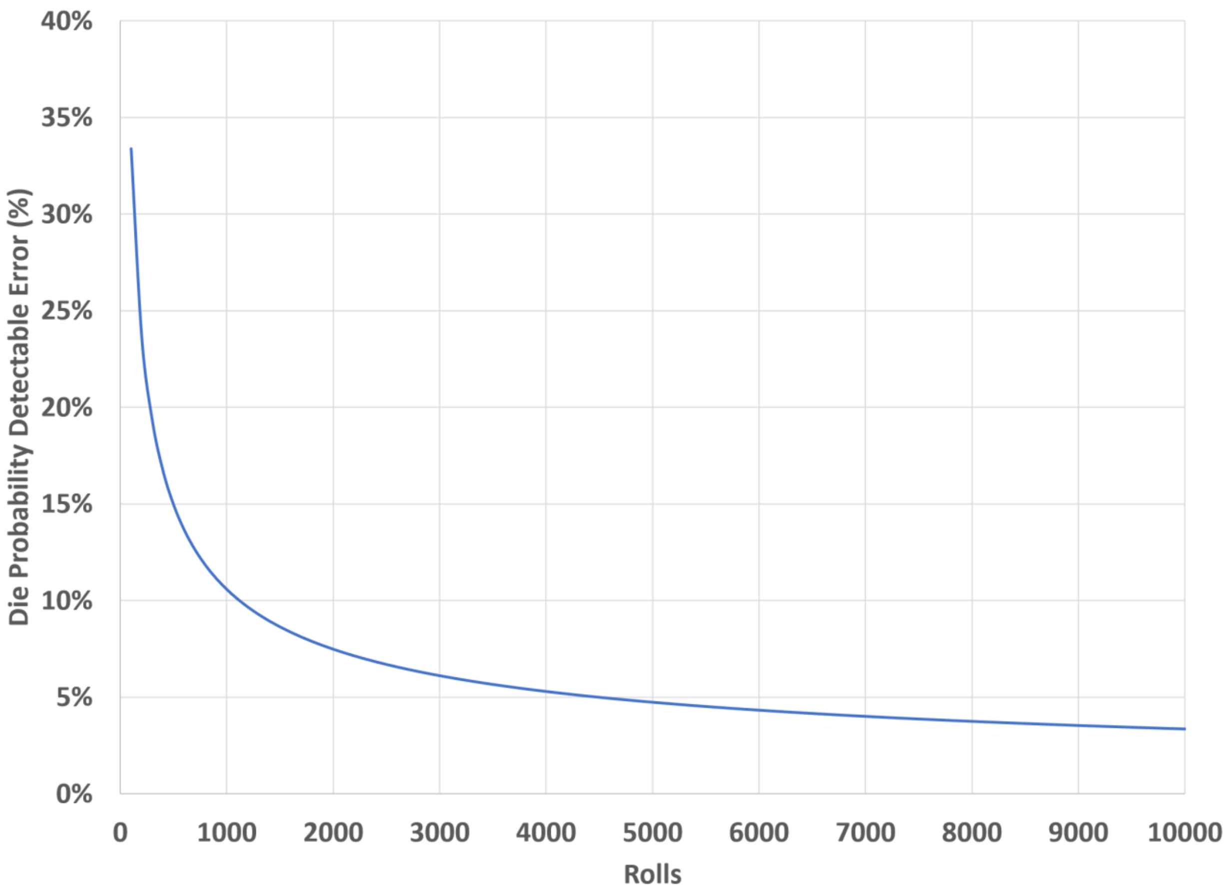

Figure 7. Detectable error for a D20 after N rolls

The power curve was developed for a very unfair die (χ^2 = 267.6), a moderately unfair die (χ^2 = 46.76) and an almost fair die (χ^2 = 33.73). The more unfair the die the farther the curve moves to the left because it takes fewer rolls to identify an unfair die. The figure shows that even for an unfair die you need more than 200 rolls to have any hope of identifying it as unfair. It takes 350 rolls to reach a 50 percent chance of detecting an unfair die. For the fairest die the probability of detection reaches 50 percent at 1750 rolls.

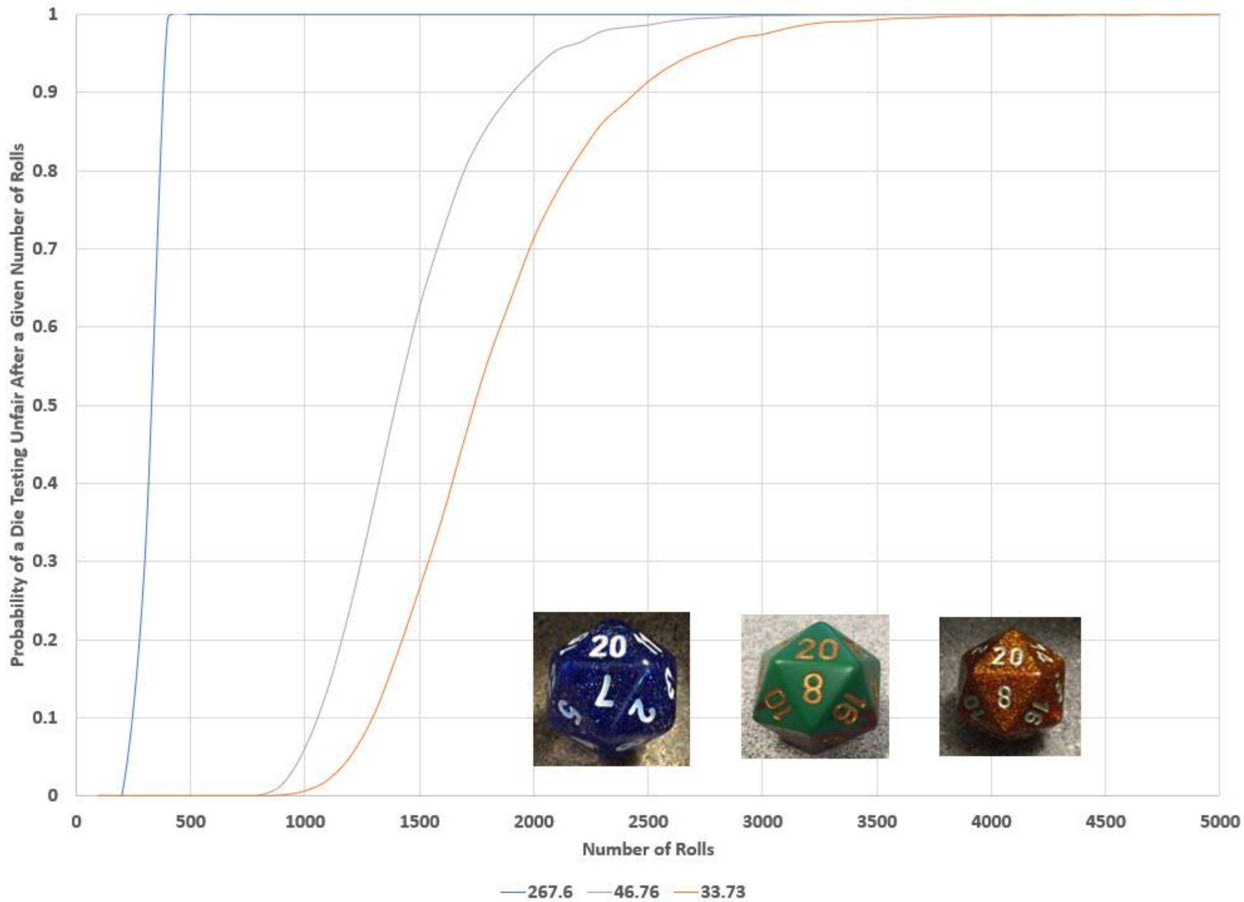

Figure 8. Power Curve as a Function of Number of Rolls

8. <u>Are metal dice more fair than plastic dice?</u>

Of the 6 metal D20s only one tested fair after 3000 rolls. This compares with only 5 of 16 plastic dice that tested fair. We tested five D6s (including Fate dice) and the four plastic dice all tested fair. The one metal Fate die tested unfair. The only advantage metal dice seem to have is that only 3 of 6 metal D20s were low rollers (more 1 – 10 than 11 – 20). This compares with 12 of 16 plastic dice that were low rollers. The average χ^2 for metal dice after 3000 rolls was 56 and for plastic dice 57, a statistically insignificant difference.

While 50 percent of the metal dice tested were low rollers and 75 percent of the plastic dice were low rollers, we still conclude that <u>metal dice are not more fair than plastic dice.</u>

Summary

A running χ^2 statistic has a linear trend if it is unfair. The slope of the trend is given by Equation 1 and is derived in Appendix B. Figure 9 is the running χ^2 statistic for a die that tested only slightly unfair in 3000 rolls. It was chosen because it was only slightly unfair so that Figure 10 could be generated. Had we used a more unfair die it would have been difficult to plot the histograms on a single chart. The statistic for a fair die asymptotically approaches a χ^2 distribution. However, if the die is unfair, the distribution of the statistic based on fair expected values is not χ^2 distributed.

Our best guess for the face probabilities for the die of Figure 8 was the number of observations of each face divided by 3000. The face probabilities are distributed according to the Dirichlet distribution. To generate Figure 10, we simulated 1000 20-dimensional random variate probabilities drawn from the Dirichlet distribution and multiplied the probabilities by 3000 rolls. We used that to calculate the chi square values in 2 ways. First we assumed fair dice so the expected observations for each face were 150. We made a histogram of these 1000 χ^2 values and that gives the blue bars on the chart. Secondly, we assumed the expected values were the actual dice side observations from the 3000 physical rolls. The histogram for these are given by the gray bars on the chart. Then we used the χ^2 distribution with 19 degrees of freedom to generate the orange bars.

The differences between distributions assuming a fair and an unfair die are stark. There is very little overlap between the histogram that assumed a fair die and the one that assumed an unfair die. The differences in the histogram for a more unfair die are even more stark. This suggests that there might be a distribution-based method to distinguish unfair dice from fair. This suggests a Kolmogorov-Smirnov type test, but we did not find any advantage to the K-S test.

Finally, we answered the 8 questions posed in the beginning of the report. Plastic dice may indeed be cursed because low rolling dice appear to be more common than high rollers. D20 dice had a lower occurrence of fairness in 3000 rolls than D6 dice. To better answer this question, we would need to repeat it for each manufacturer to see that any effects are not manufacturer dependent.

Float tests do not appear to tell us anything about which side of the die is most likely to come up. Campbell and Dolan (2019) found no correlation between the side that came up in float tests and the side that came up most often in physical dice rolls.

We provided a chart (Figure 7) that allowed an estimate of how many rolls are required to test a die of a given unfairness. 100 rolls are too few except for the most unfair dice.

Metal dice do not appear to be more fair than plastic dice, though more testing is needed to verify preliminary results.

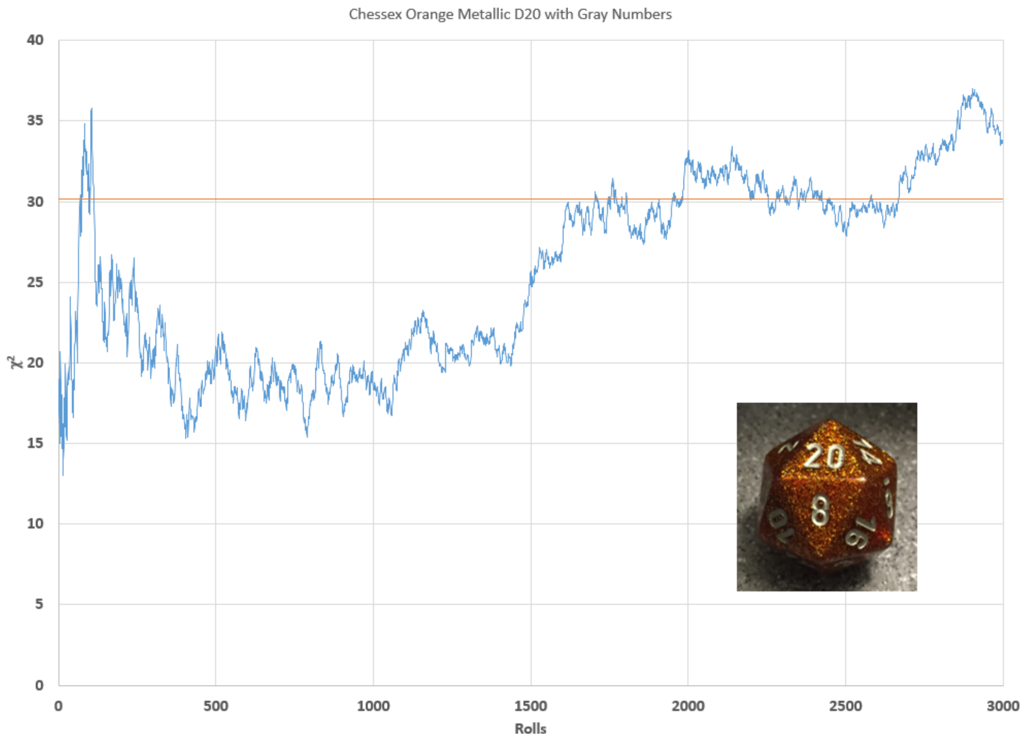

Figure 9. Die that tests slightly unfair after 3000 rolls

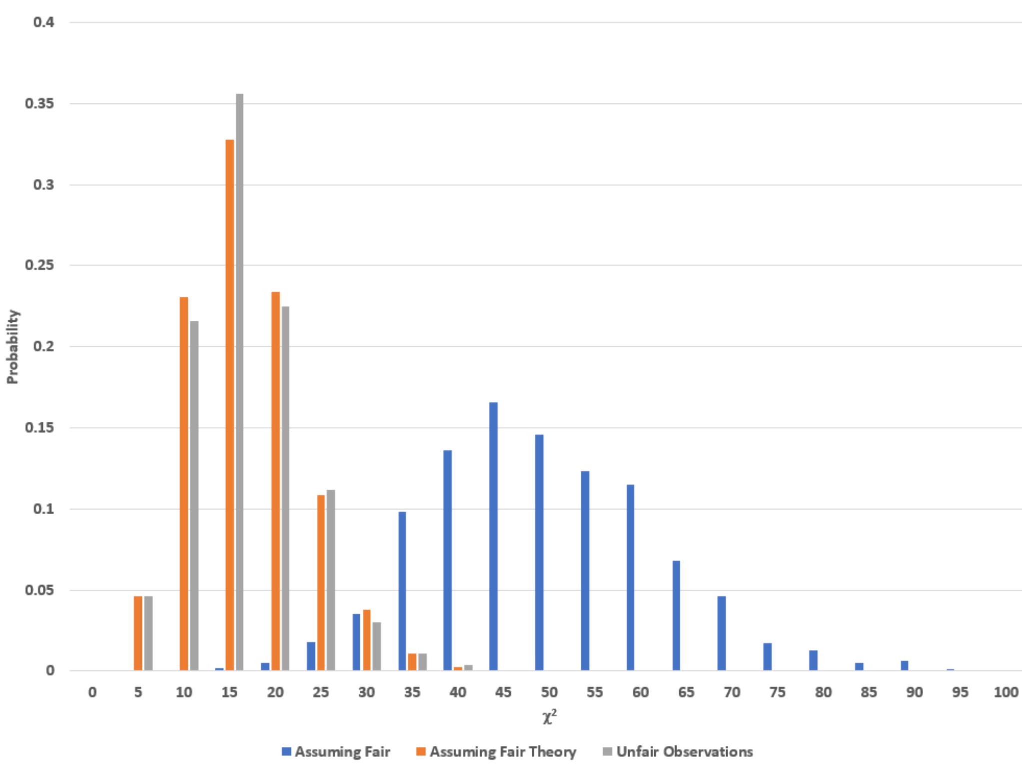

Figure 10. Distributions of χ^2 assuming a fair die and assuming it is unfair

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Appendix A. Running χ^2 Statistic for Dice

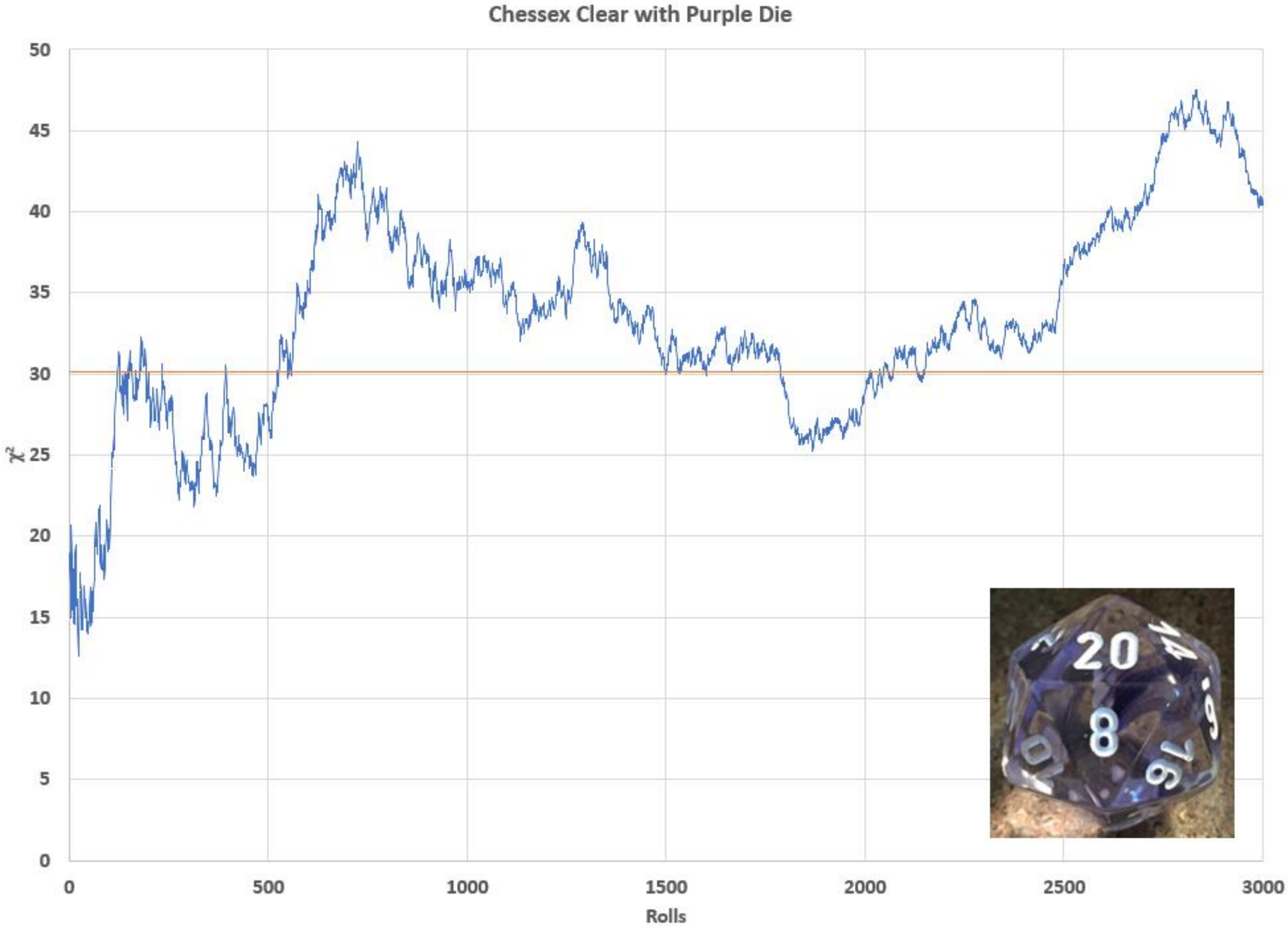


Figure A.1

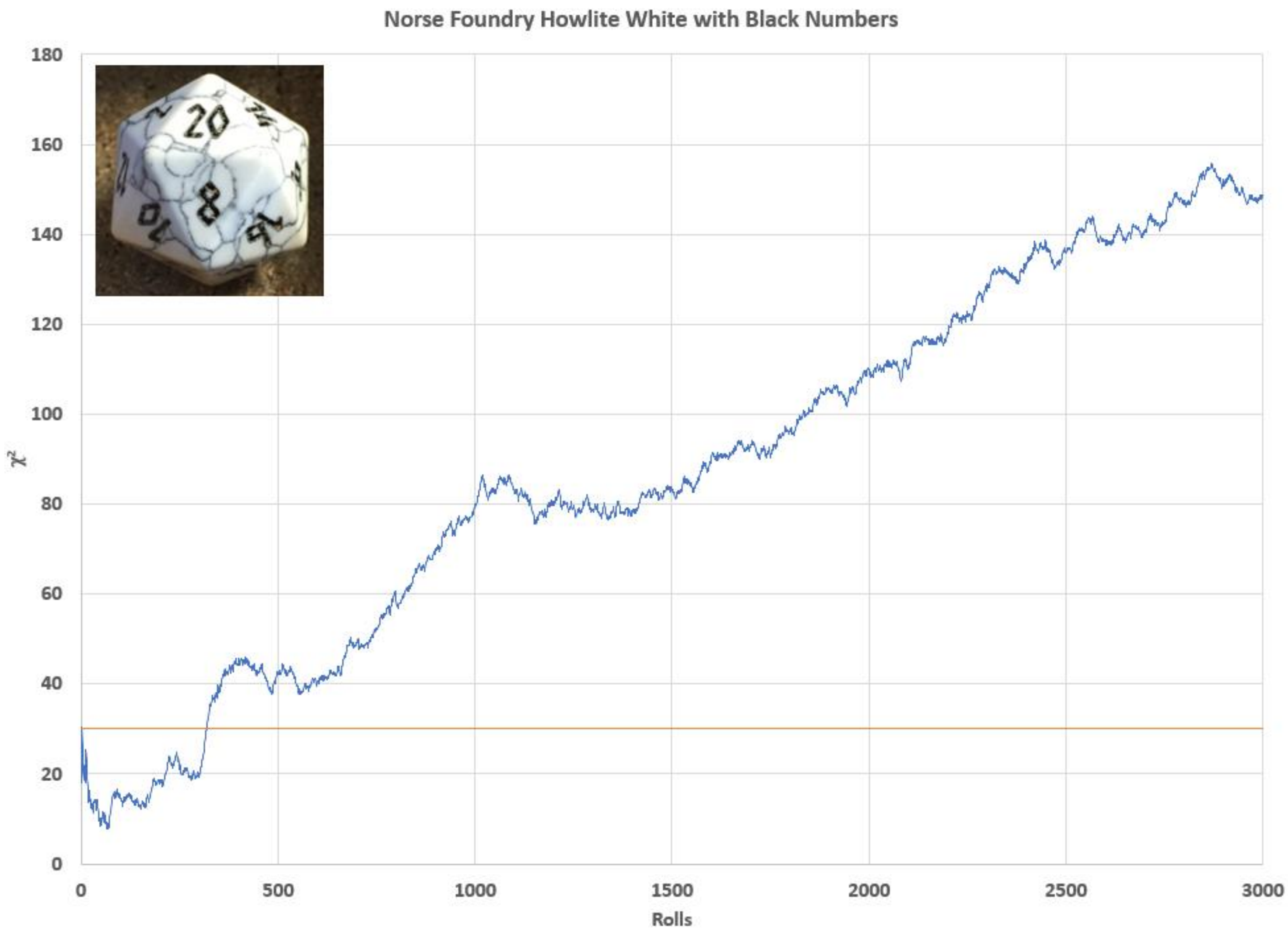


Figure A.2

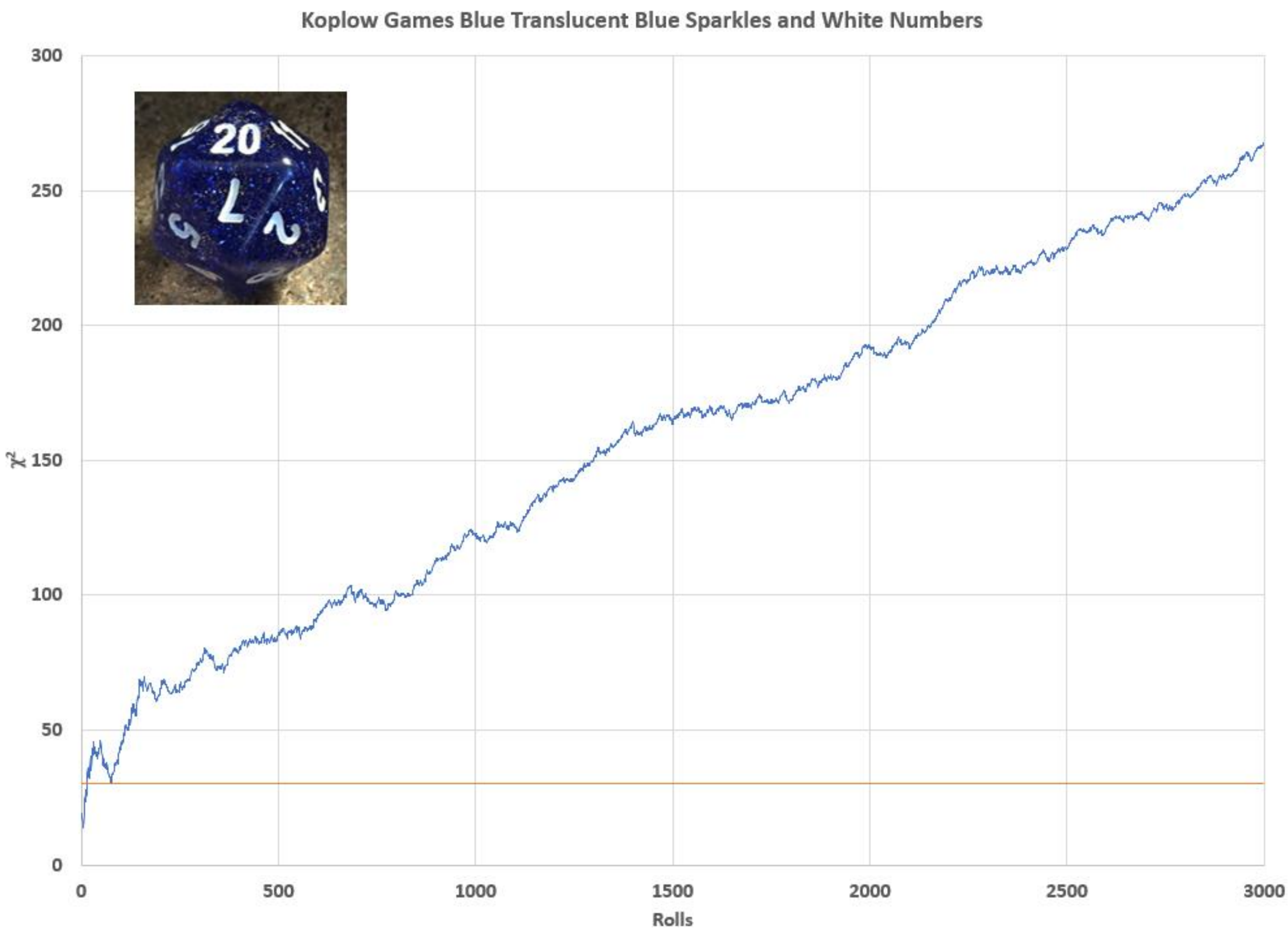

Koplow Games Blue Translucent Blue Sparkles and White Numbers
300
250
200
150
100
50
0
χ^2
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.3

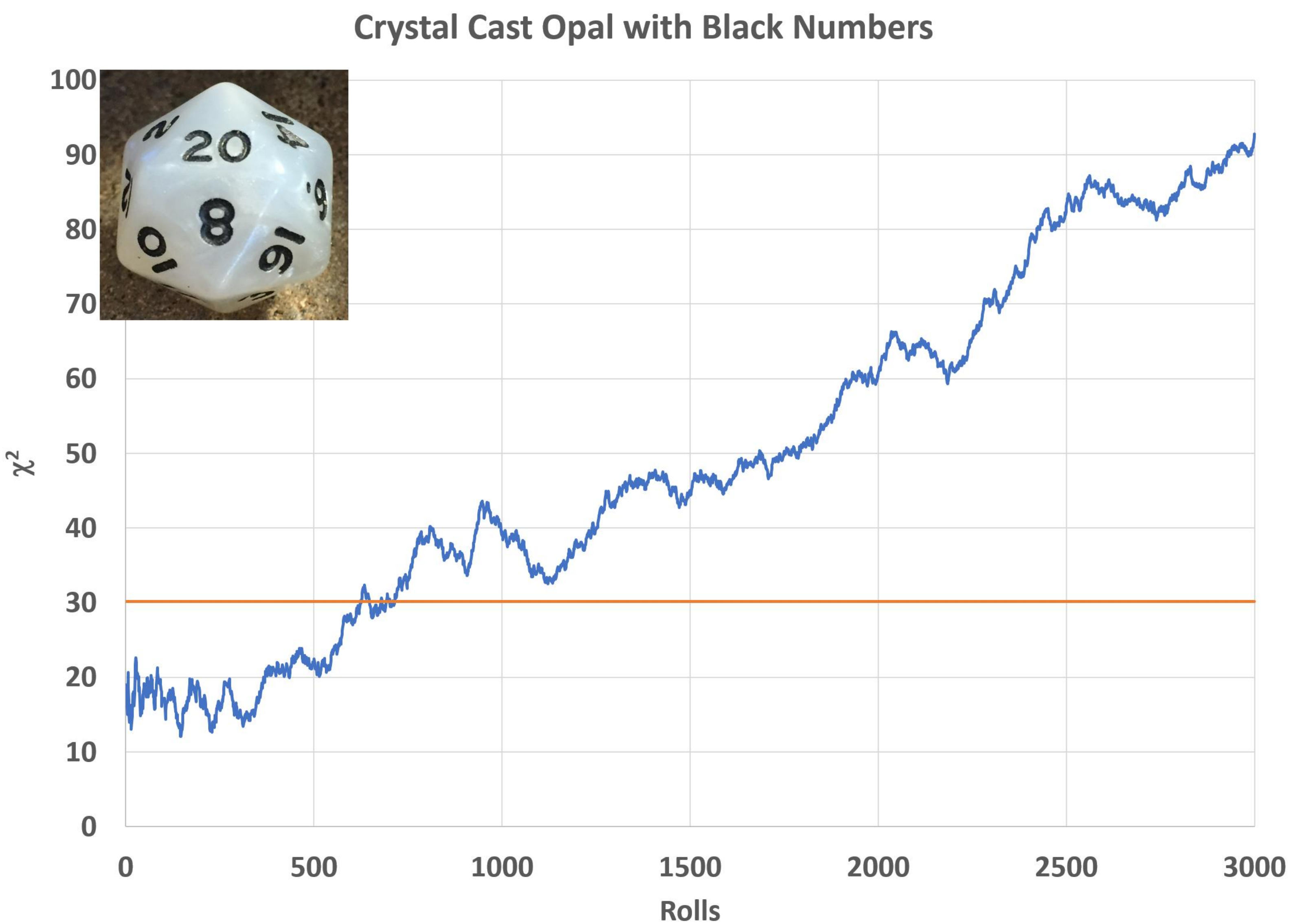

Crystal Cast Opal with Black Numbers
100
90
80
70
60
50
40
30
20
10
0
χ^2
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.4

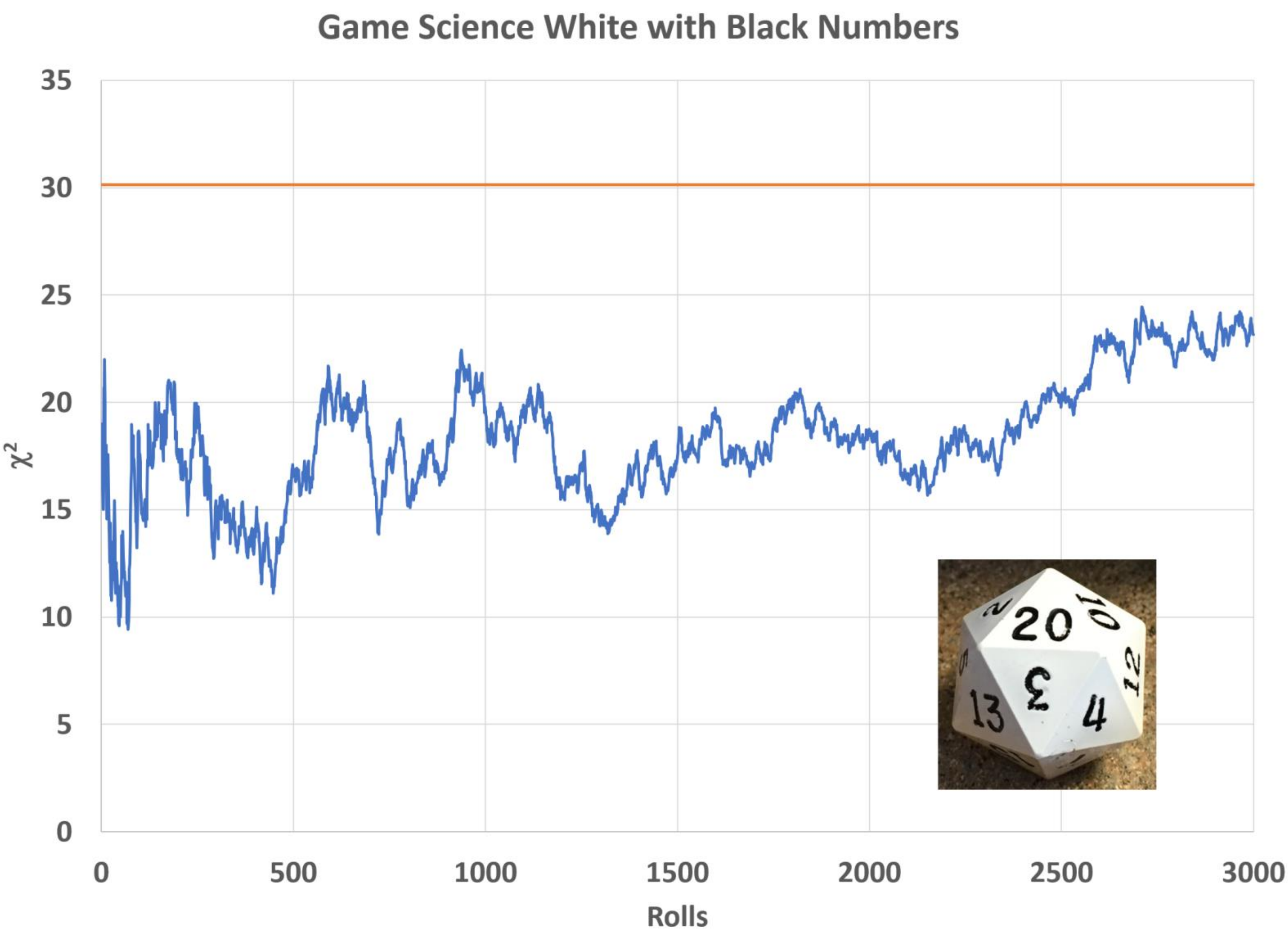

Game Science White with Black Numbers
35
30
25
20
15
10
5
0
χ²
0
500
1000
1500
2000
2500
3000
Rolls
20
10
12
3
13
4

Figure A.5

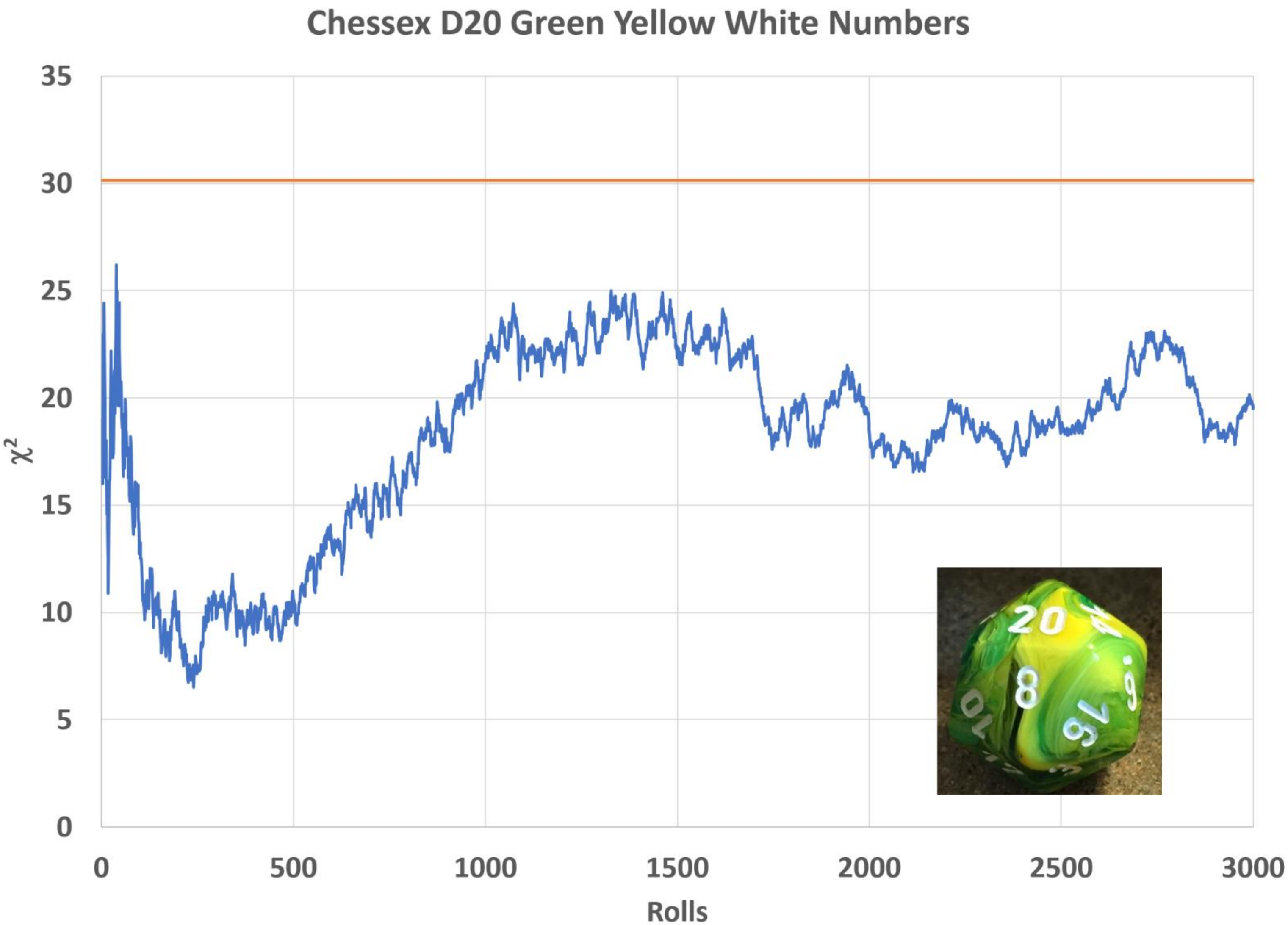

Chessex D20 Green Yellow White Numbers
35
30
25
20
15
10
5
0
χ^2
0
500
1000
1500
2000
2500
3000
Rolls
20
8
16

Figure A.6

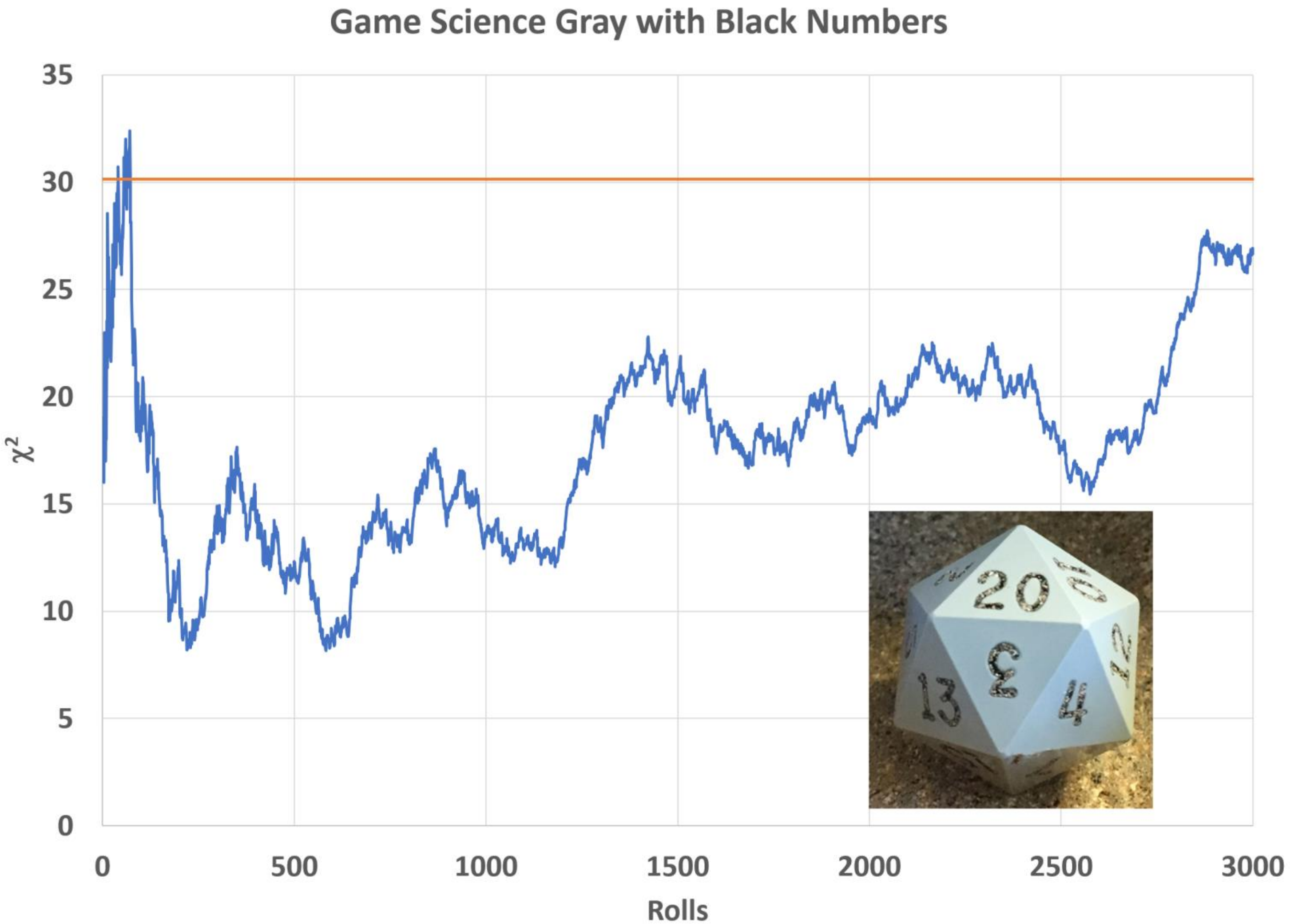

Game Science Gray with Black Numbers
35
30
25
20
15
10
5
0
χ²
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.7

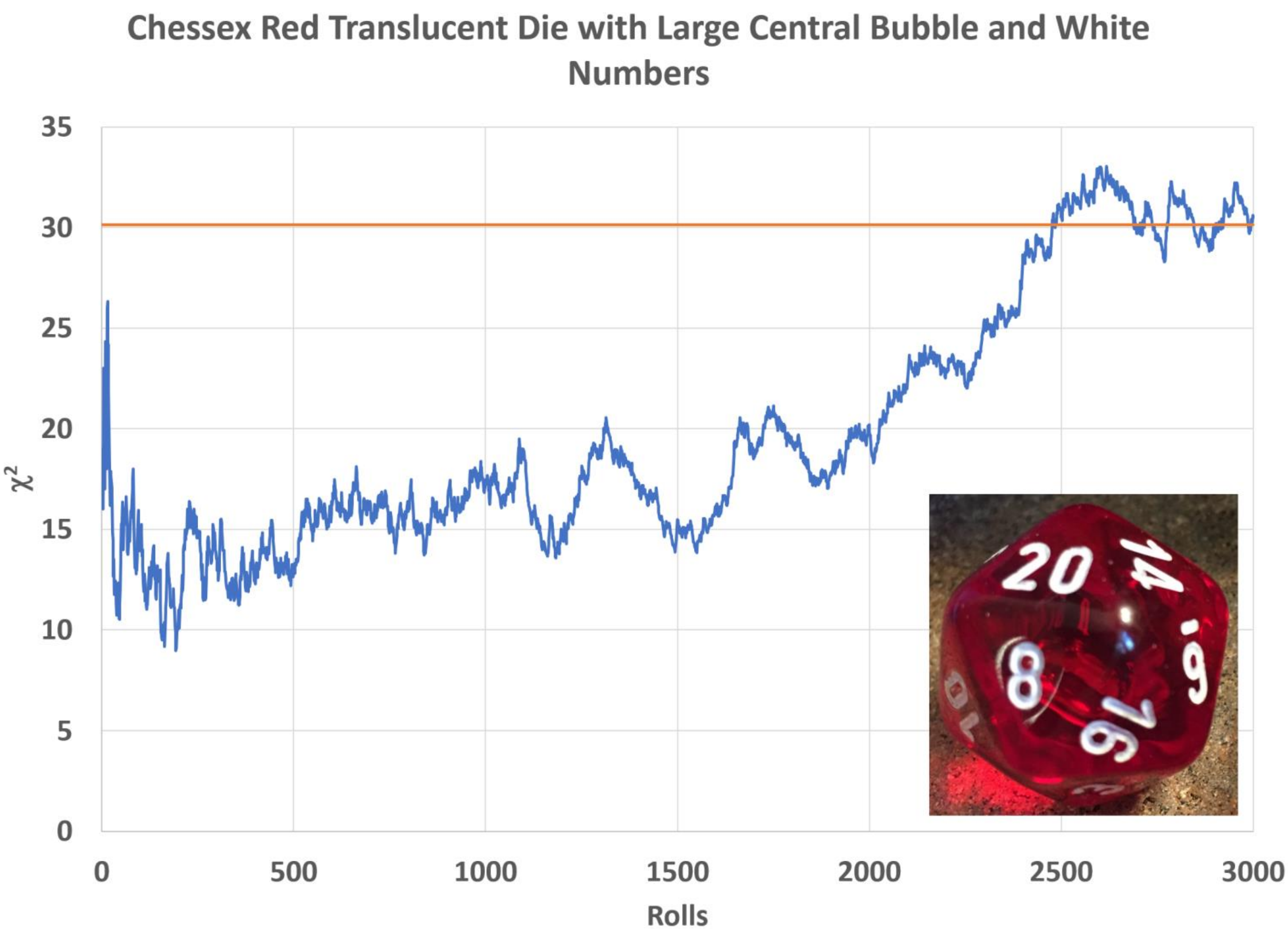

Chessex Red Translucent Die with Large Central Bubble and White Numbers
35
30
25
20
15
10
5
0
χ^2
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.8

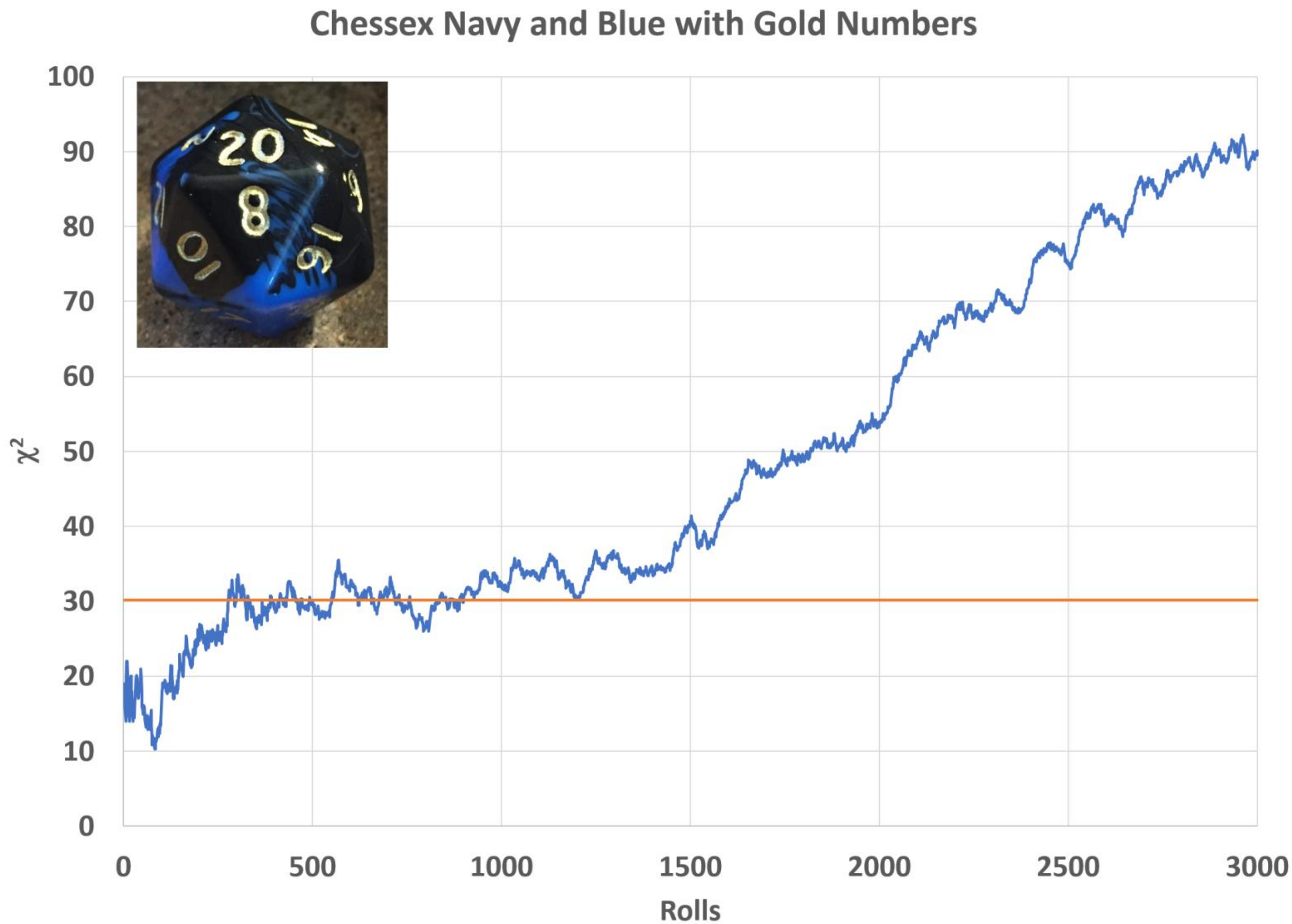

Chessex Navy and Blue with Gold Numbers
100
90
80
70
60
50
40
30
20
10
0
χ^2
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.9

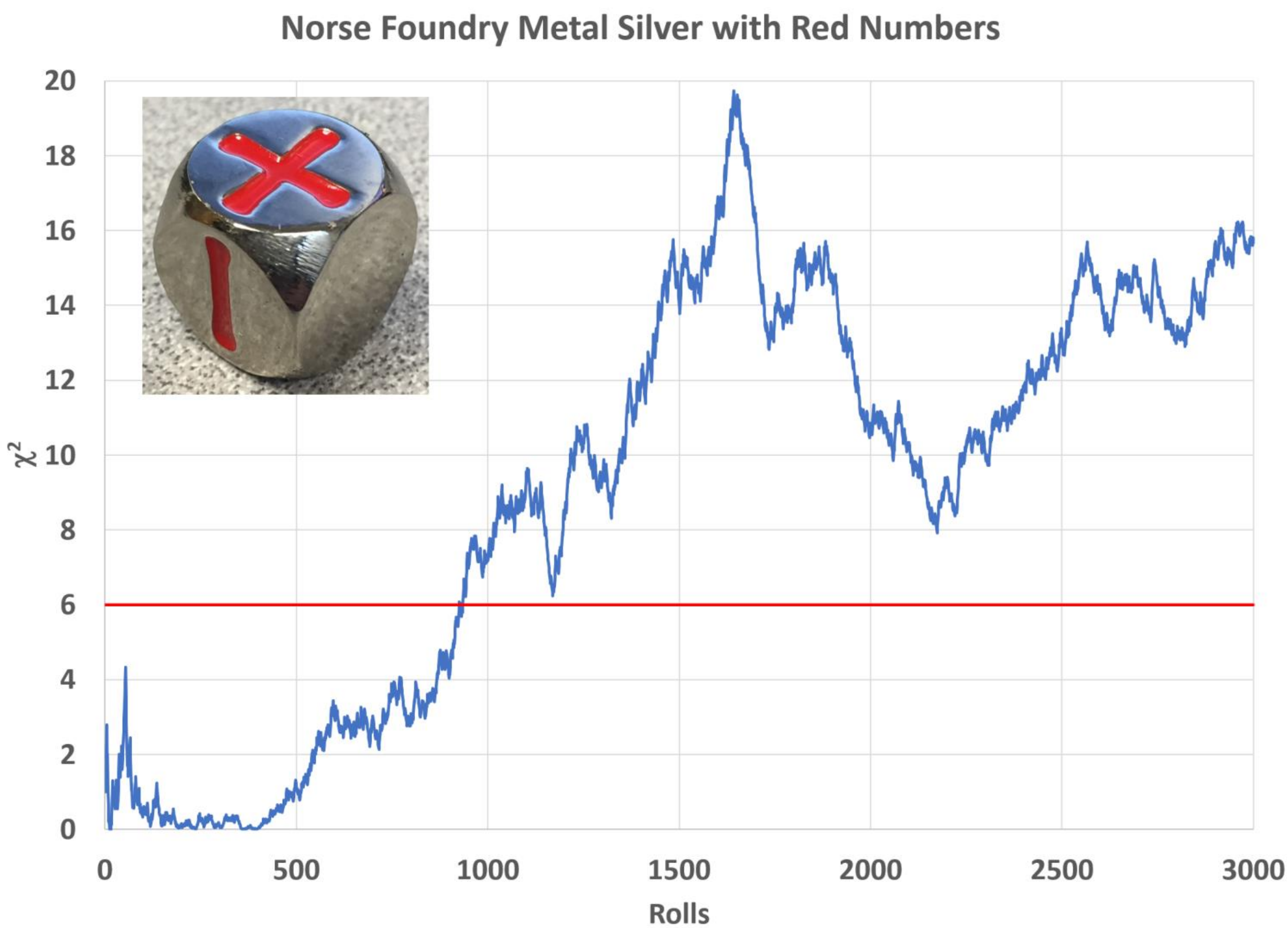
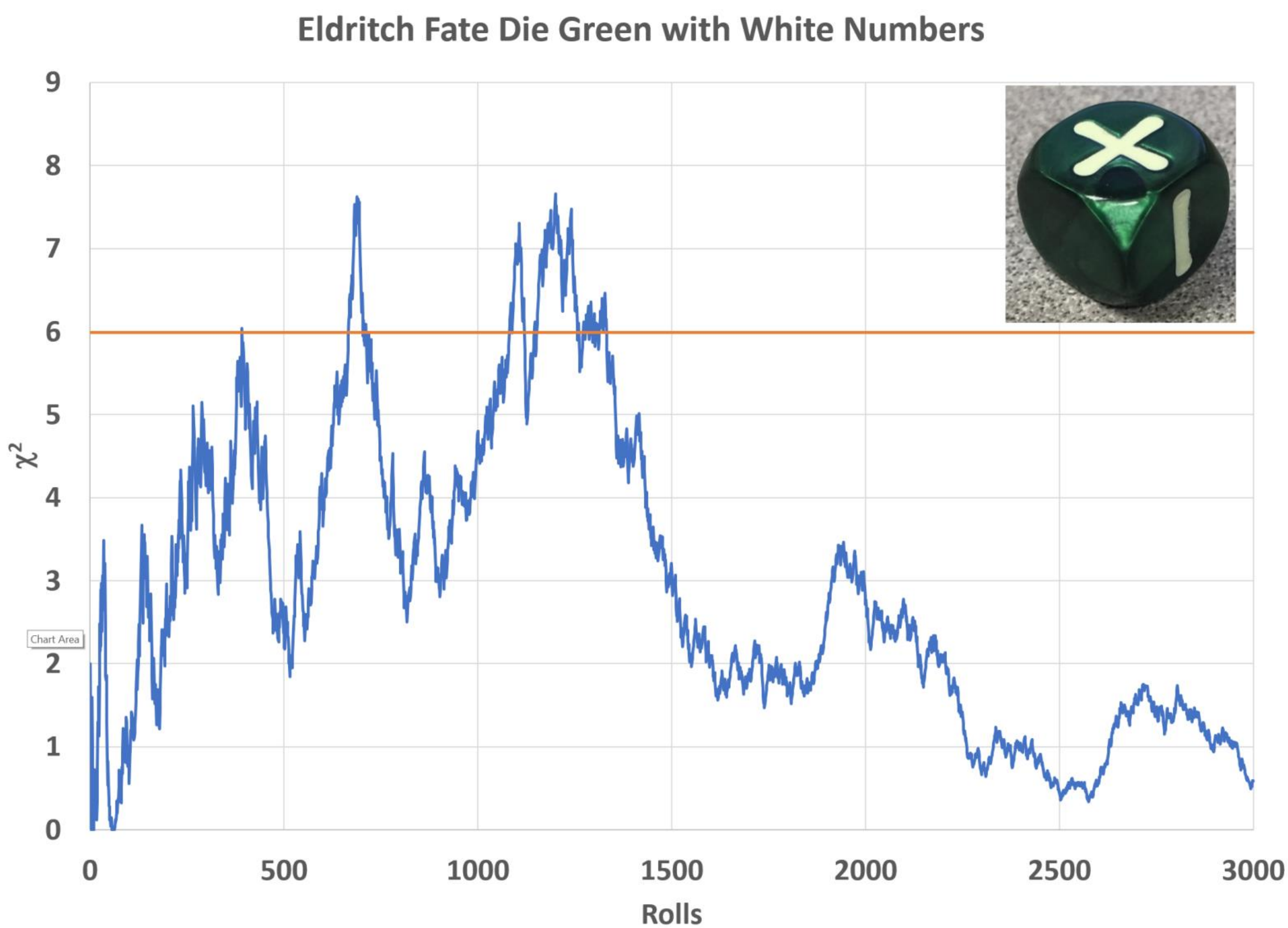

Norse Foundry Metal Silver with Red Numbers
20
18
16
14
12
χ^2 10
8
6
4
2
0
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.10



Eldritch Fate Die Green with White Numbers
9
8
7
6
5
4
3
2
1
0
χ^2
Chart Area
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.11

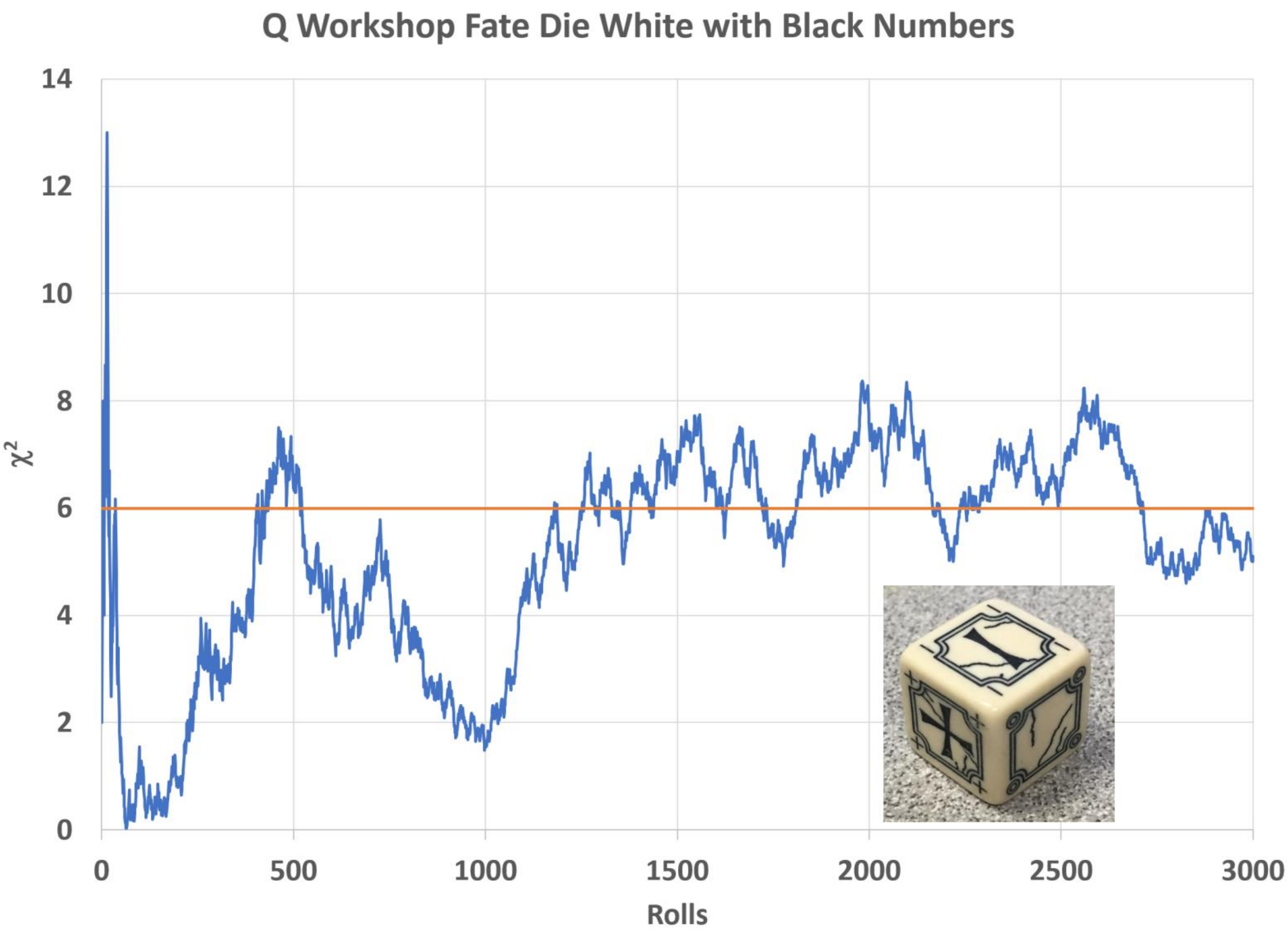

Q Workshop Fate Die White with Black Numbers
14
12
10
8
6
4
2
0
χ^2
0
500
1000
1500
2000
2500
3000
Rolls

Figure A.12

Appendix B. Linear Trend in χ^2 Statistic of Unfair Die

$$\chi_U^2 = \sum_{i=1}^{s} \frac{\left(x_i^2 + \frac{N}{s} + \delta_i \cdot N - \frac{N}{s} \right)^2}{\frac{N}{s}} = \sum_{i=1}^{s} \frac{x_i^2 + 2 \cdot x_i \cdot \delta_i \cdot N + \delta_i^2 \cdot N^2}{\frac{N}{s}}$$

$$\chi_U^2 = \chi_F^2 + \sum_{i=1}^{s} \delta_i^2 \cdot s \cdot N$$

$\chi_U^2 =$ unfair χ^2

$x_i =$ random part of the fair χ^2

$N =$ number of rolls

$s =$ number of sides of the die

$\delta_i = p_i - \frac{1}{s} =$ deviation of the face probability from fair

$p_i =$ ith face probability

$\chi_F^2 =$ fair χ^2

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Appendix D. Dirichlet Distribution

The Dirichlet distribution is closely related to the multinomial distribution. In statistical parlance it is the conjugate prior for the multinomial distribution. Mathematically, it is given by Equation D1.

$$f\left(p_1, p_2, ..., p_K; \alpha_1, \alpha_2, ..., \alpha_K\right) = \frac{\Gamma\left(\sum_{i=1}^{K} \alpha_i\right)}{\prod_{i=1}^{K} \Gamma\left(\alpha_i\right)} \cdot \prod_{i=1}^{K} p_i^{\alpha_i - 1} \quad (D1)$$

For our purposes the α_i are the observations of 1, 2, etc (faces of the die). Python SciPy has a function that can simulate drawing probabilities from the Dirichlet distribution.

```
rv = dirichlet.rvs(alpha, size = 1, random_state = None)*n
```

This line of code applied to a D20, would simulate a 20-dimensional vector of Dirichlet probabilities multiplied by the number of rolls *n* and place them in the variable rv. From that you have a single set of observations of dice faces after *n* rolls.

Wikipedia gives the method for generating random variates from the Dirichlet distributions. First you generate K gamma distribution distributions distributed as Gamma(α_1, 1), that is, as follows.

$$Gamma\left(\alpha_i, 1\right) = \frac{x_i^{\alpha_i - 1} \cdot e^{-y_i}}{\Gamma\left(\alpha_i\right)} \qquad (D2)$$

$$p_i = \frac{x_i}{\sum_{j=1}^{K} x_j} \qquad (D3)$$

The p_i are the Dirichlet-distributed 20-dimensional probabilities for a D20. You can also generate them in Excel using the inverse Gamma distribution. The following line in a cell generates the random variates in Excel.

=GAMMA.INV(RAND(), α_i ,1)

You must replace α_i in the formula with the 3000 roll observations for each face of the die α_i.

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